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A Characterization of Partial 3-Trees

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A k -tree is defined recursively as follows: The complete graph K_k on k points is a k -tree. Given a k -tree G on $n \geq k$ points, a k -tree on $n+1$ points is obtained by adding a new point u and edges connecting u to every point of a K_k in G . A graph is a *partial k -tree* if it is a subgraph of some k -tree. In this paper, we establish some interesting properties of partial 3-trees and show that a graph is a partial 3-tree if and only if it has no subgraph contractible to K_5 , $K_{2,2,2}$, $C_4(1,4)$, or $K_2 \times C_3$. A graph G is said to be *contractible* to a graph H if H can be obtained from G by a sequence of edge contractions. Hitherto, such a characterization of partial k -trees was known only for the values of $k \leq 2$.

1. INTRODUCTION

We are concerned with a class of graphs called k -trees and their subgraphs. The complete graph K_k on k points is a k -tree. Given a k -tree G on $n \geq k$ points, a k -tree on $n+1$ points is obtained by adding a new point u and edges connecting u to every point of a K_k in G . A graph is a *partial k -tree* if it is a subgraph of some k -tree. Rose [8, 9] first introduced and studied the k -trees. Subsequently, k -trees and their subgraphs received attention from several researchers [1–3, 7, 11].

In this paper, we establish several new and interesting properties of partial 3-trees. We show that a graph is a partial 3-tree if it has no subgraph contractible to any of the four graphs illustrated in Figure 1. A graph G is said to be *contractible* to a graph H if H can be obtained from G by a sequence of edge contractions. Hitherto, such a characterization of partial k -trees was known only for the values of $k \leq 2$. Notice that a 1-tree is just a tree in the usual graph-theoretic sense, and therefore a partial 1-tree is just a forest. Hence, it is obvious that a graph is a partial 1-tree if and only if it has no subgraph contractible to the complete graph K_3 on 3 points. Also, it is well known that partial 2-trees are series-parallel graphs. Duffin [5] characterized the series-parallel graphs as graphs with no subgraphs homeomorphic to the complete graph K_4 . Wald and Colbourn [11] restated Duffin's result and showed that a graph is a partial 2-tree if and only if it has no subgraph contractible to K_4 .

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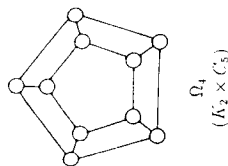
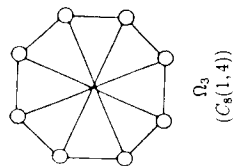

 Ω_2
($K_{2,2,1}$)


FIG. 1. Critical graphs.

The notion of critical graphs plays a central role in the characterization of partial 3-trees. A nonpartial 3-tree having no isolated points will be referred to as a *critical graph* if for every edge x of the graph, the graph obtained by deleting x and the graph obtained by contracting x are both partial 3-trees. The main result of this paper is to show that the four graphs illustrated in Figure 1 are the only critical graphs and then to prove that a graph is a partial 3-tree if and only if it has no subgraph contractible to one of these four graphs. Toward this end, a number of important properties of critical graphs are first established. We first show that a critical graph is 3-connected. Next, we prove that if G is critical with the minimum degree equal to 3, then the maximum degree is also equal to 3 and thus it is a 3-regular graph. It is then shown that every 3-regular critical graph is isomorphic to either $C_6(1,4)$ or $K_2 \times C_5$. On the other hand, if G is critical, with minimum degree greater than 3, then it is shown that it must be a 4-regular graph and, furthermore, is isomorphic to either K_5 or $K_{2,2,1}$.

Finally, by Theorem 3.4, we show that a graph is a partial 3-tree if and only if it has no subgraph contractible to one of K_5 , $K_{2,2,1}$, $C_6(1,4)$, and $K_2 \times C_5$. We conclude the paper with a discussion and some interesting results on minimal graphs. A graph is *minimal* if it is not a partial 3-tree but every proper subgraph of it is. Thus, it is obvious that every minimal graph is contractible to a critical graph. This fact together with Theorem 3.4 leads to an alternative characterization of partial 3-trees: a graph is a partial 3-tree if and only if it has no subgraph that is a minimal graph. This characterization is indeed quite different from that of Theorem 3.4. It is easily seen that any graph homeomorphic to a critical graph is minimal. However, we show by examples that there are minimal graphs that are not homeomorphic to any of the critical graphs. Thus, a characterization of minimal graphs provides a characterization of partial 3-trees that is different from that of Theorem 3.4. Though the characterization of minimal graphs is still open, we establish some important properties of minimal graphs. It is hoped that these properties are useful in the solution of this problem.

1. PRELIMINARIES

Unless defined otherwise, graph-theoretic terminology used here follows Harary [6].

Definition 2.1 (3-tree). A 3-tree is defined recursively as follows:

- (i) The complete graph K_3 is a 3-tree on 3 points.
- (ii) If G is a 3-tree on $n \geq 3$ points, and the set of points $\{a, b, c\}$ induces K_3 in G , then the graph obtained by adding a new point u to G , such that the set of points adjacent to u is $\{a, b, c\}$, is a 3-tree on $n + 1$ points.

Definition 2.2 (simplicial point). A point u of a graph G is called a *simplicial point* if its adjacency set, $\text{Adj}(u)$, induces a clique in G .

Proposition 2.1.

- (i) Every 3-tree G with four or more points has a degree-3 simplicial point. Furthermore, if $G \neq K_4$, then G has at least two nonadjacent degree-3 simplicial points.
- (ii) Every degree-3 point of a 3-tree is a simplicial point.

The proof of Proposition 2.1 is an immediate implication of Dirac's result [4].

Definition 2.3 (deletion of a point, deletion of an edge, contraction of an edge). The *deletion of a point* u from a graph G results in that subgraph $G - u$ of G consisting of all points of G except u and all edges not incident with u . On the other hand, the *deletion of an edge* x from G yields the spanning subgraph $G - x$ containing all edges of G except x . If x is an edge of G with endpoints u and v , then by the *contraction of edge* x , we mean deleting x and

identifying the points u and v to a single point. The graph possibly a pseudograph, obtained by contracting x in G is denoted by G/x . Note that all other edges with endpoints u and v in G become self-loops in G/x . Each edge of G other than x can, in a natural way, be regarded as an edge of G/x . This identification, which we henceforth assume, constitutes a 1-1 correspondence between the edge set of $G - x$ and G/x . The point of G/x that is obtained by deleting x and identifying u and v to a single point will be denoted by $u \sim v$.

Definition 2.4 (graph contractible to another graph). A graph G is said to be *contractible* to a graph H if H can be obtained from G by a sequence of edge contractions.

Definition 2.5 (partial 3-tree). A graph is a *partial 3-tree* if it is a subgraph of some 3-tree. On occasion, we even refer a multigraph G as a partial 3-tree if the graph obtained from G after repeatedly deleting multiple edges is a partial 3-tree.

The following propositions are immediate from the definition of a partial 3-tree:

Proposition 2.2. If G is a partial 3-tree, then every subgraph of G is a partial 3-tree.

Proposition 2.3. Suppose that G_1 and G_2 are the subgraphs of a graph G such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2$ is a clique with at most three points. Then G is a partial 3-tree iff both G_1 and G_2 are partial 3-trees.

Definition 2.6 (critical graph). A nonpartial 3-tree G with no isolated points is *critical* if both $G - x$ and G/x are partial 3-trees, for every edge x of G .

It is not difficult to verify that the four graphs shown in Figure 1 are critical. Indeed, as will be shown in the next section, these four are the only critical graphs, and they play a central role in the characterization of partial 3-trees. Henceforth, we refer to these four graphs with the symbols Ω_1 , Ω_2 , Ω_3 , and Ω_4 , as indicated in Figure 1. It might be of interest to note that, in the graph-theoretic jargon, Ω_1 , Ω_2 , Ω_3 , and Ω_4 are denoted, respectively, by K_3 , $K_2 \times C_3$, $C_8(1, 4)$, and $K_2 \times C_5$.

Definition 2.7 (minimal graph). A graph G is *minimal* if G is not a partial 3-tree, but every proper subgraph is a partial 3-tree.

The following proposition is immediate from Definitions 2.6 and 2.7, and Proposition 2.2.

Proposition 2.4. Any critical graph is a minimal graph.

Definition 2.8 (homeomorphic graphs). Two graphs are said to be *homeomorphic* if both can be obtained from the same graph by a sequence of subdivisions of edges.

By Definition 2.8, it can be easily verified that any graph homeomorphic to one of Ω_1 , Ω_2 , Ω_3 , and Ω_4 is minimal. The remainder of this section presents some useful properties of partial 3-trees and critical graphs.

