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ON OPTIMAL INTERPOLATION TRIANGLE INCIDENCES*

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Abstract. The problem of determining optimal incidences for triangulating a given set of vertices for the model problem of interpolating a convex quadratic surface by piecewise linear functions is studied. An exact expression for the maximum error is derived, and the optimality criterion is minimization of the maximum error. The optimal incidences are shown to be derivable from an associated Delaunay triangulation and hence are computable in $O(N \log N)$ time for N vertices.

Key words. optimal mesh, triangulation incidence, surface approximation

AMS(MOS) subject classifications. 65D05, 65L50

1. Introduction. In this paper, we study the question of an optimal choice of edge incidences for triangulating a given set of points. The study uses a model approximation problem, piecewise linear interpolation of a convex quadratic surface, and our optimality criterion is that the optimal choice of incidences minimizes the maximum error in any triangle. We establish that the optimal incidence problem can be transformed to an equivalent Delaunay triangulation problem, showing in particular that at least this model optimal incidence problem can be solved in time $O(N \log N)$ for N interpolation points.

General triangles have two independent length scales associated with them, e.g., the longest edge and the length of the perpendicular from this edge to the opposite vertex. It is common to regard the local error over a triangle T as depending on one length scale (the "size" of T, typically denoted "h") and to impose a geometric condition on the triangulation, i.e., small angles should be avoided. Strang and Fix [15] have developed an error bound that depends on the reciprocal of the sine of the minimum interior angle. However, Babuška and Aziz have shown [3] that actually small angles do not play a crucial role in approximation properties, but that limiting the largest angle is necessary and sufficient for convergence. Indeed, for a convex surface in which the curvature in the principal direction is markedly different from the curvature in the perpendicular direction, incidences producing triangles with small angles are appropriate, and are present in an optimal triangulation incidence, as shown in Example 2 below.

A commonly used incidence relation for a set of vertices is the geometrically defined Delaunay triangulation [9], [12], [13]. In § 1, we give an example of quadratic functions, and a series of sets of vertices, for which the Delaunay triangulation can be arbitrarily far from optimal. Representations of the error in linear interpolation for a general triangle and general quadratic function are surprisingly complicated (e.g., [4]–[7]). In § 2, we use a geometric argument to develop both analytic and geometric descriptions of this error, and of its maximum, over a general triangle. In § 3, the main result of the paper is presented, i.e., that an optimal triangle incidence for this interpolation problem can be obtained from a Delaunay triangulation of a transform of the given vertex set. For this result, we establish a new geometric optimality property of the Delaunay triangulation concerning minimizing the maximum circumcircle of the triangulation.

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For the quadratic model problem, the determination of optimal incidences is based on a criterion that is locally applied to a transform of the given vertices (see § 3). For more general smooth functions, this procedure can be used locally to determine appropriate triangle incidences if an adequate estimate for the Hessian matrix of the data function is available [1], [2], [14].

1.1. Example 1. To see the influence of the choice of triangle incidence on the accuracy of approximation, let us look at an example of piecewise linear approximation of the quadratic function

$$f(x, y) = \lambda_1 x^2 + \lambda_2 y^2, \qquad \lambda_1 \gg \lambda_2 > 0.$$

Consider the interpolation error over the square (see Fig. 1.1), if the y-axis is chosen as a diagonal, maximum interpolation error is

$$E_y = (\lambda_1 + \lambda_2)^2/(4\lambda_1)$$

and occurs at $P_1(P_2)$. (Expressions for the errors are developed in § 2 below.) If the x-axis is chosen, maximum interpolation error is

$$E_x = \lambda_1$$

at the origin (P_3) (see (2.5) below). The ratio of the error from these two incidences is

$$\frac{E_x}{E_y} = \frac{\lambda_1}{(\lambda_1+\lambda_2)^2/(4\lambda_1)} = \frac{4}{(1+\lambda_2/\lambda_1)^2} \leq 4.$$

Consider the same interpolation problem with extra nodes, $(ih, 1)$, $(-ih, 1)$, $(ih, -1)$, $(-ih, -1)$, $i \in \{1, 2, \cdots, N\}$ and $h = 1/N$. A standard choice of triangle incidences for a given set of vertices is the Delaunay triangulation introduced in § 3, along with some of its geometric optimality properties. If a Delaunay triangulation incidence is chosen, including the x-axis as shown in Fig. 1.2, the maximum interpolation error is dominated by E_x. Consider the triangle incidences in Fig. 1.3. The maximum interpolation error for these triangle incidences is

