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# THE GEOMETRY OF TORIC VARIETIES

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## Contents



## Introduction

**0.1.** Toric varieties (called *torus embeddings* in [26]) are algebraic varieties that are generalizations of both the affine spaces $\mathbf{A}^n$ and the projective space $\mathbf{P}^n$. Because they are rather simple in structure (although not as primitive as

$\mathbf{P}^n$), they serve as interesting examples on which one can illustrate concepts of algebraic geometry such as linear systems, invertible sheaves, cohomology, resolution of singularities, intersection theory and so on. However, two other circumstances determine the main reason for interest in toric varieties. The first is that there are many algebraic varieties which it is most reasonable to embed not in projective space $\mathbf{P}^n$ but in a suitable toric variety; it becomes more natural in such a case to compare the properties of the variety and the ambient space. This also applies to the choice of compactification of a non-compact algebraic variety. The second circumstance is closely related to the first, consisting in the fact that varieties "locally" are frequently toric in structure, or *toroidal*. As a trivial example, a smooth variety is locally isomorphic to affine space $\mathbf{A}^n$. Toroidal varieties are interesting in that one can transfer to them the theory of differential forms, which plays such an important role in the study of smooth varieties.

**0.2.** To get some idea of toric varieties, let us first consider the simplest example, the projective space $\mathbf{P}^n$. This is the variety of lines in an $(n+1)$-dimensional vector space $K^{n+1}$, where $K$ is the base field (for example $K = \mathbf{C}$, the complex number field). Let $t_0, \ldots, t_n$ be coordinates in $K^{n+1}$; then the points of $\mathbf{P}^n$ are given by "homogeneous coordinates" $(t_0 : t_1 : \ldots : t_n)$. Picking out the points with non-zero $i$th coordinate $t_i$, we get an open subvariety $U_i \subset \mathbf{P}^n$. If we consider on $U_i$ the $n$ functions $x_k^{(i)} = t_k/t_i$ (with $k = 0, 1, \ldots, \hat{i}, \ldots, n$), then these establish an isomorphism of $U_i$ with the affine space $K^n$; we call the functions $x_k^{(i)}$ coordinates on $U_i$. Projective space $\mathbf{P}^n$ is covered by charts $U_0, \ldots, U_n$, and on the intersection $U_i \cap U_j$ we have $x_k^{(j)} = t_k t_i / t_j t_i = x_k^{(i)} (x_j^{(i)})^{-1}$.

Here the important thing is that the coordinate functions $x_k^{(j)}$ on the chart $U_j$ can be expressed as Laurent monomials in the coordinates $x^{(i)}$ on $U_i$. We recall that a *Laurent monomial* in variables $x_1, \ldots, x_n$ is a product $x_1^{m_1} \ldots x_n^{m_n}$, or briefly $x^m$, where the exponent $m = (m_1, \ldots, m_n) \in \mathbf{Z}^n$. Included as the basis of the definition of toric varieties is the requirement that on changing from one chart to another the coordinate transformation is monomial. A smooth $n$-dimensional *toric variety* is an algebraic variety $X$, together with a collection of charts $x^{(\alpha)}: U_\alpha \xrightarrow{\sim} K^n$, such that on the intersections of $U_\alpha$ with $U_\beta$ the coordinates $x^{(\alpha)}$ must be Laurent monomials in the $x^{(\beta)}$.

Let $X$ be a toric variety. We fix one chart $U_0$ with coordinates $x_1, \ldots, x_n$; then the coordinate functions $x^{(\alpha)}$ on the remaining $U_\alpha$ (and monomials in them) can be represented as Laurent monomials in $x_1, \ldots, x_n$. Furthermore, if $f: U_\alpha \to K$ is a "regular" function on $U_\alpha$, that is, a polynomial in the variables $x_1^{(\alpha)}, \ldots, x_n^{(\alpha)}$, then $f$ can be represented as a *Laurent polynomial* in $x_1, \ldots, x_n$, that is, as a finite linear combination of Laurent monomials. The monomial character of the coordinate transformation is reflected in the fact



that the regularity condition for a function $f$ on the chart $U_\alpha$ can be expressed in terms of the support of the corresponding Laurent polynomial $\widetilde{f}$. We recall that the *support* of a Laurent polynomial $\widetilde{f} = \sum_{m \in \mathbf{Z}^n} c_m x^m$ is the set $\operatorname{supp}(\widetilde{f}) = \{m \in \mathbf{Z}^n \mid c_m \neq 0\}$. With each chart $U_\alpha$ let us associate the cone $\sigma_\alpha$ in $\mathbf{R}^n$ generated by the exponents of $x_1^{(\alpha)}, \ldots, x_n^{(\alpha)}$ as Laurent polynomials in $x_1, \ldots, x_n$. We regard an arbitrary Laurent polynomial $\widetilde{f}$ as a rational function on $X$; as one sees easily, regularity of this function on the chart $U_\alpha$ is equivalent to $\operatorname{supp}(\widetilde{f}) \subset \sigma_\alpha$. Thus, various questions on the behaviour of the rational function $\widetilde{f}$ on the toric variety $X$ reduce to the combinatorics of the positioning of $\operatorname{supp}(\widetilde{f})$ with respect to the system of cones $\{\sigma_\alpha\}$. The systematic realization of this remark is the essence of toric geometry.

**0.3.** As we have just said, a toric variety $X$ with a collection of charts $U_\alpha$ determines a system of cones $\{\sigma_\alpha\}$ in $\mathbf{R}^n$. The three diagrams below represent the systems of cones for the projective plane $\mathbf{P}^2$, the quadric $\mathbf{P}^1 \times \mathbf{P}^1$, and for the variety obtained by blowing up the origin in the affine plane $\mathbf{A}^2$ (Fig.1).

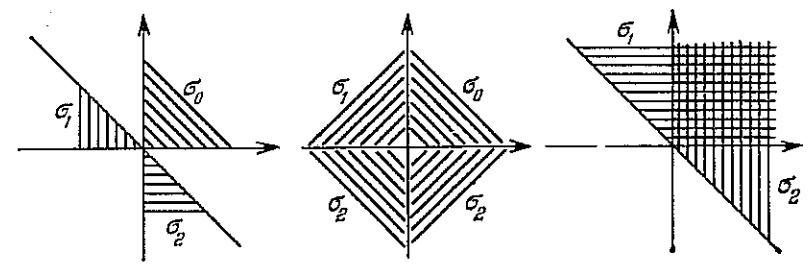


Fig. 1.

Conversely, we can construct a toric variety by specifying some system $\{\sigma_\alpha\}$ of cones satisfying certain requirements. These requirements can incidentally be most conveniently stated in terms of the system of dual cones $\check{\sigma}_\alpha$ (see the notion of *fan* in §5). Passing to the dual notions is convenient in that it re-establishes the covariant character of the operations carried out in gluing together a variety $X$ out of affine pieces $U_\alpha$.

The notion of a fan and the associated smooth toric variety was introduced by Demazure [16] in studying the action of an algebraic group on rational varieties. He also described the invertible sheaves on toric varieties and a method of computing their cohomology; these results are given in §§6 and 7. We supplement these by a description of the fundamental group (§9), the cohomology ring (for a complex toric variety, §12) and the closely related ring of algebraic cycles (§10).

**0.4.** While restricting himself to the smooth case, Demazure posed the problem of generalizing the theory of toric varieties to varieties with singularities. The foundations of this theory were laid in [26].

General toric varieties are again covered by affine charts $U_\alpha$ with monomials going into monomials on passing from one chart to another. For this one must, first of all, have a "monomial structure" on each affine chart $U_\alpha$; let us explain what this means, restricting ourselves to a single affine piece $U$. Among the regular functions on an algebraic variety $U$ a certain subset $S$ of "monomials" is singled out. Since a product of monomials is again to be a monomial, we demand that $S$ is a semigroup under multiplication. Finally, we require that the set $S$ of "monomials" forms a basis of the space of regular functions on $U$. So we arrive at the fact that the ring $K[U]$ of regular functions on $U$ is the *semigroup algebra* $K[S]$ of a semigroup $S$ with coefficients in $K$. The variety $U$ can be recovered as the *spectrum* of this $K$-algebra $U = \operatorname{Spec} K[S]$.

Lest we stray too far from the situation considered in **0.2** we suppose that $S$ is of the form $\sigma \cap \mathbf{Z}^n$, where $\sigma$ is a convex cone in $\mathbf{R}^n$. For $S = \sigma \cap \mathbf{Z}^n$ to be finitely generated, $\sigma$ must be polyhedral and rational. If $\sigma$ is generated by some basis of the lattice $\mathbf{Z}^n = \mathbf{R}^n$, then $U = \operatorname{Spec} K[\sigma \cap \mathbf{Z}^n]$ is isomorphic to $\mathbf{A}^n$. In general, $U$ has singularities. For example, let $\sigma$ be the 2-dimensional cone shown in Fig.2. If $x$, $y$, and $z$ are the "monomials" corresponding to the integral points (1,0), (1, 1), and (1, 2), then they generate the whole of $S$, and there is the single relation $y^2 = xz$ between them. Therefore, the corresponding variety $U$ is the quadratic cone in $\mathbf{A}^3$ with the equation $y^2 = xz$.

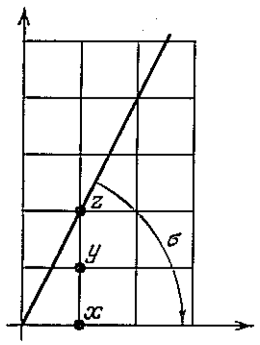

Fig. 2.

**0.5.** The properties of the semigroup rings $A_\sigma = K[\sigma \cap \mathbf{Z}^n]$ and of the corresponding affine algebraic varieties are considered in Chapter I. This chapter is completely elementary (with the exception of §3, where we prove the theorem of Hochster that rings of the form $A_\sigma$ are Cohen–Macaulay rings), and is basically concerned with commutative algebra.

**0.6.** General toric varieties are glued together from the affine varieties $U_\alpha$, as explained in **0.2**. In Chapters II and III we consider various global objects connected with toric varieties, and interpret these in terms of the corresponding fan. Unfortunately, lack of space prevents us from saying anything about the properties of subvarieties of toric varieties, which are undoubtedly more interesting objects than the frigid toric crystals.



There is a more invariant definition of a toric variety, which explains the name. A toric variety is characterized by the fact that it contains an $n$-dimensional torus $\mathbf{T}$ as an open subvariety, and the action of $\mathbf{T}$ on itself by translations extends to an action on the whole variety (see **2.7** and **5.7**). An extension of this theory would seem possible, in which the torus $\mathbf{T}$ is replaced by an arbitrary reductive group $G$; [27] and [33] give some hope in this direction.

**0.7.** Chapter IV is devoted to toroidal varieties, that is, to varieties that are locally toric in structure; this can be read immediately after Chapter I. A non-trivial example of a situation where toroidal singularities appear is the *semistable reduction theorem.* This is concerned with simplifying a singular fibre of a morphism $f: X \to \mathbf{C}$ of complex varieties by means of blow-ups of $X$ and cyclic covers of $\mathbf{C}$. The latter operation leads to the appearance of singularities of the form $z^b = x_1^{a_1} \dots x_n^{a_n}$, which are toroidal. Using a rather delicate combinatorial argument based on toric and toroidal technique, which was developed especially for this purpose, Mumford has proved in [26] the existence of a semistable reduction.

However, an idea of Steenbrink's seems even more tempting, namely, to regard singularities of the above type as being in no way "singular", and not to waste our effort in desingularizing them. Instead we only have to carry over to such singularities the local apparatus of smooth varieties, and in the first instance the notion of a differential form. It turns out that the right definition consists in taking as a differential form on a variety one that is defined on the set of smooth points (see **4.1**). Of course, this idea is nothing new and makes sense for any variety; that it is reasonable in the case of toroidal varieties (over $\mathbf{C}$) is shown by the fact that for such differential forms the Poincaré lemma continues to hold: the analytic de Rham complex

$$0 \to O_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \to \dots$$

is a resolution of the constant sheaf $\mathbf{C}_X$ over $X$ (see **13.4**). This builds a bridge between topology and algebra: the cohomology of the topological space $X(\mathbf{C})$ can be expressed in terms of the cohomology of the coherent sheaves of differential forms $\Omega_X^p$.

