



ELSEVIER

Journal of Computational and Applied Mathematics 129 (2001) 151–170

JOURNAL OF
COMPUTATIONAL AND
APPLIED MATHEMATICS

www.elsevier.nl/locate/cam

Computing an integer point of a simplex with an arbitrary starting homotopy-like simplicial algorithm

Chuangyin Dang^{a, *}, Hans van Maaren^b

^a*Department of Manufacturing Engineering and Engineering Management, City University of Hong Kong, Kowloon, Hong Kong, China*

^b*Faculty of Technical Mathematics & Informatics, Delft University of Technology, Delft, Netherlands*

Received 28 February 1999

Abstract

An arbitrary starting homotopy-like simplicial algorithm is developed for computing an integer point of an n -dimensional simplex. The algorithm is derived from the use of an integer labeling rule and a triangulation of $\mathbb{R}^n \times [0, 1]$, and consists of two interchanging phases. One phase of the algorithm constitutes a homotopy simplicial algorithm, which generates $(n + 1)$ -dimensional simplices in $\mathbb{R}^n \times [0, 1]$, and the other phase of the algorithm constitutes a pivoting procedure, which generates n -dimensional simplices in either $\mathbb{R}^n \times \{0\}$ or $\mathbb{R}^n \times \{1\}$. The algorithm varies from one phase to the other. When the matrix defining the simplex is in the so-called canonical form, starting at an arbitrary integer point in $\mathbb{R}^n \times \{0\}$, the algorithm within a finite number of iterations either yields an integer point of the simplex or proves that no such point exists. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Integer point; Simplex; Polytope; Integer programming; Integer labeling; Triangulation; Simplicial approach

1. Introduction

Determining whether a simplex contains an integer point is an NP-hard problem [17,24]. It is well known that using polynomial-time aggregation techniques, one can reduce the problem of determining whether a polytope contains an integer point to the problem of determining whether a simplex contains an integer point. Thus, computing an integer point of a polytope is equivalent to computing an integer point of a simplex. An excellent survey of the aggregation techniques can be found in [28].

The problem we consider in this paper is to find an integer point of a simplex with a homotopy-like simplicial approach. The simplicial methods were originated in [19] for computing fixed points of

* Corresponding author.

E-mail address: mecdang@cityu.edu.hk (C. Dang).

continuous mappings. Although Scarf used primitive sets in his method, the subsequential developments of the simplicial methods were based on triangulations ([1,2,8–10,12–16,19,20,25,26], etc.). In [21,22], using the primitive sets, Scarf succeeded in defining a path that either leads to an integer point of a polytope or proves that no such point exists. The question whether it is possible to find an integer point of a simplex with a simplicial approach was raised after Scarf's work. A few attempts to solve it were made in [4,5,27], etc. Although some partial results had been obtained in these papers, the question remained unsolved. We were able to obtain in [6] a positive answer to the question only after Scarf brought our attention to Pnueli's paper [18] in 1994 and we observed a beautiful property [6, Lemma 1]. In order for its convergence, the algorithm given in [6], however, can only start at some special integer points. To overcome this shortcoming, an arbitrary starting variable dimension algorithm was proposed in [7] for computing an integer point of a simplex.

Simplicial homotopy algorithms for computing fixed points were invented in [8] and further developed in [16]. Numerical results show that there is no winner between simplicial homotopy algorithms and variable dimension algorithms for computing fixed points. Therefore, it may be desired to develop an arbitrary starting homotopy-like simplicial algorithm for computing an integer point of a simplex. In order to succeed, one has to solve the following three problems:

1. What integer labeling rule should be used to assign an integer to each point in $\mathbb{R}^n \times \{0\}$ (artificial level) and what integer labeling rule should be used to assign an integer to each point in $\mathbb{R}^n \times \{1\}$ (original level)? (two integer labeling rules should be consistent so that a homotopy-like simplicial algorithm terminates within a finite number of steps)
2. Given an integer labeling rule and an arbitrary integer point in $\mathbb{R}^n \times \{0\}$, how can we obtain a complete n -dimensional simplex in $\mathbb{R}^n \times \{0\}$ with the integer point being a vertex of the complete simplex? (To start a homotopy-like simplicial algorithm, such a complete simplex is needed.)
3. When the straightforward use of a simplicial homotopy algorithm generates a complete n -dimensional simplex in either $\mathbb{R}^n \times \{0\}$ or $\mathbb{R}^n \times \{1\}$ before it meets an integer point of a simplex or proves that no such point exists, what should we do?

By using the property stated in [6, Lemma 1], we have successfully solved these three problems and developed an arbitrary starting homotopy-like simplicial algorithm for computing an integer point of a simplex. The idea of the algorithm is as follows. The algorithm first transforms the matrix defining the simplex into a matrix in the so-called canonical form through a unimodular matrix that can be obtained in polynomial time. Then it subdivides $\mathbb{R}^n \times [0, 1]$ into simplices, and assigns an integer to each integer point of $\mathbb{R}^n \times \{0, 1\}$. The algorithm consists of two phases. The first phase of the algorithm constitutes a homotopy simplicial algorithm, which generates $(n + 1)$ -dimensional simplices in $\mathbb{R}^n \times [0, 1]$, and the second phase of the algorithm constitutes a pivoting procedure, which generates n -dimensional simplices in either $\mathbb{R}^n \times \{0\}$ or $\mathbb{R}^n \times \{1\}$. The algorithm varies from one phase to the other. Starting at an arbitrary integer point in $\mathbb{R}^n \times \{0\}$, the algorithm within a finite number of iterations either yields an integer point of the simplex or proves that no such point exists. A significant difference between our algorithm and a simplicial homotopy algorithm for computing fixed points is that the simplicial homotopy algorithm for computing fixed points does not have the second phase of our algorithm.

The paper is organized as follows. We describe the problem in Section 2. We introduce an integer labeling rule in Section 3. The canonical form matrix is explained and some properties are given in

Section 4. We present the algorithm and prove its convergence in Section 5. The paper is concluded with a few numerical tests in Section 6.

2. The problem

The problem we consider in this paper is to find an integer point of P given by

$$P = \{x \in \mathbb{R}^n \mid Ax \leq b\},$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} \end{pmatrix}$$

and $b = (b_1, b_2, \dots, b_{n+1})^T$. Let $a_i = (a_{i1}, a_{i2}, \dots, a_{in})^T$ for $i = 1, 2, \dots, n + 1$. We assume throughout this paper that all the entries of A are integers, and the interior of P is nonempty and bounded. Then P is a simplex, and any $n \times n$ submatrix of A is nonsingular. Now, using the separation theorem, one can easily derive the following lemma.

Lemma 1.

- For any nonzero vector $\zeta \in \mathbb{R}^n$, there are i and j satisfying $a_i^T \zeta < 0$ and $0 < a_j^T \zeta$.
- There is a positive vector $\rho = (\rho_1, \rho_2, \dots, \rho_{n+1})^T$ satisfying $\rho^T A = 0$.

It is well known that using aggregation techniques, one can reduce the problem of finding an integer point of a polytope to the problem of finding an integer point of a simplex [28]. This implies that it may be significant if one can develop an efficient algorithm for computing an integer point of a simplex. The purpose of this paper is to present one more alternative for computing an integer point of a simplex.

