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A convex 3-complex not simplicially isomorphic to a strictly convex complex

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(I) *Introduction.* A set X in euclidean space is *convex* if the line segment joining any two points of X is in X. If X is convex, every boundary point is on an $(n-1)$-plane which contains X in one of its two closed half-spaces. Such a plane is called a *support plane* for X. A simplicial complex K in $\mathbb{R}^n$ is called *strictly convex* if $|K|$ (the underlying space of K) is convex and if, for every simplex σ in ∂K (the boundary of K) there is a support plane for $|K|$ whose intersection with $|K|$ is precisely σ. In this case $|K|$ is often called a *simplicial polytope*.

If $|K|$ is just convex it is often desired to move the vertices of K (slightly) so that the altered complex is strictly convex. In Theorem 2 we provide an example of a 3-complex where no such altering is possible even by a large move. In fact we show a bit more. There is a 3-complex K such that $|K|$ is a tetrahedron (and thus convex) and K is not simplicially isomorphic to a complex K' in $\mathbb{R}^3$, where $|K'|$ is convex and the condition for strict convexity holds at the vertices of K'.

Part of the motivation for this example was the following statement made by D. Chillingworth in (4) on page 354: '. . . we can if necessary slightly alter the positions of some of the vertices to obtain a complex simplicially isomorphic to K, which has no two vertices at the same height and is such that v_1 is strictly higher than all the other vertices'.

Our Theorem 1 provides a specific complex T, where $|T|$ is a triangle (2-simplex), such that if T is the projection (from an appropriate point below say) of a simplicially isomorphic complex T', which is part of the boundary of a convex surface in 3-space, then $|T'|$ has to be a flat triangle, so its interior vertices violate the condition of strict convexity. The condition needed for T to have the property that Theorem 1 holds, is that a certain interior triangle be turned or twisted sufficiently with respect to the outer triangle. When T satisfies this condition (defined later) we say it is *twisted*. It turns out that, if T is twisted, then any other complex T' simplicially isomorphic to T with the corresponding vertices of T' close enough to T, is twisted also. Thus if $|T|$ is a subset of the boundary of a tetrahedron, then it provides a counterexample to the statement of Chillingworth. Theorem 2 gives a way of triangulating the whole tetrahedron so that the interior simplices force the one face to be twisted, thus providing a global counterexample to Chillingworth's statement. (We say a complex K (*rectilinearly*) *triangulates* a space X, if $X = |K|$.)

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It should be borne in mind that there is no way of finding a global counterexample just using a triangulation of the boundary of a convex 3-dimensional set. A theorem of Steinitz (see (16) or (9), chapter 13, for example) says that, among other things, any abstract simplicial complex topologically homeomorphic to a 2-dimensional sphere is simplicially isomorphic to the boundary of a strictly convex complex.

Despite these counter-examples, corollaries 2 and 3 of Chillingworth are unaffected, since the starting vertex in the theorem can be chosen appropriately anyway without using the statement we quoted above. In fact, the main theorem itself is apparently still true as can be seen by a slightly different argument (provided to us by Chillingworth in private communication).

Another motivation (and the inspiration) for the example of Theorem 1, is that it can be interpreted in terms of *frameworks* in the plane. If the vertices on the boundary of the triangle are held *fixed*, for any given triangulation one can ask if it is possible to assign positive scalar *tensions* to the interior edges so that each interior vertex is in equilibrium. An interpretation of Theorem 1 via the work of J. Clerk Maxwell (13) or G. Cremona (6) (see also H. Crapo and W. Whiteley (5)) implies that if the triangulation is twisted no such positive tensions exist.

Yet another use of these 3-dimensional examples is to find analogous examples in dimensions greater than three. Previously such examples had been constructed from a related example due to Barnette (1) following Grünbaum (9), p. 218, and Grünbaum and Sreedharan (10). We construct triangulations of the boundaries of convex sets of dimensions $\geqslant 4$ such that they are not simplicially isomorphic to strictly convex complexes. We can also construct the Grünbaum–Sreedharan–Barnette type of examples as well, but with more vertices.

Lastly, we briefly discuss a hierarchy of examples such as ours, and mention some related ideas and conjectures.