$$E_h = (h^2\lambda_1 + 4\lambda_2)/4.$$

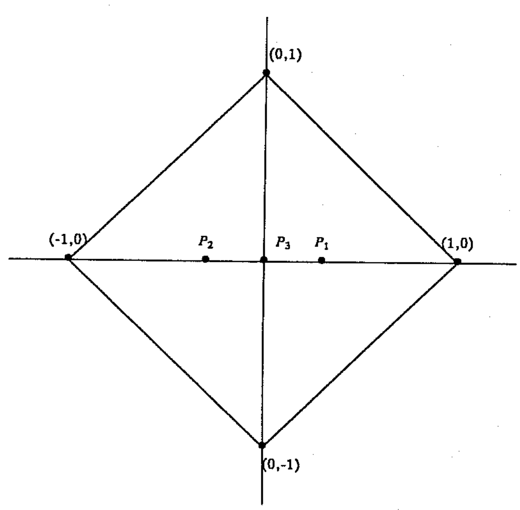

FIG. 1.1

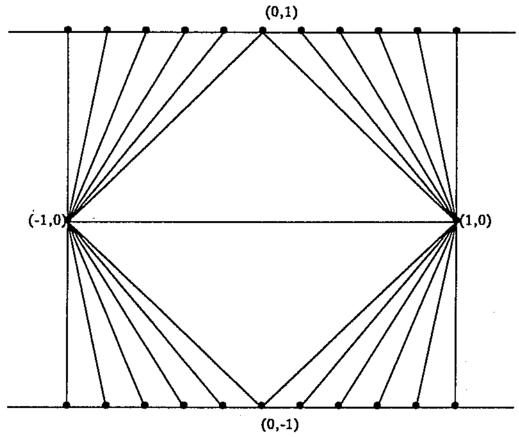

FIG. 1.2

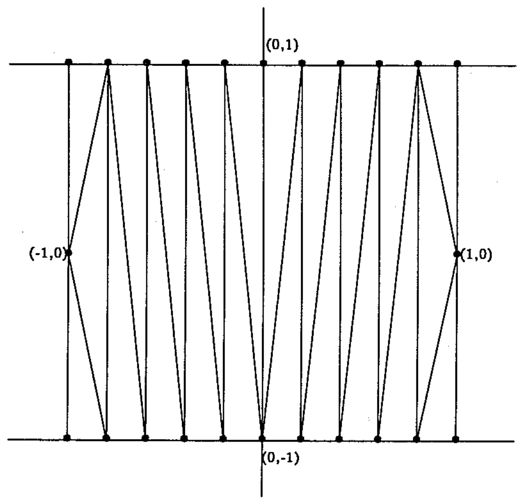

FIG. 1.3

Then the ratio of errors,

$$\frac{E_h}{E_x}=\frac{(h^2\lambda_1+4\lambda_2)}{(4\lambda_1)}=\frac{h^2}{4}+\frac{\lambda_2}{\lambda_1}.$$

For h small (N large),

$$\frac{E_h}{E_x}\approx\frac{\lambda_2}{\lambda_1}.$$

Hence the Delaunay triangle incidence can be arbitrarily far from optimal with respect to minimizing the maximum error.

1.2. Example 2. Here is a specific example of interpolation of a convex function with markedly different curvature. The data function to be interpolated is $f(x, y)=100x^2+y^2$. Figure 1.4 shows the Delaunay triangulation for an arbitrary set of points over the unit square and Fig. 1.5 shows the optimal triangulation incidence determined in this work. The errors of interpolation for both triangulations are displayed in Fig.

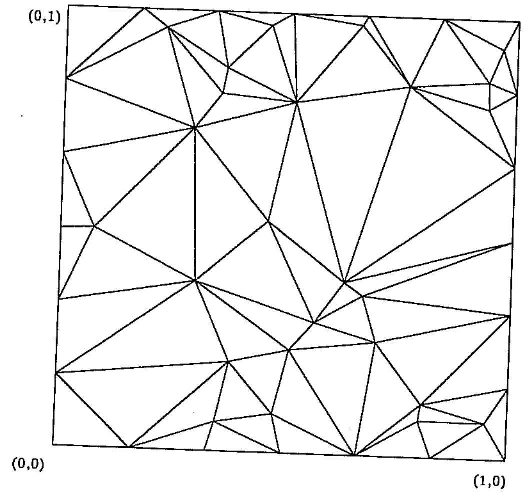

FIG. 1.4

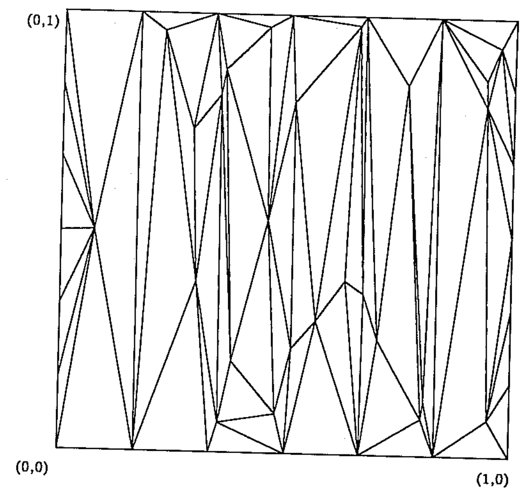

FIG. 1.5

1.6, where the maximum interpolation error over each triangle is plotted in ascending order. Note that the maximum error for the Delaunay triangulation is nearly six times larger than the one for the optimal triangulation.