To develop and implement our algorithm, we need a triangulation of $\mathbb{R}^n \times [0, 1]$ which subdivides every integer unit cube in $\mathbb{R}^n \times [0, 1]$ into integer simplices, where the integer unit cube is a unit cube having only integer vertices and the integer simplex is a simplex having only integer vertices. Observe that vertices of the simplices of the triangulation are in $\mathbb{R}^n \times \{0, 1\}$. All the triangulations that subdivide every integer unit cube into integer simplices are suitable for the purpose. For such triangulations, see [3,11,25], etc. A specific choice of the triangulation plays, however, no dominant role at all in this paper. The K_1 -triangulation given in [11] is the simplest one of them. We use it as a triangulation underlying the algorithm. Since the initialization of the algorithm depends on the underlying triangulation, we introduce for completeness the K_1 -triangulation here [11,25].

For $j = 1, 2, \dots, m$, we use u^j to denote the j th unit vector of \mathbb{R}^m , i.e.,

$$u^j = (0, \dots, \overset{j}{1}, \dots, 0)^T.$$

A simplex of the K_1 -triangulation of \mathbb{R}^m is the convex hull of $m + 1$ vectors y^0, y^1, \dots, y^m given by $y^0 = y$, and

$$y^k = y^{k-1} + u^{\pi(k)}, \quad k = 1, 2, \dots, m,$$

Table 1
Pivot rules of the K_1 -triangulation

i	\bar{y}	$\bar{\pi}$
0	$y + u^{\pi(1)}$	$(\pi(2), \dots, \pi(m), \pi(1))$
$1 < i < m$	y	$(\pi(1), \dots, \pi(i + 1), \pi(i), \dots, \pi(m))$
m	$y - u^{\pi(m)}$	$(\pi(m), \pi(1), \dots, \pi(m - 1))$

where y is an integer point of \mathbb{R}^m , and $\pi = (\pi(1), \pi(2), \dots, \pi(m))$ is a permutation of elements of $\{1, 2, \dots, m\}$. Let K_1 be the set of all such simplices. Since a simplex of the K_1 -triangulation is uniquely determined by y and π , we use $K_1(y, \pi)$ to denote it.

We say that two simplices of K_1 are adjacent if they have a common facet. We show in the following how to generate all the adjacent simplices of a simplex of the K_1 -triangulation of \mathbb{R}^m . For a given simplex $\sigma = K_1(y, \pi)$ with vertices y^0, y^1, \dots, y^m , its adjacent simplex opposite to a vertex, say y^i , is given by $K_1(\bar{y}, \bar{\pi})$, where \bar{y} and $\bar{\pi}$ are generated in Table 1.

Let $K_1[0]$ be the set of the simplices of the K_1 -triangulation of \mathbb{R}^{n+1} with $y_{n+1} = 0$. It is easy to see that $K_1[0]$ is a triangulation of $\mathbb{R}^n \times [0, 1]$. Let $\mathcal{H}_1[0]$ be the set of faces of simplices of $K_1[0]$. A q -dimensional simplex of $\mathcal{H}_1[0]$ with vertices y^0, y^1, \dots, y^q is denoted by $\langle y^0, y^1, \dots, y^q \rangle$.

For $\sigma \in K_1$, let

$$\text{grid}(\sigma) = \max\{\|x - y\| \mid x \in \sigma \text{ and } y \in \sigma\},$$

where $\|\cdot\|$ denotes the infinity norm. We define

$$\text{mesh}(K_1) = \max_{\sigma \in K_1} \text{grid}(\sigma).$$

Then it is obvious that $\text{mesh}(K_1) = 1$.

The restriction of $\mathcal{H}_1[0]$ on $\mathbb{R}^n \times \{0\}$, denoted $\mathcal{H}_1[0] \mid \mathbb{R}^n \times \{0\}$, is given by

$$\mathcal{H}_1[0] \mid \mathbb{R}^n \times \{0\} = \{\sigma \in \mathcal{H}_1[0] \mid \sigma \subset \mathbb{R}^n \times \{0\} \text{ and } \dim(\sigma) = n\},$$

and the restriction of $\mathcal{H}_1[0]$ on $\mathbb{R}^n \times \{1\}$, denoted $\mathcal{H}_1[0] \mid \mathbb{R}^n \times \{1\}$, is given by

$$\mathcal{H}_1[0] \mid \mathbb{R}^n \times \{1\} = \{\sigma \in \mathcal{H}_1[0] \mid \sigma \subset \mathbb{R}^n \times \{1\} \text{ and } \dim(\sigma) = n\},$$

where $\dim(\cdot)$ stands for the dimension. It is easy to see that $\mathcal{H}_1[0] \mid \mathbb{R}^n \times \{0\}$ is a triangulation of $\mathbb{R}^n \times \{0\}$, and $\mathcal{H}_1[0] \mid \mathbb{R}^n \times \{1\}$ is a triangulation of $\mathbb{R}^n \times \{1\}$. In fact, $\mathcal{H}_1[0] \mid \mathbb{R}^n \times \{0\}$ is the same as the K_1 -triangulation of $\mathbb{R}^n \times \{0\}$, and $\mathcal{H}_1[0] \mid \mathbb{R}^n \times \{1\}$ is the same as the K_1 -triangulation of $\mathbb{R}^n \times \{1\}$.

3. The integer labeling rule

In this section we introduce the integer labeling rule to be used in the algorithm to assign an integer to each integer point of $\mathbb{R}^n \times [0, 1]$. Let $N = \{1, 2, \dots, n\}$ and $N_0 = \{1, 2, \dots, n + 1\}$. Let $\eta = (\eta_1, \eta_2, \dots, \eta_n)^T$ be an arbitrary integer point and $d = (d_1, d_2, \dots, d_{n+1})^T = A\eta$. The point η is used as the starting point of the algorithm.

Definition 1. For $(x, t) \in \mathbb{R}^n \times \{0, 1\}$, we assign to (x, t) an integer $l(x, t)$ given by

$$l(x, t) = \begin{cases} \max \left\{ k \mid a_k^T x - d_k = \max_{j \in N_0} a_j^T x - d_j \right\} & \text{if } t = 0, \\ 0 & \text{if } x \in P \text{ and } t = 1, \\ \max \left\{ k \mid a_k^T x - b_k = \max_{j \in N_0} a_j^T x - b_j \right\} & \text{if } x \notin P \text{ and } t = 1. \end{cases}$$

Consider geometric structures of this integer labeling rule. Let $e = (1, 1, \dots, 1)^T$. For $j \in N_0$, let

$$A_{-j} = (a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_{n+1})^T$$

and

$$b_{-j} = (b_1, \dots, b_{j-1}, b_{j+1}, \dots, b_{n+1})^T.$$

Observe that A_{-j} is nonsingular for any $j \in N_0$. Clearly,

$$a_j^T A_{-j}^{-1} e < 0 \quad \text{for } j \in N_0.$$

Let ξ be any given point of \mathbb{R}^n . For $k \in N_0$, we define

$$Q_k(\xi) = \left\{ \xi + \sum_{j \neq k} \lambda_j A_{-j}^{-1} e \mid 0 \leq \lambda_j \text{ for } j \neq k \right\}.$$

Then it is obvious that

$$\bigcup_{j=1}^n Q_j(\xi) = \mathbb{R}^n$$

and for any i and j , the intersection of $Q_i(\xi)$ and $Q_j(\xi)$ is a common face of both of them. Observe that $Q_k(\xi)$, $k = 1, 2, \dots, n + 1$, are $n + 1$ cones of n dimensions, which have the same vertex ξ . Let

$$\omega = A_{-1}^{-1} b_{-1} + \vartheta A_{-1}^{-1} e,$$

where

$$\vartheta = \frac{b_1 - a_1^T A_{-1}^{-1} b_{-1}}{a_1^T A_{-1}^{-1} e - 1}.$$

Clearly, the integer labeling rule partitions $\mathbb{R}^n \times \{0\}$ and $\mathbb{R}^n \times \{1\}$ into $n + 1$ cones, respectively, i.e., for any $x \in \mathbb{R}^n$,

$$l(x, 0) = \max \{ k \mid x \in Q_k(\eta) \}$$

and

$$l(x, 1) = \begin{cases} 0 & \text{if } x \in P, \\ \max \{ k \mid x \in Q_k(\omega) \} & \text{if } x \notin P. \end{cases}$$

Therefore, the integer labeling rule assigns k to all the interior points of $Q_k(\eta)$ and to all the interior points of $Q_k(\omega)$ except those points in P .

