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A convex 3-complex not simplicially isomorphic to a strictly convex complex

What is the algebraic theory of strictly convex complexes?

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(1) *Introduction.* A set X in euclidean space is *convex* if the line segment joining any two points of X is in X. If X is convex, every boundary point is on an $(n-1)$-plane which contains X in one of its two closed half-spaces. Such a plane is called a *support plane* for X. A simplicial complex K in $\mathbb{R}^n$ is called *strictly convex* if $|K|$ (the underlying space of K) is convex and if, for every simplex σ in ∂K (the boundary of K) there is a support plane for $|K|$ whose intersection with $|K|$ is precisely σ. In this case $|K|$ is often called a *simplicial polytope*.

If $|K|$ is just convex it is often desired to move the vertices of K (slightly) so that the altered complex is strictly convex. In Theorem 2 we provide an example of a 3-complex where no such altering is possible even by a large move. In fact we show a bit more. There is a 3-complex K such that $|K|$ is a tetrahedron (and thus convex) and K is not simplicially isomorphic to a complex K' in $\mathbb{R}^3$, where $|K'|$ is convex and the condition for strict convexity holds at the vertices of K'.

Part of the motivation for this example was the following statement made by D. Chillingworth in (4) on page 354: '... we can if necessary slightly alter the positions of some of the vertices to obtain a complex simplicially isomorphic to K, which has no two vertices at the same height and is such that v_1 is strictly higher than all the other vertices'.

Our Theorem 1 provides a specific complex T, where $|T|$ is a triangle (2-simplex), such that if T is the projection (from an appropriate point below say) of a simplicially isomorphic complex T', which is part of the boundary of a convex surface in 3-space, then $|T'|$ has to be a flat triangle, so its interior vertices violate the condition of strict convexity. The condition needed for T to have the property that Theorem 1 holds, is that a certain interior triangle be turned or twisted sufficiently with respect to the outer triangle. When T satisfies this condition (defined later) we say it is *twisted*. It turns out that, if T is twisted, then any other complex T'' simplicially isomorphic to T with the corresponding vertices of T'' close enough to T, is twisted also. Thus if T is a subset of the boundary of a tetrahedron, then it provides a counterexample to the statement of Chillingworth. Theorem 2 gives a way of triangulating the whole tetrahedron so that the interior simplices force the one face to be twisted, thus providing a global counterexample to Chillingworth's statement. (We say a complex K (*rectilinearly*) *triangulates* a space X, if $X = |K|$.)

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It should be borne in mind that there is no way of finding a global counterexample just using a triangulation of the boundary of a convex 3-dimensional set. A theorem of Steinitz (see (16) or (9), chapter 13, for example) says that, among other things, any abstract simplicial complex topologically homeomorphic to a 2-dimensional sphere is simplicially isomorphic to the boundary of a strictly convex complex.

Despite these counter-examples, corollaries 2 and 3 of Chillingworth are unaffected, since the starting vertex in the theorem can be chosen appropriately anyway without using the statement we quoted above. In fact, the main theorem itself is apparently still true as can be seen by a slightly different argument (provided to us by Chillingworth in private communication).

Another motivation (and the inspiration) for the example of Theorem 1, is that it can be interpreted in terms of *frameworks* in the plane. If the vertices on the boundary of the triangle are held *fixed*, for any given triangulation one can ask if it is possible to assign positive scalar *tensions* to the interior edges so that each interior vertex is in equilibrium. An interpretation of Theorem 1 via the work of J. Clerk Maxwell (13) or G. Cremona (6) (see also H. Crapo and W. Whiteley (5)) implies that if the triangulation is twisted no such positive tensions exist.

Yet another use of these 3-dimensional examples is to find analogous examples in dimensions greater than three. Previously such examples had been constructed from a related example due to Barnette (1) following Grünbaum (9), p. 218, and Grünbaum and Sreedharan (10). We construct triangulations of the boundaries of convex sets of dimensions $\geqslant 4$ such that they are not simplicially isomorphic to strictly convex complexes. We can also construct the Grünbaum–Sreedharan–Barnette type of examples as well, but with more vertices.

Lastly, we briefly discuss a hierarchy of examples such as ours, and mention some related ideas and conjectures.

