HANDBOOK OF CONVEX GEOMETRY
(edited by P. Gruber and J. Wills)
Section 1.4

RIGIDITY

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1. INTRODUCTION

"Equal and similar solid figures are those contained by similar planes equal in multitude and magnitude.

This is Definition 10 in Book XI of Euclid's Elements. (See HEATH [1956] and LEGENDRE [1794] for a discussion.). Many authors have pointed out that this is not properly a definition but a statement that should be proved. Indeed, one should be more careful about just what the hypothesis should be and what the terms mean. The two "solid figures" in Figure 1.1 might be interpreted as a counter-example to the above statement.

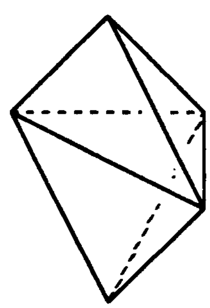
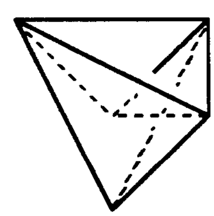

Figure 1.1

The polyhedron on the left is cut into two pieces and the top portion, a tetrahedron, is put back inside the other half to get the polyhedron on the right.

One way to avoid the difficulty of this example is to assume that all the objects being considered are convex. Indeed, the first substantial mathematical result concerning rigidity is the following.

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[CAUCHY 1813]: *Two convex polyhedra comprised of the same number of equal similarly placed faces are superposable or symmetric.*

Cauchy's Theorem was the subject of a great deal of scrutiny as well as inspiration for many generalizations, even though both were somewhat belated. The proof was in two parts, one geometric and one topological, both with certain flaws. The mistake in the geometric part was the more serious and was finally corrected by Steinitz in STEINITZ and RADEMACHER [1934]. The relatively minor mistake in the topological part of the proof was found by HADAMARD [1907] and corrected by LEBESQUE [1909]. The geometric argument is especially interesting, and we will discuss it further later.

There are two main categories for generalizations coming from Cauchy's Theorem. One is in the category of polyhedra and similar discrete objects such as frameworks and hinged plates. The other generalization is in the category of appropriately smooth surfaces. There is also the question of exactly what "kind" of rigidity one is discussing, and this applies to both categories. In Cauchy's Theorem one thinks of each of the faces of the polytope as a "rigid" plate, and one stays in the configuration space of convex objects. Then the rigidity result is a statement about uniqueness in this space. On the other hand one can also form a "linearized" definition of rigidity, called infinitesimal rigidity or first-order rigidity, in both the discrete and smooth categories. However, only in the discrete category has it been proved generally that infinitesimal rigidity implies rigidity in the sense of there being no non-trivial continuous motion of the object preserving its appropriate metric character. Despite this, there are many similarities between the ideas in the discrete category and the smooth category. Yet other "kinds" of rigidity are second-order rigidity and pre-stress stability. These are natural extensions of first-order rigidity and they are discussed in Section 5.

The study of rigid structures is large and there are many different points of view. In CONNELLY [1988], CONNELLY [1990], CRIPPEN AND HAVEL [1988], SUGIHARA [1986], and RECSKI [1989] there are some interesting applications and relations to other areas that we will not explore here. Since this is a handbook on convexity we shall try to restrict this discussion to the intersection of convexity theory and the theory of rigid structures. Nevertheless, this still leaves a great deal to be covered. Because of personal interests and lack of space we shall deal less comprehensively with the smooth category. For some older results in the category of polyhedral surfaces see STOKER [1968]. In CRAPO and WHITELEY [1995] there will be an extensive discussion of the geometry of rigid structures in the discrete category in much greater detail. In particular the chapter ROTH [1987], "The Rigidity of Frameworks Given by Convex Surfaces", will provide many of the details left out here. For a discussion of some of the classical results in the smooth category see EFIMOV [1962], NIRENBERG [1963], HICKS [1964], STOKER [1969], and SPIVAK [1975, vol. 5].

The following is an outline of this chapter:

1. Introduction

2. Early Results
   - 2.1 Cauchy's Proof of his Theorem
   - 2.2 Smooth Analogues of Cauchy's Theorem
   - 2.3 Some Counter-Examples and Flexes
   - 2.4 Alexandrov-Pogorelov Theory

3. Basic Definitions and Basic Results
   - 3.1 Frameworks and Tensegrity Frameworks
   - 3.2 Rigidity and Global Rigidity
   - 3.3 Infinitesimal Rigidity
   - 3.4 Static Rigidity
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   - 3.6 The Rigidity Map

4. Infinitesimal and Static Rigidity Related to Surfaces
   - 4.1 Grünbaum and Shephard's Conjectures
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   - 4.3 Alexandrov's Theory
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   - 4.5 E. Kann's Proof and Extension of Dehn's Theorem

2. EARLY RESULTS

We shall start with a discussion of some of the early results of Cauchy and others that are in the spirit of elementary geometry and basic differential geometry.

2.1 Cauchy's Proof of his Theorem. Cauchy's proof was in two parts, a geometric part, where the convexity assumption was used in an essential way, and a topological or combinatorial part where an Euler characteristic argument is used, as well as the underlying manifold structure. I like to call the basic geometric Lemma the "Arm Lemma" since the two polygons invlved resemble an opening arm.

> [CAUCHY 1813]: *Let A, B, ..., G represent a convex planar or spherical polygon, and let A', B', ... , G' represent another such convex polygon, where for the corresponding lengths*
>
> $$AB = A'B',\ BC = B'C',\ \ldots,\ FG = F'G'$$
>
> *and for the corresponding angles*
>
> $\angle ABC \leq \angle A'B'C'$, $\angle BCD \leq \angle B'C'D'$, ..., $\angle EFG \leq \angle E'F'G'$.
>
> *Then the for the last unmentioned length, $AG \leq A'G'$, where equality holds if and only if the two polygons are congruent. (See Figure 2.1.1)*

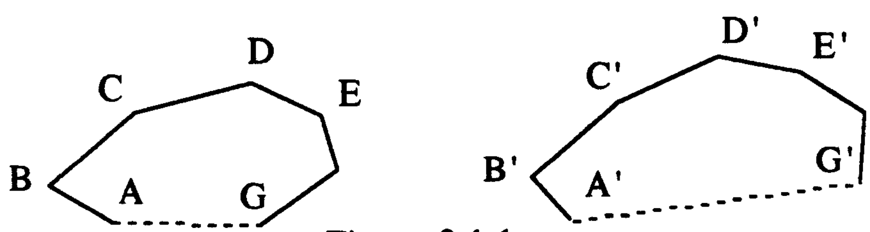


Figure 2.1.1

Cauchy attempted to prove this Lemma by induction on the number of vertices "opening" the "arm" one joint (or vertex) at a time. The trouble is that the intermediate configurations may not be convex, ruining the induction hypothesis. This is the more serious mistake mentioned in the introduction. Figure 2.1.2 shows the problem.

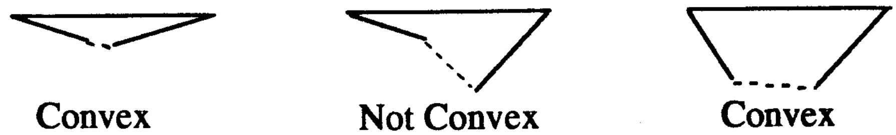


Figure 2.1.2

Nevertheless, the Lemma is still true, and there have been various alternate (correct) proofs that have been proposed. For example, see STEINITZ AND RADEMACHER [1934], LYUSTERNIK [1966], or LYUSTERNIK [1963]. In SCHOENBERG AND ZAREMBA [1967] there is an especially clever and elementary proof. Indeed the discussion in CHERN [1967] of the Lemma of SHUR [1921] is a pleasant generalization that seems to work in the case the arm is piecewise smooth, thus including both the discrete and smooth cases. See also EPSTEIN [1986] for an analogue in the hyperbolic plane. The discussion in Section 4.1 of this chapter is of a generalization replacing some of the equality constraints with inequalities, as well as with different combinatorial arrangements. This follows CONNELLY [1982] and provides yet another proof of the Arm Lemma, but this time only in the discrete category. There is a short discussion of other related references in CONNELLY [1982]. An easy corollary of this Arm Lemma is the the following:

Let P and P' be two convex planar or spherical polygons with the same number of corresponding sides of equal length. Assign a + sign or a - sign for those vertices of P, where the internal angle increases or decreases, respectively, in going from P to P'. (No sign is given to those angles where there is no change.) Then either there are at least four changes in sign as one travels around P, or there are no sign assignments at all, and P and P' are congruent. See Figure 2.1.3.

Figure 2.1.3

The idea of the proof is as follows. Suppose there are only two sign changes. Choose a line segment with its end points in the interior of two edges of P separating the two signs. The Arm Lemma applied to one side of P implies that this line segment must increase in length in going to P'. Similarly, the Arm Lemma applied to the other side in going from P' to P also implies that the line segment must have the opposite change in length, a contradiction. Thus there must be at least four sign changes, if there are any at all.

The following is Cauchy's combinatorial Lemma.

(2.1) *Suppose that + and - signs are assigned to some of the edges of a triangulated (two-dimensional) sphere so that at each vertex with some labeled edge there are at least four changes of sign as one proceeds around the vertex. Then none of the edges are labeled.*

The proof of this Lemma is an argument using the Euler characteristic of the sphere and counting the number of the number of sign changes around each vertex as well as the number of sign changes in each region defined by the edges that are labeled with a + or a -. See LYUSTERNIK [1963], for example, for a proof.

The proof of Cauchy's Theorem in the introduction now follows easily. If P and P' are the two corresponding convex polyhedra, then label an edge of P a + or a - depending on whether the dihedral angle at the edge increases or decreases respectively. By taking a small sphere centered at each vertex of P, the Corollary to the Arm Lemma implies that there must be at least four changes in sign around each vertex with at least one labeled edge coming into it. The combinatorial Lemma then implies that there are no sign changes at all. Then it is easy to see that P and P' are congruent.

These basic results of Cauchy come up again and they are applied in many other contexts.

2.2 Smooth Analogues of Cauchy's Theorem. If M is a Riemannian manifold, part of the structure that defines M is the degree of differentiability of the coordinate maps as well as the implied Riemannian metric obtained by integrating curves on the surface. The first rigidity theorems had rather strong assumptions on the differentiability of M, but after some effort these assumptions were weakened greatly. The first smooth analogue to Cauchy's Theorem was the following.

> COHN-VOSSEN [1936]: *Suppose that M and M' are two isometric, compact, closed, convex, analytic surfaces in* $\boldsymbol{R}^3$. *Then M and M' are congruent (including possibly being reflected copies of each other).*

See STOKER [1969], LIEBMANN [1900] as well as BUSEMAN [1958] for a discussion of this and related problems to follow in this section. It was later shown how to remove the condition of being analytic. Notice that this result is again a statement about the uniqueness of an object in some space of convex surfaces. Later in HERGLOTZ [1943] an extremely succinct proof of Cohn-Vossen's Theorem is presented, where the surface need only be of the class C^2 (that is, continuous second derivatives) instead of analytic. See HICKS [1964] or CHERN [1951], for example, for "modern" treatments of this proof. With some imagination, one can see certain similarities between Herglotz' proof and Cauchy's proof. For example the following Lemma comes from Herglotz.

> *Let Q(x,y) and Q'(x,y) be two positive definite quadratic forms in two variables with the same determinate when expressed as symmetric two-by-two matrices. Then the quadratic form given by the difference Q - Q' is either indefinite or identically zero.*

The idea is that the equations $Q(x,y) = 1$ and $Q'(x,y)=1$ represent two ellipses (centered at the origin) with the same area. If they are not identical, then there must be four changes in sign as one proceeds around the unit circle of directions in the plane. See Figure 2.2.1.

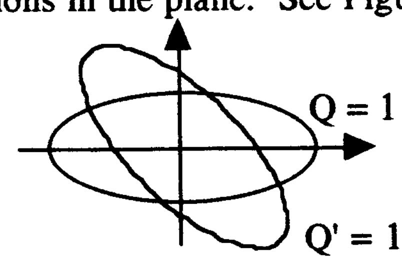


Figure 2.2.1

Loosely speaking, quadratic forms are used to represent the infinitesimal "bending" near a point in the smooth surface. The above Lemma indicates that as one proceeds around the circle of directions at a point on the surface, either there are four times when there is a change as to which surface bends the most, or the surfaces are infinitesimally isometric at that point. It seems to me that this Lemma is a smooth analogue to Cauchy's Corollary to his Arm Lemma. But in the smooth category this Lemma is quite simple. Smooth surfaces have an infinitesimal symmetry that is lacking in polyhedral surfaces. Perhaps this is why Herglotz' proof is so much easier than Cauchy's, after one gets past the fundamentals in each category. The other half of Herglotz' argument involves an integral formula, and this seems to play the role of the Euler characteristic argument in Cauchy's combinatorial lemma.

2.3 Some Counter-Examples and Flexes. In the results mentioned above, both surfaces were assumed to be convex. The consequence was that that there was only one such object in the given category. In particular, this means that there is no continuous deformation, preserving the appropriate metric structure and the convexity. A continuous deformation of the surface, preserving the metric, is what I call a *flex* . It is *non-trivial* if it is not the restriction of a rigid motion of the ambient Euclidean space.

On the other hand, if only one of the surfaces is assumed to be convex, then there can be difficulties. For example, if one considers the "piecewise linear" category, then an example similar to the example of Figure 1.1 exists even with a non-trivial continuous flex, each with the same polyhedral metric. Furthermore this flex can start with a convex surface as in Figure 2.3.1.

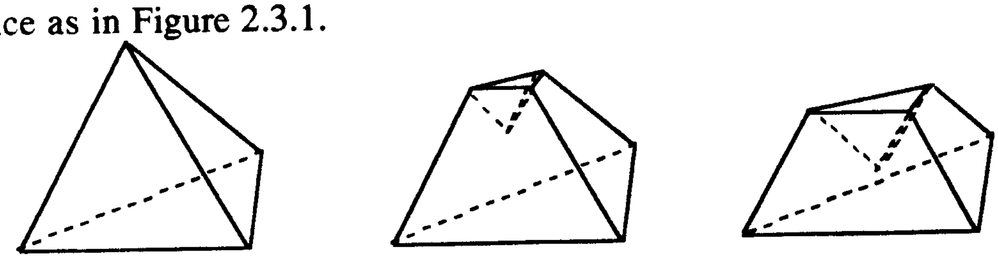


Figure 2.3.1

The top of the tetrahedron is sliced off by a continuously moving plane and then "popped in" (i.e. reflected inside through the plane). This is something like the way one might turn a paper bag inside out. There is a "crease" that moves continuously along some of the faces of the polyhedron.

Of course in Cauchy's Theorem each face of the polyhedron is assumed to be a rigid "plate", and no such continuous creasing is allowed. So for Cauchy's hypothesis in his Theorem there is no such flex. When there is no such non-trivial continuous flex, we say that the object is *rigid.*

A natural question then comes up: **Is convexity really needed for the rigidity of a polyhedral surface with rigid faces? Apparently even Euler conjectured that there was no such continuous flex.** This question and related results are discussed explicitly in GLUCK [1975]. Nevertheless a counter-example was found in CONNELLY [1978]. **This is an embedded polyhedral surface in three-space with rigid faces that flexes.** See CONNELLY [1979] for the best discussion of this example as well as a very simple example by K. Steffan roughly based on the original example in CONNELLY [1978]. See also the description in KUIPER [1978].