The following definition gives us a few notations to be used in further discussions.

Definition 2.

- For any $0 \leq q \leq n$, a q -dimensional simplex $\sigma = \langle y^0, y^1, \dots, y^q \rangle$ of $\mathcal{K}_1[0]$ is complete if $l(y^i) \neq l(y^j)$ for $0 \leq i < j \leq q$, and $l(y^k) \neq 0$ for $k = 0, 1, \dots, q$.
- For any $0 \leq q \leq n$, a q -dimensional simplex $\sigma = \langle y^0, y^1, \dots, y^q \rangle$ of $\mathcal{K}_1[0]$ is 0-complete if $l(y^i) \neq l(y^j)$ for $0 \leq i < j \leq q$, and $l(y^k) = 0$ for some k .
- For any $1 \leq q \leq n + 1$, a q -dimensional simplex $\sigma = \langle y^0, y^1, \dots, y^q \rangle$ of $\mathcal{K}_1[0]$ is almost complete if the labels of $q + 1$ vertices of σ consist of q different nonzero integers.

From this definition, it is easy to see the following property.

Lemma 2. *An almost complete simplex has exactly two complete facets.*

Let F be a subset of \mathbb{R}^n . For any $x \in \mathbb{R}^n$, the distance between x and F is given by

$$d(x, F) = \min_{y \in F} \|x - y\|.$$

For any $F \subset \mathbb{R}^n$ and any nonnegative scalar δ , let

$$\Delta(F, \delta) = \{x \in \mathbb{R}^n \mid d(x, F) \leq \delta\}.$$

For any $\xi \in \mathbb{R}^n$, let

$$\Gamma(\xi) = \{\xi + \lambda A_{-(n+1)}^{-1} e \mid 0 \leq \lambda\}.$$

Observe that

$$\Gamma(\xi) = \bigcap_{k=1}^n Q_k(\xi).$$

Since $\text{mesh}(K_1) = 1$, it is not difficult to obtain the following lemma.

Lemma 3. *There is a positive δ satisfying that*

1. *All the complete n -dimensional simplices in $\mathcal{K}_1[0] \mid \mathbb{R}^n \times \{0\}$ are contained in $\Delta(\eta, \delta) \times \{0\}$.*
2. *All the complete n -dimensional simplices in $\mathcal{K}_1[0] \mid \mathbb{R}^n \times \{1\}$ are contained in $\Delta(\omega, \delta) \times \{1\}$.*
3. *All the almost complete n -dimensional simplices in $\mathcal{K}_1[0] \mid \mathbb{R}^n \times \{0\}$ carrying only integer labels in N are contained in $\Delta(\Gamma(\eta), \delta) \times \{0\}$.*
4. *All the almost complete n -dimensional simplices in $\mathcal{K}_1[0] \mid \mathbb{R}^n \times \{1\}$ carrying only integer labels in N are contained in $\Delta(\Gamma(\omega), \delta) \times \{1\}$.*

According to this lemma, one can see that there is a finite number of complete n -dimensional simplices in $\mathbb{R}^n \times \{1\}$ and $\mathbb{R}^n \times \{0\}$.

4. The canonical form

In this section we introduce the canonical form matrix that helps us obtain some important results for the proof of convergence of the algorithm.

Definition 3. We say an $(n + 1) \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} \end{pmatrix}$$

is in canonical form if $a_{ij} \leq 0$ for $i \neq j$ and $a_{ii} > 0$, $i = 1, 2, \dots, n$.

Scarf’s special Leontief matrix in [23] is a matrix in the canonical form.

A matrix C is nonnegative if every entry of C is nonnegative. We use $0 \leq C$ to indicate that C is nonnegative. The following lemma can be found in [7].

Lemma 4. Assume that A is in the canonical form and there is a positive vector $\rho = (\rho_1, \dots, \rho_{n+1})^T$ satisfying that $\rho^T A = 0$. If $A_{-(n+1)}$ is nonsingular then

$$0 \leq A_{-(n+1)}^{-1}.$$

As a corollary of this lemma, we obtain

Corollary 1.

$$0 < A_{-(n+1)}^{-1} e.$$

Let A be an arbitrary integer $(n + 1) \times n$ matrix satisfying that there is a positive vector ρ such that $\rho^T A = 0$, and any $n \times n$ submatrix of A is nonsingular. A procedure given in [18] shows that applying the following three elementary column operations,

1. interchange two columns,
2. multiply a column by -1 ,
3. add any integer times a column to another column,

to A , one can transform A into a matrix in the canonical form. Thus, there is a unimodular matrix U such that AU has the canonical form. Note that U can be obtained in polynomial time.

The following notation is only used for obtaining without any computation the starting simplex of the algorithm.

Definition 4. We say that rows of an $(n + 1) \times n$ matrix A are in proper order if, for $i = 2, \dots, n$,

$$\sum_{j=1}^i a_{kj} \leq \sum_{j=1}^i a_{ij}, \quad k = 1, 2, \dots, i - 1.$$

Note that without loss of the canonical form, through interchanges of rows and columns of (A, b) if it is necessary, one can easily obtain a matrix such that its rows are in the proper order.

From now on, we assume, without loss of generality, that A is in the canonical form, and rows of A are in the proper order. The results in the following enable us to derive the algorithm and to prove its finite convergence.

For $\zeta \in \mathbb{R}^n$ and $K \subseteq N$, we define

$$H(\zeta, K) = \{\zeta + x \in \mathbb{R}^n \mid 0 \leq x_i, i \in K, \text{ and } x_i = 0, i \notin K\}.$$

Lemma 5. $(\eta, 0)$ is the unique point in $H(\eta, N) \times \{0\}$ carrying integer label $n + 1$.

Proof. Note that $d = A\eta$ and A is in the canonical form. Let x be an arbitrary nonnegative and nonzero vector. Using Lemma 1, we obtain that there is at least one index of N , say i , satisfying $0 < a_i^T x$ because $a_{n+1}^T x \leq 0$. Therefore, $a_{n+1}^T(\eta + x) - d_{n+1} = a_{n+1}^T x < a_i^T x = a_i^T(\eta + x) - d_i$. According to Definition 1, we obtain $l(\eta + x, 0) \neq n + 1$. Observe that $l(\eta, 0) = n + 1$. The lemma follows. \square

Note that a simplex of $\mathcal{K}_1[0] \mid \mathbb{R}^n \times \{0\}$ ($\mathcal{K}_1[0] \mid \mathbb{R}^n \times \{1\}$) is the convex hull of $n + 1$ vectors $(x^0, 0), \dots, (x^n, 0)$ ($(x^0, 1), \dots, (x^n, 1)$) given by $x^0 = x$, and

$$x^k = x^{k-1} + u^{\pi(k)}, \quad k = 1, 2, \dots, n,$$

where x is an integer vector of \mathbb{R}^n , and $\pi = (\pi(1), \dots, \pi(n))$ is a permutation of elements of N . Since a simplex of $\mathcal{K}_1[0] \mid \mathbb{R}^n \times \{0\}$ ($\mathcal{K}_1[0] \mid \mathbb{R}^n \times \{1\}$) is uniquely determined by x and π , we use $K_1(x, \pi) \times \{0\}$ ($K_1(x, \pi) \times \{1\}$) to denote it.