Proposition 2.5. If G is a partial 3-tree, then G/x is a partial 3-tree, for any edge $x = \{u, v\}$ of G .

Proof. It suffices to show that if G is a 3-tree, then for any edge $x = \{u, v\}$ of G , G/x is a partial 3-tree. The proof is by induction on the number of points of G . If G is either K_3 or K_4 , then the proposition is obvious, and this establishes the basis step. Suppose that G is a 3-tree on $n > 4$ points, and assume that the proposition holds for 3-trees having fewer points than G . If one of the endpoints of x , say u , is a degree-3 point, then the graph obtained from G/x after repeatedly deleting multiple edges is a 3-tree. Hence, assume that none of u and v is degree-3 point. Since G is a 3-tree, there is a degree-3 simplicial point in G , say p . Let $\text{Adj}(p)$ be the set of points adjacent to p in G . We know that $\text{Adj}(p)$, say $\{a, b, c\}$, induces K_3 in $G - p$. If x is not $\{a, c\}$, $\{a, b\}$, or $\{b, c\}$, then $G - p$ is a 3-tree, and by the induction hypothesis, $(G - p)/x$ is a partial 3-tree. Consider the graph H obtained by adding point p and edges $\{p, a\}$, $\{p, b\}$, and $\{p, c\}$ to $(G - p)/x$, which, by definition, is a partial 3-tree. As G/x is isomorphic to H , G/x is a partial 3-tree. Hence, suppose that x is one of $\{a, b\}$, $\{a, c\}$, or $\{b, c\}$, say $x = \{a, b\}$. Let H_1 be the graph obtained from $(G - p)/x$ by repeatedly deleting multiple edges. By the induction hypothesis, $(G - p)/x$ is a partial 3-tree, and therefore H_1 is a partial 3-tree. Consider the graph, say H_2 , obtained by adding a new point p and two edges $\{p, a + b\}$ and $\{p, c\}$ to H_1 . By Proposition 2.3, H_2 is a partial 3-tree. It is clear that H_2 is isomorphic to the graph obtained from G/x after repeatedly deleting multiple edges. Thus, G/x is a partial 3-tree. ■

Definition 2.9 (3-attached subgraph, attachment, internal point, external point). Suppose $G = (V, E)$ is a connected graph, and $\{\alpha, \beta, \gamma\}$ is a proper subset of V such that $G - \alpha - \beta - \gamma$ is disconnected. Let H' be a connected component of $G - \alpha - \beta - \gamma$ and let $V(H')$ be the point set of H' . The subgraph, say H , induced by $V(H') \cup \{\alpha, \beta, \gamma\}$ is called a *3-attached subgraph* of G . The points $\{\alpha, \beta, \gamma\}$ are the *attachments* of H , whereas the points in $V(H')$ are the *internal points* of H . A point of G that is neither an attachment nor an internal point of H is called an *external point* of H . Note that every 3-attached subgraph contains at least one internal point and at least one external point.

Definition 2.10 (trisubgraph). A 3-attached subgraph with two or more internal points is called a *nontrivial 3-attached subgraph*. Suppose H is a nontrivial 3-attached subgraph of G . If H has no proper subgraph, which again is a nontrivial

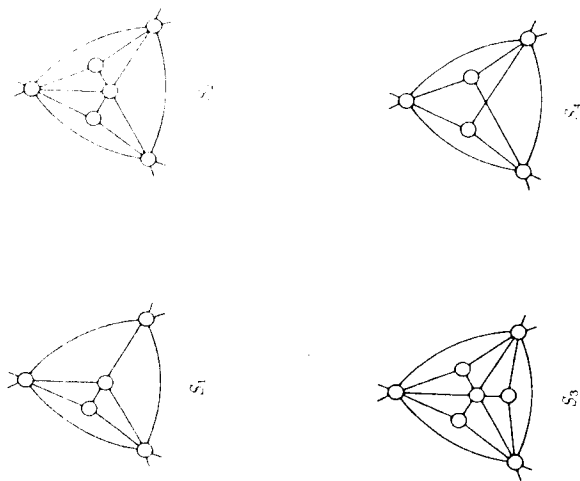


FIG. 2. Trisubgraphs of Proposition 2.6.

3-attached subgraph of G , then H is called a *trisubgraph* of G . In other words, a trisubgraph is a minimal nontrivial 3-attached subgraph.

The next proposition establishes the existence of certain trisubgraphs in 3-trees. In particular, consider the four trisubgraphs as shown in Figure 2. Henceforth, we use the symbols S_1 , S_2 , S_3 , and S_4 for these four trisubgraphs as indicated in Figure 2.

Proposition 2.6. If G is a 3-tree with $n \geq 5$ points, then G contains a trisubgraph H that is isomorphic to one of S_1 , S_2 , S_3 , and S_4 .

Proof. The proof is by induction on the number of points of G . It is easy to verify that any 3-tree on 6 points contains a trisubgraph isomorphic to either S_1 or S_4 and the basis step is established. Suppose that G is a 3-tree on $n \geq 6$ points, and assume that the result holds for all 3-trees having fewer points than G . By Proposition 2.1, G has a degree-3 simplicial point, say u . Since $G - u$ is a 3-tree, by the induction hypothesis, $G - u$ has a trisubgraph H that is isomorphic to S_i , $i = 1, \dots, 4$. Let $\text{Adj}(u)$ be the set of points adjacent to u in G .

If no internal point of H is in $\text{Adj}(u)$, then H is indeed a trisubgraph of G and we are done. Hence, $\text{Adj}(u)$ contains at least one internal point of H . Then, if u is no internal point of H is adjacent to any external point of H in $G - u$, and since u is a simplicial point of G , it follows that each point in $\text{Adj}(u)$ is either an internal point or an attachment of H . Next, we claim that $\text{Adj}(u)$ contains at least one attachment of H . This claim is justified by the fact that $\text{Adj}(u)$ induces K_3 but no three internal points of H are mutually adjacent (see Fig. 2). Now we have two cases.

Case 1. Suppose that $\text{Adj}(u)$ contains one attachment of H . Then, as the two internal points of S_1 are independent, we conclude that $H \neq S_1$. Thus, H is isomorphic to one of S_2 , S_3 , or S_4 and it follows that G has a trisubgraph that is isomorphic to S_i .

Case 2. Suppose that $\text{Adj}(u)$ contains two attachments of H . Let $\text{Adj}(u) = \{a, b, c\}$ so that a and b are attachments and c is an internal point of H . If $\deg(c) = 3$ in H , then it follows that G has trisubgraph S_1 . On the other hand, if $\deg(c) > 3$, then $H \neq S_1$. Furthermore, if H has an internal point, say d , such that $\text{Adj}(u) = \text{Adj}(d)$ in G , then G contains trisubgraph S_1 . If H has no such internal point, then $H \neq S_1$ and it follows that G has a trisubgraph isomorphic to S_2 or S_3 depending on whether H is isomorphic to S_1 or S_2 , respectively. ■

Corollary 2.1. If G is a partial 3-tree with $n \geq 5$ points, then G contains a trisubgraph H that is a subgraph of one of S_1 , S_2 , S_3 , and S_4 .

Proposition 2.7. Suppose that G is a graph and g is a trisubgraph of G with attachments $\{\alpha, \beta, \gamma\}$ such that g is isomorphic to one of g_1 , g_2 , g_3 , and g_4 as shown in Figure 3, where the broken lines correspond to edges that may or may not exist in g . Let G' be the graph obtained from G by deleting the internal points of g and adding edges between each nonadjacent pair of points in $\{\alpha, \beta, \gamma\}$. Then G is a partial 3-tree iff G' is a partial 3-tree.

Proof. Suppose that G' is a partial 3-tree. Let G'' be a 3-tree so that G' is a spanning subgraph of G'' . If $g = g_i$, $i = 1, \dots, 4$, then consider the graph $H = G' \cup S_i$, where $V(G'') \cap V(S_i) = \{\alpha, \beta, \gamma\}$ and $E(G'') \cap E(S_i) = \{\{\alpha, \beta\}, \{\beta, \gamma\}, \{\gamma, \alpha\}\}$. Clearly, H is a 3-tree, and since G is a subgraph of H , we have that G is a partial 3-tree.

Conversely, suppose that G is a partial 3-tree. It is easy to see that each g_i of Figure 3 has a subgraph contractible to a triangle whose point set is $\{\alpha, \beta, \gamma\}$. Now, the fact that G' is a partial 3-tree follows from Propositions 2.2 and 2.5. ■

III. CRITICAL GRAPHS

In this section, we first show that G is a critical graph iff G is isomorphic to one of Ω_1 , Ω_2 , Ω_3 , and Ω_4 . Next we show that G is a partial 3-tree iff G has no subgraph contractible to either one of Ω_1 , Ω_2 , Ω_3 , and Ω_4 .