What about the same question in the smooth category? Is a smooth surface convex in three-space rigid, if it must stay in the category of smooth surfaces? Here one must be careful what the definition of smooth means. The answer in the C^1 category (continuous first derivatives) is surprising. According to N. H. Kuiper, following the ideas of KUIPER [1955a] and KUIPER [1955b] (see also NASH [1954]), if one takes any C^1 smooth surface, there is a non-trivial flex in the category of C^1 surfaces. In particular, the standard unit sphere in three-space must have a non-trivial C^1 flex. It turns out (see Section 2.4) only the initial configuration can be convex. The surface must immediately move into some sort of "prune-like" shape. I know of no explicit description of such a flex or even of an explicit C^1 embedding other than the original sphere.

In the C^2 category (continuous second derivatives) of course no such examples exist. In this category one can define the Gaussian curvature of a surface. The space of C^2 closed surfaces with strictly positive Gaussian curvature is open in the space of C^2 surfaces. If the Gaussian curvature of a closed surface is positive at each point, then the surface must be convex. Hence Hergoltz' uniqueness Theorem implies that such a surface is rigid at least in the class of C^2 surfaces.

If a smooth surface is C^2, then I know of no case when a (necessarily non-convex) surface flexes staying in the C^2 category, even when the surface is immersed. (In the smooth category, a manifold is *immersed* when the differential of each coordinate map is one-to-one.)

2.4 Alexandrov-Pogorelov Theory. Although it may seem as though the smooth and piecewise-linear categories are distinct with only vague similarities, it turns out that there is a common generalization. For Cauchy-type theorems this appears to be the ultimate as far as results about the uniqueness in the class of convex surfaces. This follows the work of ALEXANDROV [1958] and POGORELOV [1973], which is a recounting of earlier work.

Given any two points p and q on a convex surface in n-space, consider any continuous arc that lies in the surface that connects p and q. The length of the shortest possible such arc is called the distance between p and q. This length is always a well-defined finite number, when, say, the convex surface is the boundary of a compact convex set. This distance function is easily seen to be a metric function for the surface. See ALEXANDROV AND ZALGALLER [1967] for a careful discussion.

If the surface is the boundary of a compact convex polytope in three-space, then intrinsically in the neighborhood of any point that is not a vertex of the polytope, the metric function is the same as the metric defined in the plane. At an interior point of an edge, intrinsically the surface is just a flat surface folded, and so its metric is the same as the metric in the plane. If any compact topological metric space has the property that all but a finite number of points have neighborhoods that are isometric to open subsets of the Euclidian plane, and the other points have neighborhoods isometric to a cone over a circle with the metric explained above, then Alexandrov defines this as a *convex polyhedral metric*. The following existence theorem of Alexandrov is related closely to a rigidity result that will be explained later.

> ALEXANDROV [1958]: *Any convex polyhedral metric given on a manifold homeomorphic to a sphere is realizable as a*

compact convex polytope, possibly degenerating into a doubly covered plane polygon.

This result is also an important steppingstone on the way to proving the following result of Pogorelov, which he called a "monotypy" theorem.

POGORELOV[1973]: *Isometric closed convex surfaces in three-space are congruent.*

This very powerful result applies equally well to the smooth category as to the piecewise-linear category. Note how these results concentrate on the intrinsic metric rather than on any other further underlying structure. In Cauchy's theorem, for instance, there was the further structure of the faces of the polytope, which act as rigid "plates". There are many more such important results in POGORELOV [1973].

3. BASIC DEFINITIONS AND BASIC RESULTS

In order to explain the results that follow and to provide a more unified and precise point of view in the discrete category, we discuss the concept of a (tensegrity) framework and the various kinds of associated rigidity. This also serves to set the notation. The discussion in Section 3.1 to Section 3.3 has evolved from the point of view of ALEXANDROV [1958], GLUCK [1975], ASIMOW AND ROTH [1978], ASIMOW AND ROTH [1979], ROTH AND WHITELEY [1981], CRAPO AND WHITELEY [1982], which was again reformulated in CONNELLY [1988a], CONNELLY [1988b].

3.1 Frameworks and Tensegrity Frameworks. A *configuration* is a finite collection $\mathbf{p}$ of n labeled points $(p_1, p_2, ..., p_n)$, where each point p_i (also called a *vertex*) is in a fixed Euclidean space $\mathbf{R}^d$. A configuration $\mathbf{p} = (p_1, p_2, ..., p_n)$ and a configuration $\mathbf{q} = (q_1, q_2, ..., q_n)$ are *congruent* if there is a congruence of $\mathbf{R}^d$ that takes p_i to q_i for all $i = 1, 2, ..., n$.

A *tensegrity graph* G is an abstract graph on the vertices $1, 2, ..., n$ where each edge is labeled as either a cable, a strut, or a bar. The idea is that cables cannot increase in length, struts cannot decrease in length, and bars cannot change in length at all. The pair G and $\mathbf{p}$ is called a *tensegrity framework* and it is denoted as $G(\mathbf{p})$.

In case all edges (or *members*) of the graph G are only bars, $G(\mathbf{p})$ will be called a *bar* framework. It often turns out that many of the geometric situations that come up are conveniently and reasonably expressed as statements about tensegrity frameworks even if the primary object of interest is a bar framework.

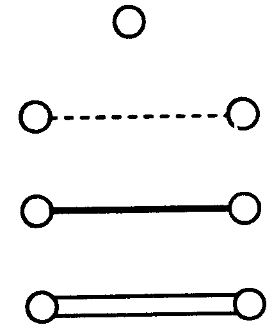
It is also pleasant to show a framework visually. We use the following notation.

A vertex in the configuration is represented by

A cable is represented by

A bar is represented by

A strut is represented by



In Figure 3.1.1 we show some examples, a tensegrity framework and two bar frameworks in the plane. Note that there is no requirement that a vertex of the configuration be placed at the intersection where two members (thought of as line segments) cross.

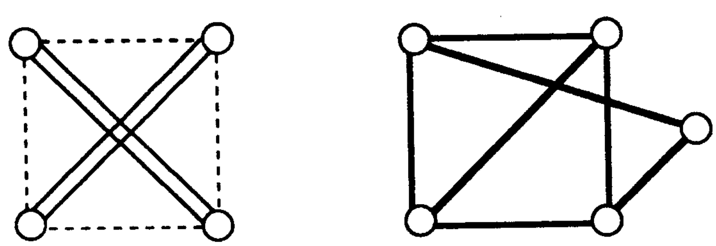


Figure 3.1.1

3.2 Rigidity and Global Rigidity. We say that a tensegrity framework $G(\mathbf{p})$ *dominates* another tensegrity framework $G(\mathbf{q})$ (with the same graph), if the length of each cable of $G(\mathbf{q})$ is no longer than the length of the corresponding cable of $G(\mathbf{p})$, the length of each strut of $G(\mathbf{q})$ is no shorter than the length of the corresponding strut of $G(\mathbf{p})$, and the length of each bar of $G(\mathbf{q})$ is the same length as the corresponding bar of $G(\mathbf{p})$. Here if $\{i,j\}$ is a member of G, then $|p_i - p_j|$ is its length, where we use vector notation throughout.

For a fixed graph G it is clear that this relation of dominance is a only a partial ordering. On the other hand, when all the members of G are bars, this relation is an equivalence relation. So for bar frameworks, we say that $G(\mathbf{p})$ is *equivalent* to another bar framework $G(\mathbf{q})$ when one of them dominates the other by the above definition. In other words, $G(\mathbf{p})$ is *equivalent* to $G(\mathbf{q})$ when the corresponding bars have the same length. Note that when two configurations $\mathbf{p}$ and $\mathbf{q}$ are congruent, then

G(**p**) always dominates G(**q**), and indeed G(**p**) is equivalent to G(**q**) for bar frameworks.

Fix a particular tensegrity framework G(**p**) in $\mathbf{R}^d$. We say that G(**p**) is *globally rigid* in $\mathbf{R}^d$ if, when G(**p**) dominates another tensegrity framework G(**q**) in $\mathbf{R}^d$, then **p** and **q** are congruent. For example, in Figure 3.1.1, the tensegrity framework on the left is globally rigid in the plane, and the other two bar frameworks are not. The framework on the right even has a continuous family of mutually non-congruent equivalent configurations. The framework in the middle has equivalent non-congruent configurations that involve the change of only the rightmost vertex, for example.

Global rigidity is a very simple and natural concept, but it is also a very strong property to impose on a framework. We will see later how it comes up in the discussion of the proof of Cauchy's Arm Lemma as well as in the resolution of some conjectures in GRÜNBAUM AND SHEPHARD [1975].

Meanwhile, a very basic property of frameworks is simply their rigidity. We say that the family of configurations **p**(t) in $\mathbf{R}^d$, for $0 \le t \le 1$, is a *(continuous) flex* of the (tensegrity) framework G(**p**), if each coordinate of each vertex is a continuous function of t, **p**(0) = **p**, and G(**p**) dominates each G(**p**(t)) for all $0 \le t \le 1$. We say that the flex **p**(t) is *trivial* if each **p**(t) is congruent to **p**. If G(**p**) admits only trivial flexes, then we say that G(**p**) is *rigid in* $\mathbf{R}^d$. For example, the left framework and the middle framework in Figure 3.1.1 are rigid in the plane, whereas the rightmost framework is not. Clearly if a (tensegrity) framework is globally rigid in $\mathbf{R}^d$, then it is rigid in $\mathbf{R}^d$. The middle framework in Figure 3.1.1 is a counter-example to the converse statement. See the discussion in CONNELLY [1988a] for some equivalent definitions.

The statement of Cauchy's Theorem can be reformulated in terms of frameworks and configurations. Each vertex of the polytope is a vertex of the configuration, and each edge of the polytope corresponds to a bar of the framework. But since each face is to be regarded as a rigid plate, it is convenient to specialize to the case when each face of the polytope is a triangle. The conclusion of Cauchy's Theorem is a statement similar to the property of global rigidity, except that the configurations are restricted to those coming from convex polytopes with triangular faces instead of all configurations in three-space. In any case, at least in the triangular face case, it turns out that Cauchy's Theorem does imply that the corresponding framework is rigid in three-space. We shall have more to say about this in Section 4.

3.3 Infinitesimal Rigidity. Even if one's main interest is only the question of whether a certain class of frameworks is rigid, one needs some methods. A very natural technique is to "linearize" the problem. That is, one tries to find some system of linear equations (or linear inequalities) that will detect rigidity. One approach is to "differentiate" the member constraints.

An *infinitesimal flex* of a framework $G(\mathbf{p})$ in $\mathbf{R}^d$ is an assignment of a vector p_i' in $\mathbf{R}^d$ to each vertex i of G such that for each member $\{i,j\}$ of G,

$$(3.1) \qquad (p_i - p_j)\cdot(p_i' - p_j') \begin{cases} \leq 0 & \text{if } \{i,j\} \text{ is a cable} \\ = 0 & \text{if } \{i,j\} \text{ is a bar} \\ \geq 0 & \text{if } \{i,j\} \text{ is a strut.} \end{cases}$$

The product that is used for the vectors above is the ordinary dot product (or inner product) in $\mathbf{R}^d$. We denote the infinitesimal flex by $\mathbf{p}' = (p'_1, p'_2, \dots, p'_n)$, another configuration. We imagine p_i' as a vector based at p_i. We say that $\mathbf{p}'$ is *trivial*, if $\mathbf{p}'$ is the derivative of a rigid congruence of all of $\mathbf{R}^d$ restricted to the vertices of $\mathbf{p}$ at $t=0$. It turns out in dimension three that $\mathbf{p}'$ is trivial if, for each i, $p_i' = r \times p_i + u$, where r and u are fixed vectors in three-space and $\times$ represents the cross product. See CONNELLY [1988a] for a further description of such trivial infinitesimal flexes. We say that a (tensegrity) framework $G(\mathbf{p})$ in $\mathbf{R}^d$ is *infinitesimally rigid* if it has only trivial infinitesimal flexes.

So $\mathbf{p}'$ is analogous to a vector field on a smooth manifold (in the smooth category). If we have a flex $\mathbf{p}(t)$ of a framework $G(\mathbf{p})$, and each of the coordinates of $\mathbf{p}(t)$ are differentiable at $t=0$, then (3.1) holds where $\mathbf{p}'$ is the derivative of $\mathbf{p}(t)$ at $t=0$. The square of the member length $\{i,j\}$ is $(p_i(t) - p_j(t))\cdot(p_i(t) - p_j(t))$. The derivative evaluated at $t=0$ gives (3.1). One must be careful though. Even if the flex $\mathbf{p}(t)$ is non-trivial, it may turn out that the infinitesimal flex $\mathbf{p}'$ *is* trivial. Simply replace the parameter t by t^2, for example.

In a visual representation of a framework, we will use a small arrow at p_i to represent an infinitesimal flex p_i'. If there is no arrow it will be understood that $p_i' = 0$. Figure 3.3.1 shows two examples of bar frameworks in the plane that have non-trivial infinitesimal flexes.

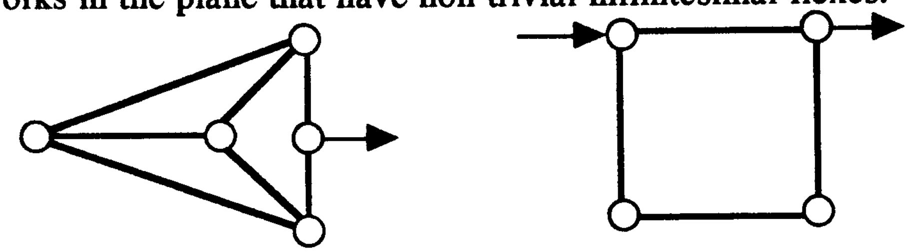


Figure 3.3.1

The infinitesimal flex for the framework on the right comes from a smooth flex of the framework. For the framework on the left, there is no such (smooth) flex. In other words, both frameworks are not infinitesimally rigid in the plane, but the one on the left is rigid nevertheless.

The following basic result helps to justify the study of infinitesimal rigidity. See CONNELLY [1988a] as well as ROTH AND WHITELEY [1981] and CONNELLY [1980] for an explicit proof.

> *If a tensegrity framework $G(p)$ is infinitesimally rigid in $\mathbf{R}^d$ then it is rigid $\mathbf{R}^d$.*

As far as I know there is no analogous result in the smooth category, although both the notions of rigidity and infinitesimal rigidity have appropriate analogues.

3.4 Static Rigidity. The concept of infinitesimal rigidity seems to be motivated by the "kinematics" of a framework, that is by its motions (even if they are "infinitesimal"). For engineering structures though, often another point of view is taken. The idea is that the "stability" of a structure should have more to do with forces or "loads" acting on it and the capability of the structure to "resolve" these forces.

We say that a *load* or *force* $\mathbf{F}$ acting on a framework $G(\mathbf{p})$ in $\mathbf{R}^d$ is an assignment of a vector F_i in $\mathbf{R}^d$ to each vertex i of G. Formally $\mathbf{F}$ is simply another configuration in $\mathbf{R}^d$, but the words are meant to suggest a physical force. Note that the force $\mathbf{F}$ is really defined for the configuration $\mathbf{p}$, independently of the graph G.