2. Model problem error expressions. Error expressions to describe the interaction between the parameters of a general triangle and the relevant parameters of a general function for even piecewise linear interpolation are surprisingly complicated (e.g., [4]-[7]). In our discussion, we model a general function by a convex quadratic polynomial; i.e., let

(2.1) $$f(\mathbf{x}) = a + \mathbf{d}^t\mathbf{x} + \mathbf{x}^t H \mathbf{x}$$

where H is symmetric with eigenvalues $\lambda_1 \geqq \lambda_2 > 0$, and without any further loss of generality, we may assume that the coordinate axes are aligned with the principal axes of H so that

(2.2) $$f(x, y) = a + d_1 x + d_2 y + \lambda_1 x^2 + \lambda_2 y^2.$$

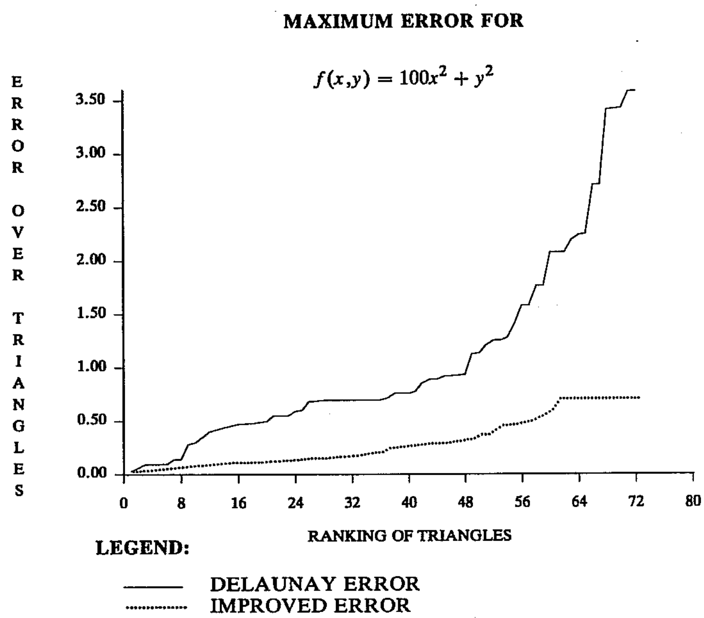

FIG. 1.6

The error function for the linear interpolation is a quadratic form in x, y with the same quadratic terms. We represent it as

$$E(x, y) = \lambda_1 x^2 + \lambda_2 y^2 + b_1 x + b_2 y + c \tag{2.3}$$

where the values of b_1, b_2, and c depend on the coordinates of the triangle vertices. Now $E(x, y)$ vanishes at the triangle vertices. Since the level curves of $E(x, y)$ are ellipses, then the zero error level curve of $E(x, y)$ is a circumscribing ellipse of the triangle. We will denote the ellipse as $e(T)$ for triangle T. The equation of the ellipse for the level curve of value $-K$ has the form

$$\begin{aligned} &E(x, y) = -K \\ &\Rightarrow \lambda_1 x^2 + \lambda_2 y^2 + b_1 x + b_2 y + c = -K \\ &\Rightarrow \lambda_1\left(x + \frac{b_1}{2\lambda_1}\right)^2 + \lambda_2\left(y + \frac{b_2}{2\lambda_2}\right)^2 = E - K \quad \text{where } E = \frac{b_1^2}{4\lambda_1} + \frac{b_2^2}{4\lambda_2} - c \\ &\Rightarrow \left(x + \frac{b_1}{2\lambda_1}\right)^2 \Big/ \left(\frac{(E-K)}{\lambda_1}\right) + \left(y + \frac{b_2}{2\lambda_2}\right)^2 \Big/ \left(\frac{(E-K)}{\lambda_2}\right) = 1. \end{aligned} \tag{2.4}$$

The parameters b_1, b_2, and c can be explicitly computed by requiring the ellipses for $K = 0$ to be the circumellipse of T. At the center, $(-b_1/(2\lambda_1), -b_2/(2\lambda_2))$ of $e(T)$, $|E(x, y)|$ attains a maximum value that can be expressed as follows:

$$E = \frac{(D_{12} D_{23} D_{31})}{16 \lambda_1 \lambda_2 A^2} \quad \text{where } D_{ij} = (\lambda_1 (x_i - x_j)^2 + \lambda_2 (y_i - y_j)^2) \tag{2.5}$$

and (x_i, y_i) $i = 1, 2, 3$ are vertices of the triangle, and A is area of the triangle. The details of the derivation can be found in Appendix A.