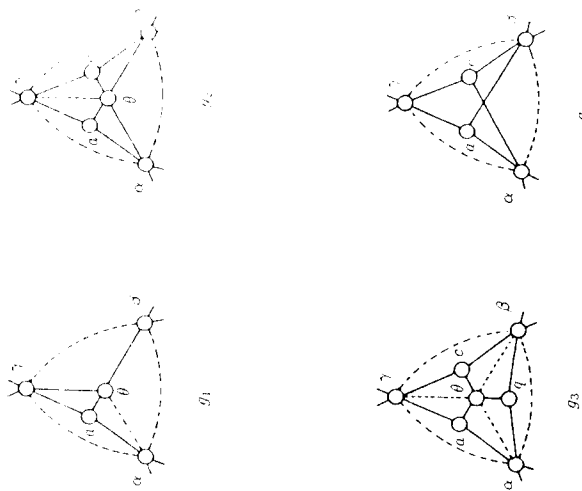


FIG. 3. Trisubgraphs of Proposition 2.7.

Theorem 3.1.¹ A graph is critical iff it is isomorphic to one of Ω_1 , Ω_2 , Ω_3 , and Ω_4 .

In preparation for the proof of Theorem 3.1, we first establish several properties of critical graphs.

Lemma 3.1. Any critical graph is 3-connected.

Proof. The proof is by contradiction. Suppose that G is a graph with as few edges as possible such that G is critical but not 3-connected.

¹After this paper was submitted, the authors learned that this result was obtained independently by S. Arnborg, A. Proskurowski, and D. Cornell. Their work appears in an unpublished technical report (CIS-TR-86-07) from the Department of Computer and Information Systems, University of Oregon, Eugene, Oregon.

Case 1. Suppose that G has a cutpoint u such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \{u\}$. If either G_1 or G_2 is not a partial 3-tree, then it follows that G is not critical. Hence, G_1 and G_2 are partial 3-trees, and by Proposition 2.3, G is a partial 3-tree, a contradiction.

Case 2. Suppose that G is nonseparable but $\{u, v\}$ is a separation pair of G . We have two subcases.

(a) Suppose that v is an edge of G such that $v = \{u, x\}$. Let G_1 and G_2 be the subgraphs of G such that $G_1 \cup G_2 = G$, $V(G_1) \cap V(G_2) = \{u, x\}$, and $E(G_1) \cap E(G_2) = \{x\}$. If either G_1 or G_2 is not a partial 3-tree, then G is not critical. Hence, G_1 and G_2 are partial 3-trees, and by Proposition 2.3, G is a partial 3-tree, a contradiction.

(b) Suppose that u and v are not adjacent in G . Let G_1 and G_2 be the subgraphs of G such that $G_1 \cup G_2 = G$, $V(G_1) \cap V(G_2) = \{u, v\}$. If either G_1 or G_2 is not a partial 3-tree, then G is not critical. Hence, G_1 and G_2 are partial 3-trees. Let G_1' and G_2' be the graphs obtained by adding edge $x = \{u, v\}$ to each of G_1 and G_2 , respectively. Note that G is 2-connected implies that G has subgraphs that are homeomorphic to G_1' and G_2' . Thus, if either G_1' or G_2' is not a partial 3-tree then, by Propositions 2.2 and 2.5, it follows that G is not critical. Hence, G_1' and G_2' are partial 3-trees. Let $H = G_1' \cup G_2'$ such that $V(H) \cap V(G) = \{u, v\}$, and $E(H) \cap E(G) = \{x\}$. By Proposition 2.3, H is a partial 3-tree. But G is a subgraph of H , and therefore by Proposition 2.2, G is a partial 3-tree. This is a contradiction that G is critical. ■

Lemma 3.2. If G is critical and u is a degree-3 point of G with $\text{Adj}(u) = \{a, b, c\}$, then $\{a, b, c\}$ is an independent set of G .

Proof. Suppose otherwise and $\{a, b\}$ is an edge of G . Consider $G \setminus x$, where the edge $x = \{u, c\}$. It is possible that $G \setminus x$ is a multigraph and let G' be the graph obtained from $G \setminus x$ after repeatedly deleting multiple edges. We know that G' is a partial 3-tree since G is critical. Furthermore, $\{a, b, c\}$ induces K_3 in G' . Let G'' be a 3-tree such that G' is a spanning subgraph of G'' . Finally, let H be the graph obtained from G'' by appending point u to G'' such that $\text{Adj}(u) = \{a, b, c\}$ in H . By Proposition 2.3, H is a 3-tree. As G is a subgraph of H , by Proposition 2.2, G is a partial 3-tree, a contradiction. ■

The following is a useful property of critical graphs:

Lemma 3.3. If G is a critical graph, then G has no trisubgraph that is isomorphic to one of g_1 , g_2 , g_3 , and g_4 as shown in Figure 3, where the broken lines correspond to edges that may or may not exist in g_i , $i = 1, \dots, 4$.

Proof. Suppose that G has a trisubgraph g_i , $i = 1, \dots, 4$, with attachments $\{a, b, \gamma\}$. Let G' be the graph obtained from G by deleting the internal points of g_i and adding edges between each nonadjacent pair of points in $\{a, b, \gamma\}$. We first claim that G' is a partial 3-tree. Note, from Figure 3, that each g_i has a

subgraph that is contractible to a K_3 with point set $\{\alpha, \beta, \gamma\}$. The claim is justified by this observation together with Propositions 2.2 and 2.3 and the fact that G is critical. Now consider the graph, say H , such that $H \cap G = g$, and $V(G) \cap V(g) = \{\alpha, \beta, \gamma\}$. We know that each g_i of Figure 3 is a partial 3-tree, and thus by Proposition 2.3, H is a partial 3-tree. Clearly, G is a subgraph of H , and therefore by Propositions 2.2, G is a partial 3-tree, a contradiction. ■

Lemma 3.4. If G is a critical graph with the minimum degree $\delta(G) = 3$, then G is 3-regular.

Let u be a degree-3 point of G and $\text{Adj}(u) = \{a, b, v\}$. Also let $x \neq u, v$ be an edge in G . It suffices if we show that a, b , and v are all degree-3 points of G . In preparation for the proof of Lemma 3.4, we first establish the following claims:

In what follows, the point of $G|x$, which is obtained by deleting x and identifying the endpoints u and v to a single point, will be denoted by $u \sim v$.

Claim 3.4.1. $G|x$ has no multiple edges. A proof of this claim follows from Lemma 3.2.

Claim 3.4.2. $G|x$ has two trisubgraphs g and h , each of which is isomorphic to one of g_1, g_2, g_3 , and g_4 , as shown in Figure 3, where the broken lines correspond to edges that may or may not exist.

Remark. If $I(g)$ and $I(h)$ are the set of internal points of g and h , respectively, then by the definition of trisubgraph, we have that $I(g) \cap I(h) = \emptyset$.

Proof of Claim 3.4.2. It can be easily verified that any graph G , with six or fewer points and having $\delta(G) = 3$, is not critical. Hence, $G|x$ has at least six points. Since $G|x$ is a partial 3-tree, by Corollary 2.1, $G|x$ has a trisubgraph, say g , with attachments $\{\alpha, \beta, \gamma\}$, which is a subgraph of one of S_1, S_2, S_3 , and S_4 . As G is 3-connected, and $G|x$ has no multiple edges, we conclude that g is isomorphic to one of g_1, g_2, g_3 , and g_4 . Furthermore, by the definition of the trisubgraph, we have that $V(G|x) - V(g) \neq \emptyset$. We have three cases.

Case 1. $|V(G|x) - V(g)| \geq 2$. Let G' be the graph obtained from $G|x$ by deleting the internal points of g and adding edges between each nonadjacent pair of points in $\{\alpha, \beta, \gamma\}$. By Proposition 2.7 G' is a partial 3-tree, and since G' has at least six points, we conclude that $G'|x$ has a trisubgraph $h \neq g$ that is isomorphic to one of g_1, g_2, g_3 , and g_4 .