We say that a *stress* ω for a framework $G(\mathbf{p})$ in $\mathbf{R}^d$ is an assignment of a real number $\omega_{ij} = \omega_{ji}$ for each member $\{i,j\}$ of G. We regard ω_{ij} as the tension or compression in the member $\{i,j\}$. We say that ω *resolves* a load $\mathbf{F}$ if the following vector equation holds for each vertex i of G,

$$F_i + \sum_j \omega_{ij}(p_j - p_i) = 0, \tag{3.2}$$

where the summation is taken over all the vertices j adjacent to i in G. (Alternatively we can define $\omega_{ij} = 0$ for each non-member of G, and sum over all j). Equation (3.2) represents an equilibrium at each vertex i.

It turns out that "most" forces cannot be resolved by any framework. So we say that a force $\mathbf{F}$ is an *equilibrium force* (or an *equilibrium load*) at the configuration $\mathbf{p}$ in $\mathbf{R}^d$ if for every trivial infinitesimal flex $\mathbf{p}'$ at the configuration $\mathbf{p}$, the following holds:

$$\sum_i F_i \cdot p_i' = 0,$$

where the summation is taken over all vertices i. Physically this means that when $\mathbf{F}$ is regarded as giving velocities at the configuration $\mathbf{p}$, it has zero linear and angular momentum. See CONNELLY [1988b] or ROTH AND WHITELEY [1981] or CRAPO AND WHITELEY [1982] for a discussion of other equivalent conditions and reformulations for what it means to be an equilibrium force.

Now it turns out that it is possible for some frameworks to resolve an equilibrium load. On the other hand, a cable cannot support compression and a strut cannot support tension, so we define one further restriction on stresses. We say that a stress ω for a framework $G(\mathbf{p})$ is *proper* if $\omega_{ij} \geq 0$ when $\{i,j\}$ is a cable, and $\omega_{ij} \leq 0$ when $\{i,j\}$ is a strut. (There is no condition when $\{i,j\}$ is a bar.)

We say that a tensegrity framework $G(\mathbf{p})$ in $\mathbf{R}^d$ is *statically rigid* if every equilibrium load $\mathbf{F}$ can be resolved by a proper stress ω. The frameworks in Figure 3.4.1 have equilibrium loads indicated. The framework on the left is not statically rigid, and the given load cannot be resolved The framework on the right is statically rigid, and the resolving proper stress is indicated.

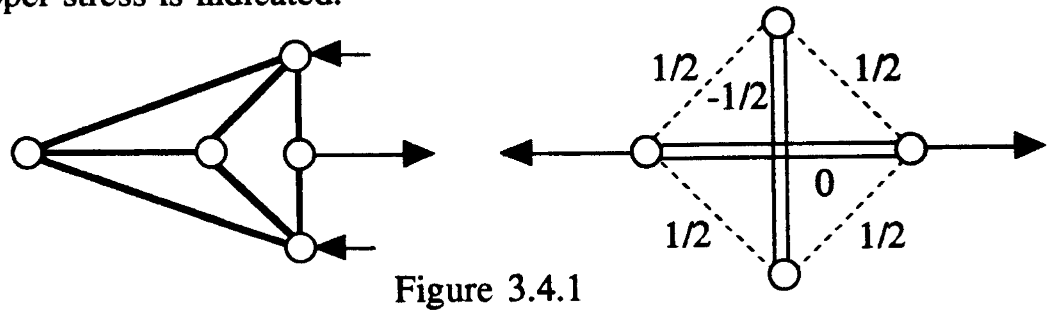


Figure 3.4.1

A very basic result now connects this definition to infinitesimal rigidity. See CONNELLY [1988b] or ROTH AND WHITELEY [1981] for a proof. Section 3.6 has some information concerning a proof.

(3.3) *A tensegrity framework $G(p)$ in R^d is infinitesimally rigid if and only if it is statically rigid.*

We also mention one more result that is very useful for relating the static rigidity (or the infinitesimal rigidity) of a tensegrity framework to the static rigidity of a bar framework. A *self stress* for a tensegrity framework $G(p)$ is a stress that resolves the zero load. So (3.2) holds with $F = 0$ The following is the main result of ROTH AND WHITELEY [1981].

(3.4) *Let $G(p)$ be a tensegrity framework in R^d. Then $G(p)$ is statically rigid in R^d if and only if there is a proper self stress that is non-zero on each cable and strut, and $G'(p)$ is statically rigid, where G' is obtained from G by replacing each member with a bar.*

3.5 Projective Invariance. It is clear that the definitions of infinitesimal rigidity and static rigidity for a framework $G(\mathbf{p})$ do not depend on the way in which the configuration $\mathbf{p}$ is realized in $\mathbf{R}^d$. In other words, if each of the points p_i of $\mathbf{p}$ is transformed by the same congruence $g: \mathbf{R}^d \to \mathbf{R}^d$, then the framework $G(\mathbf{p})$ is statically and infinitesimally rigid if and only if $G(g(\mathbf{p}))$ is statically and infinitesimally rigid, where $g(\mathbf{p}) = (g(p_1), ..., g(p_n))$. With a bit more thought it is easy to see directly that at least static rigidity is preserved when g is only affine linear. (Of course then infinitesimal rigidity is preserved as well.) However, even more is true.

Suppose that the points of one configuration **p** lie in an affine (d-1)-dimensional subspace S of some larger dimensional space. Let c be a point that does not lie in S, and let S' be another (d-1)-dimensional subspace that does not contain c. Then we can project the points of **p** from c to the points of another configuration **q** in S'. See Figure 3.5.1.

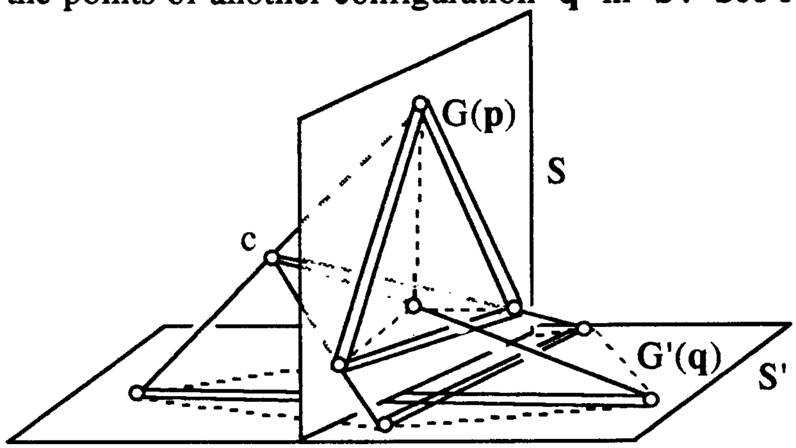


Figure 3.5.1

Notice that a cable may be transformed into either a cable or a strut. To see which, suppose $g(p_i) = q_i$ and $g(p_j) = q_j$, where $\{i,j\}$ is a member of G. If c lies between p_i and q_i but not between p_j and q_j, then $\{i,j\}$ is changed from a cable to a strut or from a strut to a cable. Otherwise, when c is not between p_i and q_i and not between p_j and q_j (or c is between p_i and q_i and is between p_j and q_j) $\{i,j\}$ is left unchanged. This defines a rule for changing the tensegrity graph G to another tensegrity graph G'. Then the framework G(**p**) is statically and infinitesimally rigid in S if and only if G'(g(**p**)) is statically and infinitesimally rigid in S'. See ROTH AND WHITELEY [1981] for a proof in this general situation of tensegrity frameworks. For the case of a bar framework there have been several proofs from several different points of view. For example see WUNDERLICH [1977], CRAPO AND WHITELEY [1982b], WHITELEY [1987a], and WEGNER [1985].

In fact, with the rule described above, it is not hard to see that one does not even have to have the vertices of the configurations lie on a hyperplane. Let c∗G(**p**) denote the framework obtained by joining the vertex c with a bar to all the vertices of some framework G(**p**) in $\mathbf{R}^d$.

Suppose that $G(\mathbf{p})$ lies in a hyperplane S as before, but the points of $\mathbf{q}$ and c are only required, for each i, to be such that p_i, q_i, and c are colinear, and c is distinct from both p_i and q_i. Then we have the following:

(3.5) *There are natural linear isomorphisms from the infinitesimal flexes of $G(p)$ in S to the infinitesimal flexes of $c*G(p)$ in R^d, fixing c, to the infinitesimal flexes of $c*G'(q)$ in R^d, fixing c.*

In particular, we can see that $G(\mathbf{p})$ is infinitesimally rigid in the hyperplane S if an only if a projection of $\mathbf{p}$ into the unit sphere in $\mathbf{R}^d$ is infinitesimally rigid in the sphere. (I.e. it is infinitesimally rigid in the sense that all the infinitesimal flexes tangent to the sphere are the time 0 derivative of a rigid congruence of the sphere.)

We can see that it is reasonable to look at the problem of static and infinitesimal rigidity, at least for bar frameworks, from the point of view of projective geometry. This is essentially what is done in CRAPO AND WHITELEY [1982b]. The property of static rigidity is reformulated entirely in terms of homogeneous coordinates of projective space. The question of the projective invariance is incorporated in the definitions and terms of a particular algebra, the Cayley Algebra, that is is especially convenient for projective geometry. One can then use the Cayley algebra to get some general algebraic information about those special configurations of bar frameworks whose space of self stress is of a larger dimension than the minimum possible for the given graph. This is done in WHITE AND WHITELEY [1983]. A similar approach is taken in WHITE AND WHITELEY [1987] but for infinitesimal motions.

But one other case arises. In projective geometry, if we regard projective space as extended Euclidean space, there are points "at infinity". Suppose some points of our configuration are at infinity and they are to be held fixed for infinitesimal flexes. Then in the linear equations (3.1) that define infinitesimal rigidity we can replace the vector $p_i - p_j$ by a unit vector if p_j is at infinity and p_i is not. Note that $p_j' = 0$. In some cases it is convenient to transform the configuration in such a way that some of the points are at infinity, and, modulo trivial infinitesimal flexes, we can often assume that these points at infinity are fixed infinitesimally. This will be useful in Section 4.5. This is also done in the smooth category where the projective transformation is called a Darboux Transformation. See DARBOUX [1896].

3.6 The Rigidity Map. We can shed a little light on what is going on in Theorem (3.3). Regard a configuration $\mathbf{p}$ of n points in $\mathbf{R}^d$ as a single point in $\mathbf{R}^{nd}$, the concatenation of the n column vectors of $\mathbf{R}^d$ into one single column vector. Suppose that a tensegrity graph G has e members . To each configuration $\mathbf{p}$ and each member $\{i,j\}$ of G we associate the square of its length $|p_j - p_i|^2$. This defines a smooth map, which we call the *rigidity map*,

$$f: \mathbf{R}^{nd} \to \mathbf{R}^e$$

given by

$$f(\mathbf{p}) = (\ldots, |p_j - p_i|^2, \ldots).$$

The differential of this map is a linear map $df_{\mathbf{p}} : \mathbf{R}^{nd} \to \mathbf{R}^e$. We define the matrix (divided by 2) of this map (with respect to the usual bases) to be the *rigidity matrix* $R(\mathbf{p})$. Each row of $R(\mathbf{p})$ corresponds to a member $\{i,j\}$ of G. The columns of $R(\mathbf{p})$ are organized into n sets of d columns. Furthermore, row $\{i,j\}$ of $R(\mathbf{p})$ consists of all zeros except for the two sets of d entries corresponding to i and j. The entries corresponding to i are the coordinates of $p_i - p_j$ and the entries corresponding to j are the coordinates of $p_j - p_i$. It is easy to see that if $\mathbf{q}$ is another configuration, again regarded as a column vector in $\mathbf{R}^{nd}$, then the $\{i.j\}$ coordinate of $R(\mathbf{p})\mathbf{q}$ is simply $(p_i - p_j)\cdot(q_i - q_j)$. Thus $\mathbf{p}'$ is an infinitesimal flex of $G(\mathbf{p})$ if and only if the $\{i,j\}$ coordinate of $R(\mathbf{p})\mathbf{p}'$ is non-positive for a cable, non-negative for a strut, and zero for a bar. For a bar framework this is simply the condition $R(\mathbf{p})\mathbf{p}' = 0$.

As mentioned in Section 3.4 any tensegrity framework $G(\mathbf{p})$ has the trivial infinitesimal flexes. These will always be in the kernel of $R(\mathbf{p})$. In $\mathbf{R}^d$ the dimension of the space of trivial infinitesimal flexes is $d(d+1)/2$. When $d=2$, the dimension of trivial infinitesimal flexes is 3, and when $d=3$, the dimension of trivial infinitesimal flexes is 6. We have the following interpretation of infinitesimal and static rigidity. See ASIMOW AND ROTH [1978], ASIMOW AND ROTH [1979], and CONNELLY [1988b] for a discussion of this result.

Let $G(p)$ be a bar framework in R^d, where G has n vertices and they do not all lie in a hyperplane. Then $G(p)$ is infinitesimally rigid if and only if the rank of $R(p)$ is $nd - d(d+1)/2$.

So in the plane, for infinitesimal rigidity, this rank must be $2n-3$, and in three-space it must be $3n-6$.

The e rows of the rigidity matrix $R(p)$ correspond to the members of G and if the rank of $R(p)$ is to be maximal (and $G(p)$ is to be infinitesimally rigid), then we must have

(3.6) $$e \geq nd - d(d+1)/2.$$

When G has all bars, if $e = nd-d(d+1)/2$, then the rows of $R(p)$ are independent if and only if $G(p)$ is infinitesimally rigid. The row relations are related to stresses.

It is easy to check that a force $\mathbf{F}$, regarded as an e-dimensional row vector, is resolved by the stress ω, also regarded as an e-dimensional row vector, if and only if $\mathbf{F} + \omega R(p) = 0$. This is one approach to the duality statement of (3.3). Recall that a stress ω for a framework $G(p)$ is a self stress if the equilibrium equation as in (3.2) holds at each vertex i with $F_i = 0$. Then $\omega R(p) = 0$ is equivalent to (3.2) for each i. Puting this information together, we get the following:

(3.7) *Let $G(p)$ be a bar framework in R^d with n vertices and e edges, where $e = nd - d(d+1)/2$. Then $G(p)$ is infinitesimally rigid if and only if $G(p)$ has only the zero self stress.*

This is one approach to an infinitesimal version of Cauchy's Theorem, which we will discuss in Section 4.2.

4. INFINITESIMAL AND STATIC RIGIDITY RELATED TO SURFACES

Now that we have the language of frameworks and some of the basic tools, we can apply them to some problems involving convexity.

4.1 Grünbaum and Shephard's Conjectures. In GRÜNBAUM AND SHEPHARD [1975] as well as GRÜNBAUM AND SHEPHARD [1978] there is a very interesting discussion of the rigidity of tensegrity frameworks in the plane. In particular, their discussion relates to some of the problems concerning Cauchy's Theorem and his Arm Lemma. Although Grünbaum and Shephard did not put their questions exactly into the language of frameworks, it is natural to do so.

Suppose that $\mathbf{p}$ is a configuration of points in the plane such that the vertices $(p_1, p_2, \ldots, p_n) = \mathbf{p}$ form the vertices, in order, of a convex polygon. (This also means of course that no three vertices are colinear.) Suppose that the edges of the polygon $\{1,2\}, \{2,3\}, \ldots, \{n\text{-}1,n\}, \{n,1\}$ are bars. Grünbaum and Shephard conjectured that for various ways of adding extra cables (which are necessarily on the inside of the polygon) the resulting framework was rigid in the plane. The two frameworks on the left in Figure 4.1.1 are some examples. The framework on the right is the same as the framework in the middle, except that the bars and cables have been interchanged.