The maximum interpolation error for a triangle T will be denoted as

$$E_{\max}(T) = \max_{(x,y)\in T} |p_f(x, y) - f(x, y)| \tag{2.6}$$

where $p_f(x, y)$ is the linear interpolant of f at the vertices of T. In the case that the center of $e(T)$ lies in T (including the boundary of T), $E_{\max}(T)=E$ and we note that the area of $e(T)$ is $A_1=\pi E/\sqrt{\lambda_1\lambda_2}$, so that

$$E_{\max}(T)=E=\sqrt{\lambda_1\lambda_2}A_1/\pi. \tag{2.7}$$

If the center of $e(T)$ is not in T (see Fig. 2.1), $E_{\max}(T)$ is attained on an edge of T. Geometrically, $E_{\max}(T)$ is the level of the osculating level curve (ellipse) tangent to this side, $|E(x,y)|=E_{\max}(T)$. The area of the tangent ellipse is from (2.4):

$$\pi(E-E_{\max}(T))/\sqrt{\lambda_1\lambda_2}=A_2.$$

Thus $E_{\max}(T)$ can be expressed in terms of the ratio $\rho=A_2/A_1$ of these areas and E can be expressed as follows:

$$E_{\max}(T)=(1-\rho)E. \tag{2.8}$$

Consider a rescaling along x-axis by $(\lambda_1/\lambda_2)^{1/2}$ to transform the elliptical level curves to concentric circles (see Fig. 2.2). Since the area of an ellipse is directly proportional to the area of the corresponding circle in the transformed plane,

$$E_{\max}(T)=(1-\rho)E=\left(1-\frac{\pi(OD)^2}{\pi(OA)^2}\right)E=\left(\frac{OA^2-OD^2}{OA^2}\right)E=\left(\frac{AD^2}{OA^2}\right)E \tag{2.9}$$

where D is midpoint of AC tangent to the inner circle.

From this result in the transformed plane, we gain a geometric interpretation of the maximum error. Let $\bigcirc(T)$ denote the circumcircle of the transformed triangle T

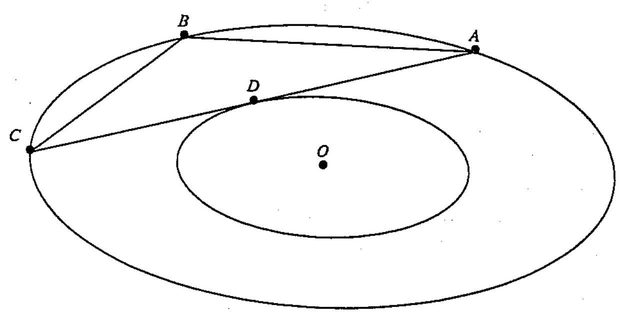

FIG. 2.1

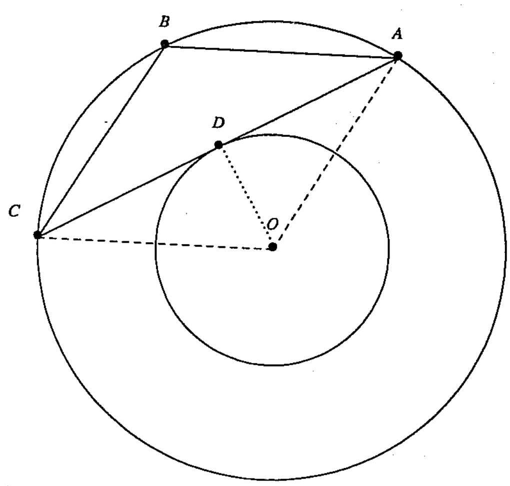

FIG. 2.2

and $|\bigcirc(T)|$ denote the area of this circumcircle. If the transformed image of a triangle has no obtuse angle, then its maximum error is proportional to the area of its circumscribing circle. If there is an obtuse angle in the transformed triangle, the maximum error occurs along the longest edge and the error is proportional to the area of circle with the longest edge as diameter.

3. Delaunay triangulation. The Delaunay triangulation of a fixed set of vertices and its related figure, the Voronoi diagram, are much studied geometric constructions (see [9], [12], [13]). Here, we briefly review some properties that are relevant for optimal interpolation incidence. The Delaunay triangulation selects triangle incidences that maximize the minimum angle in the triangulation. Lawson [9] has proposed an algorithm for converting an arbitrary triangulation to a Delaunay one by repeated application of a local edge-swapping procedure. In it, for each interior edge of the current triangulation, the two neighboring triangles are examined. If they form a convex quadrilateral, and if the replacement of the examined edge by the other diagonal of this quadrilateral would increase the minimimum angle, then a swap of diagonals is made. Lawson has shown that this criterion for picking the diagonal in a convex quadrilateral is characterized by the property that the circumcircles of either of the two triangles thus formed do not contain the fourth vertex of the quadrilateral.[1] This criterion is referred to as the empty circle criterion. Lawson has shown that the repeated application of his edge examination/edge-swapping procedure terminates in a Delaunay triangulation.

