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MINIMAL TRIANGULATION OF THE 4-CUBE*

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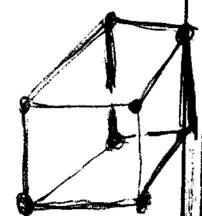

It is known that the 4-dimensional cube can be triangulated by a set of 16 simplices. This note demonstrates that the 4-dimensional cube cannot be triangulated with fewer than 16 simplices.

aim of this note is to prove that the triangulation of the unit 4-cube d by Mara [2] is minimal. Like Mara, we restrict attention to triangulation ube into simplices whose vertices are vertices of the cube itself. To say that ulation is minimal means that no fewer simplices can be used to triangu- body in question.

triangulation of the 4-cube into 16 simplices is based upon two s. The first 'slices off' 8 particular vertices and their neighbors. The plits the remaining convex body into another 8 simplices by passing three es through it. He shows that this process yields a minimal triangulation t is true that it is optimal to perform the first operation. This paper will he minimality of the resulting triangulation without such an assumption. y enough, Mara's Theorem 1 is an integral part of the minimality hould be emphasized that it is independent of the assumption he obtaining his limited minimality result. We quote Mara's Theorem im for ease of reference.

Mara [2, p. 173]). *If P_n denotes that number of simplices in the minimal n of I^n, E_n denotes the total number of exterior $(n-1)$-faces, and F_n number of interior $(n-1)$-faces, then*

$$+1)P_n = E_n + 2F_n, \quad (1)$$

$$\geq (2n)P_{n-1}, \quad (2)$$

$$\geq 2P_{n-1}. \quad (3)$$

he words 'exterior' and 'interior' in Mara's Theorem are meant he n-cube. An $(n-1)$-face (facet) of an n-simplex of a triangulation is

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exterior if and only if n of its vertices lie in some $(n-1)$-face of the
Otherwise, it is *interior*.

We now derive some consequences of Mara's Theorem for the case n

(a$_4$) $5P_4 = E_4 + 2F_4$,

(b$_4$) $E_4 \geqslant 8P_3$,

(c$_4$) $P_4 \geqslant 2P_3$.

It is not difficult to show (see Mara [2]) that $P_3 = 5$, so (b$_4$) and (c$_4$) become

$$E_4 \geqslant 40$$

and

$$P_4 \geqslant 10,$$

respectively. Thus, we obtain a crucial inequality:

$$2F_4 \leqslant 5P_4 - 40.$$

From (8) and the existence of a triangulation of I^4 into 16 simplices, we have

$$10 \leqslant P_4 \leqslant 16.$$

Useful insight into the 4-cube is given by the following observation.

Lemma 1. *The volume of any* 4-*simplex whose vertices are also vertices of* I^4 *is one of the numbers* 1/24, 2/24, 3/24.

Proof. Recall that the volume of an n-simplex is $(1/n!)\,|\det[B, e]|$ where B is the $(n+1)\times n$ matrix whose rows are the coordinates of the vertices of the simplex and e is a column vector of ones. In the case at hand ($n = 4$) we have $n! = 24$ which accounts for the denominators of the numbers listed. As for the numerators, Hadamard's determinant theorem (see [3, p.114]) implies for any n:

$$|\det[B, e]| = 2^{-n}\,|\det[2B - ee^{\mathrm{T}}, e]| \leqslant (n+1)^{(n+1)/2}.$$

For $n = 4$, it follows that $|\det[B, e]| \leqslant 3$.

We refer to a simplex (of the sort specified in Lemma 1) as being of type i if its volume is $i/24$ ($i = 1, 2, 3$). As it turns out, I^4 admits

2672 simplices of type 1,

320 simplices of type 2,

16 simplices of type 3.

Consequently, when we speak of a triangulation of I^4 (minimal or otherwise), we can let x_i denote the number of simplices of type i that it uses. So, for *any* triangulation of I^4,

$$x_1 + 2x_2 + 3x_3 = 24. \tag{11}$$

minimal triangulation of the 4-cube, we have

$$P_4 = x_1 + x_2 + x_3. \qquad (12)$$

contrast to Mara, we find it somewhat more illuminating to label the vertices he cube in accordance with the so-called Gray code rather than the base-2 e. In place of a lengthy discusion of the Gray code, we refer the reader to [1] the literature cited therein. For the present purposes, the information ided in Table 1 will suffice.

e 1
code labelling of the vertices of I^4

ordinates	Label	Coordinates	Label
0, 0, 0, 0)	0	(1, 1, 0, 0)	8
0, 0, 0, 1)	1	(1, 1, 0, 1)	9
0, 0, 1, 1)	2	(1, 1, 1, 1)	10
0, 0, 1, 0)	3	(1, 1, 1, 0)	11
0, 1, 1, 0)	4	(1, 0, 1, 0)	12
0, 1, 1, 1)	5	(1, 0, 1, 1)	13
0, 1, 0, 1)	6	(1, 0, 0, 1)	14
0, 1, 0, 0)	7	(1, 0, 0, 0)	15

t has been found empirically (using the Gray code labelling and a computer to nerate all the 4-simplices in I^4) that there are exactly 16 simplices of type 3. eir vertices bear the labels listed in Table 2.