Case 2. $|V(G|x) - V(g)| = 2$. Let $V(G|x) - V(g) = \{p, q\}$. Since $V(G|x) = V(g) \cup \{p, q\}$, and g is a trisubgraph of $G|x$ it follows that the degree of each of p and q is at most four. If one of p and q is a degree-3 point, then the graph induced by $\{\alpha, \beta, \gamma, p, q\}$ in $G|x$, say h , is isomorphic to either g_1 or g_2 , and h is a trisubgraph of $G|x$ with attachments $\{\alpha, \beta, \gamma\}$. If both p and q are degree-4 points, then $G|x$ is as shown in Figure 4. It is easy to see that each g_i of Figure 3 has a subgraph contractible to a triangle whose point set is $\{\alpha, \beta, \gamma\}$. Since g is isomorphic to one of g_1, g_2, g_3 , and g_4 , it follows that $G|x$ has a subgraph

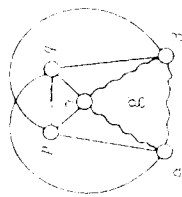


FIG. 4. Illustration for Case 2 of Claim 3.4.2

contractible to Ω_1 . This fact together with Propositions 2.2 and 2.5 implies that $G|x$ is not a partial 3-tree, a contradiction.

Case 3. $|V(G|x) - V(g)| = 1$. Let $V(G|x) - V(g) = \{p\}$. Then $\deg(p) = 3$. We know that $\deg(u + v) > 3$, and therefore $u + v \neq p$. Thus, $u + v$ is either internal to g or an attachment of g . We have four subcases.

Subcase 3.1. g is isomorphic to g_1 . First suppose that $u + v$ is an attachment of g . Then we claim that $G|x$ is isomorphic to the graph of Figure 5a. The claim is justified by the facts that g is isomorphic to g_1 , and edges $\{\alpha, \theta\}$ and $\{\alpha, \beta\}$ are not in $G|x$, for otherwise either $\text{Adj}(a)$ or $\text{Adj}(p)$ is not independent in G and we have a contradiction to Lemma 3.2. By the same reason, exactly one of a and θ is adjacent to u whereas the other is adjacent to v in G . Hence, suppose that u is adjacent to a and v is adjacent to θ . Then, again by Lemma 3.2, u is not adjacent to A , and v is not adjacent to β . This implies that u is adjacent to exactly one of p and β . If u is adjacent to β , then v must be adjacent to p and G is isomorphic to the graph of Figure 5b. Clearly, the subgraph induced by $\{u, v, a, \beta, \theta\}$ in G is a trisubgraph of G with attachments $\{v, a, \beta\}$. This trisubgraph is isomorphic to g_1 , which is a contradiction to Lemma 3.3. On the other hand, if u is adjacent to p , then $\deg(\beta) < 3$ in G , a contradiction to Lemma 3.1. Next suppose that $u + v$ is internal to g . Then, as $\deg(u + v) > 3$, we have that $u + v$ is θ and $G|x$ is as in Figure 5c. Note that $\{\alpha, \beta\}, \{\beta, \gamma\}$, and $\{\alpha, \gamma\}$ are not the edges of G , for otherwise $\text{Adj}(p)$ is not independent in G , a contradiction to Lemma 3.2. Thus, $\deg(\beta) < 3$ in G , a contradiction.

Subcase 3.2. g is isomorphic to g_2 . If $u + v$ is an attachment of g , then $G|x$ is isomorphic to the one shown in Figure 5d. Then, either $\text{Adj}(a)$ or $\text{Adj}(b)$ is not an independent set of G . This, by Lemma 3.2, is a contradiction. If $u + v$ is an internal point of g , then $G|x$ is the graph of Figure 5e. Note that there is no edge between the pairs $\{\alpha, \beta\}, \{\beta, \gamma\}$, and $\{\alpha, \gamma\}$ in $G|x$, for otherwise $\text{Adj}(p)$ will not be an independent set of G , a contradiction. Consider the subgraph, say g' , induced by $\{p, \gamma, \alpha, \beta, u + v\}$ in $G|x$. Clearly g' is a trisubgraph of $G|x$ with attachments $\{p, \gamma, u + v\}$. Furthermore, g' is isomorphic to g_1 and $G|x$ has two more points that are external to g' . We can now apply Case 2 to establish the existence of another trisubgraph h in $G|x$.

Subcase 3.3. g is isomorphic to g_3 . First assume that $u + v$ is an internal

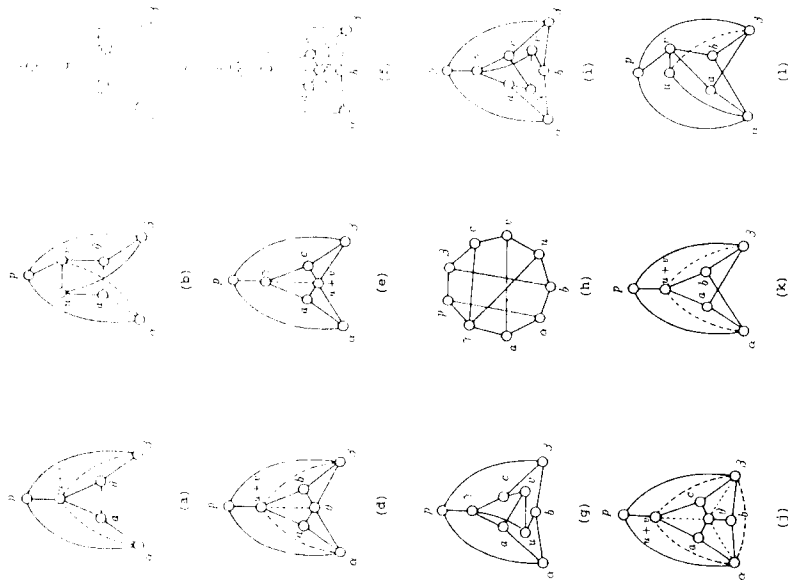


FIG. 5. Illustration for Case 3 of Claim 3.4.2.

point of g_3 . Then $G|x$ is the graph as shown in Figure 5f. If γ is adjacent to u in G , then we claim that G is isomorphic to the graph of Figure 5g. This is because of the fact that u cannot be adjacent to either a or c , for otherwise $\text{Adj}(a)$ and $\text{Adj}(c)$ are not independent in G . Thus v must be adjacent to a and c in G . Furthermore, v is not adjacent to either a and β in G , for otherwise, $\text{Adj}(a)$ or $\text{Adj}(c)$ will not be independent in G , a contradiction to Lemma 3.2. In this case, u is not adjacent to a in G , for otherwise the graph induced by $\{\alpha, \gamma, v, a, u\}$ will be a trisubgraph isomorphic to g_1 in G , and by Lemma 3.3, we

have a contradiction. By the same reason, u is not adjacent to β . Hence, u is adjacent to h in G and G is isomorphic to the graph shown in Figure 5g. Now, the graph of Figure 5h is isomorphic to that of Figure 5g, and it is clear that G is obtained from G by contracting the edges $\{p, \beta\}$, $\{c, v\}$, $\{a, b\}$, and $\{a, d\}$. This implies that G is not critical, a contradiction. Hence, assume that γ is adjacent to v in G . In this case, u must be adjacent to both a and c in G , and u is not adjacent to either a or β . Thus, G is isomorphic to the graph of Figure 5i. It is clear that the graphs of Figure 5g and i are isomorphic, and we obtain the contradiction that G is not critical.

Next suppose that $u + v$ is an attachment of g_3 . Since g_3 is symmetric, one may choose any attachment of g_3 as $u + v$. With no loss of generality, let $u + v = \gamma$, thus, $G|x$ is isomorphic to the graph shown in Figure 5j. Then, $G|x$ has another trisubgraph isomorphic to g_3 with attachments $\{\alpha, \beta, \delta\}$ and internal points $\{u + v, a, c, p\}$. Note that $u + v$ is an internal point of this trisubgraph, hence, we can obtain a contradiction using the preceding argument.

Subcase 3.4. g is isomorphic to g_4 . Then, $\deg(u + v) \geq 3$ implies that $u + v$ is an attachment of g . As g_4 is symmetric, we can pick one of $\{\alpha, \beta, \gamma\}$, say γ , as $u + v$. Hence, $G|x$ is isomorphic to the graph shown in Figure 5k. Then, we claim that G is isomorphic to the graph of Figure 5l. As p is adjacent to exactly one of u and v in G , first suppose that p is adjacent to v . Then, by Lemma 3.2, it follows that v is adjacent to both a and b , and u is adjacent to a and β in G . Thus, G is isomorphic to the graph shown in Figure 5l. Consider the subgraph induced by $\{\alpha, \beta, a, b, v\}$, say g' . Clearly, g' is a trisubgraph isomorphic to g_1 in G , a contradiction to Lemma 3.3. Finally, suppose that p is adjacent to u in G . By Lemma 3.2, u is not adjacent to either a or β in G . Hence, v is adjacent to a and v is not adjacent to a or b . But we know that $\deg(u) = 3$ in G ; hence, u is adjacent to exactly one of a and b . In either case, it follows that there is a degree-2 point in G . By Lemma 3.1, this is a contradiction.