The proof of this lemma is based on the following simple and attractive description of the module of $p$-differentials $\Omega^p$ in the toric case (§4). The ring $A_\sigma = \mathbf{C}[\sigma \cap \mathbf{Z}^n]$ has an obvious $\mathbf{Z}^n$-grading; being canonical, the module $\Omega^p$ is also a $\mathbf{Z}^n$-graded $A_\sigma$-module $\Omega^p = \bigoplus_{m \in \mathbf{Z}^n} \Omega^p(m)$. Then the space $\Omega^p(m)$ depends only on the smallest face $\Gamma(m)$ of $\sigma$ containing $m$. More precisely, $\Omega^1(m)$ is the subspace of $\mathbf{C}^n = \mathbf{R}^n \otimes \mathbf{C}$ spanned by the face $\Gamma(m)$, and $\Omega^p(m) = \Lambda^p(\Omega^1(m))$ is the $p$th exterior power.

This interpretation reduces many assertions on the modules of differentials to facts on the exterior algebra of a vector space. We extend to the toroidal case the notions of a form with logarithmic poles and its Poincaré residue, which are useful in the study of the cohomology of "open" algebraic varieties.

**0.8.** The theory of toric varieties reveals the existence of a close connection between algebraic geometry and linear Diophantine geometry (integral linear programming), which is concerned with the study of integral points in polyhedra. Thus, the number of integral points in a polyhedron is given by the Riemann–Roch formula (see §11). This connection was clearly realized in [3]. We mention the following articles on linear Diophantine geometry: [2], [19], [29].

**0.9.** Since the appearance of Mumford's book [26] many papers on toric geometry have appeared; we mention only [1], [3], [9], [18], [25], [28], [34]. Apart from the articles of Demazure [16], Mumford [26], and Steenbrink [31] already mentioned, the author has been greatly influenced by discussions with I. V. Dolgachov, A. G. Kushnirenko, and A. G. Khovanskii.

**0.10.** In this paper we keep to the following notation:

$K$ is the ground field,

$M$ and $N$ are lattices dual to one another,

$\sigma$ and $\tau$ are cones; $\langle v_1, \ldots, v_k \rangle$ is the cone generated by the vectors $v_1, \ldots, v_k$;

$\check{\sigma}$ is the cone dual to $\sigma$,

$A_\sigma = K[\sigma \cap M]$ is the semigroup algebra of $\sigma \cap M$,

$X_\sigma = \operatorname{Spec} A_\sigma$ is an affine toric variety,

$\mathbf{T} = \operatorname{Spec} K[M]$ is an algebraic torus,

$\Sigma$ is a fan in the vector space $N_{\mathbf{Q}}$,

$\Sigma^{(k)}$ is the set of $k$-dimensional cones of $\Sigma$,

$X_\Sigma$ is the toric variety associated with $\Sigma$,

$\Omega^p$ is the module (or the sheaf) of $p$-differentials,

$\Delta$ is a polyhedron in the vector space $M_{\mathbf{Q}}$,

$L(\Delta)$ is the space of Laurent polynomials with support in $\Delta$,

$l(\Delta) = \dim L(\Delta)$ is the number of integral points of $\Delta$.

# CHAPTER I

# AFFINE TORIC VARIETIES

In this chapter we study affine toric varieties associated with a cone $\sigma$ in a lattice $M$.

## §1. Cones, lattices, and semigroups

**1.1 Cones.** Let $V$ be a finite-dimensional vector space over the field $\mathbf{Q}$ of rational numbers. A subset of $V$ of the form $\lambda^{-1}(\mathbf{Q}_+)$, where $\lambda\colon V \to \mathbf{Q}$ is a non-zero linear functional and $\mathbf{Q}_+ = \{r \in \mathbf{Q} \mid r \geqslant 0\}$, is called a *half-space* of $V$. A *cone* of $V$ is an intersection of a finite number of half-spaces; a cone is always convex, polyhedral, and rational. For cones $\sigma$ and $\tau$ we denote by $\sigma \pm \tau$ the cones $\{v \pm v' \mid v \in \sigma, v' \in \tau\}$, respectively. Thus, $\sigma - \sigma$ is the smallest subspace of $V$ containing $\sigma$; its dimension is called *the dimension of* $\sigma$ and is denoted by $\dim \sigma$.



A subset of $\sigma$ of the form $\sigma \cap \lambda^{-1}(0)$, where $\lambda\colon V \to \mathbf{Q}$ is a linear functional that is positive on $\sigma$, is called a *face* of $\sigma$. A face of a cone is again a cone. The intersection of a number of faces is again a face. For $v \in \sigma$ we denote by $\Gamma_\sigma(v)$, or simply $\Gamma(v)$ the smallest face of $\sigma$ containing $v$. $\Gamma(0)$ is the greatest subspace of $V$ contained in $\sigma$, and is called the *cospan* of $\sigma$. If $\Gamma(0) = \{0\}$, we say that $\sigma$ *has a vertex*.

For $v_1, \ldots, v_k \in V$ let $\langle v_1, \ldots, v_k \rangle$ denote the smallest cone containing $v_1, \ldots, v_k$. Any cone is of this form. A cone is said to be *simplicial* if it is of the form $\langle v_1, \ldots, v_k \rangle$ with linearly independent $v_1, \ldots, v_k$.

**1.2. Lattices.** By a lattice we mean a free Abelian group of finite rank (which we call the dimension of the lattice). For a lattice $M$ the lattice $N = \operatorname{Hom}(M, \mathbf{Z})$ is called the *dual* of $M$. By a cone in $M$ we mean a cone in the vector space $M_{\mathbf{Q}} = M \otimes \mathbf{Q}$. If $\sigma$ is a cone in $M$, then $\sigma \cap M$ is a commutative subsemigroup in $M$.

**1.3.** LEMMA (Gordan). *The semigroup $\sigma \cap M$ is finitely generated.*

PROOF. Breaking $\sigma$ up into simplicial cones, we may assume that $\sigma$ is simplicial. Let $\sigma = \langle m_1, \ldots, m_k \rangle$, where $m_1, \ldots, m_k$ belong to $M$ and are linearly independent. We form the parallelotope

$$P = \{\sum_{i=1}^{k} \alpha_i m_i \mid 0 \leqslant \alpha_i < 1\}.$$

Obviously, any point of $\sigma \cap M$ can be uniquely represented as $p + \sum_{i=1}^{k} n_i m_i$, where $p \in P \cap M$, and the $n_i \geqslant 0$ are integers. In particular, the finite set $(P \cap M) \cup \{m_1, \ldots, m_k\}$ generates $\sigma \cap M$.

**1.4. Polyhedra.** We define a polyhedron in $M_{\mathbf{Q}}$ as the intersection of finitely many *affine* half-spaces. Thus, a polyhedron is always convex; furthermore, in what follows we only consider *bounded* polyhedra. Just as for cones, polyhedra can be added and subtracted, and can be multiplied by rational numbers. A polyhedron is said to be *integral* (relative to $M$) if its vertices belong to $M$.

We define $l(\Delta)$ as the number of integral points in a polyhedron $\Delta$, that is, $l(\Delta) = \#(\Delta \cap M)$. The connection between the integers $l(\Delta)$, $l(2\Delta)$, and so on is buried in the Poincaré series

$$P_\Delta(t) = \sum_{k \geqslant 0} l(k\Delta) \cdot t^k.$$

The arguments in the proof of Lemma 1.3 show that *$P_\Delta(t)$ is a rational function of $t$*. Restricting ourselves to integral polyhedra, we state a more precise assertion which will be proved in §3.

**1.5.** LEMMA. *Let $\Delta$ be an integral polyhedron of dimension $d$. Then*

$$P_\Delta(t) = \Phi_\Delta(t) \cdot (1 - t)^{-d-1},$$

*where $\Phi_\Delta(t)$ is a polynomial of degree $\leqslant d + 1$ with non-negative integral coefficients.*

The polynomial $\Phi_\Delta(t)$ is a very important characteristic of the polyhedron $\Delta$.

Thus, $\Phi_\Delta(1) = d!\, V_d(\Delta)$, where $V_d(\Delta)$ is the $d$-dimensional volume of $\Delta$ (relative to the induced lattice). We note that the Hodge numbers of hyperplane sections of the toric varieties $\mathbf{P}_\Delta$ can be expressed in terms of $\Phi_\Delta(t)$.

## §2. The definition of an affine toric variety

**2.1.** Let us fix some field $K$. Let $M$ be a lattice, and $\sigma$ a cone in $M$. Let $A_{(\sigma,M)}$, or $A_\sigma$, denote the semigroup $K$-algebra $K[\sigma \cap M]$ of $\sigma \cap M$. It consists of all expressions $\sum_{m \in \sigma \cap M} a_m x^m$, with $a_m \in K$ and almost all $a_m = 0$. Two such expressions are added and multiplied in the usual way; for example, $x^m \cdot x^{m'} = x^{m+m'}$.

The $K$-algebra $A_\sigma$ has a natural grading of type $M$. According to Lemma 1.3, it is finitely generated.

**2.2.** DEFINITION. The affine scheme Spec $K[\sigma \cap M]$ is called an *affine toric variety*; it is denoted by $X_{(\sigma,M)}$, or $X_\sigma$, or simply $X$.

The reader who is unfamiliar with the notion of the spectrum of a ring can think of $X_\sigma$ naively as a set of points (see 2.3).

**2.3. Points.** Let $L$ be a commutative $K$-algebra; by an $L$-valued point of $X_\sigma$ we mean a morphism of $K$-schemes Spec $L \to X_\sigma$, that is, a homomorphism of $K$-algebras $K[\sigma \cap M] \to L$. The latter is given by a homomorphism of semigroups $x\colon \sigma \cap M \to L$, where $L$ is regarded as a multiplicative semigroup. For each $m \in \sigma \cap M$ the number $x(m) \in L$ should be understood as a coordinate of $x$.

**2.4.** EXAMPLE. Let $\sigma$ be the whole vector space $M_{\mathbf{Q}}$; then $X_\sigma = \operatorname{Spec} K[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ is a *torus* of dimension $n = \dim M$.

If $\sigma$ is the positive orthant in $\mathbf{Z}^n$, then $X_\sigma = \operatorname{Spec} K[X_1, \ldots, X_n]$ is the $n$-dimensional *affine space* over $K$.

**2.5.** With each face $\tau$ of $\sigma$ we associate a closed subvariety of $X_\sigma$ analogous to a coordinate subspace in the affine space $\mathbf{A}^n$. Let $\chi$ be the characteristic function of the face $\tau$, that is, the function that is 1 on $\tau$ and 0 outside $\tau$. The map $x^m \mapsto \chi(m)\cdot x^m$ (for $m \in \sigma \cap M$) extends to a surjective homomorphism of $M$-graded $K$-algebras $K[\sigma \cap M] \to K[\tau \cap M]$, which defines a closed embedding of affine varieties

$$i\colon X_\tau \to X_\sigma.$$

This embedding is easily described in terms of points. Let $x\colon \tau \cap M \to L$ be an $L$-valued point of $X_\tau$; then $i(x)$ is the extension by 0 of the homomorphism $x$ from $\tau \cap M$ to $\sigma \cap M$.

**2.6. Functoriality.** The process of associating the toric variety $X_{(\sigma,M)}$ with the pair $(M, \sigma)$, where $\sigma$ is a cone in $M$, is a contravariant functor. For let $f\colon (M, \sigma) \to (M', \sigma')$ be a morphism of such pairs, that is, a homomorphism of lattices $f\colon M \to M'$ for which $f_{\mathbf{Q}}(\sigma) \subset \sigma'$. Then the semigroup homomorphism $f\colon \sigma \cap M \to \sigma' \cap M'$ gives a homomorphism of $K$-algebras $K[\sigma \cap M] \to K[\sigma' \cap M']$ and a morphism of $K$-schemes



$${}^a f\colon X_{(\sigma',M')} \to X_{(\sigma,M)}$$

On the level of points: if $x'\colon \sigma' \cap M' \to L$ is a point of $X_{(\sigma',M')}$, then ${}^a f(x')$ is the composite of $f\colon \sigma \cap M \to \sigma' \cap M'$ with $x'$.

Let us consider some particular cases.

**2.6.1.** Suppose that the lattices $M$ and $M'$ are the same. Then an inclusion $\sigma \subset \sigma'$ of cones leads to a morphism of schemes $f_{\sigma',\sigma}\colon X_{\sigma'} \to X_\sigma$. Especially important is the case when $\sigma' = \sigma - \langle m \rangle$, where $m \in \sigma \cap M$. In this case the ring $A_{\sigma'}$ can be identified with the localization of $A_\sigma$ with respect to $x^m$, $A_{\sigma'} = A_\sigma[x^{-m}]$, and the morphism $f_{\sigma',\sigma}\colon X_{\sigma'} \to X_\sigma$ is the *open immersion* of $X_{\sigma'}$ onto the complement of $\bigcap_{\tau \ni m} i(X_\tau)$ in $X_\sigma$. The converse is also true: *if $f_{\sigma',\sigma}$ is an immersion, then $\sigma'$ is of the form $\sigma - \langle m \rangle$, where $m \in \sigma$.* The easiest way to check this is by using points (see **2.3**).