The complete simplex given in the following lemma is used as the starting simplex of the algorithm.

Lemma 6. There is only one complete n -dimensional simplex in $H(\eta, N) \times \{0\}$, which is given by $K_1(\eta, (1, \dots, n)) \times \{0\} = \langle (x^0, 0), \dots, (x^n, 0) \rangle$.

Proof. Observe that $x^0 = \eta$, and

$$x^k = x^{k-1} + u^k, \quad k = 1, 2, \dots, n.$$

Since rows of A are in the proper order and A is in the canonical form, according to Definition 1, one has that $l(x^0, 0) = n + 1$, and $l(x^k, 0) = k$ for $k = 1, 2, \dots, n$. Thus, $K_1(\eta, (1, \dots, n)) \times \{0\}$ is a complete simplex.

According to Lemma 5, $(\eta, 0)$ is the unique point of $H(\eta, N) \times \{0\}$ carrying integer label $n + 1$. Thus, any complete n -dimensional simplex in $H(\eta, N) \times \{0\}$ must have $(\eta, 0)$ as a vertex. Observe that any simplex in $H(\eta, N) \times \{0\}$ having $(\eta, 0)$ as a vertex is given by $K_1(\eta, \pi) \times \{0\}$ for some unique π . Let $\bar{\sigma} = K_1(\eta, \pi) \times \{0\} = \langle (\bar{x}^0, 0), \dots, (\bar{x}^n, 0) \rangle$ be a complete simplex in $H(\eta, N) \times \{0\}$. Then $\bar{x}^0 = \eta$, and

$$\bar{x}^k = \bar{x}^{k-1} + u^{\pi(k)}, \quad k = 1, 2, \dots, n.$$

In the following, we show that $\pi = (1, 2, \dots, n)$. Noting that $d = A\eta$ and A is in the canonical form, one must have according to Definition 1 that $l(\bar{x}^0) = n + 1$, and for $k = 1, 2, \dots, n$, $l(\bar{x}^k) = \pi(k)$ since $a_i^T \bar{x}^k \leq 0$ for $i = \pi(k + 1), \dots, \pi(n)$. No matter what π is, one always has that

$$\bar{x}^n = \eta + \sum_{j=1}^n u^{\pi(j)} = \eta + e = x^n.$$

Thus, $l(\bar{x}^n) = l(x^n) = n$. From the integer labeling rule and the completeness of $\bar{\sigma}$, we obtain $\pi(n) = n$. Observe that $\bar{x}^{n-1} = \bar{x}^n - u^{\pi(n)} = x^n - u^n = x^{n-1}$. Then $l(\bar{x}^{n-1}) = l(x^{n-1}) = n - 1$. From the integer

labeling rule and the completeness of $\bar{\sigma}$, we get $\pi(n - 1) = n - 1$. Continuing this process, we have $\pi(k) = k$, $k = 1, 2, \dots, n$. The lemma follows. \square

Lemma 7. For any $K \subseteq N$, $(x, 0) \in H(\eta, K) \times \{0\}$ is labeled either $n + 1$ or some integer of K .

Proof. For $(x, 0) \in H(\eta, K) \times \{0\}$, let $\lambda = x - \eta$. Then $0 \leq \lambda_j$, $j \in K$, and $\lambda_j = 0$, $j \notin K$. Thus, for $i \notin K$,

$$\begin{aligned} a_i^T x - d_i &= a_i^T \eta - d_i + a_i^T \lambda \\ &= a_i^T \lambda = \sum_{j \in K} a_{ij} \lambda_j \\ &\leq 0, \end{aligned}$$

because A is in the canonical form. Note that if $x = \eta$ then $l(x, 0) = n + 1$, and if $x \neq \eta$ then $\lambda \neq 0$. Thus, using Lemma 1, we obtain that there is an integer of K , say k , satisfying that $0 < a_k^T \lambda$ because $a_i^T \lambda \leq 0$ for $i \notin K$. Observe that

$$a_k^T x - d_k = a_k^T \eta - d_k + a_k^T \lambda = a_k^T \lambda > 0.$$

Then $l(x, 0) \in K$ according to Definition 1. The lemma follows. \square

As a corollary of this lemma, we obtain

Corollary 2. For any $j \in N$, there is no complete $(n - 1)$ -dimensional simplex in $H(\eta, N \setminus \{j\}) \times \{0\}$ carrying only integer labels in N .

The following lemma can be found in [6].

Lemma 8. Assume that P has at least one integer point. Let z^0 be an integer point of P . For any $K \subseteq N$, $(x, 1) \in H(z^0, K) \times \{1\}$ is labeled either 0 or an integer of K .

Proof. Note that $Az^0 \leq b$. For $(x, 1) \in H(z^0, K) \times \{1\}$, let $\lambda = x - z^0$. Then $0 \leq \lambda_j$, $j \in K$, and $\lambda_j = 0$, $j \notin K$. Thus, for $i \notin K$,

$$\begin{aligned} a_i^T x &= a_i^T z^0 + a_i^T \lambda \\ &\leq b_i + a_i^T \lambda = b_i + \sum_{j \in K} a_{ij} \lambda_j \\ &\leq b_i, \end{aligned}$$

because A is in the canonical form. According to Definition 1, the lemma follows. \square

As a corollary of this lemma, we obtain

Corollary 3. There is no complete n -dimensional simplex in $H(z^0, N) \times \{1\}$, and for any $j \in N$, there is no complete $(n - 1)$ -dimensional simplex in $H(z^0, N \setminus \{j\}) \times \{1\}$ carrying only integer labels in N .

The following example shows that there exist perhaps more than one complete n -dimensional simplices in $\mathbb{R}^n \times \{0\}$ and $\mathbb{R}^n \times \{1\}$, respectively.

Example 1. Let

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 4 & -6 \\ 0 & -2 & 4 \\ -2 & 0 & 0 \end{pmatrix}$$

and $b = (1/5, 1/5, 1/5, -1/10)^T$. It is obvious that any 3×3 submatrix of A is nonsingular. Let $P = \{x \mid Ax \leq b\}$. Then $(1/10, 1/20, 1/20)^T$ is an interior point of P . Since $\rho^T A = 0$ for $\rho = (1, 1/2, 1, 1)^T$, P is bounded. Observe that A has the canonical form. Let $\eta = (0, 0, 0)^T$. Then $d = A\eta = 0$. For $x = (0, -1, -1)^T$ and $\pi = (3, 2, 1)$, we obtain

$$x^0 = (0, -1, -1)^T, \quad x^1 = (0, -1, 0)^T, \quad x^2 = (0, 0, 0)^T, \quad x^3 = (1, 0, 0)^T.$$

According to Definition 1, we have

$$l(x^0, 0) = 2, \quad l(x^1, 0) = 3, \quad l(x^2, 0) = 4, \quad l(x^3, 0) = 1$$

and

$$l(x^0, 1) = 2, \quad l(x^1, 1) = 3, \quad l(x^2, 1) = 4, \quad l(x^3, 1) = 1.$$

For $\bar{x} = (0, 0, 0)^T$ and $\bar{\pi} = (1, 2, 3)$, we obtain

$$\bar{x}^0 = (0, 0, 0)^T, \quad \bar{x}^1 = (1, 0, 0)^T, \quad \bar{x}^2 = (1, 1, 0)^T, \quad \bar{x}^3 = (1, 1, 1)^T.$$

According to Definition 1, we have

$$l(\bar{x}^0, 0) = 4, \quad l(\bar{x}^1, 0) = 1, \quad l(\bar{x}^2, 0) = 2, \quad l(\bar{x}^3, 0) = 3$$

and

$$l(\bar{x}^0, 1) = 4, \quad l(\bar{x}^1, 1) = 1, \quad l(\bar{x}^2, 1) = 2, \quad l(\bar{x}^3, 1) = 3.$$

Thus, $K_1((0, -1, -1)^T, (3, 2, 1)) \times \{t\}$ and $K_1((0, 0, 0)^T, (1, 2, 3)) \times \{t\}$ are two complete three-dimensional simplices in $\mathbb{R}^3 \times \{t\}$ for $t = 0$ and $t = 1$, respectively.