(II) *The examples.* Let A_1, A_2, A_3 be the vertices of a triangle Δ in the plane. Let B_1, B_2, B_3 be three points inside Δ such that the triangles $\Delta_1 = A_2A_3B_1$, $\Delta_2 = A_3A_1B_2$, and $\Delta_3 = A_1A_2B_3$ do not overlap except on common vertices as in Fig. 1. Consider the three angles $\angle B_2A_1B_3$, $\angle B_3A_2B_1$, $\angle B_1A_3B_2$ regarded as (closed) subsets of Δ (shaded in Figure 1). If Δ_1, Δ_2, and Δ_3 are part of a rectilinear triangulation T of Δ and the three angles above do not have a point in common, we say T is *twisted*. Fig. 2 shows one such simple triangulation.

In what follows we shall regard a surface (with boundary) as *convex with respect to a point p* in 3-space if the 3-dimensional solid, obtained by joining all possible line segments from p to the surface, is convex, and each ray from p intersects the surface in at most one point.

THEOREM 1. *Let T be a twisted triangulation of Δ, and p a point not in the plane of Δ. If T' is a triangulated surface, convex with respect to p, such that T is the projection from p of T', then T' is planar.*

Proof. Let Δ_1', Δ_2', Δ_3' be the triangles in T' carried on to Δ_1, Δ_2, Δ_3 of T, respectively. If any two of these are co-planar, then T' is planar since it is convex. (The vertices on the boundary of T' would determine a support plane.)

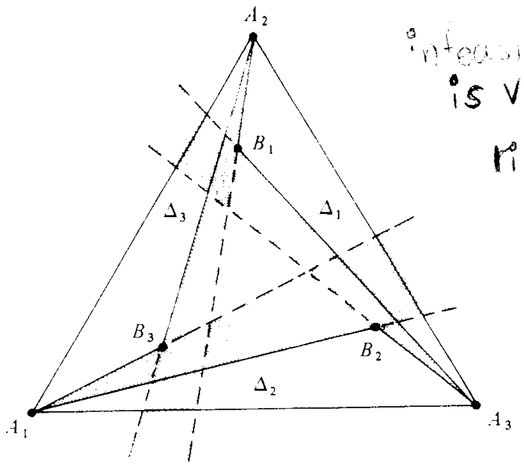

Fig. 1

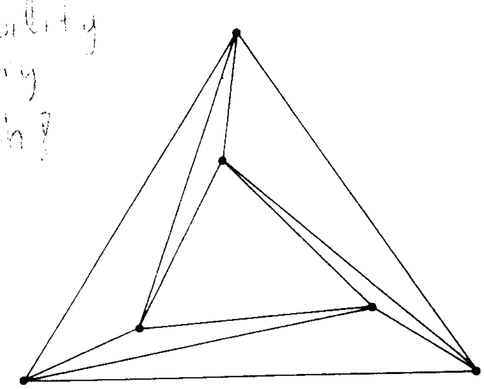

Fig. 2

If no two of Δ'_1, Δ'_2, Δ'_3 are co-planar, the planes determined by them must intersect at a point q'. Let q be the projection of q' into the plane of Δ, the twisted face.

Note that the planes determined by Δ'_{i-1} and Δ'_{i+1} (indices mod 3) must be support planes which interesect in the line $q'A'_i$, where A'_i projects on to A_i.

The plane determined by p, q', A_i ($i = 1, 2, 3$) separates Δ'_{i-1} and Δ'_{i+1}. (If not, then one of Δ'_{i-1} or Δ'_{i+1} is on the opposite side from p of the support plane determined by the other, which contradicts convexity.) So the projection from p of the line $q'A'_i$ must lie in the shaded angle $\angle B_{i-1}A_iB_{i+1}$ in the plane of Δ. Hence q must lie in the intersection of the shaded angles, which does not exist. Then the only possibility is that T' is planar, as was to be shown.