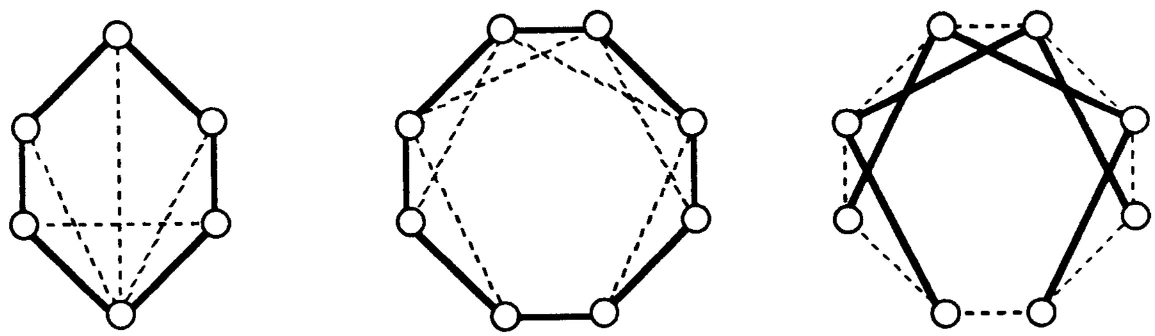


Figure 4.1.1

It turns out that all three of the above frameworks are rigid in the plane. Grünbaum and Shephard conjectured the following:

(4.1) *Let G(p) be a rigid tensegrity framework in the plane, where the vertices form a convex polygon, all the external edges are bars, and cables are the only other members. Then the framework, obtained by replacing all the bars by cables and the cables by bars, is also rigid in the plane.*

For example, this conjecture says that since the framework in the middle of Figure 4.1.1 is rigid, then the framework on the right is rigid. On the other hand the converse statement is not true in general. In Figure 4.1.2 consider the following frameworks in the plane based on a regular hexagon. The framework on the right with cables on the outside is (globally) rigid in the plane, whereas the framework on the left, obtained by interchanging the cables and bars, is not even rigid in the plane. In rigidity there is an important asymmetry between the roles that cables and bars play (or the roles that cables and struts play).

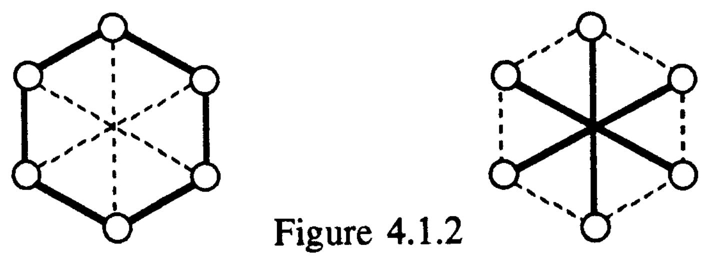


Figure 4.1.2

In CONNELLY [1982] Conjecture (4.1) was proved. It is a corollary of the following theorem. See Section 5.4 for a further discussion.

(4.2) *Let G(p) be a tensegrity framework in the plane, where the vertices form a convex polygon, all the external edges are cables, struts are the only other members, and G(p) has a proper non-zero self stress. Then G(p) is globally rigid in* R^d, *for all* $d \geq 2$.

The framework on the left in Figure 4.1.3 is an example.

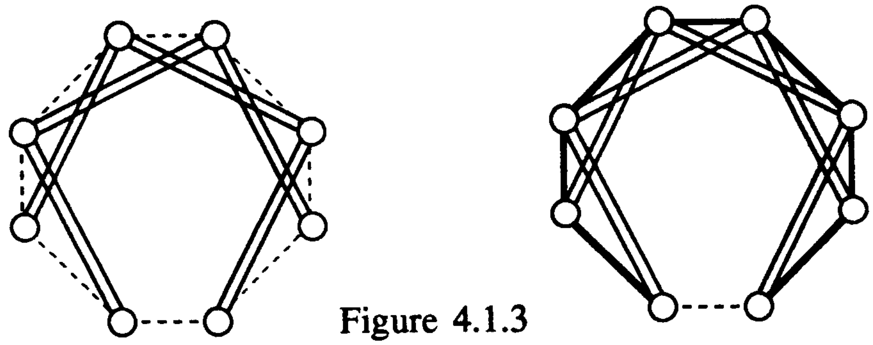


Figure 4.1.3

The framework on the right is also globally rigid in any dimension, since it is obtained from the framework on the left by simply by replacing all but one of the cables by a bar.

It is also shown in CONNELLY [1982] that if a tensegrity framework $G(\mathbf{p})$ is rigid in $\mathbf{R}^d$ and G has at least one cable or strut, then $G(\mathbf{p})$ has a proper non-zero self stress (although it may be zero on the given cable or strut). Thus for the proof of (4.1) we obtain a non-zero self stress ω from the rigidity of $G(\mathbf{p})$. Then $-\omega$ serves as the proper self stress for the framework in (4.2), where the cables and bars are interchanged.

For many frameworks coming from convex polygons it is possible to tell directly that the framework has a proper self stress and thus is globally rigid in the case of (4.2). Let $(p_1, p_2, ..., p_n) = \mathbf{p}$ be a configuration in the plane consisting of the vertices of a convex polygon as before. Let all of the external edges $\{1,2\}, \{2,3\}, ..., \{n\text{-}1,n\}, \{n,1\}$ be the cables, and let $\{1,3\}, \{2,4\}, ..., \{n\text{-}2,n\}$ be the struts. (Notice that two consecutive struts in the cyclic sequence are missing.) I called such a tensegrity framework $G(\mathbf{p})$ a *Cauchy polygon*. The framework on the left in Figure 4.1.3 is an example of such a Cauchy polygon. It turns out (see CONNELLY [1982] and ROTH AND WHITELEY [1981]) that any Cauchy polygon has a non-zero proper self stress. Thus it is always globally rigid.

Consider a polygonal framework such as the one on the right in Figure 4.1.3. When a pair of bars, adjacent to a vertex p_i, have a strut connecting the other two vertices p_{i-1} and p_{i+1}, it is equivalent to the condition that the internal angle at p_i is not permitted to decrease. This is precisely the condition for Cauchy's Arm Lemma. So if there were another configuration (even in three-space), where each angle at $p_2, p_3, ...,$ p_{n-1} increased or stayed the same, the distance between p_i and p_{i+1}

would be fixed for $i=1, 2,...,n-1$, and the distance between p_1 and p_n would decrease, then this would contradict the global rigidity of Cauchy polygons. Thus we have yet another proof and generalization of Cauchy's Arm Lemma as mentioned in Section 2.1.

Many of the frameworks mentioned in GRÜNBAUM AND SHEPHARD [1975] were rigid but not globally rigid. For example, the middle and the left frameworks of Figure 4.1.1 are infinitesimally rigid in the plane by applying (3.4). Thus they are rigid, but it can be easily seen that they are not globally rigid. Also any Cauchy polygon is infinitesimally rigid in the plane. Then, if one changes every cable to a strut and every strut to a cable, it is easy to see that infinitesimal rigidity is preserved. (This is true for the infinitesimal rigidity of any framework.) The frameworks in Figure 4.1.2 naturally are not infinitesimally rigid.

Ben Roth in ROTH AND WHITELEY [1981] suggested another approach to Conjecture (4.1). He conjectured the following:

(4.3) *Let $G(p)$ be a rigid tensegrity framework in the plane, where the vertices form a convex polygon, all the external edges are bars, and cables are the only other members. Then $G(p)$ is infinitesimally rigid in the plane.*

As a corollary we find that the framework $G'(\mathbf{p})$, obtained by reversing the cables and bars, is infinitesimally rigid in the plane. (The bars can be replaced by struts in this case for the infinitesimal theory.) Thus $G'(\mathbf{p})$ is rigid in the plane. So this conjecture by itself implies Grünbaum and Shephard's conjecture (4.1).

In CONNELLY AND WHITELEY [1990] Conjecture (4.3) is proved. It is also related to the result (4.2), but depends on a higher order analysis. An application of this result, for example, is that the hexagon on the left in Figure 4.1.2 is not rigid, since it has too few members to be infinitesimally rigid. We will say more about this in Section 5.

4.2 The Infinitesimal Version of Cauchy's Theorem. In view of our discussion of infinitesimal rigidity, it is natural to ask whether convex polyhedra are infinitesimally rigid in three-space. Given a compact convex polytope P in three-space, one may associate to P a bar framework

G(**p**). The vertices of G(**p**) are the vertices of P, and the edges of G(**p**) are the edges of P. One must be careful though. With this description, it is not automatic that the faces of P are held rigid by the bars of the framework G(**p**). Indeed, if any face of P is not a triangle, the framework G(**p**) is not rigid. (See Section 4.4.) For example, the bar framework associated to a cube is very flexible. Nevertheless, the following result of M. Dehn does show that when the faces are triangular there is no problem.

(4.4) *Let P be a compact convex polytope in three-space with all faces triangles. Then the associated bar framework G(p) is infinitesimally rigid in three-space.*

See DEHN [1916] for his original proof. See also WEYL [1917]. We will also describe a very simple proof that is basically in ALEXANDROV [1958], repeated and clarified in GLUCK [1975] and ASIMOW AND ROTH [1979].

We sketch Dehn's original proof of (4.4). His idea was to calculate the rank of the rigidity matrix R(**p**) (as defined in Section 3.5) by means of a Lagrange expansion of the determinate of the submatrix obtained by removing three rows corresponding to the edges of a triangle of a face of P and removing the nine rows corresponding to the three vertices of the same triangle. (This amounts to fixing those three points.) Each term in this expansion corresponds to an oriented graph on the remaining vertices of G (and conversely) such that there are exactly three edges coming into each vertex. Then it is possible to perform an operation on this oriented graph that creates another oriented graph with the same sign on the corresponding term in the Lagrange expansion. The proof is completed when it is shown that any such oriented graph can be obtained from any other by a sequence of such operations. All the terms must have the same sign; the determinate is non-zero; and the rank of R(**p**) is 3n-6 (where n in the number of vertices of G), which is what is needed to show that G(**p**) is infinitesimally rigid in three-space.

Alexandrov's idea, as discussed in GLUCK [1975], was to calculate the the row rank of R(**p**). Since each face of P is a triangle, if there are e edges in P (and G), then $2e = 3f$, where f is the total number of triangular faces in P. Then Euler's formula for the surface of P states:

$$n - e + 2e/3 = 2,$$

where n is the number of vertices of P. This implies that

$$e = 3n - 6.$$

In dimension three this is precisely the minimum number of edges needed for a graph with n vertices to be infinitesimally rigid. By (3.7) G(**p**) is infinitesimally rigid in $\mathbf{R}^3$ if and only if it has only the zero self stress.

There is a very simple argument to show that a non-zero self stress does not exist. At each vertex of G(**p**) the equilibrium condition (3.2) (with $F_i = 0$) implies that either all the stresses at that vertex are zero, or there are at least four changes in the sign of the stresses ω_{ij} as one proceeds around the point p_i. This is because if there were exactly two changes in sign, then a plane through p_i could be found that separates those edges with a positive stress from those edges with a negative stress. See Figure 4.2.1.

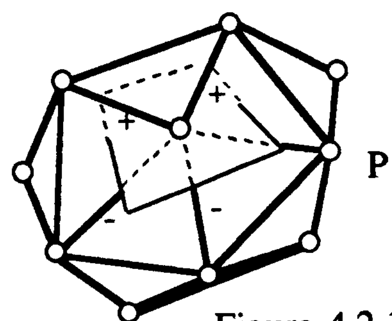


Figure 4.2.1

But that contradicts the equilibrium condition for the vertex i, since in the equilibrium equation (3.2), the vector components perpendicular to the plane all have the same sign Hence the conditions for Cauchy's combinatorial Lemma (2.1) are satisfied. None of the stresses on the edges of G are actually strictly positive or strictly negative. Thus $\omega = 0$, and G(**p**) is statically rigid and infinitesimally rigid.

Notice how the above argument takes the place of Cauchy's somewhat more difficult Arm Lemma. Also, in both cases this is precisely the point where convexity is used in an essential way.

4.3 Alexandrov's Theory. The restriction in Section 4.2 to polyhedra with only faces that are triangles is a bit unsatisfying. Cardboard models with non-triangular faces seem to hold their shape quite well, and there should be some reasonable theory to explain this. On the other hand, we must be careful in the way that we extend Dehn's Theorem (4.4). We wish have the framework G(**p**) based on the polytope P in that the members of P should at least be on the surface of P. A natural idea is to find a triangulation K of the surface of P, and the vertices and edges of K will give the framework G(**p**). By a triangulation of P we mean a simplicial complex K (in the sense of piecewise linear topology), whose underlying space is exactly the set P. (In this circumstance, a *simplicial complex* K is simply a finite collection of triangles together with their edges and vertices, where any two of these triangles intersect at a common vertex, edge, or empty set. The *underlying space of K*, denoted by |K|, is the union of the closed triangles.) Unfortunately, though, if any of the vertices of G(**p**) lie in the interior of a face of P, then G(**p**) will not be infinitesimally rigid. For example, choose an infinitesimal flex **p'** where the only non-zero vector of **p'** corresponds to a vertex in the interior of a face of P, and is perpendicular to the plane of that face. See Figure 4.3.1.

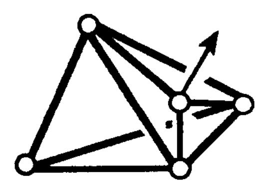


Figure 4.3.1

Notice that in Dehn's Theorem (4.2), since each face is a triangle, there is a well-defined triangulation associated to the polytope P, assuming that there are no vertices in K and G(**p**) beyond those in P.

The following theorem of ALEXANDROV [1958] gives a good description of when convex polytopes can be used for infinitesimal rigidity.

(4.5) *Let K be a triangulation of a convex polytope P in three-space such that no vertex of K lies in the interior of a face of P. Then the bar framework G(p), obtained from the vertices and edges of K, is infinitesimally rigid in three-space.*

For example, Figure 4.3.2 shows two triangulations of a cube that are infinitesimally rigid by Alexandrov's Theorem. The framework on the left has only the original vertices of P as vertices of G(**p**). The framework on the right has some extra vertices placed on the edges of P.

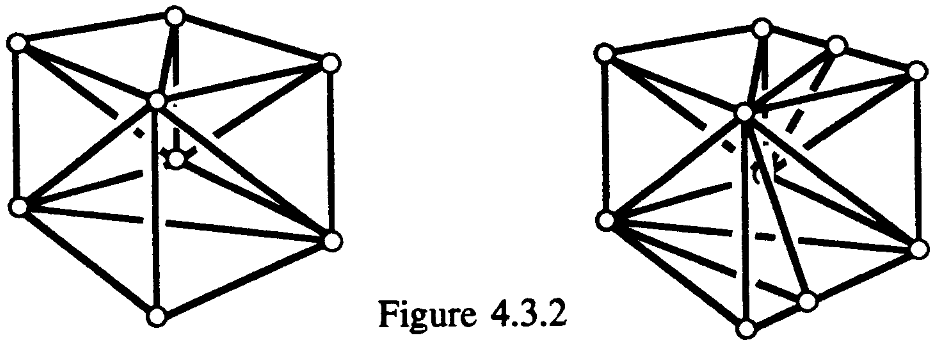


Figure 4.3.2

The proof of Alexandrov's Theorem (4.5) is more delicate than one might first imagine. In this situation the argument that is outlined in Section 4.2 has some difficulties in that the separating plane is not evident. There might possibly be exactly two changes in sign for the self stress, where all the non-zero stresses occur in one face of P. Nevertheless, it is still possible to prove infinitesimal rigidity in this situation using roughly similar ideas. See ASIMOW AND ROTH [1979] as well as WHITELEY [1984] for careful proofs. There are also some extensions of Alexandrov's Theorem to higher dimensions in WHITELEY [1984]. (See also Section 4.8.)