(II) *The examples.* Let A_1, A_2, A_3 be the vertices of a triangle Δ in the plane. Let B_1, B_2, B_3 be three points inside Δ such that the triangles $\Delta_1 = A_2A_3B_1$, $\Delta_2 = A_3A_1B_2$, and $\Delta_3 = A_1A_2B_3$ do not overlap except on common vertices as in Fig. 1. Consider the three angles $\angle B_2A_1B_3$, $\angle B_3A_2B_1$, $\angle B_1A_3B_2$ regarded as (closed) subsets of Δ (shaded in Figure 1). If Δ_1, Δ_2, and Δ_3 are part of a rectilinear triangulation T of Δ and the three angles above do not have a point in common, we say T is *twisted*. Fig. 2 shows one such simple triangulation.

In what follows we shall regard a surface (with boundary) as *convex with respect to a point p* in 3-space if the 3-dimensional solid, obtained by joining all possible line segments from p to the surface, is convex, and each ray from p intersects the surface in at most one point.

THEOREM 1. *Let T be a twisted triangulation of Δ, and p a point not in the plane of Δ. If T' is a triangulated surface, convex with respect to p, such that T is the projection from p of T', then T' is planar.*

Proof. Let Δ_1', Δ_2', Δ_3' be the triangles in T' carried on to Δ_1, Δ_2, Δ_3 of T, respectively. If any two of these are co-planar, then T' is planar since it is convex. (The vertices on the boundary of T' would determine a support plane.)

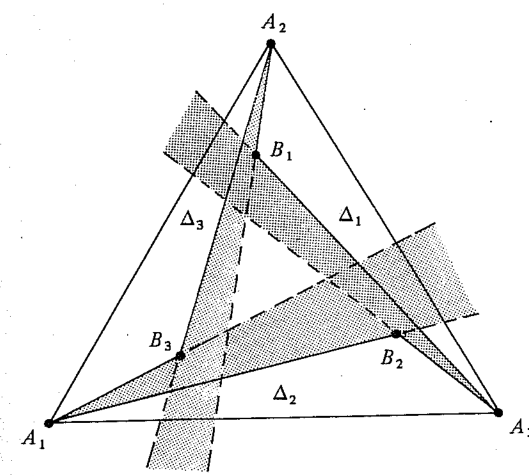

Fig. 1

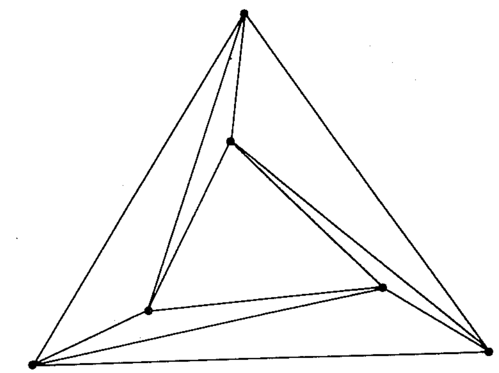

Fig. 2

If no two of Δ_1', Δ_2', Δ_3' are co-planar, the planes determined by them must intersect at a point q'. Let q be the projection of q' into the plane of Δ, the twisted face.

Note that the planes determined by Δ_{i-1}' and Δ_{i+1}' (indices mod 3) must be support planes which interesect in the line $q'A_i'$, where A_i' projects on to A_i.

The plane determined by p, q', A_i $(i = 1, 2, 3)$ separates Δ_{i-1}' and Δ_{i+1}'. (If not, then one of Δ_{i-1}' or Δ_{i+1}' is on the opposite side from p of the support plane determined by the other, which contradicts convexity.) So the projection from p of the line $q'A_i'$ must lie in the shaded angle $\angle B_{i-1}A_iB_{i+1}$ in the plane of Δ. Hence q must lie in the intersection of the shaded angles, which does not exist. Then the only possibility is that T' is planar, as was to be shown.

Remark 1. In Shephard (15) and Supnick (17) criteria or algorithms are given for when a 'spherical complex' is the central projection of a convex polytope. See in particular Theorem 3 of Shephard (15). If one takes the triangulation T as above and incorporates it as one face of a tetrahedron, then T will be the projection of a strictly convex complex if and only if the spherical complex obtained by projecting T (from a point inside the tetrahedron) and the rest of the triangulation of the tetrahedron into the 2-sphere is the central projection of a polytope. Thus Shephard and Supnick's criteria must be violated if T is twisted. Perhaps this can be seen also from just looking at Shephard's criterion, but we think that our method is simpler for our case.

COROLLARY. *If, as in Theorem 1, T is twisted, and T' is convex and simplicially isomorphic to T, with the vertices of T' sufficiently close to the corresponding vertices of T (but not necessarily projecting onto T), then T' is planar.*

Proof. T' projects on to some other twisted triangulation of Δ in the plane of Δ since the property of being twisted, as we defined it, is open.