Claim 3.4.3. Let g be a trisubgraph of $G|x$ such that g is isomorphic to one of g_1, g_2, g_3 , and g_4 as in Figure 3. Then, $u + v$ is an attachment of g .

Proof. Let $\{\alpha, \beta, \gamma\}$ be the set of attachments of g . By Claim 3.4.2, there is a trisubgraph h in $G|x$ such that $h \neq g$. By the definition of trisubgraph, we know that if $\text{In}(g)$ and $\text{In}(h)$ are the set of internal points of g and h , respectively, then $\text{In}(g) \cap \text{In}(h) = \emptyset$. If $u + v \notin \{\alpha, \beta, \gamma\}$, then $u + v$ is either an external point of g or an internal point of g . If it is external to g , then g is a trisubgraph of G , which is a contradiction by Lemma 3.3. On the contrary, if $u + v$ is internal, then it is external to h , and h is a trisubgraph of G , a contradiction again. Thus, $u + v$ is an attachment of every trisubgraph isomorphic to one of g_1, g_2, g_3 , and g_4 .

Claim 3.4.4. Let g be a trisubgraph of $G|x$ such that g is isomorphic to one of g_1, g_2, g_3 , and g_4 as in Figure 3. Suppose that $\text{Adj}(u) = \{a, b, v\}$ so that $x = \{u, v\}$. Then, exactly one of a and b is an internal point of g .

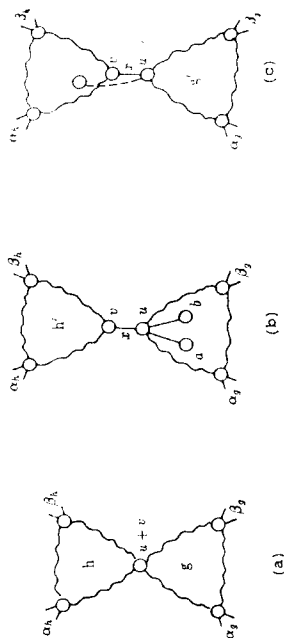


FIG. 6. Illustration for the Proof of Claim 3.4.4.

Proof. Suppose otherwise, and assume that both u and b are internal to g . By Claim 3.4.2, $G|_x$ has a trisubgraph $h \neq g$, and by Claim 3.4.3, $u + v$ is an attachment of both g and h as illustrated in Figure 6a. Hence, let $\{\alpha_g, \beta_g, u + v\}$ and $\{\alpha_h, \beta_h, u + v\}$ be the sets of attachments of h and g , respectively. Consider the subgraph of G , say h' , induced by the edge set of h . It is clear that h' is isomorphic to h and it is a trisubgraph of G having attachments $\{\alpha_g, \beta_g, v\}$ (see Fig. 6b). This, by Lemma 3.3, is a contradiction. Hence, assume that none of a and b is internal to g . Consider the subgraph of G , say g' , induced by the edges set of g . In this case, g' is isomorphic to g and it is a trisubgraph of G having attachments $\{\alpha_g, \beta_g, v\}$ as shown in Figure 6c. By Lemma 3.3, we again have a contradiction.

Claim 3.4.5. $G|_x$ has two trisubgraphs g and h that satisfy the following:

- (i) g and h are isomorphic to one of h_1, h_2 , and h_3 , as shown in Figure 7, where the broken lines correspond to edges that may or may not exist in the trisubgraphs.
- (ii) $u + v$ is an attachment of both g and h .
- (iii) If $\text{Adj}(u) = \{a, b, v\}$ so that $x = \{u, v\}$, then exactly one of a and b is an internal point of g whereas the other is an internal point of h .

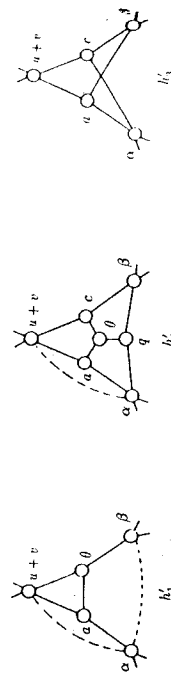


FIG. 7. Trisubgraph of Claim 3.4.5.

Proof. In view of Claims 3.4.3 and 3.4.4, we need only to prove (i). By Claim 3.4.2, $G|_x$ has two trisubgraphs that are isomorphic to one of $g_1, \dots, g_4$ of Figure 3. We show that each of these two subgraphs is isomorphic to one of h_1, h_2 , and h_3 . Let g be one of the trisubgraphs, and let $\{\alpha, \beta, \gamma\}$ be the set of attachments of g . By Claim 3.4.4, exactly one of a and b , say a , is internal to g . We have four cases.

Case 1. g is isomorphic to g_1 . In this case, $u + v$ is neither α nor β , for otherwise $\text{Adj}(a)$ will not be an independent set of G , which is a contradiction to Lemma 3.2. Hence, by Claim 3.4.3, $u + v$ is γ . Now, we claim that $\{\alpha, \theta\}$ and $\{\gamma, \beta\}$ are not the edges of g , and therefore g is isomorphic to h_1 . If $\{\alpha, \theta\}$ is an edge of g , then $\text{Adj}(a)$ is not independent in G , a contradiction to Lemma 3.2. Next, suppose that $\{\gamma, \beta\}$ is an edge of g . Since $\deg(u) = 3$ in G , it follows, from Claim 3.4.4, that v is adjacent to both β and θ , and therefore $\text{Adj}(\theta)$ is not independent in G . This is a contradiction to Lemma 3.2.

Case 2. g is isomorphic to g_2 . Then $u + v$ is neither α nor β , for otherwise either $\text{Adj}(a)$ or $\text{Adj}(c)$ will not be independent in G . Also, $u + v = \gamma$, otherwise both $\text{Adj}(a)$ and $\text{Adj}(c)$ are nonindependent sets of G . Thus, $u + v$ is not an attachment of g , which is a contradiction to Claim 3.4.3.

Case 3. g is isomorphic to g_3 . Since g_3 is symmetric, one may choose any attachment of g_3 as $u + v$. With no loss of generality, let $u + v = \gamma$. We claim that $\{\alpha, \theta\}$, $\{\beta, \theta\}$, $\{\alpha, \beta\}$, $\{\gamma, \theta\}$, and $\{\gamma, \beta\}$ are not the edges of g so that g is isomorphic to h_2 . Since $\text{Adj}(q)$ is independent in G , $\{\alpha, \theta\}$, $\{\beta, \theta\}$, and $\{\alpha, \beta\}$ are not the edges of G . As $\deg(u) = 3$ and a is an internal point of g , by Claim 3.4.4 it follows that u is not adjacent to any other point of g in G . Thus, v is adjacent to c in G , which, together with Lemma 3.2, implies that v is not adjacent to any of β and θ . Hence, $\{\gamma, \theta\}$ and $\{\gamma, \beta\}$ are not the edges of g and g is isomorphic to h_2 .

Case 4. g is isomorphic to g_4 . As g_4 is symmetric, we can pick one of $\{\alpha, \beta, \gamma\}$, say γ , as $u + v$. In this case, we claim that $\{\alpha, \beta\}$, $\{\gamma, \alpha\}$, and $\{\gamma, \beta\}$ are not the edges of g so that g is isomorphic to h_3 . Since $\deg(u) = 3$ and a is internal to g , by claim 3.4.4 it follows that u is not adjacent to any other point of g . Hence, v is adjacent to c in G , and as $\text{Adj}(c)$ is independent in G , we have that $\{\alpha, \beta\}$, $\{\gamma, \alpha\}$, and $\{\gamma, \beta\}$ are not the edges of g and g is isomorphic to h_3 .

Proof of Lemma 3.4. We show that every point in $\text{Adj}(u)$ is a degree-3 point of G . Recall that $x = \{u, v\}$ and $\text{Adj}(u) = \{a, b, v\}$. By Claim 3.4.4, $G|_x$ has two trisubgraphs, say g and h , such that a is internal to g while b is internal to h . By Claim 3.4.5, g and h are graphs isomorphic to one of h_1, h_2 , and h_3 as shown in Figure 7. Since the degrees of the internal points of h_1, h_2 , and h_3 are degree-3 points, we have that $\deg(a) = 3$ and $\deg(b) = 3$. Let $y = \{u, a\}$ be another edge of G incident on u . Using the same argument on $G|_y$, we conclude that $\deg(v) = 3$. ■

The following lemma, a slightly different version of Claim 3.4.5, is immediate from Claim 3.4.5 and the fact that G is 3-regular. This result is used in the proof of Theorem 3.2.