This last fact, and a number of others, can most conveniently be stated in terms of the dual cone. When $\sigma$ is a cone in $M$, we write

$$\check{\sigma} = \{\lambda \in N_{\mathbf{Q}} \mid \lambda(\sigma) \geqslant 0\}$$

to denote the *dual cone* in the dual vector space $N_{\mathbf{Q}}$. The condition $\sigma \subset \sigma'$ is equivalent to $\check{\sigma}' \subset \check{\sigma}$, and *the morphism ${}^a f\colon X_{\sigma'} \to X_\sigma$ is an open immersion if and only if $\check{\sigma}'$ is a face of $\check{\sigma}$.*

**2.6.2.** Let $M \subset M'$ be lattices of the same dimension, and let $\sigma' = \sigma$. In the spirit of the proof of Lemma 1.3 we can verify easily that the morphism of schemes ${}^a f\colon X_{(\sigma,M')} \to X_{(\sigma,M)}$ is *finite* and *surjective*. If $K$ is an algebraically closed field of characteristic prime to $[M'\colon M]$, then ${}^a f\colon X' \to X$ is a Galois cover (ramified, in general) with Galois group $\operatorname{Hom}(M'/M, K^*)$.

**2.6.3.** The variety $X_{(\sigma \times \sigma', M \times M')}$ is the direct product of $X_{(\sigma,M)}$ and $X_{(\sigma',M')}$.

**2.7.** Suppose that $M_{\mathbf{Q}}$ is generated by the cone $\sigma$. Applying **2.6.1** to the cone $\sigma' = \sigma - \sigma = M_{\mathbf{Q}}$ we obtain an open immersion of the "big" torus $\mathbf{T} = \operatorname{Spec} K[M]$ in $X_\sigma$. It is easy to check that the action of $\mathbf{T}$ on itself by translations extends to an action of $\mathbf{T}$ on the whole of $X_\sigma$; again it is simplest to see this by using points. Algebraically, the action of $\mathbf{T}$ is reflected in the presence of an $M$-grading of the affine ring $A_\sigma$. The orbits of the action of $\mathbf{T}$ on $X_\sigma$ are tori $\mathbf{T}_\tau$ belonging to the closed subschemes $X_\tau \subset X_\sigma$, as $\tau$ ranges over the faces of $\sigma$.

The converse is also true (see [26]): if $\mathbf{T} \subset X$ is an open immersion of a torus $\mathbf{T}$ in a normal affine variety $X$ and the action of $\mathbf{T}$ on itself by translations extends to an action on $X$, then $X$ is of the form $X_{(\sigma,M)}$ where $M$ is the lattice of characters of $\mathbf{T}$.

**2.8.** REMARK. If we do not insist on the "rationality" of the cone $\sigma$, then we obtain rings $K[\sigma \cap M]$ that are no longer Noetherian, but present some interest (see [18]).

## §3. Properties of toric varieties

Let $M$ be an $n$-dimensional lattice, and $\sigma \subset M$ an $n$-dimensional cone. We consider properties of the variety $X_\sigma$.

**3.1. Dimension.** The ring $A = K[\sigma \cap M]$ has no zero-divisors, hence the embedding of the big torus $\mathbf{T} \hookrightarrow X_\sigma$ is dense and $\dim X_\sigma = \dim \mathbf{T} = n$. Similarly, $\dim X_\tau = \dim \tau$ for any face $\tau$ of $\sigma$. From this it is also clear that $X_\sigma$ is a *rational* variety.

**3.2.** PROPOSITION. *$X_\sigma$ is normal.*

PROOF. Let us show that $A = A_\sigma$ is integrally closed. Let $\tau_1, \ldots, \tau_k$ be the faces of $\sigma$ of codimension 1, and $\sigma_i = \sigma - \tau_i$. Obviously, $\sigma = \cap\, \sigma_i$, hence, $A = \cap\, A_i$, where $A_i = K[\sigma_i \cap M]$. Therefore, it is sufficient to show that each $A_i$ is integrally closed. But $\sigma_i$ is a half-space, so that $A_i \cong K[X_1, X_2, X_2^{-1}, \ldots, X_n, X_n^{-1}]$.

**3.3. Non-singularity.** Let us now see under what conditions $X$ is a smooth variety, that is, has no singularities. It is enough to do this for a cone $\sigma$ having a vertex, since the general case differs only by taking the product with a torus, which does not affect smoothness. The answer is: *for a cone $\sigma$ with vertex, the variety $X_\sigma$ is smooth if and only if $\sigma$ is generated by a basis of the lattice $M$.*

The "if" part is obvious. For the converse, suppose that $A = A_\sigma$ is a regular ring, and let $\mathfrak{m} = \bigoplus_{m \neq 0} K \cdot x^m$ be the maximal ideal of $A$. Since the local ring $A_\mathfrak{m}$ is regular, the ideal $\mathfrak{m}A_\mathfrak{m}$ can be generated by $n$ elements. We may assume that these are of the form $x^{m_1}, \ldots, x^{m_n}$, $m_i \in M$. But then any element of $\sigma \cap M$ can be expressed as a non-negative integral combination of $m_1, \ldots, m_n$. It follows that $m_1, \ldots, m_n$ generate $M$ and $\sigma = \langle m_1, \ldots, m_n \rangle$.

This criterion looks even nicer in its dual form: for any cone $\sigma$ *the variety $X_\sigma$ is smooth if and only if the dual cone $\check{\sigma}$ is generated by part of a basis of* $N$. Quite generally, we say that a cone generated by part of a basis of a lattice is *basic* for the lattice.

**3.4.** THEOREM (Hochster [23]). *$A = K[\sigma \cap M]$ is a Cohen–Macaulay ring* (see Appendix 1).

Our proof is a combination of the arguments of Hochster himself and an idea due to Kushnirenko. First of all, we may assume that $\sigma$ has a vertex, and prove that $A$ is of depth $n$ at the vertex. By induction we may assume that for cones $\tau$ with $\dim \tau < n$ the theorem holds.

Let $\mathfrak{A}$ denote the ideal of $A$ generated by all monomials $x^m$, with $m$ strictly inside $\sigma$.

**3.4.1.** LEMMA. *The $A$-module $A/\mathfrak{A}$ is of depth $n-1$.*

PROOF. Let $\partial\sigma$ denote the boundary of $\sigma$; then $A/\mathfrak{A} = \bigoplus_{m \in \partial\sigma \cap M} K \cdot x^m$.

Using the fact that $\partial\sigma$ can be covered by faces of $\sigma$, we form an $M$-graded resolution of $A/\mathfrak{A}$:

$$0 \to A/\mathfrak{A} \to C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \to \ldots \xrightarrow{d_1} C_0 \to 0.$$



Here $C_k = \bigoplus_\tau A_\tau$, where $\tau$ ranges over the $k$-dimensional faces of $\sigma$; the differential $d$ is defined in a combinatorial manner (for details see [28], 2.11, or §12 below). It is enough to check the exactness of this sequence "over each $m \in \partial\sigma \cap M$", and this follows from the fact that $\partial\sigma$ is a homology manifold at $m$.

Now let us prove by induction that prof(Ker $d_{k-1}$) $= k$ for each $k$. For $k = n-1$ we then get the lemma. We consider the short exact sequence

$$0 \to \operatorname{Ker} d_k \to C_k \to \operatorname{Ker} d_{k-1} \to 0.$$

By the inductive hypothesis, prof(Ker $d_{k-1}$) $= k-1$. Furthermore, since $\dim \tau = k < n$, we have prof $A_\tau$ = prof $C_k = k$. It follows (see Appendix 1) that prof(Ker $d_k$) $= k$. The lemma is proved.

If $\mathfrak{A}$ was a principal ideal, it would follow from Lemma 3.4.1 that prof $A = n$. Let us show how to reduce the general case to this.

**3.4.2.** LEMMA. *Let $M \subset \overline{M}$ be lattices of the same dimension. If $\overline{A} = K[\sigma \cap \overline{M}]$ is a Cohen–Macaulay ring, then so is $A = K[\sigma \cap M]$.*

PROOF. $\overline{A}$ is a finite $A$-algebra, (see 2.6.2), and its depth as an $A$-module is $n$. Thus, it is enough to show that $A$ is a direct summand of the $A$-module $\overline{A}$.

Let $\chi$ be the characteristic function of $M$. The map $x^{\overline{m}} \mapsto \chi(\overline{m})x^m$ extends to an $A$-linear homomorphism $\rho\colon \overline{A} \to \overline{A}$, which is a projection onto $A \subset \overline{A}$. This proves Lemma 3.4.2.

**3.4.3.** It remains to show how to find for our lattice $M$ a superlattice $\overline{M} \supset M$ such that the corresponding ideal $\overline{\mathfrak{A}}$ in $\overline{A} = K[\sigma \cap \overline{M}]$ is principal. Then $\overline{A}$, and therefore also $A$, is a Cohen–Macaulay ring.

For this purpose let us choose a basis $e_1, \ldots, e_n$ of $M$ such that $e_n$ lies strictly inside $\sigma$. Suppose that the faces of $\sigma$ of codimension 1 have equations of the form $x_n = \sum_{i=1}^{n-1} r_{ij} x_i$, with $j$ an index for the faces. The $r_{ij}$ are rational, so that we can find an integer $d > 0$ for which all the $dr_{ij}$ are integers. It remains to take $\overline{M}$ to be the lattice generated by the vectors $e_1, \ldots, e_{n-1}$, $\overline{e}_n = \frac{1}{d} e_n$. It is easy to check that if $\overline{m} \in \overline{M}$ lies strictly inside $\sigma$, then it is of the form $\overline{e}_n$ + an element of $\sigma \cap \overline{M}$. In other words, $\overline{\mathfrak{A}} = x^{\overline{e}_n} \cdot \overline{A}$. This proves Theorem 3.4.

**3.5.** COROLLARY. *The ideal $\mathfrak{A}$ is also of depth $n$.*

In fact, we only have to apply the corollary in Appendix 1 to the short exact sequence $0 \to \mathfrak{A} \to A \to A/\mathfrak{A} \to 0$.

**3.6.** REMARK. In the following §4 we shall see that $\mathfrak{A}$ is isomorphic to the *canonical module* of $A$, so that if $\mathfrak{A}$ can be generated by one element, then $A$ is a *Gorenstein* ring. The rings $K[\sigma \cap M]$ give examples of both Cohen–Macaulay rings that are not Gorenstein rings, and Gorenstein rings that are not complete intersections. This final condition can be checked using the local Picard group.

**3.7.** PROOF OF LEMMA 1.5. We recall that in Lemma 1.5 we were concerned with the Poincaré series $P_\Delta(t) = \sum_{k \geq 0} l(k\Delta)t^k$. We form the auxiliary lattice $M' = M \oplus \mathbf{Z}$ and consider in $M'_{\mathbf{Q}}$ the cone $\sigma$ given by

$$\sigma = \{(m, r) \in M_{\mathbf{Q}} \oplus \mathbf{Q} \mid m \in r\Delta\}.$$

If $A = \mathbf{C}[\sigma \cap M']$ is $\mathbf{Z}$-graded by means of the projection $M \oplus \mathbf{Z} \to \mathbf{Z}$, then $P_\Delta(t) = P_A(t)$ is the Poincaré series of $A$. Suppose that we have managed to find homogeneous elements $a_0, \ldots, a_d \in A$ of degree 1 that form a regular sequence (see Appendix 1). Then $P_\Delta(t)\cdot(1-t)^{d+1}$ is the Poincaré series of the finite-dimensional ring $A/(a_0, \ldots, a_d)$, and it follows that it is a polynomial with non-negative integral coefficients. The assertion about its degree can be obtained from the proof of Lemma 1.3.

Since $A$ is a Cohen-Macaulay ring, it is enough to find elements $a_0, \ldots, a_d$ of degree 1 such that the quotient ring $A/(a_0, \ldots, a_d)$ is finite-dimensional (see Appendix 1). We claim that for $a_0, \ldots, a_d$ we can take "general" elements of degree 1. To prove this we consider the subring $A' \subset A$ generated by the elements of degree 1. Then $A$ is finite over $A'$, as follows from the arguments in the proof of Lemma 1.3 and the fact that $\dim A' = d + 1$. It is clear that $d + 1$ general elements of degree 1 in $A'$ generate an ideal of finite codimension (since the intersection of the variety $\operatorname{Spec} A' \subset \mathbf{A}^N$ with a general linear subspace of codimension $d + 1$ is zero-dimensional). Thus, the ideal $(a_0, \ldots, a_d)\cdot A \subset A$ is also of finite codimension.