3.1. Optimal incidence for the model problem. We shall define a triangle incidence to be globally optimal if it minimizes the maximum interpolation error. Each interior edge in a triangulation is associated with a quadrilateral with that edge as diagonal. We shall also define a triangular mesh incidence to be locally optimal if for each convex quadrilateral associated by an interior edge in the triangulation, the incidence minimizes the maximum interpolation error over the quadrilateral.

Here we show that the problem of constructing a locally optimal mesh incidence for N points can be transformed to the problem of generating a Delaunay triangulation which provides an $O(N \log N)$ algorithm for solving this problem. We also show that a locally optimal incidence is globally optimal.

The rescaling of the x-axis introduced at the end of § 2, that results in error ellipses being mapped into circles will be used here to define the transform plane for which we do not explicitly introduce coordinates.

THEOREM 1. *A locally optimal interpolation triangulation incidence of N vertices is defined by a Delaunay triangulation in the transformed plane* (*and hence is computable in* $O(N \log N)$ *time, which is optimal*).

To simplify our discussion, we shall use the notation that all references to angles are based on the labelling in Fig. 3.1. Moreover, we shall always assume vertex A to be exterior to $\bigcirc(\triangle BCD)$. The diagonal BD is the incidence selected by the empty circle criterion. By elementary geometry, we have the following:

$$\angle CAD = \theta_1 < \theta_2 = \angle CBD, \qquad \angle BAC = \phi_1 < \phi_2 = \angle BDC,$$

$$\angle DCA = \gamma_1 < \gamma_2 = \angle DBA, \qquad \angle ACB = \eta_1 < \eta_2 = \angle ADB.$$

LEMMA 1. *Given a convex quadrilateral ABCD with vertex A exterior to* $\bigcirc(\triangle BCD)$, *then* $\max(|\bigcirc(\triangle ABC)|, |\bigcirc(\triangle ADC)|) \geqq \max(|\bigcirc(\triangle BCD)|, |\bigcirc(\triangle ABD)|)$.

[1] If the fourth vertex lies on the boundary of a circumcircle, then no change in the minimum angle occurs from edge swapping and either choice of edge results in a Delaunay triangulation.

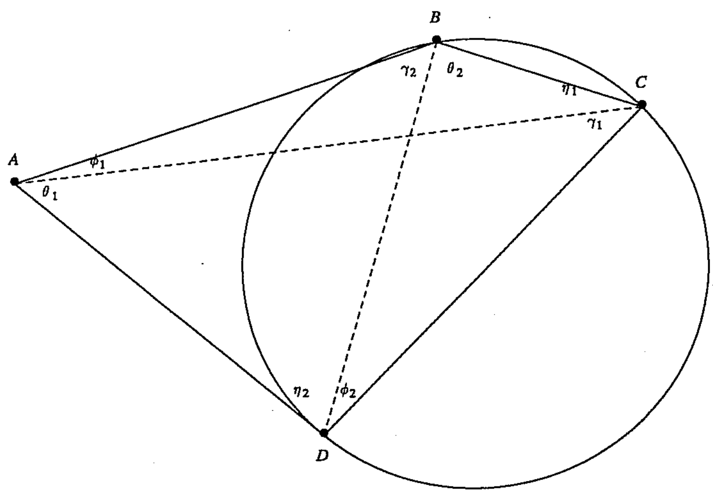

FIG. 3.1

Proof of Lemma 1.

Case 1. Assume that $\theta_2 \leqq \pi/2$. Let E be the midpoint of CD; let O_1, O_2 be centers for $\bigcirc(\triangle ADC)$ and $\bigcirc(\triangle BCD)$, respectively. The O_1 and O_2 lie on the perpendicular to DC through E, so O_1, O_2, and E are collinear. Let A', B' be points where this line meets $\bigcirc(\triangle ADC)$ and $\bigcirc(\triangle BCD)$ (see Fig. 3.2).

$$\angle EA'C = \frac{\theta_1}{2}, \qquad \angle EB'C = \frac{\theta_2}{2},$$

(3.1)

$$\tan\left(\frac{\theta_2}{2}\right) = \frac{EC}{EB'} = \frac{EC}{EO_2 + O_2B'} = \frac{EC}{EO_2 + O_2C}.$$