Remark 2. With the triangulation of Fig. 2 it is easy to see that it is the projection of a strictly convex triangulation if and only if the intersection of the *open* angles is

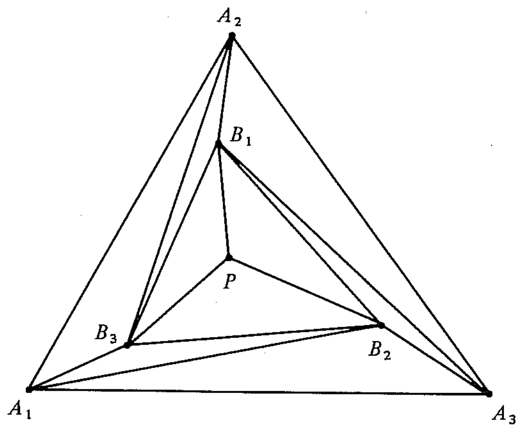

Fig. 3

not empty. One could view this as a precise statement of how far the triangulation must move to guarantee the global conclusion of Steinitz's theorem.

In the following let T be the particular twisted triangulation shown in Figure 3 where P is to the left of $A_i B_{i+1}$, $i = 1, 2, 3$.

THEOREM 2. *The triangulation T can be extended to a subdivision $\bar{T}$ of a tetrahedron such that, if $\bar{T}$ is simplicially isomorphic to a convex complex $\bar{T}'$, the corresponding twisted face T' is planar (and thus the condition for strict convexity at the vertices is violated).*

Proof. Let C be any point not in the plane of Δ, and consider the tetrahedron $CA_1A_2A_3$. Let D_i, $i = 1, 2, 3$, be any point in the relative interior of the triangle $CA_{i-1}A_{i+1}$ (indices mod 3) such that the line determined by C, D_i intersects the edge $A_{i-1}A_{i+1}$ in the angle $\angle B_i A_i P$ and such that the lines $D_i A_i$ ($i = 1, 2, 3$) are disjoint. Then all the faces of

$$T \cup \{CP\} \cup \bigcup_{i=1}^{3} \{CD_i A_{i+1}, CD_i A_{i-1}, D_i A_{i-1} A_{i+1}, A_i D_i, CB_i\}$$

form a complex in 3-space, which is a triangulation of the boundary of $CA_1A_2A_3$ together with seven spanning 1-simplices. By the lemma of J. H. C. Whitehead (19) or lemma 6 of R. H. Bing (2), this complex can be extended to a rectilinear triangulation $\bar{T}$ of $CA_1A_2A_3$. See Figs. 4 and 5.

Suppose we have a convex complex $\bar{T}'$ in $\mathbb{R}^3$ simplicially isomorphic to $\bar{T}$. Primes will label corresponding vertices. Project T', the image of T (the twisted face) from C into the plane of $A_1'A_2'A_3'$. Call the projection T''. We claim this projection is twisted also. $C'A_i'$ and $C'B_i'$ project to $A_i'' = A_i'$ and B_i'', and $A_i'D_i'$ ($i = 1, 2, 3$) must project into the angle $\angle B_i'' A_i' P''$. Also, D_i' will project outside of or on the boundary of $A_1'A_2'A_3'$, because T' is convex; and $A_i'D_i'$ must be between $C'P'$ and $C'B_i'$. To see this note, for example, that the loop C–B_i–P–C links $A_i D_i$ in $CA_1A_2A_3$ and linking is preserved because the correspondence from $\bar{T}$ to $\bar{T}'$ is a homeomorphism. This ensures that T'' is twisted. Thus by Theorem 1, T' is planar. (It may be helpful to construct a model from pipe cleaners.)