3.8. REMARK. Rings similar to $A/\mathfrak{A}$ in the proof of Theorem 3.4 and "composed of" toric rings $A_\sigma$ are often useful. For example, they appear in [28], §2 in the study of Newton filtrations. Here is another example. (See Stanley [35]).

Suppose that we are given a triangulation of the sphere $S^n$ with the set of vertices $S$. With each simplex $\sigma = \{s_0, \ldots, s_k\}$ of this triangulation we associate the polynomial ring $A_\sigma = \mathbf{C}[U_{s_0}, \ldots, U_{s_k}]$. If $\sigma'$ is a face of $\sigma$, then $A_\sigma$ has a natural projection onto $A_{\sigma'}$. Let $A$ be the projective limit of the system $\{A_\sigma\}$, as $\sigma$ ranges over all simplices of the triangulation, including the empty simplex. In other words, $A$ is the universal ring with homomorphisms $A \to A_\sigma$. Now $A$ can be described more explicitly as the quotient ring $\mathbf{C}[U_s, s \in S]/I$ of the polynomial ring in the variables $U_s$, $s \in S$, by the ideal $I$ generated by the monomials $U_{s_0} \ldots U_{s_k}$ for which $\{s_0, \ldots, s_k\}$ is *not* a simplex of the triangulation.

For $A$ we have a resolution analogous to that considered in Lemma 3.4.1,

$$0 \to A \to C_n \to C_{n-1} \to \ldots \to C_{-1} \to 0,$$

where $C_k$ is the direct sum of the rings $A_\sigma$ over $k$-dimensional simplices $\sigma$. The exactness is checked as in Lemma 3.4.1. From this we obtain two corollaries:

a) if we equip $A$ with the natural $\mathbf{Z}$-grading, then the Poincaré series of $A$ is given by the expression

$$P_A(t) = \frac{a_n}{(1-t)^{n+1}} - \frac{a_{n-1}}{(1-t)^n} + \ldots + (-1)^{n+1},$$



where $a_k$ denotes the number of $k$-dimensional simplices in our triangulation of $S^n$;

b) $A$ is a Cohen–Macaulay ring. If we choose a regular sequence $\bar{a}$ as in 3.7, consisting of elements of degree 1, we find that

$$P_A(t)\cdot(1-t)^{n+1} = a_n + a_{n-1}(t-1) + \ldots + (t-1)^{n+1}$$

has *non-negative* coefficients, as the Poincaré polynomial of the ring $A/\bar{a}$. Later we interpret $A/\bar{a}$ as the cohomology ring of a certain smooth variety, and from this, using Poincaré duality we deduce that $P_{A/\bar{a}}(t)$ is *reflexive.*

## §4. Differential forms on toric varieties

Before reading this section the reader may find it useful to peruse Appendices 2 and 3.

**4.1.** DEFINITION. Let $X$ be a normal variety, $U = X - \operatorname{Sing} X$, and $j\colon U \to X$ the natural embedding; we define the *sheaf of differential p-forms,* or of *p-differentials* (in the sense of Zariski–Steenbrink) to be $\Omega^p_X = j_*(\Omega^p_X)$.

In other words, a $p$-form on $X$ is one on the variety $U$ of smooth points of $X$. The sheaves $\Omega^p_X$ are coherent. Note also that in the definition we can take for $U$ any open smooth subvariety of $X$ with $\operatorname{codim}(X - U) \geq 2$.

**4.2. The modules $\Omega^p_A$.** For the remainder of this section, $\sigma$ is a cone generating $M_{\mathbf{Q}}$, $A = K[\sigma \cap M]$, and $X = \operatorname{Spec} A$.

The sheaf $\Omega^p_X$ on the toric variety $X$ corresponds to a certain $A$-module; we will now construct this module explicitly. For this purpose we introduce a notation that will be in constant use in what follows.

Let $V$ denote the vector space $M \otimes_{\mathbf{Z}} K$ over $K$. For each face $\tau$ of $\sigma$ we define a subspace $V_\tau \subset V$. If $\tau$ is of codimension 1, then we set

$$V_\tau = (M \cap (\tau - \tau)) \otimes_{\mathbf{Z}} K. \tag{4.2.1}$$

In general, we set

$$V_\tau = \bigcap_{\theta \supset \tau} V_\theta, \tag{4.2.2}$$

where $\theta$ ranges over the faces of $\sigma$ of codimension 1 that contain $\tau$. The principal application of differentials is to varieties over a field of characteristic 0, and then $V_\tau$ for any $\tau$ is given by (4.2.1).

Now we define the $M$-graded $K$-vector space $\Omega^p_A$ as

$$\Omega^p_A = \bigoplus_{m \in \sigma \cap M} \Lambda^p(V_{\Gamma(m)})\cdot x^m. \tag{4.2.3}$$

In other words, if $\Omega^p_A(m)$ is the component of $\Omega^p_A$ of degree $m$, then

$$\Omega^p_A(m) = \Lambda^p(V_{\Gamma(m)})\cdot x^m = \Lambda^p\big(\bigcap_{\theta \ni m} V_\theta\big)\, x^m.$$

Now $\Omega^p_A$ is naturally embedded in the $A$-module $\Lambda^p(V) \otimes_K A$ and is thus equipped with the structure of an $M$-graded $A$-module.

**4.3.** PROPOSITION. *The sheaf $\Omega^p_X$ is isomorphic to the sheaf $\widetilde{\Omega}^p_A$ associated with the $A$-module $\Omega^p_A$.*

We begin the proof by constructing a sheaf morphism

$$\alpha_p\colon \widetilde{\Omega}^p_A \to \Omega^p_X,$$

which will later turn out to be an isomorphism. To do this we must indicate for an open set $U$ as in **4.1** a homomorphism of $A$-modules

$$\alpha_p\colon \Omega^p_A \to \Gamma(U, \Omega^p_U).$$

For $U$ we take the union of the open sets $U_\theta$, as $\theta$ ranges over the faces of $\sigma$ of codimension 1, and $U_\theta = X_{\sigma-\theta} = \operatorname{Spec} A_{\sigma-\theta}$. Clearly, the $U_0$ are smooth and $\operatorname{codim}(X - U) \geqslant 2$.

We consider the inclusions

$$\Omega^p_A \subset \Omega^p_{A_{\sigma-\theta}} \subset \Omega^p_{K[M]}$$

and

$$\Gamma(U, \Omega^p_U) \subset \Gamma(U_\theta, \Omega^p_U) \subset \Gamma(\mathbf{T}, \Omega^p_{\mathbf{T}}),$$

where $\mathbf{T} = \operatorname{Spec} K[M]$ is the big torus of $X$. We define the map $\alpha_p$ as the restriction of a homomorphism of $K[M]$-modules

$$\alpha_p : \Omega^p_{K[M]} \to \Gamma(\mathbf{T}, \Omega^p_{\mathbf{T}}).$$

Note that the left-hand side is $\Lambda^p(V) \underset{K}{\otimes} K[M] = \Lambda^p(M \otimes K[M])$, and the right-hand side is $\Lambda^p(\Gamma(\mathbf{T}, \Omega^1_{\mathbf{T}}))$. Thus, it suffices to specify $\alpha_1$ (and to set $\alpha_p = \Lambda^p(\alpha_1)$), indeed just on the elements of the form $m \otimes x^{m'}$, where $m, m' \in M$. We set

$$\alpha_1(m \otimes x^{m'}) = dx^m \cdot x^{m'-m}.$$

Now we have to check that $\alpha_p$ takes $\Omega^p_A$ into $\Gamma(U, \Omega^p_U)$, that is into a $p$-form on $\mathbf{T}$ that is regular on each $U_\theta$. Since $\Gamma(U, \Omega^p_U) = \bigcap_\theta \Gamma(U_\theta, \Omega^p_U)$, and since $\sigma - \theta$ is a half-space, this follows from the more precise assertion:

**4.3.1.** LEMMA. *If $\sigma$ is a half-space, then $\alpha_p$ establishes an isomorphism of $\Omega^p_{A_\sigma}$ with $\Gamma(X_\sigma, \Omega^p_X)$.*

PROOF. We choose coordinates $x_1, \ldots, x_n$ so that $A_\sigma = K[x_1, x_2, x_2^{-1}, \ldots]$. It can be checked immediately that $\Omega^p_{A_\sigma}$ is the $p$-th exterior power of $\Omega^1_{A_\sigma}$, therefore, we may assume that $p = 1$. But $\Omega^1_{A_\sigma}$ is generated by the expressions $e_1 \otimes x_1, e_2 \otimes 1, \ldots, e_n \otimes 1$, and $\Gamma(X_\sigma, \Omega^1_{X_\sigma})$ by the forms $dx_1 = \frac{dx_1}{x_1} \cdot x_1, \frac{dx_2}{x_2}, \ldots, \frac{dx_n}{x_n}$. This proves the lemma.

Now we turn to the general case. Using the lemma, to show that $\alpha_p$ is an isomorphism it is enough to establish that $\Omega^p_A = \bigcap_\theta \Omega^p_{A_{\sigma-\theta}}$, where $\theta$ ranges over the faces of $\sigma$ of codimension 1, that is, we have to check that for every $m \in M$

$$\Omega^p_A(m) = \bigcap_\theta \Omega^p_{A_{\sigma-\theta}}(m). \tag{4.3.1}$$

If $m \notin \sigma$, then both sides are 0, since $\sigma = \bigcap_\theta (\sigma - \theta)$. Suppose then that $m \in \sigma$. If $m$ does not belong to the face $\theta$, then $m$ is strictly inside $\sigma - \theta$, and hence $\Omega_{A_{\sigma-\theta}}(m) = \Lambda^p(V)x^m$. Therefore, (4.3.1) reduces to the equality (see Appendix 2)

$$\Lambda^p\Big(\bigcap_{\theta \ni m} V_\theta\Big) = \bigcap_{\theta \ni m} \Lambda^p(V_\theta).$$

The proposition is now proved.

**4.4. Exterior derivative.** We return to Definition 4.1. Applying $j_*$ to the exterior derivative $d\colon \Omega^p_U \to \Omega^{p+1}_U$, we obtain the derivative $d\colon \Omega^p_X \to \Omega^{p+1}_X$. On the level of the modules $\Omega^p_A$ this corresponds to a homogeneous $K$-linear homomorphism $d\colon \Omega^p_A \to \Omega^{p+1}_A$ (preserving the $M$-grading). When we identify $\Omega^p_A(m)$ with $\Lambda^p(V_{\Gamma(m)})$, then the derivative $d$ becomes left multiplication by $m \otimes 1 \in V_{\Gamma(m)}$.

Now $d$ is a derivation (of degree + 1) on the skew-symmetric algebra $\Omega^*_A = \bigoplus_{p \geqslant 0} \Omega^p_A$, and $d \circ d = 0$. We define the *de Rham complex* of $A$ as

$$\Omega^\bullet_A = (\Omega^0_A \xrightarrow{d} \Omega^1_A \xrightarrow{d} \ldots \to \Omega^p_A \to \ldots).$$

The identification of $K$ with $\Omega^0_A(0)$ determines an augmentation $K \to \Omega^\bullet_A$.

**4.5.** LEMMA. *Suppose that $K$ is of characteristic 0. Then the de Rham complex $\Omega^\bullet_A$ is a resolution of $K$.*

PROOF. Let $\lambda\colon m \to \mathbf{Z}$ be a linear function, and suppose that $\lambda(m) > 0$ for any non-zero $m \in \sigma \cap M$. We consider the homomorphism of $M$-graded $A$-modules

$$h\colon \Omega^{p+1}_A \to \Omega^p_A,$$

which "on the $m$th piece" is the inner product with $\lambda \in V^*$, $\llcorner\lambda\colon \Lambda^{p+1}(V_{\Gamma(m)}) \to \Lambda^p(V_{\Gamma(m)})$ (see Appendix 2). Then $d \circ h + h \circ d$ consists in multiplication by $\lambda(m)$ (see Appendix 2) and is invertible for $m \neq 0$, so that the complex $\Omega^\bullet_A(m)$ is acyclic for $m \neq 0$, and $\Omega^\bullet_A(0)$ degenerates to $\Omega^0_A(0) = K$.