Now from $\triangle O_2EC$ we have

$$(O_2C)^2 = (O_2E)^2 + (EC)^2.$$

Thus (3.1) can be rewritten as follows:

$$\tan\left(\frac{\theta_2}{2}\right) = \frac{EC}{O_2C + ((O_2C)^2 - (EC)^2)^{1/2}}.$$

Similarly,

$$\tan\left(\frac{\theta_1}{2}\right) = \frac{EC}{O_1C + ((O_1C)^2 - (EC)^2)^{1/2}},$$

(3.2)

$$\theta_1 \leqq \theta_2 \leqq \frac{\pi}{2} \Rightarrow \tan\left(\frac{\theta_1}{2}\right) \leqq \tan\left(\frac{\theta_2}{2}\right).$$

Therefore

$$\frac{EC}{O_1C + ((O_1C)^2 - (EC)^2)^{1/2}} \leqq \frac{EC}{O_2C + ((O_2C)^2 - (EC)^2)^{1/2}}$$

(3.3)

$$\Rightarrow O_2C + ((O_2C)^2 - (EC)^2)^{1/2} \leqq O_1C + ((O_1C)^2 - (EC)^2)^{1/2}.$$

Since $x + (x^2 - a)^{1/2}$ is a monotone increasing function of x, (3.3) implies $O_2C \leqq O_1C$. Therefore $\bigcirc(\triangle ADC)$ has larger radius than $\bigcirc(\triangle BCD)$. Similarly, if $\phi_2 \leqq \pi/2$, then $\bigcirc(\triangle ABC)$ has larger radius than $\bigcirc(\triangle BCD)$.

Case 2. Assume $\theta_2 > \pi/2$. From $\triangle BCD$, we have $\phi_2 + \theta_2 < \pi$, and thus $\phi_2 \leqq \pi/2$ if $\theta_2 > \pi/2$. From the result in Case 1, $|\bigcirc(\triangle ABC)| \geqq |\bigcirc(\triangle BCD)|$. Similarly, by symmetry, C would be exterior to $\bigcirc(\triangle ABD)$.

Thus either $\bigcirc(\triangle ACD)$ or $\bigcirc(\triangle ABC)$ would be larger than $\bigcirc(\triangle ABD)$. Since Cases 1 and 2 are exhaustive, the lemma is proved.

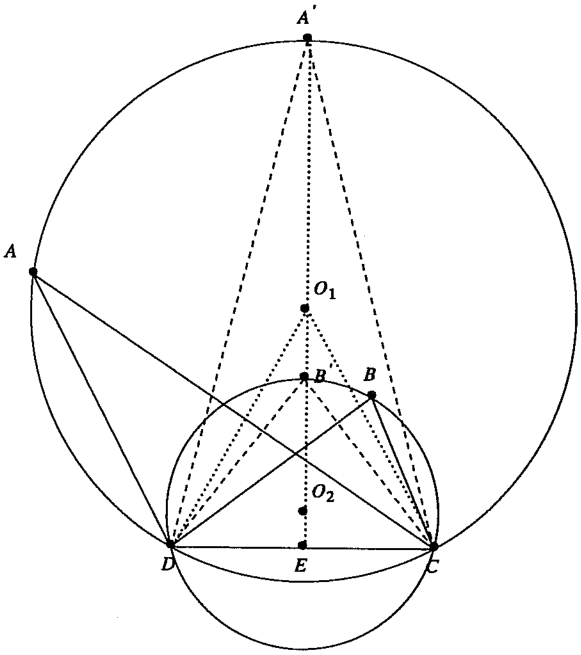

FIG. 3.2

COROLLARY. *The empty circle criterion applied to a convex quadrilateral selects the diagonal that minimizes the maximum circumcircle of the corresponding triangles.*

LEMMA 2. *Given a convex quadrilateral ABCD with vertex A exterior to $\triangle BCD$ the empty circle criterion selects the triangulation incidence that minimizes the area of the circle corresponding to the maximum interpolation error over the quadrilateral.*

Proof of Lemma 2. Recall from § 2 that the maximum interpolation error for a triangle is proportional to the area of the circumcircle of the transformed image of the triangle, if this image contains no obtuse angle. Otherwise, the maximum interpolation error is proportional to the area of the circle with diameter equal to the edge opposite the obtuse angle, i.e., the longest edge of the image triangle.

The first case was dealt with in Lemma 1 and its corollary in which we established that the diagonal BD is selected by the empty circle criterion. We now carry out a case-by-case study of configurations of image triangles with obtuse angles to show that the empty circle criterion also minimizes the maximum error circles. Hence we shall consider only cases where $\triangle ABC$ or $\triangle ACD$ contains an obtuse angle.