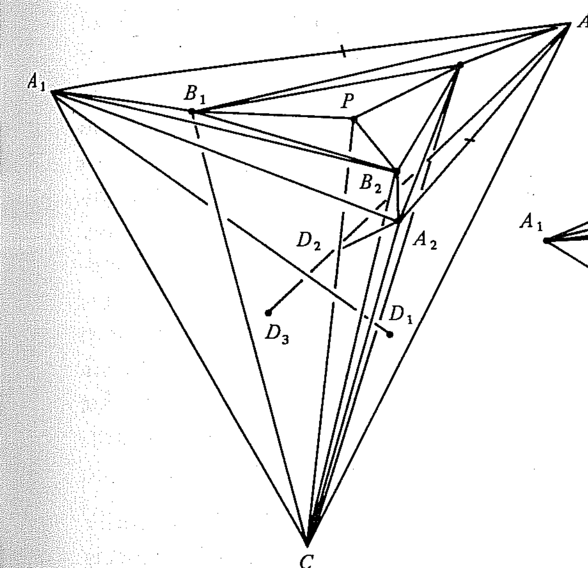

Fig. 4

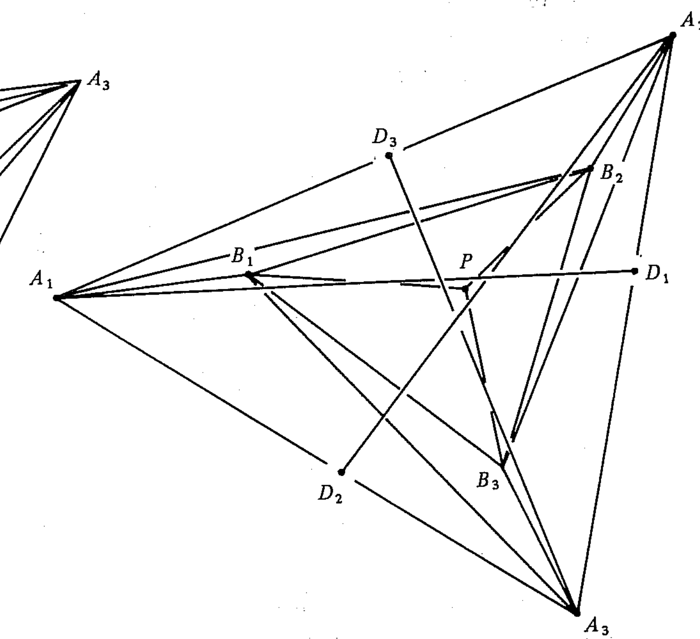

Fig. 5. View from C.

Remark 3. If $A_i D_i$ ($i = 1, 2, 3$) are woven as in Fig. 4, then the proof will go through even if P is deleted from T. In fact it is not hard to see that Theorem 2 holds for any twisted face T.

Remark 4. We can use the examples of Theorem 2 to provide alternative generators for examples asserted to exist by the following theorem (stated in our terms) due to P. Mani (12) (his Proposition 2). This is a higher-dimensional version of Theorem 2, but here only the sphere boundary is needed.

For a closed $(n-1)$-dimensional surface in $\mathbb{R}^n$, we say it is convex (or strictly convex) if the bounded domain enclosed by the surface is convex (or strictly convex). (The interior simplices are not needed for strict convexity.)

THEOREM 3 (*Mani*). *For each $n \geqslant 4$ there is a convex simplicial complex (a simplicial $(n-1)$-sphere) K^{n-1} in $\mathbb{R}^n$ such that K^{n-1} is not simplicially isomorphic to a strictly convex complex in $\mathbb{R}^n$.*

Actually the $(n-1)$-spheres guaranteed by Mani have only $n+4$ vertices, and our sphere will surely have many more but our methods are somewhat different.

Briefly the idea is to start with an example of Theorem 2 and then take the cone over its boundary from a point in $\mathbb{R}^4$ not in $\mathbb{R}^3$. This defines K^3 in R^4. Triangulating the bounded domain of $|K^3|$ without adding any new vertices to K^3 and taking the cone over K^3 from a point in $\mathbb{R}^5$ not in $\mathbb{R}^4$ gives K^4, etc. It is easy to show that these K^n's have the desired properties for Theorem 3.

In Mani (12) K^3 is created from a complex due basically to Grünbaum and Sreedharan (10) and simplified (and changed slightly) by D. Barnette (1). This is a simplicial subdivision of a tetrahedron (with eight vertices) that is not simplicially isomorphic to a strictly convex complex in $\mathbb{R}^4$. See Grünbaum (9) chapter 11 for another version (the first). To rephrase this, there are examples of d-diagrams for $d = 3$ (simplicial subdivisions of d-simplices) that are not Schlegel diagrams (projections onto one face of the rest of the boundary) of a polytope in $\mathbb{R}^{d+1}$ (a strictly convex triangulated d-dimensional surface). In all of our examples coming from Theorem 2 at least one $(d-1)$-dimensional face has a support plane that intersects the convex set in just that face. So we could project from a point close to the face on the opposite side of the rest of the complex to get our own examples of d-diagrams that are not Schlegel diagrams. It is intriguing to compare the twisting in our examples with the twisting in Barnette's (1). Our examples, however, will have many more than eight vertices.