This example tells us that any straightforward use of a simplicial homotopy algorithm may terminate at a complete simplex either in $\mathbb{R}^n \times \{0\}$ or $\mathbb{R}^n \times \{1\}$ before it meets an integer point of a simplex or proves no such point exists.

5. The algorithm

In this section we describe the algorithm and prove its convergence. Before doing it, we need the following notation. Let

$$x^{\max} = (x_1^{\max}, x_2^{\max}, \dots, x_n^{\max})^T,$$

where

$$x_i^{\max} = \max_{x \in P} x_i$$

for $i = 1, 2, \dots, n$. Obviously, if $x \in P$ then $x \leq x^{\max}$. For any number α , let $\lceil \alpha \rceil$ denote the smallest integer greater than or equal to α . We define $x^u = (x_1^u, x_2^u, \dots, x_n^u)^T$ by

$$x_i^u = \begin{cases} \lceil x_i^{\max} \rceil & \text{if } x_i < \lceil x_i^{\max} \rceil, \\ 1 + \lceil x_i^{\max} \rceil & \text{otherwise} \end{cases}$$

for $i = 1, 2, \dots, n$. Then $x < x^u$ for any $x \in P$.

The algorithm can be stated as follows.

Initialization. Let $y = (\eta, 0)$ and $\pi = (1, 2, \dots, n + 1)$. Then $\sigma_0 = K_1(y, \pi)$ is the unique $(n + 1)$ -dimensional simplex in $\mathbb{R}^n \times [0, 1]$ having the complete n -dimensional simplex $K_1(\eta, (1, \dots, n)) \times \{0\}$, denoted by τ_0 , as a facet. Let $k = 0$ and go to Step 1.

Step 1: Let y^+ be the vertex of σ_k opposite to τ_k . If $l(y^+) = 0$ then the algorithm terminates, and an integer point of P has been found. Otherwise, let y^- be the vertex of σ_k other than y^+ and carrying integer label $l(y^+)$, and τ_{k+1} the facet of σ_k opposite to y^- . Go to Step 2.

Step 2: If $\tau_{k+1} \subset \mathbb{R}^n \times \{0\}$ then $t = 0$ and go to Step 3, if $\tau_{k+1} \subset \mathbb{R}^n \times \{1\}$ then $t = 1$ and go to Step 3, and otherwise, do as follows: Let σ_{k+1} be the unique simplex that is adjacent to σ_k and has τ_{k+1} as a facet. Let $k = k + 1$, and go to Step 1.

Step 3: Let $\sigma_{k+1} = \tau_{k+1}$, and y^- be the vertex of σ_{k+1} carrying integer label $n + 1$. Let τ_{k+2} be the facet of σ_{k+1} opposite to y^- . Let $k = k + 1$, and go to Step 4.

Step 4: Let σ_{k+1} be the simplex in $\mathbb{R}^n \times \{t\}$ that is adjacent to σ_k and has τ_{k+1} as a facet. Let y^+ be the vertex of σ_{k+1} opposite to τ_{k+1} . Let $k = k + 1$, and go to Step 5.

Step 5: If $l(y^+) = 0$ then the algorithm terminates, and an integer point of P has been found. If $t = 1$ and $(x^u, 1) \leq y^+$ then the algorithm terminates, and P has no integer point. If $l(y^+) = n + 1$ then let $\tau_{k+1} = \sigma_k$, and σ_{k+1} be the unique $(n + 1)$ -dimensional simplex in $\mathbb{R}^n \times [0, 1]$ having τ_{k+1} as a facet. Let $k = k + 1$, and go to Step 1. If $l(y^+) \neq n + 1$ then let y^- be the vertex of σ_k other than y^+ and carrying integer label $l(y^+)$, and τ_{k+1} the facet of σ_k opposite to y^- . Go to Step 4.

Theorem 1. *Within a finite number of iterations, the algorithm either yields an integer point of P or proves that no such point exists.*

The proof of this theorem follows the standard arguments given in [25]. First of all, we need to introduce an undirected graph. The way of defining the graph is similar to that in [25]. Let G be the graph, which is given as follows:

- The nodes of G consist of
 1. all the 0-complete n -dimensional simplices contained in $\mathbb{R}^n \times \{1\}$ and carrying all integer labels in N ,
 2. all the 0-complete $(n + 1)$ -dimensional simplices,
 3. all the almost complete n -dimensional simplices contained in $\mathbb{R}^n \times \{0\}$ and carrying only integer labels in N ,
 4. all the almost complete n -dimensional simplices contained in $\mathbb{R}^n \times \{1\}$ and carrying only integer labels in N ,

- 5. all the complete n -dimensional simplices contained in $\mathbb{R}^n \times \{0\}$,
- 6. all the complete n -dimensional simplices contained in $\mathbb{R}^n \times \{1\}$, and
- 7. all the almost complete $(n + 1)$ -dimensional simplices,
- There is an edge between two nodes of G if one is both a complete simplex and a facet of the other or they have a common complete facet carrying all integer labels in N or N_0 .

The degree of a node of G is equal to the number of nodes of G adjacent to it. Let us consider the degree of each node of G :

1. Consider node σ of a 0-complete n -dimensional simplex contained in $\mathbb{R}^n \times \{1\}$ and carrying all integer labels in N . One can observe that node σ is only adjacent to the node given by one of
 - (a) the almost complete n -dimensional simplex contained in $\mathbb{R}^n \times \{1\}$ and carrying only integer labels in N ,
 - (b) the complete n -dimensional simplex contained in $\mathbb{R}^n \times \{1\}$, or
 - (c) the 0-complete n -dimensional simplex contained in $\mathbb{R}^n \times \{1\}$ and carrying all integer labels in N .

Thus, node σ has degree one.

2. Consider node σ of a 0-complete $(n + 1)$ -dimensional simplex. One can observe that node σ is only adjacent to the node given by one of
 - (a) the almost complete $(n + 1)$ -dimensional simplex,
 - (b) the complete n -dimensional simplex contained in $\mathbb{R}^n \times \{0\}$, or
 - (c) the 0-complete $(n + 1)$ -dimensional simplex.

Thus, node σ has degree one.

3. Consider node σ of an almost complete n -dimensional simplex contained in $\mathbb{R}^n \times \{0\}$ and carrying only integer labels in N . Note that σ has exactly two complete facets. Then one can observe that node σ is only adjacent to two nodes given by one of the following three pairs:
 - (a) two almost complete n -dimensional simplices contained in $\mathbb{R}^n \times \{0\}$ and carrying only integer labels in N ,
 - (b) an almost complete n -dimensional simplex contained in $\mathbb{R}^n \times \{0\}$ and carrying only integer labels in N and a complete n -dimensional simplex contained in $\mathbb{R}^n \times \{0\}$, or
 - (c) two complete n -dimensional simplices contained in $\mathbb{R}^n \times \{0\}$.