**4.6. The canonical module.** Let $n = \dim \sigma$. We consider $\Omega^n_A = \omega_A$, the module of differentials of the highest order $n$. Since

$$\Omega^n_A(m) = \begin{cases} \Lambda^n(V)x^m, & \text{when } m \text{ is strictly inside } \sigma; \\ 0 & \text{otherwise,} \end{cases}$$

we see that $\Omega^n_A = \Lambda^n(V) \underset{K}{\otimes} \mathfrak{A}$, where $\mathfrak{A}$ is the ideal in **3.4**. Now $\Lambda^n(V)$ is a one-dimensional vector space, so that the module $\Omega^n_A$ is (non-canonically) isomorphic to $\mathfrak{A}$.

On the other hand, it is shown in [20] that $\Omega^n = \omega$ is a *canonical dualizing module* for $A$. In other words, for any $A$-module $F$ the pairing

$$H^i_{(0)}(F) \times \mathrm{Ext}^{n-i}_A(F, \omega) \to I$$

is perfect (see [22], 6.7), where $I = H^n_{(0)}(\omega)$ is the injective hull of the residue field of $A$ at the vertex. In particular, if $F$ is an $A$-module of depth $n$, then $H^i_{(0)}(F) = 0$ for $i < n$ (see Appendix 1), and $\mathrm{Ext}^k_A(F, \omega) = 0$ for $k > 0$. The exterior product gives a pairing of $A$-modules $\Omega^p_A \otimes \Omega^{n-p}_A \to \Omega^n_A = \omega_A$, or equivalently, a homomorphism

$$\varphi\colon \Omega^p_A \to \mathrm{Hom}_A(\Omega^{n-p}_A, \omega_A).$$

It is well known that for smooth varieties this is an isomorphism.

4.7. PROPOSITION. $\varphi\colon \Omega^p_A \to \mathrm{Hom}_A(\Omega^{n-p}_A, \omega_A)$ *is an isomorphism.*

PROOF. As was shown in 4.3, $\Omega^p_A$ is isomorphic to $\Gamma(U, \Omega^p_U)$. On the other hand, $U$ is smooth so that $\Omega^p_U \xrightarrow{\sim} \mathcal{H}om_{O_U}(\Omega^{n-p}_A, \Omega^n_U)$, and hence $\Omega^p_A \xrightarrow{\sim} \mathrm{Hom}_{O_U}(\Omega^{n-p}_U, \Omega^n_U)$. Consider the commutative diagram

$$\begin{tikzcd}
\Omega^p_A \arrow[rr, "\varphi"] \arrow[dr, "\sim"'] & & \mathrm{Hom}_A(\Omega^{n-p}_A, \Omega^n_A) \arrow[dl, "\psi"] \\
& \mathrm{Hom}_{O_U}(\Omega^{n-p}_U, \Omega^n_U) &
\end{tikzcd}$$

where $\psi$ is the restriction homomorphism from $X$ to $U$. Since $\Omega^n_A = \Gamma(U, \Omega^n_U)$, we see that $\psi$ is injective and $\varphi$ is an isomorphism.

4.8. PROPOSITION. *Suppose that $K$ is of characteristic* 0 *and that $\sigma$ is simplicial. Then* prof $\Omega^p_A = n$ *for all* $p$.

The proof is based on the same device as that of Lemma 3.4.2. Let $\overline{M}$ be a lattice containing $M$ with respect to which $\sigma$ is basic. Using the characteristic function of $M$, as in Lemma 3.4.2, we form an $A$-linear homomorphism $\rho\colon \Omega^p_{\overline{A}} \to \Omega^p_{\overline{A}}$, which is a projection onto $\Omega^p_A \subset \Omega^p_{\overline{A}}$. Hence we find that $\Omega^p_A$ is a direct summand of $\Omega^p_{\overline{A}}$, and $\mathrm{prof}_A\, \Omega^p_A \geqslant \mathrm{prof}_A\, \Omega^p_{\overline{A}} = \mathrm{prof}_{\overline{A}}\, \Omega^p_{\overline{A}}$. But $\overline{A} = K[X_1, \ldots, X_n]$ is a regular ring, and $\Omega^p_{\overline{A}}$ is a free $\overline{A}$-module, and is therefore of depth $n$.

Applying local duality (see 4.6) we get the following corollary.

4.9. COROLLARY. *Under the hypotheses of Proposition* 4.8 *we have* $\mathrm{Ext}^k_A(\Omega^p_A, \Omega^n_A) = 0$ *for all* $p \geqslant 0$ *and* $k > 0$.

## CHAPTER II

## GENERAL TORIC VARIETIES

General toric varieties are obtained by gluing together affine toric varieties; the scheme describing the gluing is given by a certain complex of cones, which we call a fan. We show how to describe in terms of a fan and a lattice the invertible sheaves on toric varieties, their cohomology, and also their unramified coverings.



### §5. Fans and their associated toric varieties

5.1. DEFINITION. A *fan* in a $\mathbf{Q}$-vector space in a collection $\Sigma$ of cones satisfying the following conditions:

a) every cone of $\Sigma$ has a vertex;

b) if $\tau$ is a face of a cone $\sigma \in \Sigma$, then $\tau \in \Sigma$;

c) if $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma'$ is a face both of $\sigma$ and of $\sigma'$.

Here are some more definitions relating to fans. The *support* of a fan $\Sigma$ is the set $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$. A fan $\Sigma'$ is *inscribed* in $\Sigma$ if for any $\sigma' \in \Sigma'$ there is a $\sigma \in \Sigma$ such that $\sigma' \subset \sigma$. If, furthermore, $|\Sigma'| = |\Sigma|$, then $\Sigma'$ is said to be a *subdivision* of $\Sigma$. A fan $\Sigma$ is said to be *complete* if its support $|\Sigma|$ is the whole space. A fan $\Sigma$ is said to be *simplicial* if it consists of simplicial cones. Finally, $\Sigma^{(i)}$ denotes the set of $i$-dimensional cones of a fan $\Sigma$.

5.2. Let $M$ and $N$ be lattices dual to one another, and let $\Sigma$ be a fan in $N_{\mathbf{Q}}$. We fix a field $K$. With each cone $\sigma \in \Sigma$ we associate an affine toric variety $X_{(\check{\sigma}, M)} = \mathrm{Spec}\, K[\check{\sigma} \cap M]$. If $\tau$ is a face of $\sigma$, then $X_{\check{\tau}}$ can be identified with an open subvariety of $X_{\check{\sigma}}$ (see 2.6.1). These identifications allow us to glue together from the $X_{\check{\sigma}}$ (as $\sigma$ ranges over $\Sigma$) a variety over $K$, which is denoted by $X_\Sigma$ and is called the *toric variety associated with* $\Sigma$.

The affine varieties $X_{\check{\sigma}}$ are identified with open pieces in $X_\Sigma$, which we denote by the same symbol. Here $X_{\check{\sigma}} \cap X_{\check{\tau}} = X_{(\sigma \cap \tau)^\vee}$.

5.3. EXAMPLE. Let $N = \mathbf{Z}^n$; $e_1, \ldots, e_n$ a basis of $N$, and $e_0 = -(e_1 + \ldots + e_n)$. We consider the fan $\Sigma$ consisting of cones $\langle e_{i_1}, \ldots, e_{i_k}\rangle$, with $k \leqslant n$ and $0 \leqslant i_j \leqslant n$. As is easy to check, the variety $X_\Sigma$ is the projective space $\mathbf{P}^n_K$.

We meet other examples of toric varieties later. Sometimes one has to consider varieties associated with infinite fans (see [18]); however, we restrict ourselves to the finite case. Local properties of toric varieties were considered in Chapter I, and from the results there it follows that $X_\Sigma$ is a normal Cohen–Macaulay variety of dimension dim $N_{\mathbf{Q}}$. Also $X_\Sigma$ is a smooth variety if and only if all the cones of $\Sigma$ are basic relative to $N$; such a fan is called *regular*.

5.4. PROPOSITION. *The variety $X_\Sigma$ is separated.*

The proof uses the separation criterion [21], 5.5.6. $X_\Sigma$ is covered by affine open sets $X_{\check{\sigma}}$, and since the intersection $X_{\check{\sigma}} \cap X_{\check{\sigma}'}$ is again affine (and isomorphic to the spectrum of $K[(\sigma \cap \sigma')^\vee \cap M]$), it remains to verify that the ring $K[(\sigma \cap \sigma')^\vee \cap M]$ is generated by its subrings $K[\check{\sigma} \cap M]$ and $K[\check{\sigma}' \cap M]$, that is, that $(\sigma \cap \sigma')^\vee$ is generated by $\check{\sigma}$ and $\check{\sigma}'$.

Since $\sigma \cap \sigma'$ is a face of $\sigma$ and of $\sigma'$, we can find an $m \in M$ such that when $m$ is regarded as a linear function on $N_{\mathbf{Q}}$, then $m \geqslant 0$ on $\sigma$, $m \leqslant 0$ on $\sigma'$, and the hyperplane $m = 0$ meets $\sigma$ along $\sigma \cap \sigma'$. Now let $m' \in (\sigma \cap \sigma')^\vee$, that is, $m'$ is a linear function that is positive on $\sigma \cap \sigma'$. We can find an integer $r \geqslant 0$ such that $m' + rm$ is positive on $\sigma$, that is, $m' + rm \in \check{\sigma}'$. Then $m' = (m' + rm) + (-rm) \in \in \check{\sigma} + \check{\sigma}'$.

**5.5. Functoriality.** Let $f: N' \to N$ be a morphism of lattices, $\Sigma'$ a fan in $N'_{\mathbf{Q}}$, and suppose that for each $\sigma' \in \Sigma'$ we can find a $\sigma \in \Sigma$ such that $f(\sigma') \subset \sigma$. In this situation there arises a morphism of varieties over $K$

$$^a f\colon X_{\Sigma',N'} \to X_{\Sigma,N}.$$

Locally $^a f$ is constructed as follows. Under the dual map $\check{f}: M \to M'$ to $f$, we have $\check{f}(\check{\sigma}) \subset \check{\sigma}'$, hence (see **2.6**) we have a morphism of affine varieties $X_{(\check{\sigma}',M')} \to X_{(\check{\sigma}, M)}$. Now $^a f$ is obtained by gluing together these local morphisms. Let us consider some particular cases.

**5.5.1.** The case most frequently occurring is $N' = N$ and $\Sigma'$ is inscribed in $\Sigma$. Then $^a f$: $X_{\Sigma'} \to X_\Sigma$ is a birational morphism. According to **2.6.1**, $^a f$ is an open immersion if and only if $\Sigma' \subset \Sigma$. At the opposite extreme, $^a f$ is proper if and only if $\Sigma'$ is a subdivision of $\Sigma$ (see **5.6**).

**5.5.2.** Let $N' \subset N$ be lattices of the same dimension, and $\Sigma' = \Sigma$. The morphism $X_{\Sigma,N'} \to X_{\Sigma, N}$ is finite and surjective.

**5.5.3.** A lattice homomorphism $f$: $\mathbf{Z} \to N$ defines a morphism

$$^a f\colon \mathbf{G}_m = \operatorname{Spec} K[\mathbf{Z}] \to \mathbf{T} \subset X_{\Sigma, N}.$$

This extends to a map of $\mathbf{A}^1 = X_{\langle 1 \rangle, \mathbf{Z}}$ to $X_{\Sigma, N}$ if and only if $f(1) \in |\Sigma|$.

**5.5.4.** Let $\Sigma$ be a fan in $N$ and $\Sigma'$ a fan in $N'$. The direct product of $K$-varieties $X_{\Sigma, N} \underset{K}{\times} X_{\Sigma', N'}$ is again a toric variety associated with the fan $\Sigma \times \Sigma'$ in $N \times N'$.

**5.5.5. Blow-up.** Let $\sigma = \langle e_1, \ldots, e_k \rangle$, where $e_1, \ldots, e_k$ is part of a basis of $N$, and let $e_0 = e_1 + \ldots + e_k$. We consider the fan $\Sigma$ in $N$ consisting of the cones $\langle e_{i_1}, \ldots, e_{i_r} \rangle$ with $r \leqslant k$ and $\{i_1, \ldots, i_r\} \neq \{1, \ldots, k\}$. Then the morphism $X_\Sigma \to X_{\check{\sigma}}$ is the blow-up in $X_{\check{\sigma}}$ of the closed smooth subscheme $X_{\operatorname{cospan} \check{\sigma}}$.

**5.5.6.** PROPOSITION. *In the notation of* **5.5**, *the morphism* $^a f$: $X_{\Sigma', N'} \to X_{\Sigma, N}$ *is proper if and only if* $|\Sigma'| = f^{-1}(|\Sigma|)$. *In particular, the completeness of* $X_\Sigma$ *is equivalent to that of* $\Sigma$.