According to Barnette, Grünbaum's example and his have the property that they cannot be 'inverted'. That is they could be realized in various ways with different tetrahedra as the outside tetrahedron, but not any tetrahedron can be the outside of a representation. He even conjectures that an invertible 3-diagram is a Schlegel diagram. It would be interesting to know if our examples are invertible. However, see (24).

(III) *The hierarchy and some questions*. The examples discussed above fit naturally into a hierarchy of complexes that have more and more convexity. First, there is Cairns' example (3) (see (9) or (18) also) of a 3-complex with a subdivision isomorphic to a subdivision of a 3-simplex, but not isomorphic itself to any complex in $\mathbb{R}^3$.

Second, there are examples due to Goodrick (8) of complexes in $\mathbb{R}^3$ which have subdivisions isomorphic to a subdivision of a 3-simplex, but are not themselves isomorphic to any convex complex in $\mathbb{R}^3$. These are the cubes with a knotted plug, and it is possible for them to be simplicially collapsible, also, as long as the 'bridge number' of the knot is 2. (See Lickorish and Martin (11).) If the bridge number of the knot is high, then it turns out that these examples cannot simplicially collapse (see Goodrick (8)) and from Chillingworth's theorem they, *therefore*, cannot be simplicially isomorphic to a convex complex in $\mathbb{R}^3$.

Third, our example of Theorem 2 is convex but not strictly convex, and in higher dimensions it need only be defined on the boundary.

In view of our results, it is natural to make the following definition. A convex simplicial complex K in $\mathbb{R}^n$ is said to be *k-strictly convex* if, for every k-simplex σ^k in the boundary of K, there is a support plane for $|K|$ intersecting $|K|$ in only σ^k.

It is easy to check that if K is k-strictly convex, then it is k'-strictly convex for every $k' \leqslant k$, and if K is $(n-2)$-strictly convex, it is $(n-1)$-strictly convex. So $(n-1)$-strict convexity is what we have been calling strict convexity.

Note that even if for every $(n-3)$-simplex, σ^{n-3}, in the boundary of K, a support plane of $|K|$ intersects $|K|$ only in σ^{n-3}, then K might be non-convex.

Theorem 2 can be interpreted as saying that the complex defined there is not simplicially isomorphic to a 0-strictly convex complex. However, Theorem 3 seems to need $(n-1)$-strict convexity.

Question. Let K be a convex 0-strictly convex complex in $\mathbb{R}^n$. Is K simplicially isomorphic (by a small move) to a complex K' that is $(n-1)$-strictly convex?

If the answer to the question were affirmative, then there would be little essential difference between the types of strict convexity.

For $n = 3$ the answer to the question is yes. A detailed proof is out of place here, but very briefly the idea is to find a triangular face, if one is available, that is the intersection of a support plane with $|K|$. By projecting the rest of K into this face one can view the projection as a framework in equilibrium with non-negative tensions as mentioned in the introduction.

The edges with 0-tensions correspond to the 'flat' edges of K. By adding a very small amount of tension to these 0-tension edges the framework then will have another equilibrium near to the original. Then a 1-strictly convex surface (thus strictly convex) can be recovered close to the original K. If K has no triangular face (2-simplex) that is the intersection of a support plane with $|K|$, then it can be shown that K has a vertex v with only three 'bent' edges. Then one can 'slice off' v to create a triangular face and apply the above procedure leaving the tensions zero on the 0-tension edges that touch this new face. This will create a triangular face outside the star of v. Then put v back in by extending the nearby faces. The altered complex will now have a triangular face and the above argument applies.

Remark 5. A natural question is: can a convex complex be subdivided to allow it to be altered to a strictly convex embedding? It is easy to see that if one takes a 'stellar' subdivision of a strictly convex complex, then there is a small motion of the star points that makes that subdivision strictly convex. (See Ewald and Shephard (7), theorem 4, for this same observation.) Any convex n-complex K of $\mathbb{R}^n$ has a subdivision L (not necessarily stellar) which is isomorphic to a stellar subdivision of the n-simplex (see Zeeman (20) or Rourke and Sanderson (14) for instance). Thus L has a strictly convex embedding in $\mathbb{R}^n$. So the answer to this question is yes.