Remark 1. In Shephard (15) and Supnick (17) criteria or algorithms are given for when a 'spherical complex' is the central projection of a convex polytope. See in particular Theorem 3 of Shephard (15). If one takes the triangulation T as above and incorporates it as one face of a tetrahedron, then T will be the projection of a strictly convex complex if and only if the spherical complex obtained by projecting T (from a point inside the tetrahedron) and the rest of the triangulation of the tetrahedron into the 2-sphere is the central projection of a polytope. Thus Shephard and Supnick's criteria must be violated if T is twisted. Perhaps this can be seen also from just looking at Shephard's criterion, but we think that our method is simpler for our case.

COROLLARY. *If, as in Theorem* 1, T *is twisted, and* T' *is convex and simplicially isomorphic to* T, *with the vertices of* T' *sufficiently close to the corresponding vertices of* T *(but not necessarily projecting onto* T*), then* T' *is planar.*

Proof. T' projects on to some other twisted triangulation of Δ in the plane of Δ since the property of being twisted, as we defined it, is open.

Remark 2. With the triangulation of Fig. 2 it is easy to see that it is the projection of a strictly convex triangulation if and only if the intersection of the *open* angles

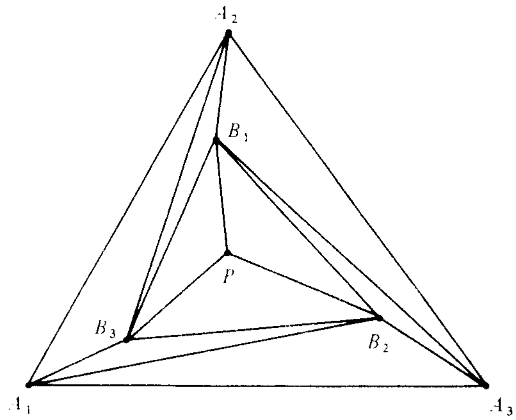

Fig. 3

not empty. One could view this as a precise statement of how far the triangulation must move to guarantee the global conclusion of Steinitz's theorem.

In the following let T be the particular twisted triangulation shown in Figure 3 where P is to the left of $A_i B_{i+1}$, $i = 1, 2, 3$.

THEOREM 2. *The triangulation T can be extended to a subdivision $\overline{T}$ of a tetrahedron such that, if $\overline{T}$ is simplicially isomorphic to a convex complex $\overline{T}'$, the corresponding twisted face T' is planar (and thus the condition for strict convexity at the vertices is violated).*

Proof. Let C be any point not in the plane of Δ, and consider the tetrahedron $CA_1A_2A_3$. Let D_i, $i = 1, 2, 3$, be any point in the relative interior of the triangle $CA_{i-1}A_{i+1}$ (indices mod 3) such that the line determined by C, D_i intersects the edge $A_{i-1}A_{i+1}$ in the angle $\angle B_i A_i P$ and such that the lines $D_i A_i$ ($i = 1, 2, 3$) are disjoint. Then all the faces of

$$T \cup \{CP\} \cup \bigcup_{i=1}^{3} \{CD_i A_{i+1}, CD_i A_{i-1}, D_i A_{i-1} A_{i+1}, A_i D_i, CB_i\}$$

form a complex in 3-space, which is a triangulation of the boundary of $CA_1A_2A_3$ together with seven spanning 1-simplices. By the lemma of J. H. C. Whitehead (19) or lemma 6 of R. H. Bing (2), this complex can be extended to a rectilinear triangulation $\overline{T}$ of $CA_1A_2A_3$. See Figs. 4 and 5.

Suppose we have a convex complex $\overline{T}'$ in $\mathbb{R}^3$ simplicially isomorphic to $\overline{T}$. Primes will label corresponding vertices. Project T', the image of T (the twisted face) from C' into the plane of $A_1'A_2'A_3'$. Call the projection T''. We claim this projection is twisted also. $C'A_i'$ and $C'B_i'$ project to $A_i'' = A_i'$ and B_i'', and $A_i'D_i'$ ($i = 1, 2, 3$) must project into the angle $\angle B_i'' A_i' P''$. Also, D_i' will project outside of or on the boundary of $A_1'A_2'A_3'$, because T' is convex; and $A_i'D_i'$ must be between $C'P'$ and $C'B_i'$. To see this note, for example, that the loop C–B_i–P–C links A_iD_i in $CA_1A_2A_3$ and linking is preserved because the correspondence from $\overline{T}$ to $\overline{T}'$ is a homeomorphism. This ensures that T'' is twisted. Thus by Theorem 1, T' is planar. (It may be helpful to construct a model from pipe cleaners.)