Thus, node σ has degree 2.

4. Consider node σ of an almost complete n -dimensional simplex contained in $\mathbb{R}^n \times \{1\}$ and carrying only integer labels in N . Note that σ has exactly two complete facets. Then one can observe that node σ is only adjacent to two nodes given by one of the following six pairs:
 - (a) two almost complete n -dimensional simplices contained in $\mathbb{R}^n \times \{1\}$ and carrying only integer labels in N ,
 - (b) an almost complete n -dimensional simplex contained in $\mathbb{R}^n \times \{1\}$ and carrying only integer labels in N and a complete n -dimensional simplex contained in $\mathbb{R}^n \times \{1\}$,
 - (c) an almost complete n -dimensional simplex contained in $\mathbb{R}^n \times \{1\}$ and carrying only integer labels in N and a 0-complete n -dimensional simplex contained in $\mathbb{R}^n \times \{1\}$ and carrying all integer labels in N ,
 - (d) two complete n -dimensional simplices contained in $\mathbb{R}^n \times \{1\}$,
 - (e) two 0-complete n -dimensional simplices contained in $\mathbb{R}^n \times \{1\}$ and carrying all integer labels in N , or

- (f) a complete n -dimensional simplex contained in $\mathbb{R}^n \times \{1\}$ and a 0-complete n -dimensional simplex contained in $\mathbb{R}^n \times \{1\}$ and carrying all integer labels in N .

Thus, node σ has degree 2.

5. Consider node σ of a complete n -dimensional simplex contained in $\mathbb{R}^n \times \{0\}$. One can observe that node σ is only adjacent to two nodes given by one of the following four pairs:
 - (a) an almost complete n -dimensional simplex contained in $\mathbb{R}^n \times \{0\}$ and carrying only integer labels in N and an almost complete $(n + 1)$ -dimensional simplex,
 - (b) an almost complete n -dimensional simplex contained in $\mathbb{R}^n \times \{0\}$ and carrying only integer labels in N and a 0-complete $(n + 1)$ -dimensional simplex,
 - (c) a complete n -dimensional simplex contained in $\mathbb{R}^n \times \{0\}$ and an almost complete $(n + 1)$ -dimensional simplex, or
 - (d) a complete n -dimensional simplex contained in $\mathbb{R}^n \times \{0\}$ and a 0-complete $(n + 1)$ -dimensional simplex.

Thus, node σ has degree 2.

6. Consider node σ of a complete n -dimensional simplex contained in $\mathbb{R}^n \times \{1\}$. One can observe that node σ is only adjacent to two nodes given by one of the following three pairs:
 - (a) an almost complete n -dimensional simplex contained in $\mathbb{R}^n \times \{1\}$ and carrying only integer labels in N and an almost complete $(n + 1)$ -dimensional simplex,
 - (b) a complete n -dimensional simplex contained in $\mathbb{R}^n \times \{1\}$ and an almost complete $(n + 1)$ -dimensional simplex, or
 - (c) a 0-complete n -dimensional simplex contained in $\mathbb{R}^n \times \{1\}$ and carrying all integer labels in N and an almost complete $(n + 1)$ -dimensional simplex.

Thus, node σ has degree 2.

7. Consider node σ of an almost complete $(n + 1)$ -dimensional simplex. Note that an almost complete simplex has exactly two complete facets. Then one can observe that node σ is only adjacent to two nodes given by one of the following seven pairs:
 - (a) two almost complete $(n + 1)$ -dimensional simplices,
 - (b) an almost complete $(n + 1)$ -dimensional simplex and a 0-complete $(n + 1)$ -dimensional simplex,
 - (c) an almost complete $(n + 1)$ -dimensional simplex and a complete n -dimensional simplex contained in $\mathbb{R}^n \times \{0\}$,
 - (d) an almost complete $(n + 1)$ -dimensional simplex and a complete n -dimensional simplex contained in $\mathbb{R}^n \times \{1\}$,
 - (e) two 0-complete $(n + 1)$ -dimensional simplices,
 - (f) a 0-complete $(n + 1)$ -dimensional simplex and a complete n -dimensional simplex contained in $\mathbb{R}^n \times \{0\}$, or
 - (g) a 0-complete $(n + 1)$ -dimensional simplex and a complete n -dimensional simplex contained in $\mathbb{R}^n \times \{1\}$.

Thus, node σ has degree 2.

Note that each of the nodes of G belongs to one of these seven categories. Combining the above cases, we obtain that the degree of each node of G is at most two. Thus, we come to the following result.

Lemma 9. Each connected component of G has one of the following forms:

- A simple circuit, each of whose nodes has degree 2.
- A path, each of whose endpoints (if it has any) has degree one and is given by one of a 0-complete n -dimensional simplex contained in $\mathbb{R}^n \times \{1\}$ and carrying all integer labels in N or a 0-complete $(n + 1)$ -dimensional simplex.

It is obvious that all the simplices in a path of the graph are different from one another.

Proof of Theorem 1. According to the algorithm, it can be seen that all the simplices generated by the algorithm are nodes of the graph and form a connected component. Using Lemmas 5, 6 and Corollary 2, one can obtain that the starting complete n -dimensional simplex of the algorithm belongs to a path of the graph. Thus, all the simplices generated by the algorithm are different from one another.

For any positive integer μ , let

$$A(\eta, \mu) = \{x \in \mathbb{R}^n \mid \|x - \eta\| \leq \mu\}.$$

Then the boundary of $A(\eta, \mu)$, denoted $\partial A(\eta, \mu)$, is the union of a finite number of integer unit cubes of $n - 1$ dimensions. Thus, $\partial A(\eta, \mu) \times [0, 1]$ is the union of a finite number of integer unit cubes of n dimensions. For any nonzero vector $\xi \in \mathbb{R}^n$, we define

$$f(\xi) = \min_{i \in N_0} a_i^T \xi.$$

Then f is continuous on \mathbb{R}^n . Using Lemma 1, we obtain that $f(\xi) < 0$ for any $\xi \neq 0$. Thus,

$$\max_{\xi \in \{\xi \mid \|\xi\|=1\}} f(\xi) < 0.$$

Then, according to Definition 1 and the geometrical structures of the integer labeling rule, there exists a sufficiently large integer μ_0 such that when $\mu_0 \leq \mu$, for each integer unit cube φ contained in $\partial A(\eta, \mu)$, there is some $i \in N_0$ satisfying that no point of $\varphi \times \{0, 1\}$ is labeled i . Thus, when $\mu_0 \leq \mu$, for each integer unit cube in $\partial A(\eta, \mu) \times [0, 1]$, there is some $i \in N_0$ satisfying that no integer point of the integer unit cube is labeled i , i.e. there is no complete n -dimensional simplex in the integer unit cube. Therefore, when $\mu_0 \leq \mu$, there is no complete n -dimensional simplex in $\partial A(\eta, \mu) \times [0, 1]$.

Assume that Steps 1 and 2 of the algorithm are being implemented. Then all the simplices generated by the algorithm in these steps are contained in $A(\eta, \mu_0) \times [0, 1]$. Since $A(\eta, \mu_0) \times [0, 1]$ is bounded, there is a finite number of simplices in it. Thus, within a finite number of implementations of Steps 1 and 2, the algorithm either yields an integer point of P , or starts implementing Steps 3–5.