Addendum. R. Stanley has pointed out to us that if one takes an example of M. E. Rudin (22), which is a non-shellable triangulation of tetrahedron (see (22) for the definition of shellable), and cones over its boundary from a point in $\mathbb{R}^4$, not in the 3-space spanned by the tetrahedron, then one obtains another example of a convex triangulated three-sphere in $\mathbb{R}^4$ that is not simplicially isomorphic to a strictly convex embedding. This is because if there were a strictly convex embedding in $\mathbb{R}^4$ an argument of P. McMullen (23) (p. 182 in the middle), following H. Bruggesser and P. Mani (21), implies that the complement of the star of the cone point, which is the Rudin complex, would shell, a contradiction. Thus there is no strictly convex embedding. So this provides an alternate example for Theorem 3.

However, there is more. Rudin's complex has the property that all its vertices are on the boundary of the tetrahedron, and it is not hard to show that the vertices can be moved slightly so that Rudin's example can be taken to be strictly convex in R^3. This can be seen since the subdivision of the tetrahedron when restricted to the boundary is a stellar subdivision and Remark 5 above applies. If one now takes the cone over the boundary of this complex, Stanley's argument still applies, and so this triangulated three-sphere is also *not* simplicially isomorphic to a strictly convex complex in $\mathbb{R}^4$. However, it is easily seen that this three-sphere in $\mathbb{R}^4$ is 0-strictly convex. Thus the answer to our question above in the beginning of this section is that K is not always simplicially isomorphic to a strictly convex embedding, even if it is 0-strictly convex.

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Collapsing three-dimensional convex polyhedra: correction

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An ingenious construction due to Connelly and Henderson (2) has shown that there exists a rectilinearly triangulated convex polyhedron P in $\mathbb{R}^3$ having the property that at least one vertex of the triangulation lies in the interior of a face of P, and yet there is no isomorphic triangulation of a convex polyhedron P' all of whose vertices are vertices of P'. Thus the assertion beginning on the top line of p. 354 of (1) is false, which leaves a gap in the proof of essentially the main result of (1), namely that *any rectilinearly triangulated convex polyhedron in* $\mathbb{R}^3$ *can be simplicially collapsed onto its boundary minus a 2-simplex* σ. The purpose of this note is to show that the theorem is nevertheless still true. In any case the Corollaries 2 and 3 in (1) are unaffected by the error.

We refer to (1) for notation and terminology, here taking *collapse* to mean *simplicial collapse* and renaming σ as σ_0. The basic idea of the proof in (1) was to choose an appropriate direction in $\mathbb{R}^3$ to measure 'height', and to carry out the required collapse by considering the vertices of the triangulation K in order of decreasing height. The mistake lay in supposing that without loss of generality one vertex of σ_0 could be taken to be the highest of all the vertices of K. Since we cannot now assume this, we have to proceed more carefully to deal with the face T of P which contains σ_0.

Choose a direction to measure height as before, in such a way that (i) all vertices of K are at different heights, and (ii) each vertex in T is higher than all vertices not in T. Let S be the subcomplex of K consisting of those closed simplexes that meet T. Observe that S is topologically a closed 3-ball, being homeomorphic to the part of P lying in or above a horizontal plane that passes below T but above all other vertices of K. Let ∂S denote the triangulation of the boundary of this 3-ball that it inherits from K.

PROPOSITION. *S collapses to $\partial S - \sigma_0$.*

The truth of this Proposition implies the collapsibility of K to $\partial K - \sigma_0$, and hence the validity of the main Theorem in (1), since once all the simplexes of $\operatorname{int} S = S - \partial S$ have been collapsed away, the inductive procedure in (1) can then be applied to the rest of K with no further difficulty.

Proof. For convenience in this proof we now re-select our height direction in order to make T horizontal. Let τ be a 1-simplex of K in T, and let $\mathrm{St}(\tau, K)$ be the smallest subcomplex of K containing all 3-simplexes having τ as an edge. Topologically $\mathrm{St}(\tau, K)$ is a 3-ball, whose boundary $\partial\,\mathrm{St}(\tau, K)$ has a triangulation as a subcomplex of K. Let $\mathrm{Lk}(\tau, K)$ be the union of those simplexes of $\mathrm{St}(\tau, K)$ that do not share any vertex with τ, and let $\mathrm{Lk}_L(\tau, K) \subset \mathrm{Lk}(\tau, K)$ consist of those simplexes all of whose vertices lie below T. Finally let $\mathrm{St}_L(\tau, K)$ be the join of τ (and its vertices) to $\mathrm{Lk}_L(\tau, K)$. The following lemma is an elementary exercise.