PROOF. Let $V$ be a discrete valuation ring with field of fractions $F$ and valuation $\nu$: $F^* \to \mathbf{Z}$. A criterion for $^a f$ to be proper (see [21], 7.3.8) is that any commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} F \arrow[r, phantom, "\subset"] \arrow[d] & \operatorname{Spec} V \arrow[d] \arrow[dl, dashed] \\
X_{\Sigma',N'} \arrow[r, "^a f"] & X_{\Sigma,N}
\end{tikzcd}$$

can be completed by a morphism $\operatorname{Spec} V \to X_{\Sigma', N'}$ that leaves the diagram commutative.

Without loss of generality we may assume that $\operatorname{Spec} F$ falls into the "big" torus $\mathbf{T}' = \operatorname{Spec} K[M'] \subset X_{\Sigma', N'}$, hence also into the torus $\mathbf{T} = \operatorname{Spec} K[M] \subset X_{\Sigma, N}$. So we obtain $F$-valued points of $\mathbf{T}'$ and $\mathbf{T}$, that is, semigroup homomorphisms $\varphi'$: $M' \to F^*$ and $\varphi = \varphi' \circ \check{f}$: $M \to F^*$. The image of $\operatorname{Spec} V$ lies in one of the affine pieces $X_{\check{\sigma}}$, $\sigma \in \Sigma$. This means that $\varphi(\check{\sigma} \cap M) \subset V^*$, or $(\nu \circ \varphi)(\check{\sigma} \cap M) \geqslant 0$, that is, $\nu \circ \varphi$ regarded as a linear function on $M$ belongs to $\sigma$.

Arguing similarly with $V$-valued points of $X_{\Sigma', N'}$ we find that the existence of a $V$-valued point of $X_{\Sigma', N'}$ extending a given $F$-valued point is equivalent to the existence of a cone $\sigma' \in \Sigma'$ such that $\nu \circ \varphi' \in \sigma'$. The remaining calculations are obvious.



**5.7. Stratification.** The torus $\mathbf{T} = \operatorname{Spec} K[M] \subset X_\Sigma$ has a compatible action on the open pieces $X_{\check{\sigma}}$, and this determines its action on $X_\Sigma$. It can be shown (see [26]) that this property characterizes toric varieties (and this explains the name): if a normal variety $X$ contains a torus $\mathbf{T}$ as a dense open subvariety, and the action of $\mathbf{T}$ on itself extends to an action on $X$, then $X$ is of the form $X_\Sigma$.

The orbits of the action of $\mathbf{T}$ on $X_\Sigma$ are isomorphic to tori and correspond bijectively to elements of $\Sigma$. More precisely, with a cone $\sigma \in \Sigma$ we associate a unique closed orbit in $X_{\check{\sigma}}$, namely, the closed subscheme $X_{\operatorname{cospan} \check{\sigma}} \subset X_{\check{\sigma}}$ (see **2.5**). Its dimension is equal to the codimension of $\sigma$ in $N_{\mathbf{Q}}$.

The closure of the orbit associated with $\sigma \in \Sigma$ is henceforth denoted by $F_\sigma$; subvarieties of this form are to play an important role in what follows. $F_\sigma$ is again a toric variety; the fan with which it is associated lies in $N_{\mathbf{Q}}/(\sigma - \sigma)$ and is obtained as the projection of the *star* $\operatorname{St}(\sigma) = \{\sigma' \in \Sigma \mid \sigma' \supset \sigma\}$ of $\sigma$ in $\Sigma$.

**5.8.** It is sometimes convenient to specify a toric variety by means of a polyhedron $\Delta$ in $M_{\mathbf{Q}}$. With each face $\Gamma$ of $\Delta$ we associate a cone $\sigma_\Gamma$ in $M_{\mathbf{Q}}$: to do this we take a point $m \in M_{\mathbf{Q}}$ lying strictly inside the face $\Gamma$, and we set

$$\sigma_\Gamma = \bigcup_{r \geqslant 0} r \cdot (\Delta - m).$$

The system $\{\check{\sigma}_\Gamma\}$, as $\Gamma$ ranges over the faces of $\Delta$, is a complete fan, which we denote by $\Sigma_\Delta$. The toric variety $X_{\Sigma_\Delta}$ is also denoted by $\mathbf{P}_\Delta$ to emphasize the analogy with projective space $\mathbf{P}$.

This construction is convenient, for example, in that if $\sigma \in \Sigma_\Delta$ corresponds to a face $\Gamma$ of $\Delta$, then the subvariety $F_\sigma$ is isomorphic to $\mathbf{P}_\Gamma$, and $\mathbf{P}_\Gamma \cap \mathbf{P}_{\Gamma'} = \mathbf{P}_{\Gamma \cap \Gamma'}$.

## §6. Linear systems

In this section we consider invertible sheaves on $X_\Sigma$ and their sections. As always, $M$ and $N$ are dual lattices, $\Sigma$ is a fan in $N_{\mathbf{Q}}$, and $X = X_\Sigma$.

**6.1.** We begin with a description of the group $\operatorname{Pic}(X)$ of invertible sheaves on $X$. Let $\mathscr{E}$ be an invertible sheaf on $X$; we restrict it to the big torus $\mathbf{T} \subset X$. Since $\operatorname{Pic}(\mathbf{T}) = 0$, we see that $\mathscr{E}_{|\mathbf{T}}$ is isomorphic to $O_{\mathbf{T}}$. An isomorphism $\varphi$: $\mathscr{E}_{|\mathbf{T}} \xrightarrow{\sim} O_{\mathbf{T}}$, to within multiplication by an element of $K^*$, is called a trivialization of $\mathscr{E}$. The group $M$ acts on the set of trivializations (multiplying them by $x^m$), and this action is transitive.

Let Div inv($X$) denote the set of pairs $(\mathcal{E}, \varphi)$, where $\mathcal{E} \in \operatorname{Pic} X$ and $\varphi$ is a trivialization of $\mathcal{E}$. Obviously,

$$\operatorname{Pic}(X) \simeq \operatorname{Div}\ \operatorname{inv}(X)/M,$$

so that it is enough to describe the group Div inv($X$).

**6.2.** Let $(\mathcal{E}, \varphi) \in \operatorname{Div}\operatorname{inv}(X)$, and let $\sigma \in \Sigma$ be a cone. If we restrict the pair $(\mathcal{E}, \varphi)$ to the affine piece $X_{\check{\sigma}} = \operatorname{Spec} A$, with $A = \operatorname{Spec} K[\check{\sigma} \cap M]$, we obtain an invertible $A$-module $E$ together with an isomorphism $E \underset{A}{\otimes} K[M] \rightrightarrows K[M]$.

Temporarily we denote the ring $K[M]$ by $B$; then we have an inclusion $E \subset B$ with $E \cdot B = B$. According to [6] (Chapter 2, §5, Theorem 4), if we set $E' = (A : E) = \{b \in B \mid bE \subset A\}$, then $E \cdot E' = A$, and $E'$ is the unique $A$-submodule of $B$ having this property. It is easy to check that $A : E$ is an $M$-graded submodule of $B$. Since in its turn $E = A : E'$, we find that $E$ is an $M$-graded $A$-submodule of $B$. Therefore, from the fact that $E$ is invertible we deduce the relation $\Sigma\, e_i e_i' = 1$ where the $e_i$ (and $e_i'$) are homogeneous elements of $E$ (and $E'$). But then we can also find a relation $e \cdot e' = 1$ with homogeneous $e$ and $e'$, hence $E \rightrightarrows A \cdot e$.

Thus, $E$ is of the form $A \cdot x^{m_\sigma}$ for some $m_\sigma \in M$. This element $m_\sigma$ is uniquely determined modulo the cospan of $\check{\sigma}$. Or if when we denote by $M_\sigma$ the group $M/M \cap (\operatorname{cospan} \check{\sigma})$, we see that the pair $(\mathcal{E}, \varphi)$ determines a collection $(m_\sigma)_{\sigma \in \Sigma}$, that is, an element of $\prod_\sigma M_\sigma$. This collection is not arbitrary, it satisfies an obvious compatibility condition. Namely, if $\tau$ is a face of $\sigma$, then under the projection $M_\sigma \to M_\tau$ the element $m_\sigma$ goes into $m_\tau$. In other words, the group Div inv($X$) is the projective limit of the system $\{M_\sigma | \sigma \in \Sigma\}$,

$$\operatorname{Div}\operatorname{inv}(X_\Sigma) = \varprojlim_{\sigma \in \Sigma} M_\sigma.$$

**6.3.** The same arguments also allow us to describe the space $\Gamma(X, \mathcal{E})$ of global sections of an invertible sheaf $\mathcal{E}$. Once more, let us fix a trivialization $\varphi$. If $s$ is a section of $\mathcal{E}$, then $\varphi(s)$ is a section of $O_{\mathbf{T}}$, that is, a certain Laurent polynomial $\Sigma a_m x^m \in K[M]$. For $\sigma \in \Sigma$ the condition for $s$ to be regular on the open piece $X_{\check{\sigma}}$ is that the support of $\Sigma a_m x^m$ should be contained in $m_\sigma + \check{\sigma}$. We obtain an identification of $\Gamma(X, \mathcal{E})$ with $L(\Delta)$, the space of Laurent polynomials with support in the polyhedron $\Delta = \bigcap_{\sigma \in \Sigma} (m_\sigma + \check{\sigma})$.

**6.4.** Of course, it would be more consistent to describe Div inv($X_\Sigma$) entirely in terms of $\Sigma$ and $N$. To do this we must represent $m_\sigma \in M_\sigma$ as a function on $\sigma$. The compatibility condition of **6.2** then takes the form: if $\tau$ is a face of $\sigma$, then $m_{\sigma|\tau} = m_\tau$. In other words, the functions $m_\sigma$ on the cones $\sigma \in \Sigma$ glue together to a single function on $|\Sigma|$, which we denote by $g = \operatorname{ord}(\mathcal{E}, \varphi)$. Clearly, $g$ takes integer values on elements of $|\Sigma| \cap N$. Thus we obtain another description of the group of invariant divisors:



$$\operatorname{Div}\operatorname{inv}(X_\Sigma) = \left\{ \begin{array}{l} \text{functions } g : |\Sigma| \to \mathbf{Q} \text{ such that} \\ \text{a) } g|_\sigma \text{ is linear on each } \sigma \in \Sigma, \\ \text{b) } g \text{ takes integer values on } |\Sigma| \cap N. \end{array} \right\}$$

The group operation on Div inv($X$) corresponds to addition of functions; the principal divisors $\operatorname{div}(x^m)$ are represented by global linear functions $m|_{|\Sigma|}$. The condition that the monomial $x^m$ belongs to the space $\Gamma(X, \mathcal{E})$ can then be rewritten as: $m \geqslant \operatorname{ord}(\mathcal{E}, \varphi)$ on $|\Sigma|$, where $m \in M$ is regarded as a linear function on $N_{\mathbf{Q}}$.

**6.5.** With each invertible ideal $J = (\mathcal{E}, \varphi) \in \operatorname{Div}\operatorname{inv}(X)$ there is associated a divisor $D$ on $X$, that is, an integral combination of irreducible subvarieties of $X$ of codimension 1. Being **T**-invariant, this divisor $D$ does not intersect the torus **T**, and hence consists of subvarieties $F_\sigma$, with $\sigma \in \Sigma^{(1)}$: $D = \Sigma n_\sigma F_\sigma$. The integers $n_\sigma$ can be expressed very simply in terms of the function $\operatorname{ord}(J)$ on $|\Sigma|$. Namely, if $e_\sigma$ is the primitive vector of $N$ on the ray $\sigma \in \Sigma^{(1)}$, then $n_\sigma = \operatorname{ord}(J)(e_\sigma)$. To verify this relation we can restrict ourselves to the open piece $X_{\check{\sigma}}$; in this case the situation is essentially one-dimensional, and the assertion is obvious.

**6.6. The canonical sheaf.** To stay within the framework of smooth varieties let us suppose here that the fan $\Sigma$ is regular. In this case the canonical sheaf $\Omega_X^n$ (where $n = \dim N_{\mathbf{Q}}$) is invertible. The invariant $n$-form $\omega = \dfrac{dX_1}{X_1} \wedge \ldots \wedge \dfrac{dX_n}{X_n}$ gives a trivialization $\omega : \Omega_{\mathbf{T}}^n \rightrightarrows O_{\mathbf{T}}$. The corresponding divisor is $\sum_{\sigma \in \Sigma^{(1)}} F_\sigma$. We can check this, again restricting ourselves to the essentially one-dimensional case $X_{\check{\sigma}}$, $\sigma \in \Sigma^{(1)}$ (see also **4.6**).