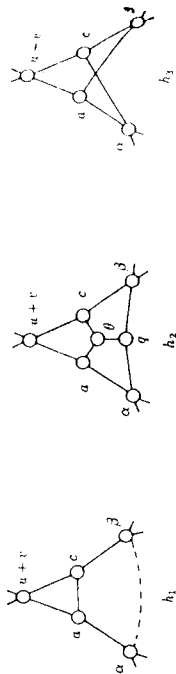


FIG. 8. Trisubgraph of Lemma 3.5.

Lemma 3.5. Suppose that G is a 3-regular and critical graph, and let x be an edge of G . Then $G|x$ has two trisubgraphs g and h that satisfy the following:

- (i) g and h are isomorphic to one of h_1 , h_2 , and h_3 , as shown in Figure 8, where the broken line in h_1 corresponds to the edge $\{\alpha, \beta\}$ that may or may not exist in h_1 .
- (ii) $u + v$ is an attachment of both g and h .
- (iii) If $\text{Adj}(u) = \{a, b, v\}$ so that $x = \{u, v\}$, then exactly one of a and b is an internal point of g whereas the other is an internal point of h .

Lemma 3.6. Suppose that G is a critical graph with minimum degree $\delta(G) = 3$. Then there exists an edge x in G such that $G|x$ is 3-connected.

Proof. By Lemmas 3.1 and 3.4, G is 3-connected and 3-regular. Hence, for any edge x , graph $G - x$ is not 3-connected. We also know that $G \neq K_4$. Now the lemma is immediate from a result of Tutte [10] that asserts that if G is a 3-connected graph and for every edge x of G , $G - x$ and $G|x$ are not 3-connected, then G is isomorphic to K_4 .

Theorem 3.2. If G is a critical graph with minimum degree $\delta(G) = 3$, then G is isomorphic to either Ω_3 or Ω_4 .

Proof. By Lemma 3.4, G is 3-regular, and by Lemma 3.6, G has an edge, say $x = \{u, v\}$, such that $G|x$ is 3-connected. Let $\text{Adj}(u) = \{a, b, v\}$ and $\text{Adj}(v) = \{c, d, u\}$. By Lemma 3.5, $G|x$ has trisubgraphs, say g and h , such that $u + v$ is an attachment of g as well as h . Furthermore, g and h are isomorphic to one of h_1 , h_2 , and h_3 as shown in Figure 8.

We first claim that $\text{Adj}(u + v) \subseteq V(g) \cup V(h)$ in $G|x$. By Lemma 3.5, exactly one of the points a and b is internal to g whereas the other is internal to h . Suppose that a is internal to g and b is internal to h . Likewise, again by the same lemma, one of c and d , say c , is internal to g , and d is internal to h . Thus, $\text{Adj}(u + v) \subseteq V(g) \cup V(h)$.

Let $\{u + v, \alpha_g, \beta_g\}$ and $\{u + v, \alpha_h, \beta_h\}$ be the sets of attachments of g and h , respectively. We have three cases.

Case 1. Suppose that $\{u + v, \alpha_g, \beta_g\} = \{u + v, \alpha_h, \beta_h\}$. With no loss of generality,

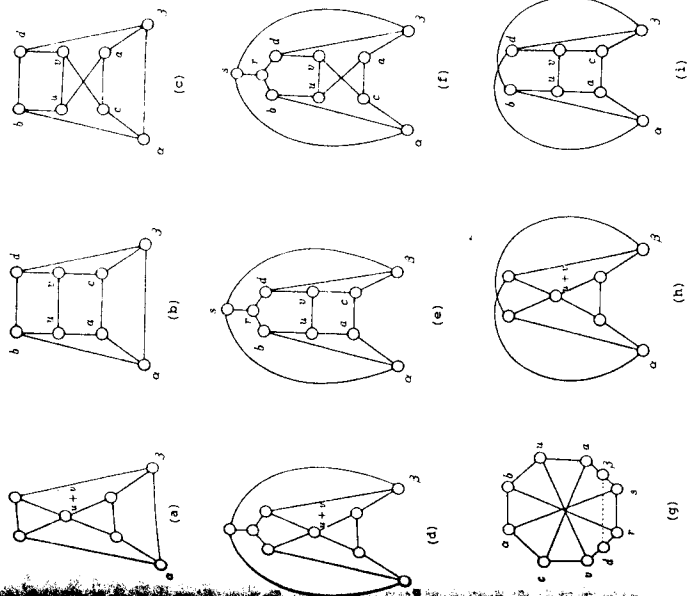


FIG. 9. Illustration for Case 1 of Theorem 3.2.

by, let $\alpha = \alpha_g = \alpha_h$, and $\beta = \beta_g = \beta_h$. We claim that $V(G|x) = V(g) \cup V(h)$. In other words, every point of $G|x$ is in either g or in h . Suppose otherwise, and let p be a point of $G|x$ such that $p \notin V(g) \cup V(h)$. We know that $\text{Adj}(u + v) \subseteq V(g) \cup V(h)$. Hence, every path from $u + v$ to p must contain either α or β , which is a contradiction to the fact that $G|x$ is 3-connected. Thus, $G|x = g \cup h$. Next we claim that one of g and h is isomorphic to h_1 , for otherwise $\deg(a) > 3$, a contradiction to the fact that G is 3-regular. Without loss of generality, suppose that g is isomorphic to h_1 . We have three subcases.

Subcase 1.1. h is isomorphic to h_1 . Then $G|x$ is isomorphic to the graph shown in Figure 9a, and therefore G must be isomorphic to either the graph of Figure 9b or that of Figure 9c. However, it can be easily shown that the

graph of Figure 9b is a partial 3-tree. Hence, G is the graph of Figure 9c, which is isomorphic to Ω_3 .

Subcase 1.2. h is isomorphic to h_2 . Then $G|x$ is isomorphic to the graph shown in Figure 9d, and therefore G must be isomorphic to either the graph of Figure 9e or the graph of Figure 9f. Clearly, the graph of Figure 9e is isomorphic to Ω_4 . We claim that G is not isomorphic to the graph of Figure 9f. This is due to the fact that the graph of Figure 9f has a subgraph contractible to Ω_3 . Specifically, deleting edge $\{d, \beta\}$ and contracting edges $\{s, \beta\}$ and $\{d, r\}$ in the graph of Figure 9f yields Ω_3 as shown in Figure 9g. This by Propositions 2.2 and 2.5, leads to the contradiction that G is not critical.

Subcase 1.3. h is isomorphic to h_3 . Then $G|x$ is isomorphic to the graph shown in Figure 9h, and therefore G must be isomorphic to the graph of Figure 9i, which is isomorphic to Ω_3 .

Case 2. Suppose that $\{|u + v, \alpha, \beta, \gamma\} \cap \{u + v, \alpha, \beta, \gamma\} = 2$. With no loss of generality, let $\alpha = \alpha_x = \alpha_u$. Then, as each of h_1, h_2 , and h_3 has odd number of points, we must have that $|V(g) \cup V(h)|$ is even. However, G is 3-regular, and therefore has even number of points. Thus, $G|x$ must have at least one point, say p , such that $p \notin V(g) \cup V(h)$. Furthermore, as α and $u + v$ are attachments to both g and h , it follows that none of g and h is isomorphic to either h_2 or h_3 . If this is not the case, then $\deg(\alpha) > 3$, a contradiction. Thus, g and h are isomorphic to h_1 . Now, as $G|x$ is 3-connected, we have three point disjoint paths from p to $u + v$. Since g and h are trisubgraphs of $G|x$, and $\text{Adj}(u + v) \subseteq V(g) \cup V(h)$, it follows that any path between p and $u + v$ must contain one of $\{\alpha, \beta, \gamma, \beta_u\}$. Thus, $G|x$ has point disjoint paths from p to each of α, β, γ , and β_u as illustrated in Figure 10a. Hence, G has a subgraph homeomorphic to the one, say H , as shown in Figure 10b. It is clear that H is homeomorphic to Ω_3 . Further, $H|e$ is also homeomorphic to Ω_3 where the edge $e = \{u, \beta_u\}$. Thus, $G|e$ has a subgraph that is not a partial 3-tree. This, by Proposition 2.2, is a contradiction since G is critical.

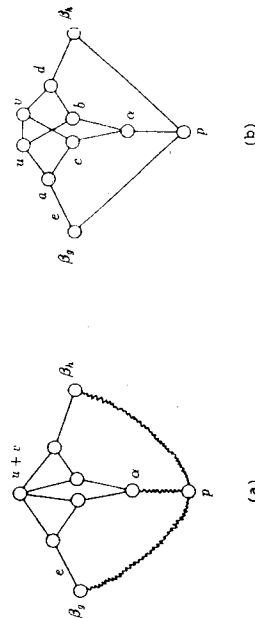


FIG. 10. Illustration for Case 2 of Theorem 3.2.