Assume that Steps 3–5 of the algorithm are being implemented. There are two cases to be considered:

1. Consider $t = 0$. Using Lemma 6 and Corollary 2, we obtain that all the simplices generated by the algorithm in these steps are in $\mathbb{R}^n \setminus H(\eta, N) \times \{0\}$. According to Lemma 3 and Corollary 1, we know that all the almost complete n -dimensional simplices in $\mathbb{R}^n \times \{0\}$ carrying only integer labels in N are contained in $\Delta(\Gamma(\eta), \delta) \times \{0\}$ and $0 < A_{-(n+1)}^{-1}e$. Then there is a finite number of

almost complete n -dimensional simplices in $\mathbb{R}^n \setminus H(\eta, N) \times \{0\}$ carrying only integer labels in N because the intersection of both $\mathbb{R}^n \setminus H(\eta, N) \times \{0\}$ and $\Delta(\Gamma(\eta), \delta) \times \{0\}$ is bounded. Thus, within a finite number of implementations of Steps 3–5, the algorithm starts implementing Steps 1 and 2.

2. Consider $t = 1$. There are two situations to be investigated.
 - (a) Assume that P has at least one integer point. Let z^0 be an integer point of P . Using Lemma 8 and Corollary 3, we obtain that all the simplices generated by the algorithm in these steps are in $\mathbb{R}^n \setminus H(z^0, N) \times \{1\}$. According to Lemma 3 and Corollary 1, we know that all the almost complete n -dimensional simplices in $\mathbb{R}^n \times \{1\}$ carrying only integer labels in N are contained in $\Delta(\Gamma(\omega), \delta) \times \{1\}$ and $0 < A_{-(n+1)}^{-1}e$. Then there is a finite number of almost complete n -dimensional simplices in $\mathbb{R}^n \setminus H(z^0, N) \times \{1\}$ carrying only integer labels in N because the intersection of both $\mathbb{R}^n \setminus H(z^0, N) \times \{1\}$ and $\Delta(\Gamma(\omega), \delta) \times \{1\}$ is bounded. Note that $z^0 < x^u$. Thus, within a finite number of implementations of Steps 3–5, the algorithm either yields an integer point of P or starts implementing Steps 1 and 2.
 - (b) Assume that P has no integer point. Then all the simplices generated by the algorithm in these steps are contained in $\mathbb{R}^n \setminus H(x^u, N) \times \{1\}$. According to Lemma 3 and Corollary 1, we know that all the almost complete n -dimensional simplices in $\mathbb{R}^n \times \{1\}$ carrying only integer labels in N are contained in $\Delta(\Gamma(\omega), \delta) \times \{1\}$ and $0 < A_{-(n+1)}^{-1}e$. Then there is a finite number of almost complete n -dimensional simplices in $\mathbb{R}^n \setminus H(x^u, N) \times \{1\}$ carrying only integer labels in N because the intersection of both $\mathbb{R}^n \setminus H(x^u, N) \times \{1\}$ and $\Delta(\Gamma(\omega), \delta) \times \{1\}$ is bounded. Thus, within a finite number of implementations of Steps 3, 4 and 5, the algorithm either yields a vertex that is greater than or equal to $(\geq) (x^u, 1)$ or starts implementing Steps 1 and 2. Considering (a) and noting that $x < x^u$ for any $x \in P$, we obtain that if the algorithm yields in Steps 3–5 a vertex that is greater than or equal to $(\geq) (x^u, 1)$ then P has no integer point.

According to Lemma 3, we know that there is a finite number of complete n -dimensional simplices in $\mathbb{R}^n \times \{0\}$ and $\mathbb{R}^n \times \{1\}$. Observe that Steps 3–5 will be executed only when a complete n -dimensional simplex in $\mathbb{R}^n \times \{0\}$ or $\mathbb{R}^n \times \{1\}$ is generated in Steps 1 and 2. Thus, the algorithm will start implementing Steps 3–5 only a finite number of times. The theorem follows. \square

6. Numerical tests

In this section we give a few numerical tests. In our implementation of the algorithm, for $(x, t) \in \mathbb{R}^n \times \{0, 1\}$, we assign to (x, t) an integer $l(x, t)$ given by

$$l(x, t) = \begin{cases} \max \left\{ k \mid \frac{a_k^T x - d_k}{a_k^T a_k} = \max_{j \in N_0} \frac{a_j^T x - d_j}{a_j^T a_j} \right\} & \text{if } t = 0, \\ 0 & \text{if } x \in P \text{ and } t = 1, \\ \max \left\{ k \mid \frac{a_k^T x - b_k}{a_k^T a_k} = \max_{j \in N_0} \frac{a_j^T x - b_j}{a_j^T a_j} \right\} & \text{if } x \notin P \text{ and } t = 1. \end{cases}$$

Example 2. Consider the simplex given by $S = \{y \mid By \leq b\}$, where

$$B = \begin{pmatrix} t_1 - r & t_1 & \cdots & t_1 \\ t_2 & t_2 - r & \cdots & t_2 \\ \vdots & \vdots & \ddots & \vdots \\ t_n & t_n & \cdots & t_n - r \\ -1 & -1 & \cdots & -1 \end{pmatrix}$$

and

$$b = (t_1 - \varepsilon, t_2 - \varepsilon, \dots, t_n - \varepsilon, -1 - \varepsilon)^T$$

with $t_j, j = 1, 2, \dots, n$, being integers in $[1, 50]$, which are generated randomly, $\varepsilon = 0.00001$, and $r = -1 + \sum_{j=1}^n t_j$. We sort t so that $t_1 \leq t_2 \leq \dots \leq t_n$. This problem is provided by H.E. Scarf (private communication). Let

$$U_1 = \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & & \ddots & & \\ & & & -1 & 1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ \vdots & \vdots & \ddots & & \\ 1 & 1 & \cdots & 1 & \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

$$U_3 = \begin{pmatrix} 1 & & & 1 \\ & 1 & & 1 \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix}, \quad U_4 = \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & -1 & \\ & & & & 1 \end{pmatrix}.$$

Note that U_1, U_2, U_3 and U_4 are unimodular matrices. Thus, $U = U_1 U_2 U_3 U_4$ is a unimodular matrix. Computing the product of $A = BU$, we obtain

$$A = BU = \begin{pmatrix} r & & & t_1 - r \\ & \ddots & & \vdots \\ & & r & t_{n-1} - r \\ -r & \cdots & -r & t_n + (n-2)r \\ 0 & \cdots & 0 & -1 \end{pmatrix},$$

which is in the canonical form. Interchanging rows $n - 1$ and n and columns $n - 1$ and n , we obtain

$$A = \begin{pmatrix} r & & & t_1 - r & 0 \\ & \ddots & & \vdots & \vdots \\ & & r & t_{n-2} - r & 0 \\ -r & \cdots & -r & t_n + (n-2)r & -r \\ 0 & \cdots & 0 & t_{n-1} - r & r \\ 0 & \cdots & 0 & -1 & 0 \end{pmatrix}$$

Table 2

n	Number of iterations	n	Number of iterations
20	13 453	120	502 234
40	58 210	140	628 837
60	131 056	160	896 979
80	206 631	180	1 111 159
100	357 444	200	1 309 810

and

$$b = (t_1 - \varepsilon, \dots, t_{n-2} - \varepsilon, t_n - \varepsilon, t_{n-1} - \varepsilon, -1 - \varepsilon)^T$$

Let $P = \{x \mid Ax \leq b\}$.

Starting at the greatest integer point less than or equal to the interior point of P that has the same distance to every facet of P , the algorithm proves that P does not contain any integer point for $n = 20, 40, 60, 80, 100, 120, 140, 160, 180, 200$. The number of iterations required are given in Table 2.