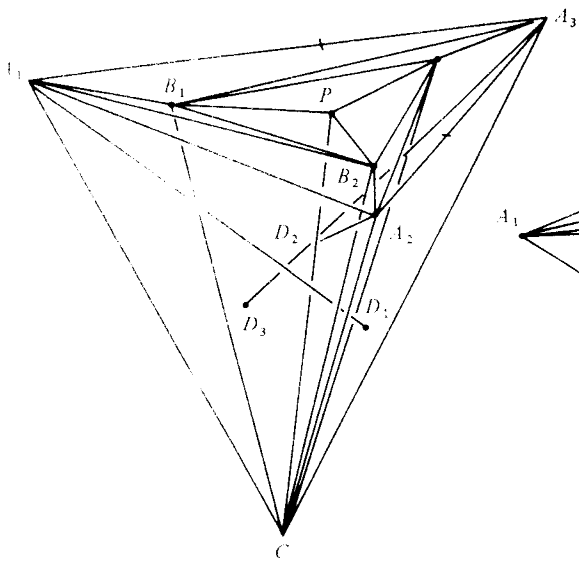

Fig. 4

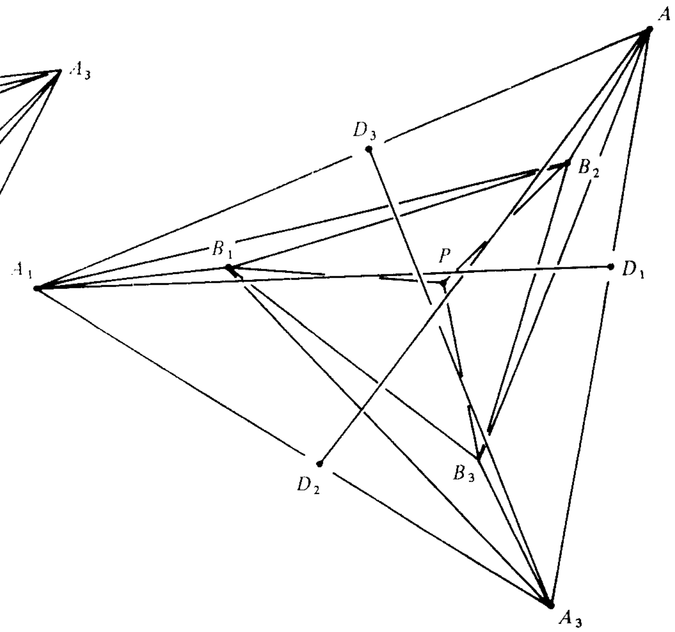

Fig. 5. View from C.

Remark 3. If $A_i D_i$ $(i = 1, 2, 3)$ are woven as in Fig. 4, then the proof will go through even if P is deleted from T. In fact it is not hard to see that Theorem 2 holds for any twisted face T.

Remark 4. We can use the examples of Theorem 2 to provide alternative generators for examples asserted to exist by the following theorem (stated in our terms) due to P. Mani (**12**) (his Proposition 2). This is a higher-dimensional version of Theorem 2, but here only the sphere boundary is needed.

For a closed $(n-1)$-dimensional surface in $\mathbb{R}^n$, we say it is convex (or strictly convex) if the bounded domain enclosed by the surface is convex (or strictly convex). (The interior simplices are not needed for strict convexity.)

THEOREM 3 (*Mani*). *For each $n \geqslant 4$ there is a convex simplicial complex* (*a simplicial* $(n-1)$*-sphere*) K^{n-1} *in* $\mathbb{R}^n$ *such that* K^{n-1} *is not simplicially isomorphic to a strictly convex complex in* $\mathbb{R}^n$.

Actually the $(n-1)$-spheres guaranteed by Mani have only $n+4$ vertices, and our sphere will surely have many more but our methods are somewhat different.

Briefly the idea is to start with an example of Theorem 2 and then take the cone over its boundary from a point in $\mathbb{R}^4$ not in $\mathbb{R}^3$. This defines K^3 in R^4. Triangulating the bounded domain of $|K^3|$ without adding any new vertices to K^3 and taking the cone over K^3 from a point in $\mathbb{R}^5$ not in $\mathbb{R}^4$ gives K^4, etc. It is easy to show that these K^n's have the desired properties for Theorem 3.