**6.7.** PROPOSITION. *Let $\Sigma$ be a complete fan, and let $(\mathcal{E}, \varphi) \in \operatorname{Div}\operatorname{inv}(X_\Sigma)$. The sheaf $\mathcal{E}$ is generated by its global sections if and only if the function* $\operatorname{ord}(\mathcal{E}, \varphi)$ *is upper convex.*

PROOF. Suppose first that $\operatorname{ord}(\mathcal{E}, \varphi)$ is convex. Let $\sigma \in \Sigma^{(n)}$ and let $m_\sigma \in M_\sigma = M$ be the element defined in **6.2**. Then the local section $x^{m_\sigma}$ generates $\mathcal{E}$ on the open piece $X_{\check{\sigma}} \subset X$. Since such open sets cover $X$, it is enough to show that $x^{m_\sigma}$ is a global section of $\mathcal{E}$. But according to the definition of $g = \operatorname{ord}(\mathcal{E}, \varphi)$ we have $g|_\sigma = m_{\sigma|\sigma}$, and since $g$ is convex, $m_\sigma \geqslant g$ on the whole of $|\Sigma|$. It remains only to use **6.4**.

Conversely, suppose that $\mathcal{E}$ is generated by its global sections and let $\hat{g}$ denote the convex hull of $g = \operatorname{ord}(\mathcal{E}, \varphi)$. Replacing, if necessary, the invertible sheaf $\mathcal{E}$ by a power $\mathcal{E}^{\otimes k}$ we see that $\hat{g}$ satisfies the conditions of **6.4**, and $\hat{g}$ determines an invertible sheaf $\hat{\mathcal{E}}$, a subsheaf of $\mathcal{E}$. From the description of the global sections in **6.4** it follows that $\Gamma(X, \hat{\mathcal{E}}) = \Gamma(X, \mathcal{E})$. Since $\Gamma(X, \mathcal{E})$ generates $\mathcal{E}$, we obtain $\hat{\mathcal{E}} = \mathcal{E}$ and $\hat{g} = g$.

It is also easy to see that in the convex case the polyhedron $\Delta$ of **6.3** is the convex hull of the elements $m_\sigma$ with $\sigma \in \Sigma^{(n)}$.

As we have already said, each Laurent monomial $f \in L(\Delta)$ can be interpreted as a section of the invertible sheaf $\mathscr{E}$. Hence it determines a closed subvariety $D_f$ of $X$, the variety of "zeros" of $f \in \Gamma(X, \mathscr{E})$. As $f$ ranges over the set of non-zero sections of $\mathscr{E}$ (or the non-zero elements of $L(\Delta)$), the effective divisors $D_f$ form the linear system $|D_f|$ on $X$. When $\mathscr{E}$ is generated by its global sections, this system $|D_f|$ is without base points and it follows from Bertini's theorem (in characteristic 0, the last restriction can be lifted) that the "generic" element of this system $D_f$ has singularities only at the singular points of $X$. In particular, for the "generic" element $f \in L(\Delta)$ the variety $D_f \cap \mathbf{T}$ is smooth.

This last result can be made more precise. Let us call an orbit of the action of $\mathbf{T}$ a *stratum* of $X$ (see 5.7).

**6.8.** PROPOSITION. *Suppose that $K$ has characteristic 0 and that the invertible sheaf $\mathscr{E}$ on $X_\Sigma$ is generated by its global sections. Then for the general section $f \in \Gamma(X, \mathscr{E})$ the variety $D_f$ is transversal to all strata of $X$.*

PROOF. We recall that the strata of $X$ corresponds to cones $\sigma \in \Sigma$, and that their closure $F_\sigma$ are again toric varieties. Hence, using the consequence of Bertini's theorem above it is enough to prove the following assertion.

**6.8.1.** LEMMA. *Let $\mathscr{E} \in \operatorname{Pic} X_\Sigma$ be generated by its global sections, and let $\sigma \in \Sigma$. Then the restriction homomorphism*

$$\Gamma(X, \mathscr{E}) \to \Gamma(F_\sigma, \mathscr{E}|_{F_\sigma})$$

*is surjective.*

PROOF. We choose compatible trivializations of $\mathscr{E}$ and $\mathscr{E}|_{F_\sigma}$; this is possible, since $\operatorname{Pic} X_{\breve{\sigma}} = 0$ (see **6.2**). Then the function $\operatorname{ord}(\mathscr{E})$ is zero on $\sigma$. The sections of $\mathscr{E}|_{F_\sigma}$, or more precisely, a basis of them, are given by the elements $m \in M$ that are zero on $\sigma$ and $\geqslant \operatorname{ord}(\mathscr{E})$ on $\operatorname{St}(\sigma)$. But then from the convexity of $\operatorname{ord}(\mathscr{E})$ we deduce that $m \geqslant \operatorname{ord}(\mathscr{E})$ on the whole of $|\Sigma|$, that is, $x^m \in \Gamma(X, \mathscr{E})$. The lemma and with it Proposition 6.8 are now proved.

The varieties $D_f$ are of great interest, since they are generalizations of hypersurfaces in $\mathbf{P}^n$. Their cohomology is closed connected with the polyhedron $\Delta$; we hope to return to this question later.

**6.9. Projectivity.** As before, let $\Sigma$ be a complete fan. We say that a convex function $g$ on $|\Sigma|$ is *strictly convex with respect to* $\Sigma$ if $g$ is linear on each cone $\sigma \in \Sigma^{(n)}$ and if distinct cones of $\Sigma^{(n)}$ correspond to distinct linear functions. In other words, $g$ suffers a break on passing from one chamber of $\Sigma$ to another. We also recall that an invertible sheaf $\mathscr{E}$ on $X$ is said to be *ample* if some tensor power $\mathscr{E}^{\otimes k}$ $(k > 0)$ of it is generated by its global sections and defines an embedding of $X$ in the projective space $\mathbf{P}(\Gamma(X, \mathscr{E}^k))$.

**6.9.1.** PROPOSITION. *If $\mathscr{E}$ is ample, then $\operatorname{ord}(\mathscr{E})$ is strictly convex with respect to $\Sigma$.*

That $\operatorname{ord}(\mathscr{E})$ is convex follows from **6.7**, so that it remains to check that $\operatorname{ord}(\mathscr{E})$ suffers a break on each face $\sigma \in \Sigma^{(n-1)}$, that is, that it is strictly convex on the stars of the cones of $\Sigma^{(n-1)}$. Thus, the question reduces to the one-dimensional case, that is, to ample sheaves on the projective line $\mathbf{P}^1$, where



it is obvious.

Note that the converse assertion also holds (see [26]).

This criterion for ampleness allows us to construct examples of complete, but not projective toric varieties. Without going into detailed explanations, let us just say that $X_\Sigma$ is prevented from being projective by the presence in $\Sigma$ (or more precisely, in the intersection of $\Sigma$ with a sphere) of fragments such as these (Fig. 3):

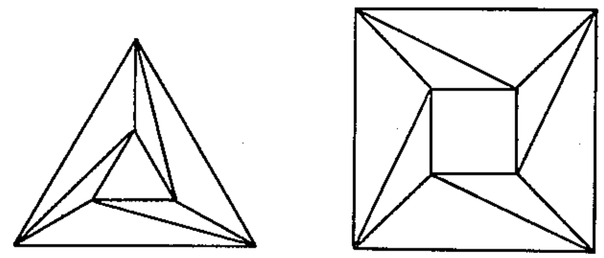

Fig. 3.

The point is that a convex function on such a complex cannot possibly be strictly convex. However, we do have the following toric version of Chow's lemma:

**6.9.2.** LEMMA. *For every complete fan $\Sigma$ there exists a subdivision $\Sigma'$ that is projective, that is, admits a strictly convex function.*

For example, we can extend each cone in $\Sigma^{(n-1)}$ to a hyperplane in $N_{\mathbf{Q}}$ and take the resulting subdivision.

## §7. The cohomology of invertible sheaves

**7.1.** We keep to the notation of the previous section. Let $\mathscr{E}$ be an invertible sheaf on $X = X_\Sigma$. A choice of trivialization $\varphi\colon \mathscr{E}|_{\mathbf{T}} \xrightarrow{\sim} O_{\mathbf{T}}$ defines a $\mathbf{T}$-linearisation of $\mathscr{E}$ and hence an action of $\mathbf{T}$ on the cohomology spaces $H^i(X, \mathscr{E})$. Therefore, these spaces have a weight decomposition, that is, an $M$-grading

$$H^i(X, \mathscr{E}) = \bigoplus_{m \in M} H^i(X, \mathscr{E})(m).$$

This decomposition can also be understood from the description in **6.2** and the computation of the cohomology using a Čech covering. Let us show how a weighted piece $H^i(X, \mathscr{E})(m)$ can be expressed in terms of $g = \operatorname{ord}(\mathscr{E}, \varphi)$.

Here it is convenient to regard the $\sigma$ as cones in the real vector space $N_{\mathbf{R}} = N \otimes \mathbf{R}$; then $\sigma$ and $|\Sigma|$ are closed subsets of $N_{\mathbf{R}}$. We introduce the closed subset $Z_m = Z_m(g)$ of $|\Sigma|$:

$$Z_m = \{x \in N_{\mathbf{R}} \mid m(x) \geqslant g(x)\}$$

(here $m \in M$ is regarded as a linear function on $N_{\mathbf{R}}$).

7.2. THEOREM. (Demazure [16]). $H^i(X, \mathcal{E})(m) = H^i_{Z_m}(|\Sigma|; K)$.

The proof is based on representing both sides as the cohomology of natural coverings of $X_\Sigma$ and $|\Sigma|$, which also turn out to be acyclic.

We begin with the left-hand side. $X_\Sigma$ is covered by the affine open pieces $X_{\check\sigma}$, $\sigma \in \Sigma$. An intersection of such affines is of the same form, in particular, is affine. By Serre's theorem this covering is acyclic, and the cohomology $H^i(X, \mathcal{E})$ is the same as that of the covering $\mathfrak{X} = \{X_{\check\sigma}\}_{\sigma\in\Sigma}$, that is, the cohomology of the complex

$$C^*(\mathfrak{X}, \mathcal{E}) = (\dots \xrightarrow{\partial} \bigoplus_\sigma H^0(X_{\check\sigma}, \mathcal{E}) \xrightarrow{\partial} \dots),$$

whose construction is well known. Each term of this complex has a natural $M$-grading (see **6.2**), the differential preserves the grading, and the $H^i(X, \mathcal{E})(m)$ are equal to the $i$-dimensional cohomology of the complex $C^*(\mathfrak{X}, \mathcal{E})(m)$ put together from the $H^0(X_{\check\sigma}, \mathcal{E})(m)$.

Now we consider the right-hand side, using the closed covering of $|\Sigma|$ by the cones $\sigma$ with $\sigma \in \Sigma$. The intersection of cones in $\Sigma$ is again a cone of $\Sigma$. Let us check that our covering is acyclic, so that we can then use Leray's theorem ([7], II, 5.2.4) to represent $H^i_{Z_m}(|\Sigma|; K)$ as the $i$-dimensional cohomology of the complex

$$C^*_{Z_m}(\{\sigma\}_{\sigma\in\Sigma}; K) = (\dots \xrightarrow{\partial} \bigoplus_\sigma H^0_{Z_m}(\sigma; K) \xrightarrow{\partial} \dots).$$

Thus, let us check that $H^i_{Z_m}(\sigma; K) = 0$ for $i > 0$. For this purpose we use the long exact sequence

$$\dots \to H^{i-1}(\sigma - Z_m; K) \to H^i_{Z_m}(\sigma; K) \to H^i(\sigma; K) \to \dots$$

Since $\sigma$ and $\sigma - Z_m$ are convex, we come to the required assertion.

It is now enough to verify that the two spaces $H^0(X_{\check\sigma}, \mathcal{E})(m)$ and $H^0_{Z_m}(\sigma; K)$ are equal. Bearing **6.2** in mind, we see that the first of these is equal either to $K$ or to 0, depending on whether $m$ belongs to the set $m_\sigma + \check\sigma$ or not (where $m_\sigma$ is as in **6.2**), that is, whether or not $m \geqslant m_\sigma = \operatorname{ord}(\mathcal{E}, \varphi)$ as functions on $\sigma$. To find the second space we use the same exact sequence as before:

$$0 \to H^0_{Z_m}(\sigma; K) \to H^0(\sigma; K) \to H^0(\sigma - Z_m; K).$$

From this we find that $H^0_{Z_m}(\sigma; K)$ is isomorphic to $K$ or to 0 depending on whether $\sigma - Z_m$ is empty or non-empty, that is, again whether or not $m \geqslant m_\sigma$ holds on $\sigma$.