Case 1. Assume $\phi_1 \geqq \pi/2$. (See Fig. 3.3.) We show that BC is the longest edge and it determines the error circle, regardless of the choice of diagonals. The area of the error circle for $\triangle ABD$ is $\pi(BD/2)^2$, and for $\triangle BCD$ it is $\pi(BC/2)^2$. Note that $\phi_2 \geqq \phi_1 \geqq \pi/2$. $|\bigcirc(\triangle ABC)|$ is $\pi(BC/2)^2$, and $|\bigcirc(\triangle ADC)|$ is $\pi(AC/2)^2$. Now $\phi_1 < \phi_2$, and thus $|BC| > |BD|$. Since BD cannot be the longest chord, the lemma holds. By symmetry, the same applies if θ_1, η_1, γ_1 is obtuse.

Case 2. Assume $\angle ABC$ and $\angle ADC$ are both obtuse (see Fig. 3.4). The error circle for each of $\triangle ABC$ and $\triangle ACD$ would be the circle with AC as diameter. Let this

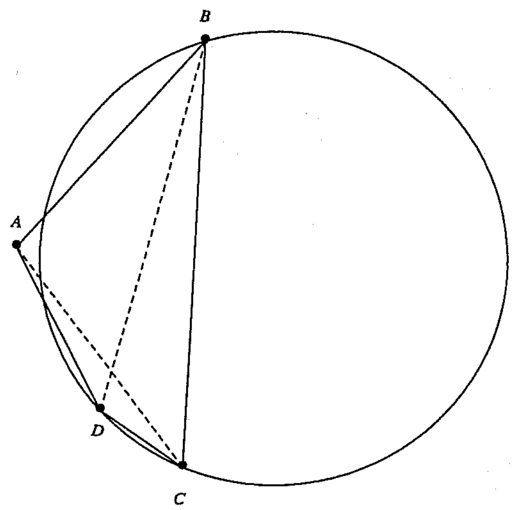

FIG. 3.3

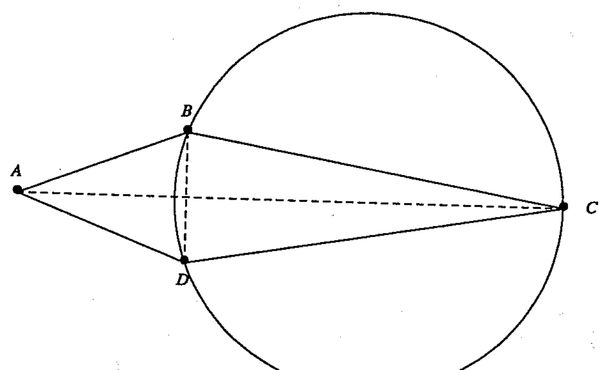

FIG. 3.4

circle be denoted by Γ. Both vertices B and C are contained in Γ. If triangle BCD has an obtuse angle, its error is indicated by its longest edge, which is entirely interior to Γ. If $\triangle BCD$ has no obtuse angle, its circumcircle is smaller than Γ. Hence error for $\triangle BCD$ is smaller than error for triangle ABC. The same reasoning applies for $\triangle ABD$. Therefore diagonal BD should be chosen, and the circle criterion holds.

Case 3. Assume $\angle ABC \leqq \pi/2$ is acute but $\angle ADC > \pi/2$ is obtuse, and ϕ_1, θ_1, η_1, γ_1 are all acute (see Fig. 3.5). The error for $\triangle ABC$ is proportional to the area of its circumcircle,

$$|\bigcirc(\triangle ABC)| = \frac{\pi(AC/2)^2}{\sin^2(\gamma_2+\theta_2)} = \frac{\pi(BC/2)^2}{\sin^2(\phi_1)} = \frac{\pi(AB/2)^2}{\sin^2(\eta_1)}. \tag{3.4}$$

If η_2 is obtuse, $|\bigcirc(\triangle ABD)|$ is $\pi(AB/2)^2 < |\bigcirc(\triangle ABC)|$; otherwise, it is bounded by the area of the circumcircle $\pi(AB/2)^2/\sin^2(\eta_1)$. If $\eta_2 \leqq \pi/2$, then $\eta_1 \leqq \eta_2$ and $\sin(\eta_1) \leqq \sin(\eta_2)$. Therefore, the error for $\triangle ABD$ is less than that of $\triangle ABC$.