Example 3. Consider the simplex given by $P = \{x \mid Ax \leq b\}$, where

$$A = \begin{pmatrix} r - t_1 & -t_1 & \cdots & -t_1 \\ -t_2 & r - t_2 & \cdots & -t_2 \\ \vdots & \vdots & \ddots & \vdots \\ -t_n & -t_n & \cdots & r - t_n \\ -1 & -1 & \cdots & -1 \end{pmatrix}$$

and

$$b = (t_1 + 1, t_2 + 1, \dots, t_n + 1, r)^T$$

with $t_j, j = 1, 2, \dots, n$, being integers in $[1, 50]$, which are generated randomly, and $r = 1 + \sum_{j=1}^n t_j$. We sort t so that $t_1 \geq t_2 \geq \dots \geq t_n$. This problem is a modification of the problem given in the previous example.

- Starting at the greatest integer point less than or equal to the vertex of the simplex opposite to the hyperplane $a_{n/4}^T x = b_{n/4}$, the algorithm yields a feasible integer point for $n = 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000$. The number of iterations required are given in Table 3.
- Starting at the greatest integer point less than or equal to the vertex of the simplex opposite to the hyperplane $a_{n/2}^T x = b_{n/2}$, the algorithm yields a feasible integer point for $n = 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000$. The number of iterations required are given in Table 4.
- Starting at the greatest integer point less than or equal to the vertex of the simplex opposite to the hyperplane $a_{3n/4}^T x = b_{3n/4}$, the algorithm yields a feasible integer point for $n = 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000$. The number of iterations required are given in Table 5.
- Starting at the greatest integer point less than or equal to the interior point of P that has the same distance to every facet of P , the algorithm yields a feasible integer point for $n = 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000$. The number of iterations required are given in Table 6.

Table 3

n	Number of iterations	n	Number of iterations
100	223	600	1348
200	448	700	1573
300	673	800	1798
400	898	900	2023
500	1123	1000	2248

Table 4

n	Number of iterations	n	Number of iterations
100	248	600	1498
200	498	700	1748
300	748	800	1998
400	998	900	2248
500	1248	1000	2498

Table 5

n	Number of iterations	n	Number of iterations
100	273	600	1648
200	548	700	1923
300	823	800	2198
400	1098	900	2473
500	1373	1000	2748

Table 6

n	Number of iterations	n	Number of iterations
100	1218	600	18 744
200	3938	700	37 878
300	6138	800	21 644
400	6160	900	15 894
500	23 348	1000	32 878

- Starting at the greatest integer point less than or equal to an arbitrary vertex, i , of the simplex, the algorithm yields a feasible integer point for $n = 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000$. The number of iterations required are given in Table 7.

Comparing these numerical results with those in [6], one may realize that no improvement can be made by arbitrary starting when a simplex has no integer point, but significant improvement can be made by arbitrary starting when a simplex has an integer point.

Table 7

n	i	Number of iterations	n	i	Number of iterations
100	18	216	600	261	1459
200	140	538	700	221	1619
300	193	791	800	609	2207
400	254	1052	900	279	2077
500	26	1024	1000	497	2495

References

- [1] E.L. Allgower, K. Georg, Simplicial and continuation methods for approximating fixed points and solutions to systems of equations, *SIAM Rev.* 22 (1980) 28–85.
- [2] E.L. Allgower, K. Georg, *Numerical Continuation Methods*, Springer, Berlin, 1990.
- [3] C. Dang, The D_1 -triangulation of R^n for simplicial algorithms for computing solutions of nonlinear equations, *Math. Oper. Res.* 16 (1991) 148–161.
- [4] C. Dang, H. van Maaren, A simplicial approach to integer programming — Part I, REPORT 93-06, Faculty of Technical Mathematics and Informatics, Delft University of Technology, Delft, The Netherlands.
- [5] C. Dang, H. van Maaren, A simplicial approach to integer programming — Part II, *Ann. Oper. Res. Soc. New Zealand* 29 (1993) 131–138.
- [6] C. Dang, H. van Maaren, A simplicial approach to the determination of an integer point of a simplex, *Math. Oper. Res.* 23 (1998) 403–415.
- [7] C. Dang, H. van Maaren, An arbitrary starting variable dimension algorithm for computing an integer point of a simplex, *Comput. Optim. Appl.* 14 (1999) 133–155.
- [8] B.C. Eaves, Homotopies for the computation of fixed points, *Math. Programming* 3 (1972) 1–22.
- [9] B.C. Eaves, R. Saigal, Homotopies for the computation of fixed points on unbounded regions, *Math. Programming* 3 (1972) 225–237.
- [10] W. Forster, Homotopy methods, in: R. Horst, P. Pardalos (Eds.), *Handbook of Global Optimization*, Kluwer Academic Publishers, Dordrecht, 1995, pp. 669–750.
- [11] H. Freudenthal, Simplicialzerlegungen von Beschränkter Flachheit, *Ann. Math.* 43 (1942) 580–582.
- [12] M. Kojima, Y. Yamamoto, Variable dimension algorithms: basic theory, interpretation, and extensions of some existing methods, *Math. Programming* 24 (1982) 177–215.
- [13] H.W. Kuhn, Simplicial approximation of fixed points, *Proc. Nat. Acad. Sci.* 61 (1968) 1238–1242.
- [14] G. van der Laan, A.J.J. Talman, A restart algorithm for computing fixed points without an extra dimension, *Math. Programming* 17 (1979) 74–84.
- [15] G. van der Laan, A.J.J. Talman, A class of simplicial restart fixed point algorithms without an extra dimension, *Math. Programming* 20 (1981) 33–48.
- [16] O.H. Merrill, Applications and extensions of an algorithm that computes fixed points of certain upper semi-continuous point to set mappings, Ph.D. Thesis, Department of Industrial and Operations Engineering, University of Michigan, Ann Arbor, MI.
- [17] G.L. Nemhauser, L.A. Wolsey, *Integer and Combinatorial Optimization*, Wiley, New York, 1988.
- [18] A. Pnueli, A method of truncated relaxation for integer programming, RC 2267, IBM Research, Research Division, Yorktown Heights, New York, 1968.
- [19] H.E. Scarf, The approximation of fixed points of a continuous mapping, *SIAM J. Appl. Math.* 15 (1967) 1328–1343.
- [20] H.E. Scarf (Collaboration with T. Hansen), *The Computation of Economic Equilibria*, Yale University Press, New Haven, 1973.
- [21] H.E. Scarf, Production sets with indivisibilities — part I: generalities, *Econometrica* 49 (1981) 1–32.
- [22] H.E. Scarf, Production sets with indivisibilities — part II: the case of two activities, *Econometrica* 49 (1981) 395–423.
- [23] H.E. Scarf, Neighborhood systems for production sets with indivisibilities, *Econometrica* 54 (1986) 507–532.

- [24] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, New York, 1986.
- [25] M.J. Todd, *The Computation of Fixed Points and Applications*, *Lecture Notes in Economics and Mathematical Systems*, vol. 124, Springer, Berlin, 1976.
- [26] Y. Yamamoto, A new variable dimension algorithm for the fixed point problem, *Math. Programming* 25 (1983) 329–342.
- [27] Z. Yang, A simplicial algorithm for testing the integral properties of polytopes: a revision, *CentER Discussion paper No. 9489*, Center for Economic Research, Tilburg University, Tilburg, The Netherlands, 1994.
- [28] N. Zhu, *Some theory, computations and applications of knapsack problems*, Ph.D. Thesis, The University of Waikato, Hamilton, New Zealand, 1997.