Case 3. Suppose that $\{|u + v, \alpha, \beta, \gamma\} \cap \{u + v, \alpha, \beta, \gamma\} = 1$. Consider the four points $\alpha, \beta, \gamma, \alpha_u$ and β_u . We claim that $G|x$ has paths, say P_1 and P_2 with the following properties: (i) P_1 is between α_x and α_u whereas P_2 is between β_x and β_u ; (ii) P_1 and P_2 are point disjoint; (iii) P_1 and P_2 do not contain $u + v$ or any internal point of g and h . This claim is justified by the facts that $G|x$ is 3-connected, and g and h are trisubgraphs of $G|x$. Thus, $G|x$ has a subgraph, say H , homeomorphic to the one shown in Figure 11a or b depending on the endpoints of P_1 and P_2 . Indeed, the graph of Figure 11b is obtained from that of Figure 11a by interchanging the roles of α_x and β_x . Hence, assume that H is as in Figure 11a. Now we have the following subcases.

Subcase 3.1. Both g and h are isomorphic to h_1 . Suppose that $G|x$ has paths, say P_3 and P_4 , such that P_3 is between α_u and β_u , and P_4 is between α_x and β_x . Furthermore, suppose that no point of P_3 is in $\{u + v, \alpha, \beta, \gamma, \alpha_x, \beta_x\}$ and no point of P_4 is in $\{u + v, \alpha, \beta, \gamma, \alpha_u, \beta_u\}$. Then $G|x$ has a subgraph, say H' that is homeomorphic to the one shown in Figure 11c. On the contrary, suppose that at least one of the paths, say P_4 , does not exist in $G|x$. Then as $G|x$ is 3-connected, we have three point disjoint paths from α_x to β_x . One of these paths must use a and c , and the other two paths use exactly one each of α_u and β_u . Note that the paths that use α_x and β_u contain no point of $\{u + v, \alpha, \beta, \gamma, \alpha_x, \beta_x\}$. Thus, if $G|x$ has no P_4 , then $G|x$ has a subgraph, say H' , that is homeomorphic to the one shown in Figure 11d.

If $G|x$ has a subgraph homeomorphic to H' , then G has a subgraph homeomorphic to either the graph of Figure 11e or that of f. Clearly, the graph of Figure 11e is isomorphic to Ω_4 , and if G is not Ω_4 , then we have a contradiction to the fact that G is critical. If G has a subgraph homeomorphic to the one of Figure 11f, then it is a contradiction, since Ω_3 is obtained from the graph of Figure 11f by deleting the edge $\{\alpha_x, \beta_x\}$ and contracting the edges $\{a, \beta_x\}$ and $\{c, \alpha_x\}$ as illustrated in Figure 11g.

If $G|x$ has a subgraph homeomorphic to H' , then G has a subgraph homeomorphic to either the graph of Figure 11h or that of Figure 11i. It can be easily seen that the graphs of Figures 11h and i are isomorphic to those of Figures 11f and e, respectively. Hence, in this case, we conclude that G is Ω_4 .

Subcase 3.2. g is isomorphic to h_1 , and h is isomorphic to h_2 . Then H is as shown in Figure 11j. Consider the graph, say H' , as shown in Figure 11k, which is obtained from H by contracting the edges of paths P_1 and P_2 . Note that H' is isomorphic to the graph of Figure 9d. This is a contradiction as already established in Subcase 1.2.

Subcase 3.3. g is isomorphic to h_1 , and h is isomorphic to h_3 . Then H is as shown in Figure 11l. Let H' be the graph obtained from H by contracting the edges of paths P_1 and P_2 (see Fig. 11m). As H' is isomorphic to the graph of Figure 9h, an application of the argument of Subcase 1.3 yields a contradiction.

Subcase 3.4. g is isomorphic to h_2 , and h is isomorphic to h_2 . Then H is as shown in Figure 11n. By Subcase 3.2, this leads to a contradiction since the graph of Figure 11j can be obtained from H by deleting point q and contracting the edge $\{a, \theta\}$.

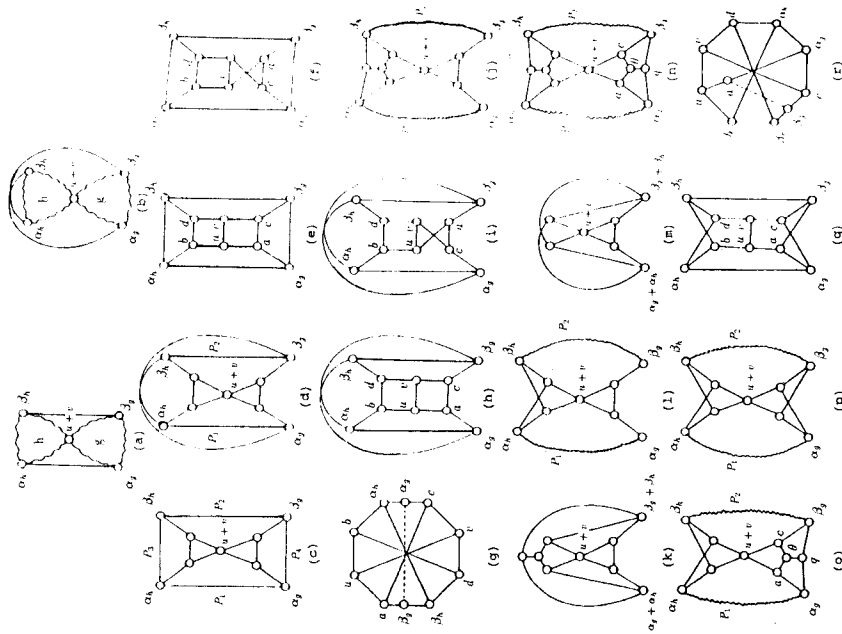


FIG. 11. Illustration for Case 3 of Theorem 3.2.

Subcase 3.5. g is isomorphic to h_2 , and h is isomorphic to h_3 . Then H is as shown in Figure 11o. This, by Subcase 3.3, is again a contradiction, since the graph of Figure 11l is obtained from H by deleting point q and contracting edge $\{a, \theta\}$.

Subcase 3.6. Both g and h are isomorphic to h_3 . Then H is as shown in

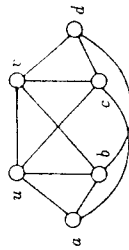


FIG. 12. Illustration for Case 1 of Theorem 3.3.

Figure 11p, and therefore G has a subgraph homeomorphic to the graph, say H' , as shown in Figure 11q. This is a contradiction to the fact that G is critical, since Ω_1 is obtained from H' by deleting edge $\{b, a\}$ and contracting edges $\{c, b\}$ and $\{c, b\}$ as illustrated in Figure 11r. ■

Theorem 3.3. If G is a critical graph with the minimum degree $\delta(G) \geq 3$, then G is isomorphic to either Ω_1 or Ω_2 .

Proof. If G is K_5 , then we are done. Hence, assume that $G \neq K_5$. We first claim that for any edge $y = \{p, q\}$ of G , $\text{Adj}(p) \cap \text{Adj}(q)$ contains at least two nonadjacent degree-4 points. Suppose otherwise, and let G' be the graph obtained by repeatedly deleting multiple edges from $G|y$. It is easy to see that G' has at most one degree-3 point. This is a contradiction, since $G|y$ and therefore G' is a partial 3-tree, and by Proposition 2.1, G' must have at least two points of degree no more than 3.

Next we claim that G has an edge $x = \{u, v\}$ such that $\deg(u) = \deg(v) = 4$. Let $G - w$ be the graph obtained by deleting an arbitrary point w from G . By Proposition 2.5, $G - w$ is a partial 3-tree. Hence, by Proposition 2.1 and Lemma 3.1, it follows that $G - w$ has two nonadjacent degree-3 points. Thus, w is adjacent to two nonadjacent degree-4 points in G . As the statement is true for any point w , we can pick a degree-4 point, say u , and it follows that u is adjacent to another degree-4 point so that $x = \{u, v\}$ is an edge of G . Now we have two cases.