Similarly, if ϕ_2 is obtuse, the area for $\bigcirc(\triangle BCD)$ is $\pi(BC/2)^2 < |\bigcirc(\triangle ABC)|$; otherwise, it is bounded by the area of its circumcircle $\pi(BC/2)^2/\sin^2(\phi_1)$. If $\phi_2 \leqq \pi/2$,

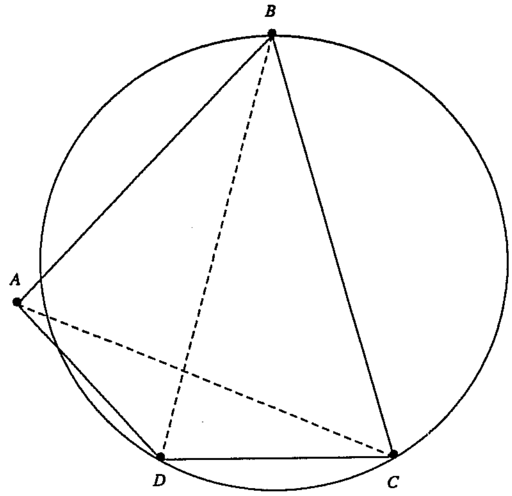

FIG. 3.5

then $\phi_1 \leqq \phi_2$ and $\sin(\phi_1) \leqq \sin(\phi_2)$. Therefore, the error for $\triangle BCD$ is less than that of $\triangle ABC$, and the empty circle criterion selects the incidence minimizing the maximum error.

Cases 1, 2, and 3 exhaust all possibilities for an obtuse angle in $\triangle ABC$ or $\triangle ACD$. The lemma is proved.

Proof of Theorem 1. For the image of the interpolation vertices in the transform plane, a triangulation that satisfies the empty circle criterion can be constructed in $O(N \log N)$ time [12, Thm. 5.18, p. 215]. The convex quadrilaterals of this Delaunay triangulation and of the triangulation induced on the interpolation vertices are in one-to-one correspondence. But by Lemmas 1 and 2, the incidences of the Delaunay triangulation minimize the interpolation error circles with respect to diagonal interchanges, and hence the induced triangulation is locally optimal.

COROLLARY. *The locally optimal triangle incidence of Theorem* 1 *defines a globally optimal triangle incidence.*

Proof. Starting with any globally optimal triangle incidence, we can apply the local edge-swapping procedure in the transformed plane to obtain a locally optimal incidence with the same maximum error. However, the size of the maximum error circle is uniquely determined for Delaunay triangulation in the transformed plane, so the maximum error in the original globally optimal incidence cannot be smaller than that of a locally optimal incidence, and hence a Delaunay triangulation also defines a globally optimal incidence. Note, however, a globally optimal triangulation incidence need not be a Delaunay triangulation.

The use of the transform plane appears in several related contexts in the literature. Nadler [10] uses it to establish the shape of triangulation for optimal L_2 linear interpolation for model quadratic data. Also in [11], Peraire et al. use it to support an adaptive remeshing scheme for the finite-element method for compressible flow computations.

Appendix A.

Derivation of interpolation error. Let the interpolation error over the triangle be given by

$$E(x, y) = \lambda_1 x^2 + \lambda_2 y^2 + b_1 x + b_2 y + c \qquad \text{(A1)}$$

and by the interpolation condition

$$E(x_1, y_1) = E(x_2, y_2) = E(x_3, y_3) = 0$$

at the three vertices of the triangle. The unknowns b_1, b_2, c can easily be obtained by solving the following system of linear equations:

(A2)
$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ c \end{bmatrix} = \begin{bmatrix} -r_1 \\ -r_2 \\ -r_3 \end{bmatrix}$$

where $r_i = \lambda_1 x_i^2 + \lambda_2 y_i^2$. By Cramer's rule,

(A3)
$$b_1 = \frac{\det \begin{bmatrix} -r_1 & y_1 & 1 \\ -r_2 & y_2 & 1 \\ -r_3 & y_3 & 1 \end{bmatrix}}{D}, \quad b_2 = \frac{\det \begin{bmatrix} x_1 & -r_1 & 1 \\ x_2 & -r_2 & 1 \\ x_3 & -r_3 & 1 \end{bmatrix}}{D}, \quad c = \frac{\det \begin{bmatrix} x_1 & y_1 & -r_1 \\ x_2 & y_2 & -r_2 \\ x_3 & y_3 & -r_3 \end{bmatrix}}{D},$$

where A is area of triangle and

$$D = \det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 2A.$$

Now by (2.4)

(A4)
$$E_{\max} = \frac{b_1^2}{4\lambda_1} + \frac{b_2^2}{4\lambda_2} - c.$$

The substitution of (A3) into (A4) and its simplification was obtained through the algebraic computation system MAPLE [8]:

(A5)
$$E = \frac{(D_{12} D_{23} D_{31})}{16\lambda_1 \lambda_2 A^2} \quad \text{where } D_{ij} = (\lambda_1 (x_i - x_j)^2 + \lambda_2 (y_i - y_j)^2).$$

$E_{\max}$ represents the global maximum interpolation error obtained at $(-b_1/(2\lambda_1), -b_2/(2\lambda_2))$. It can be shown by calculus that the local maximum error along the boundary is attained at the midpoint of each edge. The maximum error along edge (x_i, y_i), (x_j, y_j) is $|D_{ij}/4|$.

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