The techniques used in GLUCK [1975] are not adequate to show this version of Alexandrov's Theorem. Gluck defines a simplicial complex K, triangulating the convex polytope P, as *strictly convex* if for each vertex p_i of K there is a plane meeting P only at p_i. (So the triangulation on the left of Figure 4.3.2 is strictly convex, but the one on the right is not strictly convex.) Unfortunately in Gluck's proof of his separating Lemma 5.3, his definition of strict convexity is not enough to insure that the edges with positive and negative stresses can be separated by a plane. (Branko Grünbaum was the first to point out this problem with GLUCK [1975].) However, when each face of P is a triangle, Gluck's description is more than adequate. This is also enough for Gluck's main result that G(**p**) is rigid for an open dense set of configurations **p**.

4.4 Consequences of Alexandrov's Theory. There are several corollaries/extensions of Alexandrov's result (4.5). Most of these results follow from the statement of (4.5) without too much extra work, or one can extend one's favorite proof of (4.5). The following are some examples. For more details see CONNELLY [1980], WHITELEY [1984], ASIMOW AND ROTH [1979], and ROTH [1987].

(4.6) *Let $G(p)$ be a (tensegrity) framework where its vertices include the vertices and lie on the edges of a convex polytope P in three-space; all members of $G(p)$ lie on the surface of P; and for each face F of P the framework, determined by the vertices and members of $G(p)$ that lie in F, is infinitesimally rigid in the plane of F. Then $G(p)$ is infinitesimally rigid in three-space.*

For example, in Figure 4.4.1 shows some tensegrity frameworks in three-space that are infinitesimally rigid by this result. The planar tensegrity frameworks are infinitesimally rigid by the results in Section 4.1. Only the near faces of the dodecahedron on the right are shown.

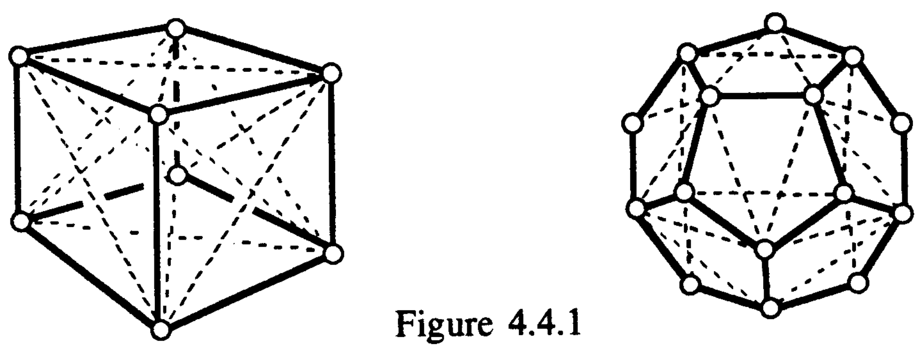


Figure 4.4.1

In these examples the bars can be replaced by struts and the frameworks will still be infinitesimally rigid. These examples were taken from GRÜNBAUM AND SHEPHARD [1975].

The principle used for the result (4.6) is that the infinitesimal rigidity of whole structure does not depend on the particular framework that "holds" the vertices of each face infinitesimally rigid in its plane. This "subframework" provides the appropriate implied bars needed. (A bar is *implied* between two vertices i and j if adding the corresponding condition for $\mathbf{p}'$ for the bar $\{i,j\}$ does not change the space of infinitesimal flexes.) Alexandrov's Theorem (4.5) provides one such framework that is infinitesimally rigid in three-space, and so we get the result (4.6). Other cell complexes with planar faces would work as well, as long as the analogue of (4.5) holds.

Another interesting application of (4.5) is show that certain frameworks are *not* rigid. Suppose that G(p) is a bar framework in $\mathbf{R}^d$ (with $n \geq d+1$) which is infinitesimally rigid, but with the just the minimum number of edges needed, namely $nd-d(d+1)/2$. When $d=3$ this number is $3n-6$, of course. What happens when one of the bars is removed? Certainly the framework does not remain infinitesimally rigid, but we can say more. From the discussion in Section 3.6, the rigidity map f for G into the space of edge lengths is locally onto. This is because the differential of f, the rigidity matrix R(p), is onto since its rank $nd-d(d+1)/2$ is equal to the number of edges of G. This means that any perturbation of the edge lengths, that is small enough, will correspond to a configuration with those given lengths. So if one edge length is shortened or lengthened continuously, there will be a configuration realizing those given lengths. It is easy to show (using the inverse function theorem) that the configuration can be chosen to vary continuously as well. Thus when one bar is removed from G(p) the resulting framework is flexible. For example, when one bar is removed from either of the frameworks of Figure 4.3.2, it becomes flexible. Also ideas similar to the those above show that if any one of the cables of the cube on the left in Figure 4.4.1 is removed, then the framework becomes flexible. These ideas essentially come from ALEXANDROV [1958] and are used extensively in ASIMOW AND ROTH [1978]. See also WHITELEY [1988].

4.5 E. Kann's Proof and Extension of Dehn's Theorem. In the last several years there has been some interest in proving the infinitesimal version of Cauchy's Theorem, Dehn's Theorem. This has not been because of any lack of confidence in the result, but because there was interest in finding a generalization in some particular direction or because the result followed from some other techniques.

In KANN [1970] there is a proof of the infinitesimal rigidity of C^2 smooth convex surfaces with positive Gaussian curvature. This follows the results of MINAGAWA AND RADO [1952] and BLASCHKE [1967].

In KANN [1990] this idea is applied to "polyhedral" surfaces in three-space. We define a *polygonal region* as a compact polygonal disk in a plane. We say that K is a *polyhedral network* if it is a finite collection of polyhedral regions such that they have disjoint interiors, and any two such intersect either at a common vertex, a common edge, or in the empty set. Each polygonal region in K has its own affine linear structure, and any two of these structures agree on the overlap on their boundaries. We may think of K as something like a generalized simplicial complex. The *underlying space* of K, denoted by |K|, is what you get by gluing together the polyhedral regions along their boundaries. Figure 4.5.1 shows such a polyhedral network when the underlying space is naturally identified with a disk in the plane.

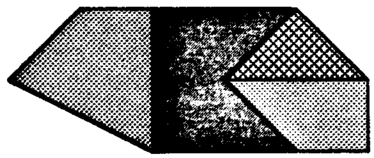


Figure 4.5.1

Suppose for simplicity that the underlying space |K| is a connected orientable two dimensional surface with boundary, i.e. a sphere with handles and holes. Let g be a continuous function from the underlying space of K to three-space, where g is affine linear on each polygonal region. We will say that g represents a *polyhedral cap* if

i. The affine map g is one-to-one when restricted to each polygonal region;

ii. The directed normals to the singular surface given by g lie in a fixed hemisphere. In other words, for each polygonal region the outward pointing normal N_i has $N_i \cdot e > 0$, where e is a fixed vector in three-space;

iii. The images under g of adjacent polygonal regions are *edge-convex*. That is the interior one of the regions is contained in the open positive half-space determined by the normal to the plane of the other region.

Notice that condition iii is milder than the usual global form of convexity. We regard the image under g as a singular surface in that there can be a variety of branch points and self-intersections. Figure 4.5.2 shows some examples. The surface on the left is an example of singularity. The surface on the right is a spiral trough.

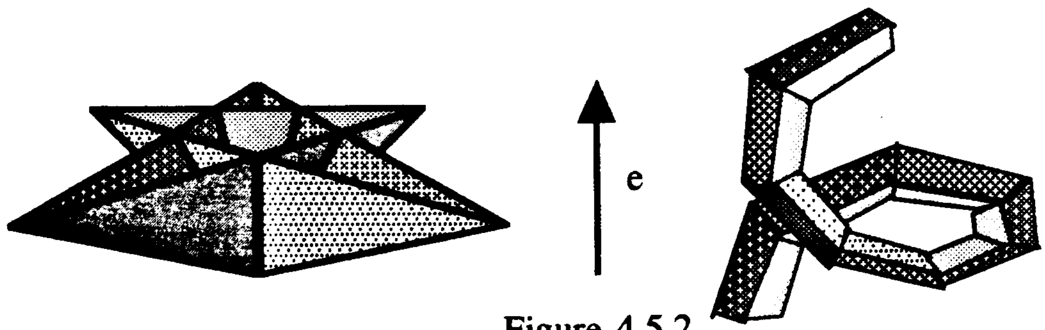


Figure 4.5.2

We can now state one form of E. Kann's Theorem that is a generalization of Dehn's Theorem about infinitesimal rigidity.

(4.7) *Let C be a polyhedral cap satisfying i, ii, and iii, and let* $\mathbf{p}'$ *be an infinitesimal flex such that* $\mathbf{p}'$ *is trivial on each polygonal region and* $p'_i \cdot e$ *= constant for every vertex* p_i *on each boundary component of C separately. Then* $\mathbf{p}'$ *is a trivial infinitesimal flex.*

For example, if the boundary of C has just one component, then by adding a trivial flex to $\mathbf{p}'$ we may assume that $p'_i \cdot e = 0$ for each boundary component. This is called the *glidebending condition.*

Despite the special role that the boundary plays, the statement (4.7) still does specialize to Dehn's result about the infinitesimal rigidity of triangulated polyhedra (4.4). (Indeed, it is even possible to prove the more refined Alexandrov version (4.5) using these techniques.) To see this start with one vertex of a triangulated convex polytope. As we mentioned in Section 3.5, the infinitesimal rigidity of a bar framework is invariant under projective transformations, even when a point is sent to the "plane at infinity" under the transformation. It is always possible to find a projective transformation that sends exactly one vertex (and no other vertices or edges) of the polytope to the plane at infinity. Fix that point at infinity, so the infinitesimal flex is zero at that point. Remove the vertex now at infinity. The object that one obtains is now a polyhedral cap that satisfies the glidebending condition as well as the local convexity. Thus it is rigid by (4.7).

We sketch a proof of (4.7). Recall from Section 3.3 that a trivial infinitesimal flex $\mathbf{p}'$ has the form, for each i, $p'_i = r \times p_i + u$, where r and u are fixed vectors in three-space. The vector r corresponds to an infinitesimal rotation and the vector u corresponds to an infinitesimal translation. Since a triangle composed of three bars is infinitesimally rigid, the infinitesimal flexes at its vertices are the restriction of a trivial infinitesimal flex. So in a triangulation of a surface as in Theorem (4.7), each triangle τ in any infinitesimal flex will have its own infinitesimal rotation vector r_τ. If $r_\tau \cdot e > 0$, then we say that the *spin* of the triangle is counterclockwise. If $r_\tau \cdot e < 0$, then we say that the *spin* of the triangle is clockwise. Add a trivial infinitesimal flex to the given infinitesimal flex so that $r_\tau = 0$ for some face τ. We do this in such a way that there remain faces where there are both kinds of spin clockwise and counterclockwise.

Divide the triangles (with a non-zero spin) of the polytope into two sets, those with counterclockwise spin and those with clockwise spin. If two adjacent triangles have opposite spins (or one spins one way and the other has no spin), then when their common edge is oriented in the direction compatible with both spins, the convexity condition implies that the oriented edge (i,j) points "uphill", i.e. $(p_i - p_j) \cdot e > 0$. The boundary of one of the sets of triangles has all of its edges oriented in a cycle, where the edges point uphill. This is not possible, so there can be no non-zero spin and the infinitesimal flex must be trivial.

4.6 More Proofs and Extensions of Dehn's Theorem. We mention briefly some other approaches to Dehn's Theorem.

The following is a sketch of an idea of Oded Schramm that is basically in SCHRAMM [1990]. Start with an infinitesimal flex $\mathbf{p}'$ of a convex polytope P with triangular faces. Orient some of the edges of P by the following rule. If p_i and p_j are the vertices of an edge, orient the edge from p_i to p_j if $p_i' \cdot (p_j - p_i) > 0$. This means that p_i is moving infinitesimally in the direction of p_j. By the condition of being an infinitesimal flex (3.1), $(p'_i - p_j') \cdot (p_i - p_j) = 0$, we get $p_i' \cdot (p_j - p_i) = - p_j' \cdot (p_i - p_j)$, so each edge has only one possible orientation. Since P is convex, the plane defined by $\{ p \mid p_i' \cdot p = 0 \}$ can separate the edges incident to p_i into at most two sets and can contain at most two edges unless $p_i' = 0$. Thus the orientation of the edges of P has the following properties.

i. The edges oriented out of a vertex are connected as one proceeds cyclically around a vertex. Similarly the edges oriented into a vertex are connected.

ii. Either 0, 1, 2 or all of the edges incident to a vertex are unoriented. (If all the edges incident to p_i are unoriented, then $p'_i = 0$.)

iii. If two edges are unoriented, then either they separate both ends of the interval of edges that are oriented out of a vertex from the interval of edges that are oriented into a vertex, or they are next to each other and all the other edges are oriented out of or into the vertex.

Figure 4.6.1 shows an example of such an orientation of the edges in a neighborhood of a vertex.

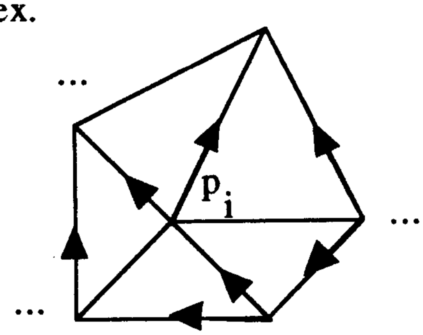


Figure 4.6.1

Fix a triangular face of P so that the infinitesimal flex is zero on all its vertices. Thus all the edges incident to those three vertices are unoriented.

Then a combinatorial lemma implies that there is no way of orienting the rest of the vertices consistent with the three properties above, other that leaving them all unoriented. This implies that the infinitesimal flex is zero on all the vertices, and thus the bar framework of the edges is infinitesimally rigid.

We mention some other approaches to Dehn's Theorem. Let $\{n_1, n_2, ..., n_k\}$ be a finite set of unit vectors in three-space. Let $a_1, a_2, ..., a_k$ be a corresponding set of positive real numbers such that

$$\sum_{1}^{k} a_i n_i = 0.$$

A theorem of Minkowskii states that there is a unique polytope (up to translation) for which n_i is the unit normal and a_i is the area of the i-th face. See STOKER [1969], Chapter X for a discussion in the smooth category as well as LYUSTERNIK [1963], Chapter 5 for a discussion in the category of polytopes. The uniqueness parts of the proofs of these results are very similar to the proofs of Cauchy's original theorem.

Another recent proof of Dehn's Theorem is due to P. Filliman in FILLIMAN [1990]. The idea is to translate the problem of the infinitesimal rigidity of the simplicial polytope P to a corresponding statement about its polar dual P^*. Then Minkowskii's Theorem and the Brunn-Minkowskii inequality is applied to obtain the rigidity result. See ALEXANDROV [1937] for a discussion of mixed volumes and the Brunn-Minkowskii inequality.