Case 1. Let $\text{Adj}(u) = \{a, b, c, v\}$ and $\text{Adj}(v) = \{b, c, d, u\}$ so that the degrees of b, c, u , and v are all equal to 4, and $\{b, c\}$ is not an edge of G . Since u is adjacent to b , it follows from the first claim that b is adjacent to a . Likewise, v is adjacent to c , it follows that c is adjacent to d . Furthermore again by the first claim, the existence of edges $\{u, c\}$ and $\{v, b\}$ in G imply that a is adjacent to c and b is adjacent to d , as shown in Figure 12. If G has a point that is not in $\{u, v, a, b, c, d\}$, then G can be disconnected by deleting a and d , which is a contradiction because G is 3-connected by Lemma 3.1. If G has no other points, then it follows that $\{a, d\}$ is an edge of G and G is isomorphic to Ω_2 .

Case 2. Suppose that $\text{Adj}(u) = \{a, b, c, v\}$, and $\text{Adj}(v) = \{a, b, c, u\}$. In other words, $\text{Adj}(u) - \{v\} = \text{Adj}(v) - \{u\}$. From the first claim, it follows that at least two nonadjacent degree-4 points of g are in $\{a, b, c\}$, say a and b . Again applying our first claim on the edge, say $y = \{u, a\}$, we must have two nonadjacent degree-

4 points of G in $\text{Adj}(u) \cap \text{Adj}(a)$. However, since a is not adjacent to b , we must have that $\text{Adj}(u) \cap \text{Adj}(a) = \{v, c\}$. This is a contradiction since v is adjacent to c in G .

Theorem 3.1 is now immediate from Theorems 3.2 and 3.3.

Theorem 3.4. A graph is a partial 3-tree iff it has no subgraph contractible to one of Ω_1 , Ω_2 , Ω_3 , and Ω_4 .

Proof. Suppose that G is a partial 3-tree. Then by Propositions 2.2 and 2.5 and the definition of critical graphs, it follows that G has no subgraph contractible to a critical graph. Thus, by Theorem 3.1, G has no subgraph contractible to Ω_1 , Ω_2 , Ω_3 , and Ω_4 .

Conversely, suppose that G is not a partial 3-tree, then G has a subgraph that is not a partial 3-tree. Let H be a subgraph of G with as few edges as possible such that H is not a partial 3-tree. Clearly, H is a minimum graph. As every minimal graph is contractible to a critical graph, the result follows by Theorem 3.1. ■

Some Results on Minimal graphs

Recall that a graph G is said to be minimal if G is not a partial 3-tree but every proper subgraph of G is. By Theorem 3.4 and the definition of a minimal graph, it is immediate that every minimal graph is contractible to a critical graph. Furthermore, a graph is a partial 3-tree iff it has no subgraph that is minimal. Thus, minimal graphs are important as forbidden subgraphs in the characterization of partial 3-trees. A characterization of minimal graphs is still open, and in what follows we establish some properties of minimal graphs.

It is easily seen that any graph homeomorphic to a critical graph is minimal. However, the converse of this is not true as illustrated by the minimal graphs of Figures 13e-f.

Lemma 3.7. If G is minimal, then $\Delta(G) \leq 4$. Furthermore, if $\delta(G) \geq 3$, then G has at most 12 points.

A proof of this lemma follows from Theorem 3.1 and the fact that every minimal graph is contractible to a critical graph.

Lemma 3.8. If G is a minimal graph and x is any edge of G , then $G|x$ is either a partial 3-tree or a minimal graph.

Proof. Suppose that $G|x$ is not a partial 3-tree, then we show that $G|x$ is minimal. We know that, for every edge y , graph $G - y$ is a partial 3-tree. By Proposition 2.5, $(G - y)|x$ is a partial 3-tree for any $y \neq x$. However, $(G|x) - y$ is isomorphic to $(G - y)|x$ and the lemma follows. ■

The following corollary is immediate from Lemmas 3.7 and 3.8.

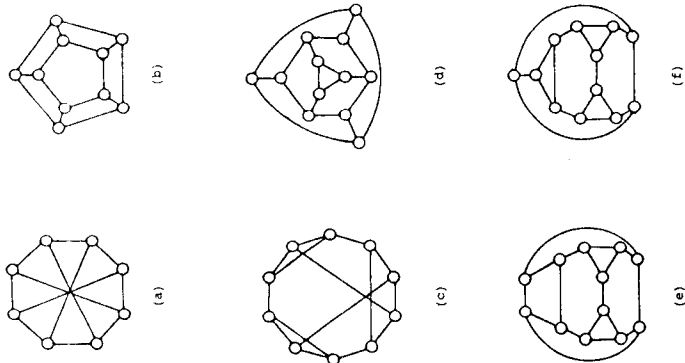


FIG. 13. Regular degree-3 minimal graphs.

Corollary 3.1. If G is minimal and $x = \{u, v\}$ is an edge of G such that $\deg(u) = 4$ and $\deg(v) \geq 3$, then $G|x$ is a partial 3-tree.

Corollary 3.1 and an exhaustive verification of all regular degree-3 graphs having 12 or fewer points establishes the following fact:

Fact 3.1. If G is a minimal graph with $\delta(G) = \Delta(G) = 3$, then G is isomorphic to one of the six graphs of Figure 13.

Lemma 3.9. Suppose G is a graph and the subset of points $\{\alpha, \beta, \gamma\}$ of G induces a triangle. Let G' be the graph obtained from G by deleting the edges $\{\alpha, \beta\}$ and $\{\alpha, \gamma\}$, and adding a degree-3 point u such that $\text{Adj}(u) = \{\alpha, \beta, \gamma\}$ in G' .

- (i) If G is minimal, then either G' is minimal or $G' - x$ is minimal, where $x = \{\beta, \gamma\}$.
 (ii) If G' is minimal, then G is a minimal graph.

Proof.

(i) Suppose that G is minimal. Consider any edge $y = \{p, q\}$ of G' such that either p or q is not in $\{u, \alpha, \beta, \gamma\}$. Clearly, y is in G and we have that $G - y$ is a partial 3-tree. Let G'' be the graph obtained from $G - y$ by adding a degree-3 point u so that $\text{Adj}(u) = \{\alpha, \beta, \gamma\}$ in G'' . By Proposition 2.3, G'' is a partial 3-tree, and as $G' - y$ is a subgraph of G'' , by Proposition 2.2, $G' - y$ is a partial 3-tree. Next consider any edge $z = \{u, p\}$ of G' such that $p \in \{\alpha, \beta, \gamma\}$. If $z = \{u, \alpha\}$, then by Proposition 2.2, $G' - z$ is a partial 3-tree iff $(G' - z) - u$ is a partial 3-tree. Since $(G' - z) - u$ is isomorphic to $(G - \{\alpha, \beta\}) - \{\alpha, \gamma\}$ it follows that $G' - z$ is a partial 3-tree. On the other hand, if one of the endpoints of z is either β or γ , then $G' - z$ is homeomorphic to $G - \{\alpha, \beta\}$ or $G - \{\alpha, \gamma\}$ and again we have that $G' - z$ is a partial 3-tree. Finally, consider the edge $x = \{\beta, \gamma\}$. If $G' - x$ is a partial 3-tree, then G' is a minimal graph and we are done. If $G' - x$ is not a partial 3-tree, then $G' - x$ is a minimal graph since $G' - y$ is a partial 3-tree for all $y \neq x$.

(ii) Suppose that G' is minimal. Consider the edge $y = \{u, \alpha\}$ of G' . It is clear that $G' - y$ is isomorphic to G . By Lemma 3.8, $G' - y$ is either a partial 3-tree or a minimal graph. We claim that $G' - y$ is not a partial 3-tree. For otherwise, let G'' be the graph obtained from G by adding a degree-3 point u such that $\text{Adj}(u) = \{\alpha, \beta, \gamma\}$. By Proposition 2.3, G'' is then a partial 3-tree, and as G' is a subgraph of G'' , we have a contradiction. ■

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Fair Dissections of Spiders, Worms, and Caterpillars

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In the present paper we deal with equipartition problems for special classes of trees, i.e., spiders, stars, worms, and caterpillars. We prove that the equipartition problem is NP-complete for spiders (and, hence, for general trees); on the other hand, we give efficient polynomial-time algorithms for stars, worms, and caterpillars.

1. INTRODUCTION

Given a connected graph $G = (V, E)$ with $n = |V|$, let $\pi = \{C_1, \dots, C_p\}$ be a partition of V into p ($1 \leq p \leq n$) subsets (the classes of π): π is said to be *feasible* in G if, for every $1 \leq k \leq p$, the subgraph $G(C_k)$ induced by C_k is connected. For the graph-theoretic terminology, see [5]. We denote by $\Pi(G, p)$ the set of all such feasible partitions.

We shall consider the following optimal graph partitioning problem: Given any function $f: \Pi(G, p) \rightarrow R$, find a $\pi^* \in \Pi(G, p)$ such that

$$f(\pi^*) = \min_{\pi \in \Pi(G, p)} \{f(\pi)\}.$$

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