In Mani (**12**) K^3 is created from a complex due basically to Grünbaum and Sreedharan (**10**) and simplified (and changed slightly) by D. Barnette (**1**). This is a simplicial subdivision of a tetrahedron (with eight vertices) that is not simplicially isomorphic to a strictly convex complex in $\mathbb{R}^4$. See Grünbaum (9) chapter 11 for another version (the first). To rephrase this, there are examples of d-diagrams for $d = 3$ (simplicial subdivisions of d-simplices) that are not Schlegel diagrams (projections onto one face of the rest of the boundary) of a polytope in $\mathbb{R}^{d+1}$ (a strictly convex triangulated d-dimensional surface). In all of our examples coming from Theorem 2 at least one $(d-1)$-dimensional face has a support plane that intersects the convex set in just that face. So we could project from a point close to the face on the opposite side of the rest of the complex to get our own examples of d-diagrams that are not Schlegel diagrams. It is intriguing to compare the twisting in our examples with the twisting in Barnette's (1). Our examples, however, will have many more than eight vertices.

According to Barnette, Grünbaum's example and his have the property that they cannot be 'inverted'. That is they could be realized in various ways with different tetrahedra as the outside tetrahedron, but not any tetrahedron can be the outside of a representation. He even conjectures that an invertible 3-diagram is a Schlegel diagram. It would be interesting to know if our examples are invertible. However, see (24).

(III) *The hierarchy and some questions*. The examples discussed above fit naturally into a hierarchy of complexes that have more and more convexity. First, there is Cairns' example (3) (see (9) or (18) also) of a 3-complex with a subdivision isomorphic to a subdivision of a 3-simplex, but not isomorphic itself to any complex in $\mathbb{R}^3$.

Second, there are examples due to Goodrick (8) of complexes in $\mathbb{R}^3$ which have subdivisions isomorphic to a subdivision of a 3-simplex, but are not themselves isomorphic to any convex complex in $\mathbb{R}^3$. These are the cubes with a knotted plug, and it is possible for them to be simplicially collapsible, also, as long as the 'bridge number' of the knot is 2. (See Lickorish and Martin (11).) If the bridge number of the knot is high, then it turns out that these examples cannot simplicially collapse (see Goodrick (8)) and from Chillingworth's theorem they, *therefore*, cannot be simplicially isomorphic to a convex complex in $\mathbb{R}^3$.

Third, our example of Theorem 2 is convex but not strictly convex, and in higher dimensions it need only be defined on the boundary.

In view of our results, it is natural to make the following definition. A convex simplicial complex K in $\mathbb{R}^n$ is said to be *k-strictly convex* if, for every k-simplex σ^k in the boundary of K, there is a support plane for $|K|$ intersecting $|K|$ in only σ^k.

It is easy to check that if K is k-strictly convex, then it is k'-strictly convex for every $k' \leqslant k$, and if K is $(n-2)$-strictly convex, it is $(n-1)$-strictly convex. So $(n-1)$-strict convexity is what we have been calling strict convexity.

Note that even if for every $(n-3)$-simplex, σ^{n-3}, in the boundary of K, a support plane of $|K|$ intersects $|K|$ only in σ^{n-3}, then K might be non-convex.

Theorem 2 can be interpreted as saying that the complex defined there is not simplicially isomorphic to a 0-strictly convex complex. However, Theorem 3 seems to need $(n-1)$-strict convexity.

Question. Let K be a convex 0-strictly convex complex in $\mathbb{R}^n$. Is K simplicially isomorphic (by a small move) to a complex K' that is $(n-1)$-strictly convex?

If the answer to the question were affirmative, then there would be little essential difference between the types of strict convexity.

For $n = 3$ the answer to the question is yes. A detailed proof is out of place here, but very briefly the idea is to find a triangular face, if one is available, that is the intersection of a support plane with $|K|$. By projecting the rest of K into this face one can view the projection as a framework in equilibrium with non-negative tensions as mentioned in the introduction.