So far most of the results have been concerned with objects that have been topologically spheres. This is natural since the idea is to use convexity to show rigidity. However, it is possible to have convexity available even for a surface other than a sphere. The only published results that I know along this line are in the smooth category. Let S and S' be two compact oriented smooth (C^3) immersed surfaces in a Riemannian three-manifold of constant sectional curvature. Assume that S and S' are both "locally convex" in the sense that the second fundamental forms of both surfaces are positive definite. Let e_3 and e_3' be unit normal vectors in S and S' respectively, and let e denote a continuous conformal vector field on an open set containing both S and S'. We assume that scalar products $e \cdot e_3 > 0$ and $e \cdot e_3' > 0$. This is analogous to the similar condition ii in Section 4.5. With this set-up the following result of HSIUNG AND LIU [1977] is a generalization of Cohn-Vossen's Theorem that applies naturally to the surfaces S and S' of arbitrarily large genus.

(4.8) *Let $f: S \to S'$ be an isometry. Then the second fundamental forms of S and S' are equal under f.*

The second fundamental form is analogous to the dihedral angle in the polyhedral category.

A suggestion was made by M. Gromov that there should be a result similar to (4.8), but for infinitesimal rigidity in the polyhedral category. For such triangulated two-manifolds, in GROMOV [1986], there is a proof of certain combinatorial inequalities that would follow from the infinitesimal rigidity of such triangulated two-manifolds. More will said about this in Section 4.7.

In FOGELSANGER [1988] there is a proof that any compact connected triangulated two-manifold is infinitesimally rigid in three-space if the configuration of vertices is generic; that is there is no polynomial relation with integer coefficients among the coordinates of the vertices. This result, however, makes no use of convexity.

See TAY AND WHITELEY [1985] for an introduction to generic rigidity in general.

4.7 Maxwell-Cremona Theory and Spider Webs. So far most of the results that have been discussed have been related to Cauchy's Theorem. There is a very simple rigidity result that on the face of it does not seem to be related to convexity or Cauchy's result, but nevertheless it does have a connection to convex polytopes in three-space.

Suppose that one has a tensegrity framework that is composed entirely of cables attached to some points fixed in the plane. In CONNELLY [1982], such a tensegrity framework is called a *spider web* in honor of the creature that so frequently makes them. When is such a spider web rigid? For example, Figure 4.7.1 shows two spider webs where the one on the left is rigid, but the one on the right is not rigid.

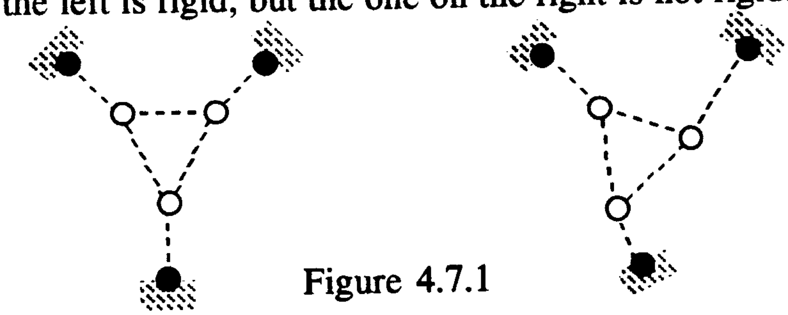


Figure 4.7.1

The black vertices are the fixed vertices. Spider webs with the graph as in Figure 4.7.1 are rigid only if the lines through the three outside cables intersect in one point. In general, a spider web is rigid if there is a proper self stress that is positive on all the cables. (The equilibrium condition (3.2) with $F_i = 0$ holds only for the vertices that are not fixed.) In fact when the self stress condition does hold, then the spider web is globally rigid. This result is in CONNELLY [1982] and is quite easy.

The question remains: When is there a positive self stress? One answer, of course, is just when there is an appropriate solution to the equilibrium equations (3.2) with the constraint that each stress is positive.

However, there is a more geometric answer. Suppose that the fixed vertices form a convex polygon F_0 in the plane, and the cables do not cross, as in Figure 4.7.1. Suppose further that there is a convex polytope P in three-space that has F_0 as a face and orthogonal projection into the plane of F_0 takes the vertices of P onto the vertices of the spider web in a one-to-one manner. Then a Theorem of J. Clerk Maxwell in MAXWELL [1864] and L. Cremona in CREMONA [1872] states that there is positive self stress in the spider web. Figure 4.7.2 shows how this is done for the for the spider web on the left in Figure 4.7.1. The polytope P is a truncated tetrahedron.

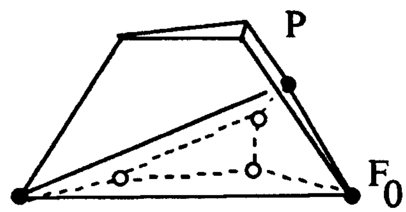


Figure 4.7.2

This result has been reformulated expanded and extended in WHITELEY [1982] and in CRAPO AND WHITELEY [1988]. In fact, the convexity of P is not relavant to the correspondence itself, which can be achieved by a purely projective construction. It is true though that since P is convex, then in this set-up the corresponding stresses will be positive.

Indeed, perhaps the proper way to think of this sort of construction is to instead consider an infinitesimal flex **p'** of a closed orientable polyhedral surface P_0. Each face of the surface is say a planar polygon, but the surface itself is allowed to be quite singular. Then there is a canonical self stress ω defined on the edges of P_0. Let θ_{ij}' be infinitesimal rate of change of the dihedral angle corresponding to the edge between the vertices p_i and p_j of P_0. Then

$$\omega_{ij} = \theta_{ij}'/|p_i - p_j|$$

serves as the self stress defined by **p'**. One must be careful to be sure that the infinitesimal flex **p'** preserves the faces of P_0 (as if they were rigid plates). This point of view is presented in GLUCK [1974]. A purely algebraic/projective-geometric description of ω is described in WHITELEY [1982] and in CRAPO AND WHITELEY [1988].

In our setting for spider webs, the polyhedral surface P_0 is the orthogonal projection of P, contained entirely in F_0. The infinitesimal flex of a vertex p_i in P_0 is simply the vector from p_i to the corresponding vertex of P. See Figure 4.7.3.

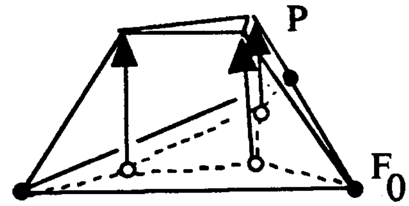


Figure 4.7.3

One can also reverse the correspondence $\mathbf{p}' \rightarrow \omega$ above if the surface P_0 is simply connected. In other words if P_0 is topologically a sphere and ω is a self stress of the edge framework of P_0, then there is an infinitesimal flex $\mathbf{p}'$ of P_0 which is non-trivial if ω is non-zero. This is described in WHITELEY [1982] and in CRAPO AND WHITELEY [1988].

See also HOPCROFT AND KAHN [1989] for an explicit description of the self stress described above as well as an application of the rigidity of spider webs to convex polytopes. In computational geometry it often occurs that a polytope P is defined, where some small unwanted edges appear. One wants to change P to another "nearby" polytope that does not have those small edges. This can be accomplished by using the correspondences above and deforming the spider web . Simply increase the stress on the corresponding unwanted edges until their end points are close enough to be considered the same. Given the fixed vertices, there will be a unique spider web with any given positive stress, and in this situation the deformed spider web will be close to the original. It turns out that such a deformation of P quite often must necessarily deform almost all of the vertices. This is why some sort of global procedure as above is necessary.

In ASH, BOLKER, CRAPO AND WHITELEY [1988] there is a survey of various aspects of the Maxwell-Cremona correspondence and its relation to things such as Dirichlet tessellations. (A *Dirichlet tessellation* is the collection of sets corresponding to a collection of points in the plane, where each set is the set of nearest points to one of the given points.) A finite part of the vertices and edges of a Dirichlet tessellation is a spider web. Section 5.3 discusses some of the ideas inherent to this technique.

See SUGIHARA [1986] for an application of the Maxwell-Cremona correspondence to scene analysis and combinatorics.

4.8 Rigidity, Convexity, and Combinatorics in Higher Dimensions. Suppose that P is a convex polytope in $\mathbf{R}^d$, where each face is a simplex. Let f_i $i = 1, 2, 3 \ldots$ be the number of simplicial faces of P of dimension i. The Lower Bound Theorem of BARNETT [1973] (see also WALKUP [1970]) says, among other things, that

$$f_k \geq \begin{cases} \binom{d}{k} f_0 - \binom{d+1}{k+1} k & \text{for } 1 \leq k \leq d-2 \\ (d-1)f_0 - (d+1)(d+2) & \text{for } k = d-1. \end{cases}$$

When $d \geq 3$, a result due to McMullen, Perles, and Walkup (See KALAI [1987]), reduces this inequality to the special case:

$$f_1 \geq f_0 d - d(d+1)/2.$$

The number of vertices of P is f_0, and the number of edges of P is f_1. But this is the same as the inequality (3.6) that is necessary for the infinitesimal rigidity of the 1-skeleton (the vertices and edges) of P in $\mathbf{R}^d$. This observation was exploited and extended in KALAI [1983] and KALAI [1987] to many other similar situations.

However, Dehn's Theorem (4.4) on the infinitesimal rigidity of convex polytopes with triangular faces is stated only when the dimension d=3. Fortunately there is a general principle that allows one to "bootstrap" rigidity results in lower dimensions up to results in higher dimensions. For example, suppose that P is a convex polytope in $\mathbf{R}^4$, where all its faces are simplices. Let p_i be any vertex of P. Then those simplices that are adjacent to p_i form a simplicial complex (called the *star* of p_i in P) which may be regarded as the cone over another complex, called the *link* of p_i in P. Let S_0 be a three-dimensional support hyperplane that intersects P at p_i only. Let S be another hyperplane that is parallel to S_0 that does not contain p_i. Project the link of p_i into S from p_i. It is easy to see that this projection is a convex simplicial three-polytope in S. Thus by Dehn's Theorem (4.4) this projection is is infinitesimally rigid in S. See Figure 4.8.1. (So that we may see things more clearly, the dimension of all the sets in the Figure are decreased by one. In Figure 4.8.1 the link is a polygonal circle in the plane, and this is not infinitesimally rigid.)

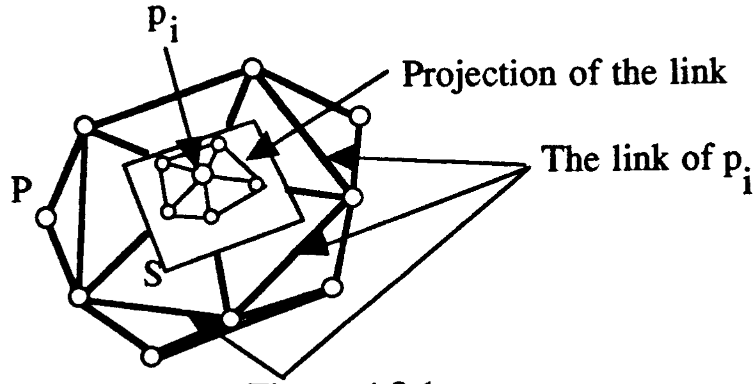


Figure 4.8.1

By our discussion in Section 3.5 about the projective invariance of infinitesimal rigidity and from Statement (3.5), we see that the cone on the projection and the cone on the link (i.e. the star of p_i in P) are both infinitesimally rigid in $\mathbf{R}^4$. But the stars of all the vertices of P overlap in such a way that the whole framework can be built by gluing infinitesimally rigid pieces together along intersections such that the union is infinitesimally rigid.

Clearly this argument extends to higher dimensions, and, by induction on the dimension d, we can show that all simplicial polytopes for $d \geq 3$ are infinitesimally rigid in $\mathbf{R}^d$. Thus we have a proof of the lower bound inequalities of Barnett.

A slightly more ambitious idea is to extend an appropriate version of Alexandrov's Theorem (4.5) to higher dimensions. The following is a result in WHITELEY [1984].

> *Suppose that P is convex polytope in R^d $d \geq 3$, and G(p) is a bar framework consisting of the vertices and edges of P, together with additional bars that triangulate all the two dimensional faces of P. Then G(p) is infinitesimally rigid in R^d.*

This result follows essentially from the bootstrap idea and the property that one can retriangulate the two-dimensional faces of P without altering its infinitesimal rigidity. As a consequence we get the following inequality:

$$f_1 + \sum_{k\geq 3} (k-3)f_2^k \geq f_0 d - d(d+1)/2,$$

where f_2^k is the number of two-dimensional faces of P which have k sides. This is again from KALAI [1987]. Previously this inequality had been only known for polytopes that had a realization with all the coordinates as rational numbers.

In FILLIMAN [1991] there is an attempt of a proof of the conditions of McMullen (generalizing the the Lower Bound Theorem of Barnett) on the face numbers f_k of a simplicial polytope. The idea is to replace a previous difficult proof with the calculation of the rank of a matrix that generalizes the rigidity matrix of Section 3.6. See BILLERA AND LEE [1981] for a discussion. This follows a suggestion of C. Lee, which has generated some interest in the various proofs of Dehn's Theorem.

Even though the proof of Dehn's Theorem (4.4) is simple enough, it seems that there should be a way of proving the combinatorial inequalities of Barnett following the general outline as suggested by Kalai, but without appealing to the geometric realization that is needed for infinitesimal rigidity. In fact, this is the case, and this is done in GROMOV [1986], Chapter 2.4.10. (Unfortunately, some of the proofs are slightly garbled.)

Another way of thinking of the proof of the higher dimensional Dehn's Theorem is to imagine a small (d-1)-dimensional sphere S^{d-1} centered at p_i. The intersection of S^{d-1} with the star of p_i in P will be a triangulated spherical polytope of dimension one less than P. If we have an appropriate theorem about the infinitesimal rigidity of such objects, then we will able to conclude the that the star of p_i in P is infinitesimally rigid in $\mathbf{R}^d$. Statement (3.5) can be interpreted as a way of transforming a rigidity result about polytopes in $\mathbf{R}^{d-1}$ to the corresponding result about polytopes in S^{d-1}.

Indeed, this correspondence has been exploited in POGORELOV [1964] to show that there is in fact a way of transforming rigidity results about pairs of objects in $\mathbf{R}^d$ to rigidity results about pairs of objects in

S^d. One can then bootstrap lower dimensional rigidity results about pairs to higher dimensional results about pairs. For example, Cauchy's original result about convex polytopes (mentioned in the introduction) is about pairs of convex polytopes, and this bootstrap method allows one to generalize it to any dimension greater than three:

> *Any pair of convex polytopes in R^d or S^d for $d \geq 3$ that have a correspondence which is a congruence on each facet of one lower dimension extends to congruence of the whole space.*

(Of course, a set in S^d is convex if, for every pair of points in the set, the shortest geodesic arc joining them is in the set.)

One can view our statements about infinitesimal rigidity as a statement about pairs of configurations, where both pairs are "almost the same" but differ by an "infinitesimal". With that interpretation Pogorelov's correspondence can be regarded as specializing to give the statements about infinitesimal rigidity. (In CONNELLY [1988a] it is shown how to deduce the statement about some pairs from the statement about infinitesimal flexes (3.5) as well.)

5. SECOND-ORDER RIGIDITY AND PRE-STRESS STABILITY

So far we have been only concerned with the linear or first-order theory of rigid mostly convex objects. There are many other convex objects that are still rigid, but are not first-order rigid.