ble 2
plices of type 3 (with Gray code label-
g of vertices)

0, 2, 4, 9, 12	1, 4, 9, 13, 15
0, 2, 6, 11, 14	1, 5, 7, 9, 12
0, 4, 6, 8, 13	2, 4, 6, 10, 15
0, 5, 8, 12, 14	2, 7, 9, 11, 15
0, 5, 9, 11, 13	2, 7, 10, 12, 14
1, 3, 5, 8, 13	3, 5, 7, 11, 14
1, 3, 7, 10, 15	3, 6, 8, 10, 14
1, 4, 8, 10, 12	3, 6, 11, 13, 15

Because of the Gray Code labelling, one interesting feature of these simplices stands out. Each simplex (of type 3) has exactly one vertex whose label has the opposite parity from the other four vertices of that simplex. We call this the *apex* of that simplex. As it turns out, each of the numbers 0, 1, . . . , 15 is the label of some apex of a simplex of type 3. Furthermore, all 16 of these type-3 simplices have the same 'structure'. The apex of each is the antipodal point (with resp the 4-cube) of a point whose neighbors are the remaining vertices of the simpl

The simplices of type 3 do not have disjoint interiors.

Lemma 2. *The point* $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ *is interior to each simplex of type* 3.

Proof. Notice that the numbers $\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{3}$ sum to 1. These 5 numbers (in so order) are the barycentric coordinates of $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ with respect to the vertic each simplex of type 3. Indeed, the apex gets 'weight' $\frac{1}{3}$, and the other verti each get weight $\frac{1}{6}$.

Corollary. *No triangulation of* I^4 *uses more than one simplex of type* 3, *i.e.,*

$$x_3 \leqslant 1. \qquad (13)$$

Further (computer-aided) analysis of these three type of simplices in I^4 reveals that each simplex of type 2 contains at least 4 interior 3-faces. It is clear that every simplex contains at least one interior 3-face, and it is easily verified that every simplex of type 3 contains exactly 5 interior 3-faces.

Since, in any triangulation, the interior 3-faces (of the 4-simplices it uses) must belong to exactly two of the 4-simplices, we have, for a minimal triangulation of I^4:

$$2F_4 \geqslant x_1 + 4x_2 + 5x_3. \qquad (14)$$

To show that a minimal triangulation of the 4-cube must consist of 16 simplices, we use (9), (10), (11), (12), (13), and (14). In particular, (9), (14) and (11) imply

$$7x_2 + 12x_3 \leqslant 56. \qquad (15)$$

Moreover, (10), (12), and (11) imply

$$8 \leqslant x_2 + 2x_2 \leqslant 14. \qquad (16)$$

Recall that x_1, x_2 and x_3 are nonnegative integers and from (13) we have $x_3 = 0$ or $x_3 = 1$.

If $x_3 = 0$, then (15) and (16) imply $x_2 = 8$, in which case (11) implies $x_1 = 8$ and (12) implies $P_4 = 16$. (This is just the sort of triangulation exhibited by Mara.) If $x_3 = 1$, then (15) and (16) imply $x_2 = 6$, in which case $x_1 = 9$ and (12) implies $P_4 = 16$. Thus, we have proved the

Theorem. $P_4 = 16$.

Discussion

This result has been obtained in a much different way by Sallee [4] who obtains an upper bound on P_n which in the case of $n = 4$ implies that the four cube has a minimal triangulation using only 16 simplices.

roof that $P_4 = 16$ given above raises the question: Do there exist triangu- f type (9, 6, 1)? The answer is in the negative as each simplex of type 3 the ball of radius $1/(2\sqrt{7})$ centered at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Four of the 3-faces of implex are tangent to this ball. These four faces also belong to simplices of But this ball meets every simplex of type 2. It follows that there can exist than 4 simplices of type 2 in any triangulation of I^4 that uses a simplex of

t from the parity of the vertices 'sliced off', the preceding remark implies ere is essentially only one minimal triangulation of I^4. It should be noted is triangulation induces minimal triangulations of the faces of I^4. The latter ngruent to I^3. It would be very interesting to know whether a minimal lation of I^n induces minimal triangulations of its $(n-1)$-faces. If so, then hypothesis (that in minimal triangulations of I^n the alternate vertices are 'sliced off') would be verified.

ences

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Some GKZ vectors for the 3-cube

[1, ?, ?, ?, ?, 3, 5, ?] *

[1, 1, 3, 3, 3, 3, 5, 5] *

[1, ?, ?, 3, 3, 4, ?, ?] *

[1, 1, 1, 3, 4, 4, 4, 6] *

[1, 1, 1, 1, 5, 5, 5, 5] *

[2, 2, 2, 2, ?, 2, ?, ?] *