The edges with 0-tensions correspond to the 'flat' edges of K. By adding a very small amount of tension to these 0-tension edges the framework then will have another equilibrium near to the original. Then a 1-strictly convex surface (thus strictly convex) can be recovered close to the original K. If K has no triangular face (2-simplex) that is the intersection of a support plane with $|K|$, then it can be shown that K has a vertex v with only three 'bent' edges. Then one can 'slice off' v to create a triangu[illegible] face and apply the above procedure leaving the tensions zero on the 0-tension [illegible] that touch this new face. This will create a triangular face outside the star of v. Th[illegible] put v back in by extending the nearby faces. The altered complex will now ha[illegible] triangular face and the above argument applies.

Remark 5. A natural question is: can a convex complex be subdivided to all[illegible] to be altered to a strictly convex embedding? It is easy to see that if one tak[illegible] 'stellar' subdivision of a strictly convex complex, then there is a small motion of [illegible] star points that makes that subdivision strictly convex. (See Ewald and Shephard [illegible] theorem 4, for this same observation.) Any convex n-complex K of $\mathbb{R}^n$ has a subdivis[illegible] L (not necessarily stellar) which is isomorphic to a stellar subdivision of the n-simp[illegible] (see Zeeman (**20**) or Rourke and Sanderson (**14**) for instance). Thus L has a stri[illegible] convex embedding in $\mathbb{R}^n$. So the answer to this question is yes.

Addendum. R. Stanley has pointed out to us that if one takes an example of M[illegible] Rudin (**22**), which is a non-shellable triangulation of tetrahedron (see (**22**) for th[illegible] nition of shellable), and cones over its boundary from a point in $\mathbb{R}^4$, not in the 3-[illegible] spanned by the tetrahedron, then one obtains another example of a convex triangula[illegible] three-sphere in $\mathbb{R}^4$ that is not simplicially isomorphic to a strictly convex embed[illegible] This is because if there were a strictly convex embedding in $\mathbb{R}^4$ an argument [illegible] McMullen (**23**) (p. 182 in the middle), following H. Bruggesser and P. Mani (**21**), [illegible] that the complement of the star of the cone point, which is the Rudin complex, [illegible] shell, a contradiction. Thus there is no strictly convex embedding. So this provi[illegible] alternate example for Theorem 3.

However, there is more. Rudin's complex has the property that all its vert[illegible] on the boundary of the tetrahedron, and it is not hard to show that the verti[illegible] be moved slightly so that Rudin's example can be taken to be strictly [illegible] This can be seen since the subdivision of the tetrahedron when restricted [illegible] boundary is a stellar subdivision and Remark 5 above applies. If one now ta[illegible] cone over the boundary of this complex, Stanley's argument still applies, and so th[illegible] triangulated three-sphere is also *not* simplicially isomorphic to a strictly [illegible] complex in $\mathbb{R}^4$. However, it is easily seen that this three-sphere in $\mathbb{R}^4$ is 0-strictl[illegible] convex. Thus the answer to our question above in the beginning of this section is that K is not always simplicially isomorphic to a strictly convex embedding, even if it is 0-strictly convex.

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and also fit the triangles of $T(B)$ in B.

All that remains to be done is to triangulate the part of the elements of V_2 not in elements of W_2', and do this so as to get a fit with the other elements of T_3 previously defined. As a step toward this, we consider a face F of a tetrahedron of V_2 and triangulate it. Now F already has a linear structure under T_2, and if F does not intersect B, we leave F as it is. Otherwise F is given a triangulation $T(F)$ into triangles under T_2 so that if an element of $T(B)$ lies in F, this element is an element of $T(F)$ — furthermore $T(F)$ is chosen so that if F_i, F_j have an edge in common $T(F_i)$, $T(F_j)$ agree on this edge and no edge of V_1 is subdivided.

Suppose v is a tetrahedron of V_2 not covered by elements of W_2' and $T(\text{Bd}\ v)$ is a triangulation of Bd v imposed by the four triangulations of the faces of v. It follows from Lemma 6 that there is a triangulation $T(v)$ of v which is rectilinear with respect to the structure of v under T_2 and such that each element of $T(B)$ in v and each element of $T(\text{Bd}\ v)$ is an element of $T(v)$. For each tetrahedron v in V_2 not covered by elements of W_2', the elements of $T(v)$ that do not lie in the sum of elements of W_2' are the remaining elements of T_3. We note that these elements agree with $T(B)$ on B and with T_2 on the faces of the tetrahedra of V_1.