5.1 The Definition of Second-Order Rigidity. Most of the definitions will be taken from CONNELLY [1980] and CONNELLY AND WHITELEY [1990]. There is also a discussion of second-order rigidity in ROSENBERG [1980]. Suppose that we have a (tensegrity) framework $G(\mathbf{p})$ in $\mathbf{R}^d$. Let $\mathbf{p}'$ be an infinitesimal flex of $G(\mathbf{p})$, which we will also call a *first-order flex* of $G(\mathbf{p})$ satisfying (3.1). Let $\mathbf{p}'' = (p_1'', p_2'', \ldots, p_n'')$ be another configuration of vectors in $\mathbf{R}^d$ such that for each member $\{i,j\}$ of G where $(p_i - p_j)\cdot(p_i' - p_j') = 0$,

$$(5.1) \qquad |p_i' - p_j'|^2 + (p_i - p_j)\cdot(p_i'' - p_j'') \begin{cases} \leq 0 & \text{if } \{i,j\} \text{ is a cable} \\ = 0 & \text{if } \{i,j\} \text{ is a bar} \\ \geq 0 & \text{if } \{i,j\} \text{ is a strut.} \end{cases}$$

There is no condition for the member $\{i,j\}$ when $(p_i - p_j)\cdot(p_i' - p_j') \neq 0$. (Of course, for a bar framework both (5.1) and (3.1) must hold.) We then say that the pair $(\mathbf{p}',\mathbf{p}'')$ is a *second-order flex* of $\mathbf{p}'$ and extends the first-order flex $\mathbf{p}'$. We can think of $\mathbf{p}'$ as the velocities of the points of the configuration and $\mathbf{p}''$ as the accelerations of the points of the configuration, since (3.1) and (5.1) are obtained by differentiating the distance constraints.

We say that the tensegrity framework $G(\mathbf{p})$ is *second-order rigid* in $\mathbf{R}^d$ if for every non-trivial first-order flex $\mathbf{p}'$ of $G(\mathbf{p})$, there is no extension to a second-order flex $(\mathbf{p}',\mathbf{p}'')$. It is important for this definition that the first-order flex be non-trivial, since any trivial first-order flex extends to some second-order flex. For example, if $\mathbf{p}' = 0$, then the equations (5.1) become identical to the first-order equations (3.1) except for $\mathbf{p}''$ replacing $\mathbf{p}'$.

The basic result for second-order rigidity is the following:

(5.2) *If a tensegrity framework $G(p)$ is second-order rigid in R^d, then it is rigid in R^d.*

This was proved for bar frameworks in CONNELLY [1980] and extended to tensegrity frameworks in CONNELLY AND WHITELEY [1990]. The proof involves an analysis of the Taylor series expansion of any non-trivial analytic flex $\mathbf{p}(t)$ of $G(\mathbf{p})$.

Consider the left framework of Figure 3.3.1, which is first-order flexible and has a first-order flex indicated. This first-order flex together with the trivial first-order flexes generate all of the first-order flexes of $G(\mathbf{p})$. But none of the non-trivial first-order flexes extend to a second-order flex. For example, for the flex indicated in Figure 3.3.1, the middle vertex must have a second-order flex, an acceleration vector, that has a non-zero component in both the positive and negative y directions. This is not possible, so the second-order flex does not exist, and the framework is rigid (which is easy to see anyway for this example).

5.2 Second-Order Rigidity and Convex Surfaces. Let P be a compact convex polytope in three-space. For Alexandrov's Theorem (4.5) it is important that no vertex be in the interior of any two-dimensional face of P. But if some of the vertices *are* chosen in the relative interior of some of the faces of P, is the resulting framework still rigid, even though it is not infinitesimally rigid? Figure 5.2.1 shows some examples of triangulated polytopes with vertices in the interior of some of the faces.

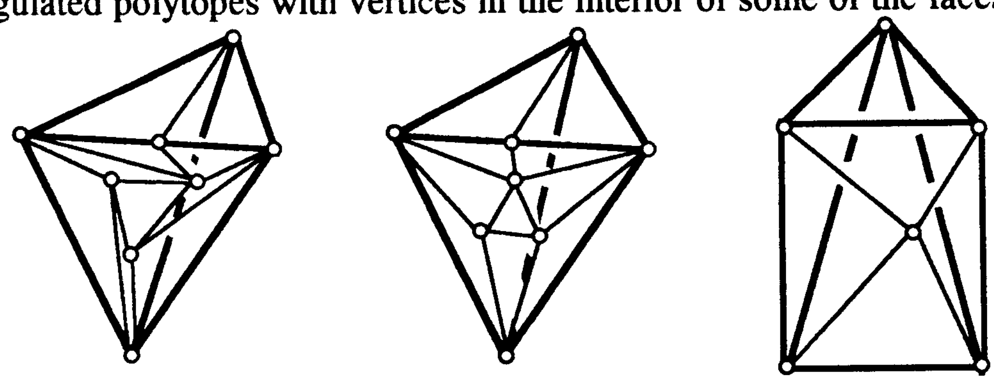


Figure 5.2.1

The lighter weight edges are bars in the interior of the faces of the polytope.

It is easy to see from Alexandrov's Theorem that any such first-order flex on a framework as in Figure 5.2.1 is trivial when restricted to the vertices and edges of the polytope P. So if **p'** is any first-order flex of the whole framework G(**p**), by adding a trivial infinitesimal flex to **p'**, we may assume that each p_i', in addition to being zero for all the vertices that do not lie in the interior of a face of P, is perpendicular to each face on which it lies.

If (**p'**,**p''**) is a second-order flex that is an extension of **p'**, then it turns out that **p''**, when restricted to the vertices of P, acts as if it were a first-order flex of a framework as in Alexandrov's Theorem. This is a tensegrity framework that just the vertices of P, cables between all of the vertices on the same face and bars along the edges of P. (The frameworks of Figure 4.4.1 are examples.) So (**p'**,**p''**) acts as if the vertices of P were fixed. Then it is easy to see that the rest of the vertices in the triangulation must have $p_i' = 0$ for the the vertices in the interior of faces of P, for much the same reason as for the middle vertex of Figure 3.3.1 as was discussed in Section 5.1. This shows that **p'** is trivial, and G(**p**) is second-order rigid. Thus we have the following, which is in CONNELLY[1980].

(5.3) *Any triangulation of a convex polytope in three-space is second-order rigid.*

In fact, the same idea can be extended to show that if a convex hole is removed from the polyhedral surface, and the resulting surface with boundary is triangulated, then it is still second-order rigid. It might look something like Figure 5.2.2.

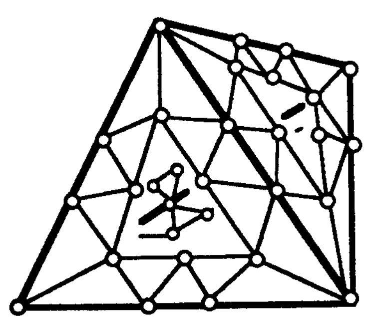


Figure 5.2.2

In the smooth category similar objects have been studied, but the analogue of the fundamental result (5.2) is not known (except for analytic surfaces). There is a discussion of second-order rigidity in the smooth category in EFIMOV [1951], which is translated from the original paper EFIMOV [1948]. See also EFIMOV [1957] and EFIMOV [1962] for a general discussion of the rigidity of smooth surfaces.

5.3 Pre-Stress Stability. Although second-order rigidity is natural mathematically, physically it may not be the concept that describes the stability of a structure most accurately. From a physical point of view, it is not enough just to declare that the distance constraints force the structure to have only one configuration locally up to rigid motions. The members of a physical framework do not behave as ideal bars or cables. There must be some way of describing the behavior of the system as it is perturbed.

A common method used to describe such behavior is to introduce energy functions. For example, if a spring at rest is displaced, then Hooke's law describes how much work it takes to displace the spring any given amount. This can be though of as changing the potential energy that is stored in the spring. Idealized springs will be our way of modeling a bar in a framework. Mathematically the idea is that when the end points of a bar are displaced, there is a well-defined (energy) function of the lengths of the bars (or the squares of the lengths) such that each function has a strict minimum at the rest length of the bar. For a cable, its energy function is strictly monotone increasing, and for a strut its energy function is strictly monotone decreasing.

In structural mechanics a basic principle used for the stability of structures is that the configuration must be at a local minimum for the total energy functional. This is often associated with "Castigliano's Principle". See PRZEMIENIECKI [1968]. Mathematically this is sound, since if the total energy function is at a unique local minimum, modulo rigid motions, then it must be at least rigid. This is because if there were another nearby non-congruent configuration that satisfied the member constraints, then each of the energy functions of the members would be no larger, and the original configuration would not be at a unique minimum. Physically, the idea is that if the structure is perturbed to a nearby configuration, then the gradient of the energy function provides the force necessary to push the configuration back to the unique minimum point, or at least keep the structure from wandering far from the desired minimum point.

The following is taken mostly from CONNELLY AND WHITELEY [1990a] and CONNELLY AND WHITELEY [1990b]. Suppose that G(p) is a tensegrity framework in $\mathbf{R}^d$ and let H_{ij} be an energy function for the member $\{i,j\}$ as described above. The total energy is

$$H(\mathbf{q}) = \sum_{i,j} H_{ij}(|q_i-q_j|^2),$$

where $\mathbf{q}$ is any configuration in $\mathbf{R}^d$. We have chosen to write H as a function of the square of the lengths to make some expressions simpler that appear later.

The conditions for a configuration $\mathbf{p}$ to be a critical point for H turn out to be that $G(\mathbf{p})$ has a proper self stress ω, where $\omega_{ij} = H_{ij}'(|p_i-p_j|^2)$. Conditions (3.2) must hold with $F_i=0$. The functional H_{ij}' is the first derivative of H_{ij}.

We also need to find conditions when H has a minimum, or at least a local minimum at a given configuration $\mathbf{p}$. We simply apply the second derivative test from multivariable calculus. The Hessian of H turns out to be the following nd-by-nd symmetric matrix (up to a scalar multiple):

$$\Omega\otimes I^d + R(\mathbf{p})^t D R(\mathbf{p}), \tag{5.4}$$

where $R(\mathbf{p})$ is the e-by-dn rigidity matrix as defined in Section 3.6, D is the e-by-e diagonal matrix such that the second derivative $H_{ij}''(|p_i-p_j|^2)$ is the ij-th diagonal entry of D, and Ω is the n-by-n *stress matrix* obtained by defining its ij-th entry as

$$\Omega_{ij} = \begin{cases} -\omega_{ij} & \text{if } i\neq j \\ \sum_k \omega_{ik} & \text{if } i=j. \end{cases} \tag{5.5}$$

Recall that $\omega_{ij} = 0$, for $\{i,j\}$ non-members of the graph G. The tensor product of Ω with the d-by-d identity matrix is denoted by $\Omega\otimes I^d$, which is obtained by replacing each 1 entry of I^d with an n-by-n block Ω. The transpose of a matrix is indicated by $(\ )^t$, n is the number of vertices of G, and e is the number of members of G.

The matrix $R(p)^t DR(p)$ is called the *stiffness matrix*, and it is always positive semi-definite, since we insist that the second derivative $H_{ij}'' > 0$ for all members $\{i,j\}$.

The second derivative test says that H has a local minimum at the configuration **p**, if the matrix (5.4) is positive semi-definite (regarding it as quadratic form). If, in addition, the kernel of (5.4) contains only the trivial infinitesimal flexes, we say that G(p) is *pre-stress stable (with* ω *as a rigidifying prestress).* Clearly if G(p) is pre-stress stable, then it is rigid.

A basic result of CONNELLY AND WHITELEY [1990] is the following:

> *If a tensegrity framework G(p) is infinitesimally rigid, then it is pre-stress stable. If G(p) is pre-stress stable, then it is second-order rigid. Neither of these implications is reversible.*

For example, many spider webs are not first-order rigid but when they have a proper self stress that is positive on all the cables, they are pre-stress stable, essentially because the stress matrix is positive semi-definite of maximal rank. (This imagines some appropriate globally rigid framework, behind the scenes, holding the pinned vertices fixed.) Also all three of the frameworks in Figure 5.2.1 are not first-order rigid, but they are pre-stress stable, which is a bit harder to show. Many of the tensegrity frameworks championed by R. Buckminster Fuller are pre-stress stable. Figure 5.3.1 shows some of these tensegrities. See FULLER [1975].

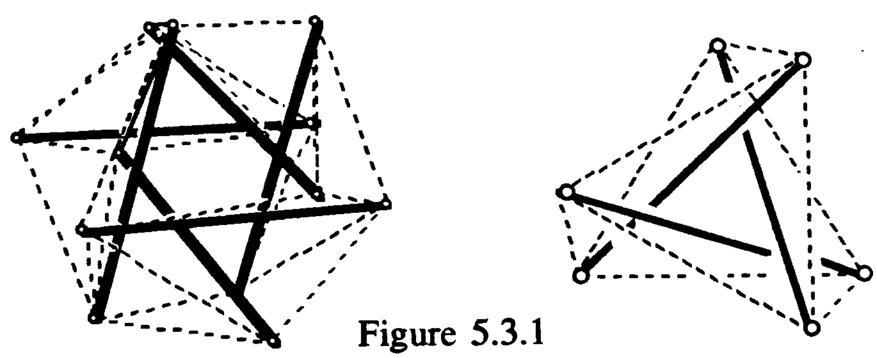
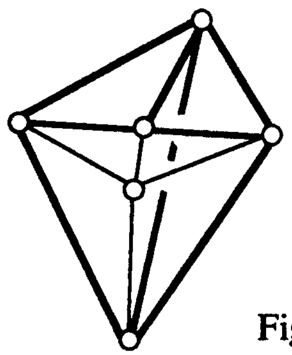


Figure 5.3.1

The framework on the right was one of the first discovered by Kenneth Snelson. See SNELSON [1973]. The bars in these tensegrity frameworks can be replaced by struts and they will still be pre-stress stable. All of the tensegrity frameworks that are rigid in GRÜNBAUM AND SHEPHARD [1975] are surely rigid because they are pre-stress stable. Similarly most of the large collection of tensegrity frameworks in PUGH [1976] are not infinitesimally rigid and must be pre-stress stable. In CONNELLY AND TERRELL [1991] an explicit computation with the stress matrix is performed for a certain collection of tensegrity frameworks with dihedral symmetry. Those with a positive definite stress matrix are identified, and it is easy to check that they are at least pre-stress stable.

When a framework is pre-stress stable with ω as the rigidifying pre-stress, then those bars $\{i,j\}$ that have $\omega_{ij} \neq 0$ can be replaced by a cable or strut. For example, the framework on the the left in Figure 5.3.2 is a triangulation of a tetrahedron and is pre-stress stable. The framework on the right replaces some of the bars by cables and struts. Both are pre-stress stable. If any member is removed or any bar is replaced by a cable or strut in the framework on the right, then it will not be rigid.



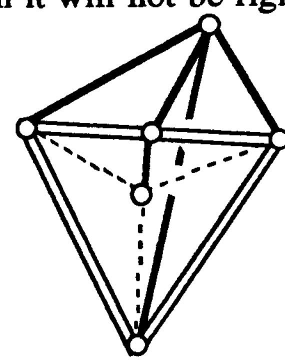


Figure 5.3.2

5.4 Global Rigidity and the Stress Matrix. It turns out that there is another geometric property associated to the stress matrix Ω that was defined in Section 5.3. It is very useful for guaranteeing that a tensegrity framework is globally rigid. This analysis however does not deal directly with the rigidity matrix or the stiffness matrix.

It is helpful instead to think of the stress ω, and its associated stress matrix Ω, first, and then look for a configuration $\mathbf{p}$ with ω as its self stress. (The graph G is superfluous.) It is easy to check that if the configuration $\mathbf{q}$ is an affine image of the configuration $\mathbf{p}$, then any self stress for $\mathbf{p}$ will be a self stress for $\mathbf{q}$. Extending the ideas discussed in Section 5.2, the following result is shown in CONNELLY [1982]. (Recall from Section 3.2, that if one tensegrity framework dominates another, that means the cables become no longer, struts become no shorter, and bars are the same length.)