LEMMA 6. *If P is a finite polyhedron in E^3 and $T(P)$ is a rectilinear triangulation of P, there is a rectilinear triangulation T of E^3 such that each simplex of $T(P)$ is a simplex of T.*

PROOF. First we build protective cushions about certain exposed parts of P. First we cover exposed faces of $T(P)$ and then we cover exposed edges of $T(P)$.

COVERING EXPOSED FACES. If abc is an exposed face of $T(P)$ (some interval intersects P only at a point, and this point belongs to Int abc), consider a point p to one side of abc and and so near the center of abc that the tetrahedron $abcp$ intersects P only in abc. We regard $abcp$ as a pyramid with base abc and apex p. Such pyramids are placed over each exposed face in $T(P)$, on both sides of a face if both sides are exposed, and such that no two of these added pyramids intersect except possibly in a subset of their bases. Denote the sum of P and all such pyramids by P_1.

COVERING EXPOSED EDGES. Let ab be an exposed edge of $T(P)$ and xyz be a small equilateral triangle such that ab is perpendicular to xyz at the center of xyz and xyz bisects ab. We suppose that the triangles xyz are taken so small that the double pyramids $xyza + xyzb$ about the various edges ab do not intersect, except possibly at the ends of the edges, and no one of these double pyramids needlessly intersects any face, edge, vertex of $T(P)$, or pyramid added in previous step. Denote the sum of P_1 and all such double pyramids by P_2.

THE TRIANGULATION T. Let T' be a triangulation of E^3 such that each isolated vertex of $T(P)$ is a vertex of T' and the 2-skeleton of T' contains each triangle abp, bcp, acp described at the first step, and each triangle xya, xyb, yza, yzb, xza, xzb described at the second stage. Each tetrahedron of T' not in P_2 is an element of T as is each tetrahedron in $T(P)$.

We now triangulate a triangle abp introduced at the first step. The intersection of it with the boundary of the double pyramid $xyza + xyzb$ is a broken line aqb broken only at q. This broken line lies in the 1-skeleton of T' and contains certain vertices of T'. Let $T(abq)$ be the triangulation of triangle abq formed by drawing edges from the center of triangle abq to the various vertices of T' on broken line aqb. Then $T(abp)$ is a triangulation of abp containing the triangles in $T(abq)$ and the triangles of T' in the closure of $abp - abq$. Triangulations $T(bcp)$ and $T(acp)$ are given to bcp and acp in a similar fashion.

We now triangulate the tetrahedron $abcp$. Suppose the triangulations $T(abp)$, $T(bcp)$, $T(abp)$ have been described as above. The tetrahedrons of $T(abcp)$ consists of cones from the center of $abcp$ to the triangle abc, and the triangles of the triangulations of the other faces of $abcp$. Each tetrahedron in each $T(abcp)$ is an element of T. Now we have described all elements of T that lie in either P_1 or $E^3 - P_2$. We note that ab is an edge and abc is a face in this triangulation.

Now consider a tetrahedron $abxy$ in the double pyramid described at the second step. If it does not lie in P_1 (elements of T have already been defined covering it if it lies in P_1) the closure of each component of $abxy - P_1$ is a pyramid $abrs$. Then abr and abs are triangulated as abq was triangulated previously (it may be that abr or abs is an abq) and ars and brs are triangulated by T'. Then $abrs$ is triangulated by taking cones from the center of $abrs$ to the various triangles on its triangulated boundary. The tetrahedrons $bcxy$ and $acxy$ are treated similarly.

LEMMA 7. *Suppose M is a 3-manifold with boundary, T is a triangulation of M, K is a closed subset of M, and f is a positive continuous function defined on M. Then there is a subdivision T' of T such that each element of T that misses K is an element of T' and each element t of T' that intersects K is of diameter less than the minimum value of f on t.*

PROOF. Let W be the set of elements of T that miss K and f' be a positive continuous function on M such that $f' < f$ and for each point x