(5.6) *Suppose $G(p)$ is a tensegrity framework in R^d with n vertices not all in a (d-1)-dimensional affine subspace, a self stress ω, and associated stress matrix Ω. If Ω is positive semi-definite of rank $n-d-1$, and if $G(q)$ is dominated by $G(p)$, then q is an affine image of p.*

In other words, if Ω is positive semi-definite of maximal rank (it turns out that if the affine span is all of $\mathbf{R}^d$, then the kernel must be at least $d+1$ dimensional), then the framework is globally rigid up to affine motions.

When is there an affine image $\mathbf{q}$ of a configuration $\mathbf{p}$ such that $G(\mathbf{p})$ dominates $G(\mathbf{q})$? Since translations are rigid motions we need only consider when the affine motion is given by some linear map given by a d-by-d matrix A in some coordinate system. If $\{i,j\}$ is a member of the graph G, then

$$|q_i - q_j|^2 = |Ap_i - Ap_j|^2 = (Ap_i - Ap_j)^t (Ap_i - Ap_j)$$

$$= (p_i - p_j)^t A^tA (p_i - p_j) \begin{cases} \leq (p_i - p_j)^t (p_i - p_j) & \text{if } \{i,j\} \text{ is a cable} \\ = (p_i - p_j)^t (p_i - p_j) & \text{if } \{i,j\} \text{ is a bar} \\ \geq (p_i - p_j)^t (p_i - p_j) & \text{if } \{i,j\} \text{ is a strut.} \end{cases}$$

This is the same as the condition

$$(5.7) \qquad (p_i - p_j)^t \; [A^tA - I^d] \; (p_i - p_j) \begin{cases} \leq 0 & \text{if } \{i,j\} \text{ is a cable} \\ = 0 & \text{if } \{i,j\} \text{ is a bar} \\ \geq 0 & \text{if } \{i,j\} \text{ is a strut.} \end{cases}$$

We can regard the symmetric matrix $A^tA - I^d$ as a quadratic form, and condition (5.7) says that all the *member directions*, defined as all scalar multiples of $(p_i - p_j)$ with $\{i,j\}$ in G, lie in the appropriate region defined by the *quadric at infinity* (regarding the coordinates in $\mathbf{R}^d$ as homogeneous coordinates for the (d-1)-dimensional real projective space "at infinity"). For example, the bar directions must lie on the quadric at infinity, where the quadric is given by the symmetric matrix $A^tA - I^d$. When the dimension $d = 2$, then the quadric at infinity is simply two directions. When $d = 3$, then the projective space at infinity is the usual real projective plane, and the quadric at infinity consists of the directions that lie on any given conic, possibly degenerating into two lines.

We could leave things at this state, but there is a further simplification that is very useful. Suppose that $G(\mathbf{p})$ is a tensegrity framework in $\mathbf{R}^d$ with a proper self stress ω. We say that a member direction $p_i - p_j$ with $\omega_{ij} \neq 0$ is a *stressed direction.*

> *There is another non-congruent configuration* q*, an affine image of* p*, such that* $G(q)$ *is dominated by* $G(p)$*, if and only if for* $G(p)$ *the stressed directions lie on a non-zero quadric* Q *at infinity, as well at the condition (5.7), with* Q *replacing* $A^tA - I^d$ *for the non-stressed directions..*

By replacing the symmetric matrix Q by εQ for $\varepsilon > 0$ small enough, it is possible to solve $Q = A^tA - I^d$ for the matrix A. See WHITELEY [1987c] as well as WHITELEY [1987b].

Suppose that ω is a proper self stress of a tensegrity framework $G(\mathbf{p})$ in $\mathbf{R}^d$ such that the associated stress matrix Ω is positive semi-definite of maximal rank $n-d-1$, and no non-congruent affine image $\mathbf{q}$ of $\mathbf{p}$ is such that $G(\mathbf{q})$ is dominated by $G(\mathbf{p})$. Then we say that $G(\mathbf{p})$ is ω *globally rigid.* For example, all spider webs are ω globally rigid, where each scalar component of ω is positive. (But some vertices are fixed by some appropriate globally rigid tensegrity framework in the background with its own self stress and corresponding positive semi-definite stress matrix of maximal rank. The two stresses then can be combined to give a positive semi-definte stress matrix of maximal rank on all of the vertices.).

We apply the stress matrix-stiffness matrix decomposition of (5.4) to convex polygons. This follows the proof in CONNELLY [1982] of the statement (4.2) about the global rigidity of convex polygons when the internal members are struts. Statement (5.8) below is the key fact needed.

(5.8) *Let* $G(p)$ *be a tensegrity framework in the plane, where the vertices form a convex polygon, all the external edges are* *cables, struts are the only other members, and* $G(p)$ *has a proper non-zero self stress* ω. *Then the stress matrix* Ω *is positive semi-definite of rank* $n-3$. *Thus* $G(p)$ *is* ω *globally rigid.*

It is clear that there are more than two stressed directions for such polygons, and thus (5.8) together with (5.6) imply that such polygons are globally rigid in any dimension greater that one. From CONNELLY [1982] it is shown that if such a framework (other than a triangle) is rigid at all, then it must have a non-zero proper self stress. Thus any such rigid polygonal framework with cables on the outside edges and struts on the inside must be globally rigid even in higher dimensional Euclidean spaces.

Many, but not all, of the tensegrity frameworks in PUGH [1976] are also ω globally rigid. Thus they are also globally rigid in higher dimensional Euclidean spaces.

5.5 Second-Order Duality. The following equivalence is useful for understanding the relation among global rigidity, second-order rigidity, and pre-stress stability. This is the second-order stress test in CONNELLY AND WHITELEY [1990]. This follows from an application of the "Farkas Alternative" as used in linear programming duality.

(5.9) *A tensegrity framework* $G(p)$ *in* $\mathbb{R}^d$ *is second-order rigid if and only if for every non-trivial first-order flex* p' *there exists a proper self stress* ω *of* $G(p)$ *such that*

$$(p')^t\, \Omega \otimes I^d\, p' > 0,$$

where Ω *is the stress matrix associated to* ω.

It is not too hard to see that the condition for being pre-stress stable is the same as the condition as in (5.9) except that a *single* proper self stress must satisfy the condition. There are examples in CONNELLY AND WHITELEY [1990] of bar frameworks that are second-order rigid but not pre-stress stable. Another such example is the complete bipartite graph $K_{3,3}$ on a line in the plane as in Figure 5.5.1. All six points of $K_{3,3}$ lie on a line with one partition of three points on the negative part of the line, and the other partition on the positive part. Each point of one partition is joined to each other point of the other partition. This bar framework in second-order rigid in the plane, and is a mechanism (i.e. a flexible framework) in three-space. In Figure 5.5.1 the bars are curved so as to be more visible.

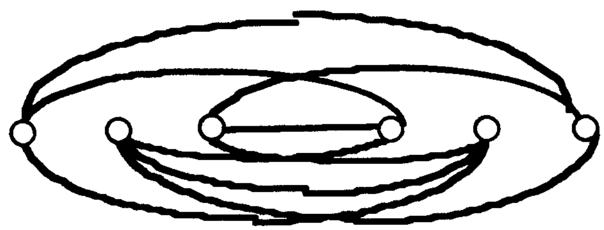


Figure 5.5.1

In such examples it is as if there is a demon in the framework that somehow senses what first-order flex (or load) will be encountered and then provides an appropriate self stress to "block" that flex. Different flexes require different self stresses. Because of this, it seems to me much more reasonable to regard a framework as truly "stable" if it is pre-stress stable, than if it is just second-order rigid.

An application of the second-order stress test (5.9) is to show Roth's Conjecture (4.3). Suppose that $G(\mathbf{p})$ is a tensegrity polygonal framework in the plane with bars on the external edges and cables on the inside. If $G(\mathbf{p})$ is not infinitesimally rigid, let $\mathbf{p}'$ be a non-trivial first-order flex of $G(\mathbf{p})$. Let ω be any proper self stress for $G(\mathbf{p})$. The result in (5.8) says that if Ω is the associated stress matrix, then $-\Omega$ is positive semi-definite of rank $n-3$. So Ω is negative definite of rank $n-3$. By an argument similar to the one sketched in Section 5.4, since there are no non-trivial infinitesimal flexes that are affine images of $\mathbf{p}$, the infinitesimal flex $\mathbf{p}'$ is not in the kernel of $\Omega \otimes I^2$. Thus

$$(\mathbf{p}')^t \, \Omega \otimes I^2 \, \mathbf{p}' < 0,$$

and $\mathbf{p}'$ must extend to a second-order flex $(\mathbf{p}', \mathbf{p}'')$ of $G(\mathbf{p})$ by the second-order stress test (5.9). An extension of the second-order stress test implies that since the inequality above is strict, any smooth flex $\mathbf{p}(t)$ of the points of $\mathbf{p}$, with the first and second derivatives at $t = 0$ the same as $\mathbf{p}'$ and $\mathbf{p}''$ respectively, will have all of the cable distances decrease. It is possible to make sure that, in fact, $\mathbf{p}(t)$ does not change the length of the bars of $G(\mathbf{p})$ as well. Then $\mathbf{p}(t)$ is a non-trivial flex of all of $G(\mathbf{p})$. If $G(\mathbf{p})$ is rigid, this non-trivial flex $\mathbf{p}(t)$ cannot exist. So $\mathbf{p}'$ cannot exist, and G(p) is infinitesimally rigid, showing Roth's Conjecture. This argument is explained in detail in CONNELLY AND WHITELEY [1991].

5.6 Polyhedral Surfaces Revisited. Another application of the second-order stress test is to triangulated polygons, thought of as "paper membranes". This in turn can be used to strengthen our result (5.3) that triangulated polyhedral surfaces are second-order rigid. First we need a different kind of stability result from CONNELLY [1991]. A *bar simplex* in $\mathbf{R}^d$ is a framework where all the vertices of the configuration are affine independent (no k+2 vertices lie in an affine k-dimensional subspace), and every pair of vertices is joined by a bar. For example, bar triangles and bar tetrahedra are bar simplices.

(5.10) *A tensegrity framework G(p) in R^d is second-order rigid in $R^{d'}$ for all $R^{d'} \supset R^d$, $d' \geq d$ if and only if either G(p) is a bar simplex in R^d or there is a proper self stress ω such that G(p) is ω globally rigid.*

This result allows one to use results about the second-order rigidity of a framework to know that there is some particular self stress ω that works for ω global rigidity. Although the proof is non-constructive, the result can be useful. The following result follows easily from the techniques in CONNELLY [1980].

(5.11) *Suppose that a bar framework G(p) in R^2 consists of the vertices and edges of some triangulation of a convex polygon in the plane, with the boundary vertices fixed. Then G(p) is second-order rigid in any R^d for $d \geq 2$. Thus there is a self stress ω such that G(p) is ω globally rigid*

At first sight this result seems like the statement about the ω global rigidity of a spider web. But for a spider web all the coordinates of the stress are positive. For the triangulation af a triangle as in Figure 5.5.2 it turns out that no proper self stress can be positive on all the internal bars. See CONNELLY AND HENDERSON [1980] for an easy proof.

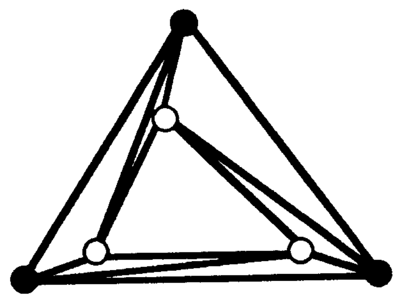


Figure 5.5.2

Nevertheless, the result (5.11) does say that there is a self stress ω that makes $G(\mathbf{p})$ ω globally rigid.

We generalize the result (5.11) to a more delicate property of ω. Suppose that $G_0(\mathbf{p})$ is the bar framework obtained from a triangulation of a convex planar polygon P_0. Attach the boundary vertices of $G_0(\mathbf{p})$ to a system of parallel bars in three-space which are in turn attached to fixed (pinned) vertices in three-space as in Figure 5.5.3. (The pinned vertices are black.) This gives us a bar framework $G(\mathbf{p})$.

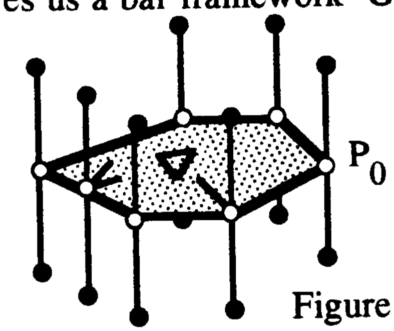


Figure 5.5.3

Suppose that the triangulation of P_0 is the orthogonal projection of a face of a convex polyhedral surface. We assume also that each vertex of in the interior of P_0 corresponds to a strictly convex vertex on the surface. (I.e. a support plane intersects the surface just at the vertex.) From the discussion of Maxwell-Cremona Theory in Section (4.6) this implies that there is a self stress ω_0 on $G_0(\mathbf{p})$ that is positive on interior edges that correspond to the edges of the surface. (It may be negative on the boundary.) Each interior vertex is adjacent to at least two members of $G(\mathbf{p})$ with a strictly positive stress. Let ω be the self stress for $G(\mathbf{p})$ obtained by adjoining to ω_0 a large positive self stress on each vertical bar.

Suppose that $\mathbf{p}'$ is a first-order flex of $G(\mathbf{p})$ in any dimension $d \geq 3$. We calculate

$$(\mathbf{p}')^t\, \Omega \otimes I^d\, \mathbf{p}' = \sum_{i,j} \omega_{ij} |p_i' - p_j'|^2.$$

On the boundary of P_0 the vertical stresses dominate, and thus the first-order flexes at those vertices must evaluate to zero. For the interior vertices of P_0 all the stresses on members adjacent to the vertices are positive, so anything non-zero will be strictly positive. Thus all the first-order flexes must be zero, and by (5.10), $G(\mathbf{p})$ is ω globally rigid.

If we further subdivide this triangulation of P_0 by only adding new vertices along the existing edges of first triangulation, then the same argument shows that the corresponding $G(\mathbf{p})$ will be ω globally rigid.

Now suppose that $G_0(\mathbf{p})$ and $G(\mathbf{p})$ come from *any* triangulation of P_0. We claim that $G(\mathbf{p})$ is second-order rigid in any dimension $d \geq 3$. If not, then there must be a second-order flex $(\mathbf{p}',\mathbf{p}'')$ of $G(\mathbf{p})$, where $\mathbf{p}'$ is non-trivial. But it is easy to see that the second-order flex must extend to any subdivision of the triangulation of P_0. But we can choose a subdivision of this triangulation that is also a triangulation coming from a convex surface as mentioned above. This contradicts the argument above. Hence, this $G(\mathbf{p})$ must also be second-order rigid in all higher dimensions. Our stability result (5.10) then says that there is a self stress ω such that $G(\mathbf{p})$ is ω globally rigid.

Using such a self stress for each face of an arbitrarily triangulated convex polytope and an argument similar to the one above gives the following basic result relating convexity and rigidity. This result and details of the argument above will appear in CONNELLY [1991].

Every triangulation of a convex polytope in three-space is pre-stress stable.

For example, the frameworks of Figure 5.3.2 and Figure 5.2.1 are seen to be pre-stress stable in three-space.

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