7.3. COROLLARY. *Suppose that $\Sigma$ is a complete fan, and that $g = \operatorname{ord}(\mathcal{E}, \varphi)$ is upper convex. Then $H^i(X, \mathcal{E}) = 0$ for $i > 0$.*

For any $m \in M$ the set $N_{\mathbf{R}} - Z_m = \{x \in N_{\mathbf{R}} \mid m(x) < g(x)\}$ is convex, hence, as in the proof of acyclicity, we obtain $H^i_{Z_m}(N_{\mathbf{R}}; K) = 0$ for $i > 0$.



7.4. COROLLARY. $H^i(X, O_X) = 0$ *for* $i > 0$.

**7.5.** We take this opportunity to say a few words on the cohomology of the sheaves of differential forms $\Omega^p_X$ (see **4.1**). These sheaves have a canonical T-linearization, and the spaces $H^i(X, \Omega^p_X)$ have a natural $M$-grading. Using the covering $\mathfrak{X}$ and Serre's theorem, we find that these $M$-graded spaces coincide with the cohomology of the complex of $M$-graded spaces

$$C^*(\mathfrak{X}, \Omega^p_X) = (\dots \xrightarrow{\partial} \bigoplus_\sigma H^0(X_{\check\sigma}, \Omega^p_X) \xrightarrow{\partial} \dots).$$

Note that the space $H^0(X_{\check\sigma}, \Omega^p_X) = \Omega^p_{A_{\check\sigma}}$ is described in (4.2.3). Since $\Omega^p_{A_{\check\sigma}}(m)$ and $\Omega^p_{A_{\check\sigma}}(km)$ are canonically isomorphic for any $k > 0$, so are $H^i(X, \Omega^p)(m)$ and $H^i(X, \Omega^p)(mk)$. This simple remark leads to a corollary:

7.5.1. COROLLARY. *If $\Sigma$ is a complete fan, then $H^i(X, \Omega^p_X)(m) = 0$ for $m \neq 0$.*

For $H^i(H, \Omega^p_X)$ is finite-dimensional.

As for the component $H^i(X, \Omega^p_X)(0)$ of weight 0, note that the spaces $\Omega^p_{A_{\check\sigma}}(0)$ needed to compute it are also of a very simple form, (see **4.2.3**), namely:

$$\Omega^p_{A_{\check\sigma}}(0) = \Lambda^p(V_{\operatorname{cospan}\check\sigma}) \tag{7.5.1}$$

(which in characteristic 0 is equal to $\Lambda^p(\operatorname{cospan}\check\sigma) \underset{\mathbf{Q}}{\otimes} K$).

Finally, in the style of **6.3** we can describe the space of sections of the sheaf $\Omega^p_X \otimes \mathcal{E}$, where $\mathcal{E} \in \operatorname{Pic} X$. We restrict ourselves to the case when $\operatorname{ord}(\mathcal{E})$ is convex. As in **6.3**,

$$\Gamma(X, \Omega^p_X \otimes \mathcal{E}) = \bigcap_\sigma x^{m_\sigma}\Omega^p_{A_{\check\sigma}}.$$

Alternatively, in terms of the polyhedron $\Delta$ connected with $\operatorname{ord}(\mathcal{E})$, for each $m \in M$ let $V_m$ denote the subspace of $V = M \otimes K$ generated by the smallest face of $\Delta$ containing $m$. Then

$$\Gamma(X, \Omega^p_X \otimes \mathcal{E}) = \bigoplus_{m\in\Delta\cap M} \Lambda^p(V_m)\cdot x^m. \tag{7.5.2}$$

The next proposition, which we give without proof, generalizes a well-known theorem of Bott:

7.5.2. THEOREM. *Let $\Sigma$ be a complete fan, and let $\mathcal{E}$ be an invertible sheaf such that $\operatorname{ord}(\mathcal{E})$ is strictly convex with respect to $\Sigma$. Then the sheaves $\Omega^p_X \otimes \mathcal{E}$ are acyclic, that is, $H^i(X, \Omega^p_X \otimes \mathcal{E}) = 0$ for $i > 0$.*

**7.6. The cohomology of the canonical sheaf.** Let $\Sigma$ be a fan that is complete and regular, so that the variety $X_\Sigma$ is complete and smooth. The canonical sheaf $\Omega^n_X$ is invertible, and the function $g_0 = \operatorname{ord}(\Omega^n_X)$ takes the value 1 on primitive vectors $e_\sigma$, $\sigma \in \Sigma^{(1)}$ (see **6.6**). According to Corollary 7.5.1, $H^i(X, \Omega^n_X) = H^i(X, \Omega^n_X)(0)$; let us compute this space, using Theorem 7.2. For $g_0$ the set $Z_0 = \{x \in N_{\mathbf{R}} \mid g_0(x) \leqslant 0\}$ degenerates to a point $\{0\}$, so that $H^i(X, \Omega^n_X)(0)$ is isomorphic to $H^i_{\{0\}}(\mathbf{R}^n; K) = H^i(\mathbf{R}^n, \mathbf{R}^n - \{0\}; K)$. We arrive finally at

$$H^i(X, \Omega_X^n) = \begin{cases} 0, & i \neq n, \\ K, & i = n. \end{cases}$$

**7.7. Serre duality.** We keep the notation and hypotheses of **7.6**. Following a suggestion of Khovanskii we show how to check Serre duality for invertible sheaves on $X_\Sigma$.

Let $\omega = \Omega_X^n$, and $\mathscr{E} \in \operatorname{Pic} X$.

7.7.1. PROPOSITION. *The natural pairing*

$$H^k(X, \mathscr{E}) \otimes H^{n-k}(X, \mathscr{E}^{-1} \otimes \omega) \to H^n(X, \omega) = K$$

*is non-degenerate.*

To prove this we fix a trivialization of $\mathscr{E}$. The above pairing is compatible with the $M$-grading, and to verify that it is non-degenerate we need only check that for each $m \in M$ the spaces $H^k(X, \mathscr{E})(m)$ and $H^{n-k}(X, \mathscr{E}^{-1} \otimes \omega)(-m)$ are dual to one another. By changing the trivialization we may assume that $m = 0$.

Now $H^k(X, \mathscr{E})(0)$ is isomorphic to $H_Z^k(\mathbf{R}^n) = H^k(\mathbf{R}^n, \mathbf{R}^n - Z)$, where

$$Z = \{x \in \mathbf{R}^n \mid 0 \geqslant \operatorname{ord}(\mathscr{E})(x)\},$$

and $H^{n-k}(X, \mathscr{E}^{-1} \otimes \omega)(0)$ is isomorphic to $H_{Z'}^{n-k}(\mathbf{R}^n) = H^{n-k}(\mathbf{R}^n, \mathbf{R}^n - Z')$, where

$$Z' = \{x \in \mathbf{R}^n \mid \operatorname{ord}(\mathscr{E})(x) \geqslant g_0(x)\}.$$

The above pairing goes over into the cup-product

$$H^k(\mathbf{R}^n, \mathbf{R}^n - Z) \otimes H^{n-k}(\mathbf{R}^n, \mathbf{R}^n - Z') \to H^n(\mathbf{R}^n, \mathbf{R}^n - Z \cap Z') = \\ = H^n(\mathbf{R}^n, \mathbf{R}^n - \{0\}),$$

and the question becomes purely topological. We restrict ourselves to the non-trivial case when both $Z$ and $Z'$ are distinct from $\mathbf{R}^n$.

We replace $\mathbf{R}^n$ by the disc $D^n$; more precisely, let

$$D^n = \{x \in \mathbf{R}^n \mid g_0(x) \leqslant 1\}.$$

We set $S = \partial D^n$; clearly, $S$ is homeomorphic to the $(n-1)$-dimensional sphere and is a topological manifold. The above pairing can be rewritten

$$H^k(D^n, S - Z) \otimes H^{n-k}(D^n, S - Z') \to \\ \to H^n(D^n, S) = K.$$

The key technical remark is that the inclusion $Z \cap S \hookrightarrow S - Z'$ is a deformation retract. The required deformation can be constructed separately on each simplex $\sigma \cap S$, taking care to ensure compatibility. We leave the details to the reader, restricting ourselves to Fig. 4. Of course, here we have to use the circumstance, which follows from the definition of $g_0$, that the vertices of a



simplex do not fall into the "belt" strictly between $Z$ and $Z'$.

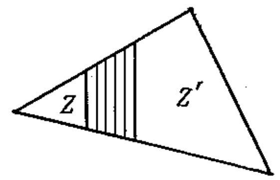


Fig. 4.

Thus, $(D^n, S - Z')$ is equivalent to $(D^n, S \cap Z)$, and we have only to check that the map

$$H^{n-k}(D^n, S \cap Z) \to H^k(D^n, S - Z)^*$$

is an isomorphism. We consider the commutative diagram

$$\begin{array}{ccccccc} \ldots H^{n-k}(D, S) & \to & H^{n-k}(D, S \cap Z) & \to & H^{n-k}(S, S \cap Z) & \to & \ldots \\ \downarrow\wr & & \downarrow & & \downarrow\wr & & \\ \ldots H^k(D)^* & \to & H^k(D, S - Z)^* & \to & H^{k-1}(S - Z) & \to & \ldots \end{array}$$

Here the top row is the long exact sequence of the triple $(D, S, S \cap Z)$, and the bottom row is the dual to the long exact sequence of the pair $(D, S - Z)$. The left-hand vertical arrow is the obvious isomorphism, and the right-hand one is also an isomorphism, by Lefschetz duality on $S$ (see [12], Ch. 6, §2, Theorem 19). Therefore, the middle vertical arrow is also an isomorphism, which completes the proof.

7.7.2. REMARK. The results of **7.6** and Proposition 7.7.1 remain true for any complete fan.

## §8. Resolution of singularities

**8.1.** Using the criterion for a toric variety to be smooth we can give a simple method of desingularizing toric varieties. We recall that a resolution of the singularities of a variety $X$ is a morphism $f\colon X' \to X$ such that a) $f$ is proper and birational, and b) $X'$ is a smooth variety.

Now let $\Sigma$ be a fan in $N_{\mathbf{Q}}$ and $X = X_\Sigma$. According to **5.5.1**, to resolve the singularities of $X$ it is enough to find a subdivision $\Sigma'$ of $\Sigma$ such that $\Sigma'$ is regular with respect to $N$. Thus, the problem becomes purely combinatorial, which is typical for "toric geometry".

We construct the required subdivision $\Sigma'$ as a sequence of "elementary" subdivisions. Let $\lambda \subset |\Sigma|$ be a ray; the *elementary subdivision associated with* $\lambda$ goes like this: if $\sigma \in \Sigma$ is a cone not containing $\lambda$, then it remains unchanged; otherwise $\sigma$ is replaced by the collection of convex hulls of $\lambda$ with the faces of $\sigma$ that do not contain $\lambda$.

**8.2.** Using the operations indicated above we carry out the barycentric subdivision of $\Sigma$: After this, all the cones become simplicial, and our subsequent actions are directed at "improving" simplicial cones. For this purpose it is convenient to introduce a certain numerical characteristic of simplicial cones, which measures their deviation from being basic.

Let $\sigma = \langle e_1, \ldots, e_k \rangle$ be a simplicial cone, where the vectors $e_i$ belong to $N$ and are primitive. We define the *multiplicity* mult$(\sigma)$ of $\sigma$ as the index in the lattice $N \cap (\sigma - \sigma)$ of the subgroup generated by $e_1, \ldots, e_k$. This number mult$(\sigma)$ is also equal to the volume of the parallelotope $P_\sigma = \{\sum_i \alpha_i e_i \mid 0 \leqslant \alpha_i < 1\}$ in $(\sigma - \sigma)$ normalized by the lattice $N \cap (\sigma - \sigma)$, and also to the number of integral points (points of $N$) in $P_\sigma$. Obviously, $\sigma$ is basic with respect to $N$ if and only if mult$(\sigma) = 1$.

Suppose that mult$(\sigma) > 1$. Then we can find a non-zero point $x \in N \cap P_\sigma$, that is, a point of the form

$$x = \sum_i \alpha_i e_i, \quad 0 \leqslant \alpha_i < 1.$$

Let us pass to the subdivision of $\Sigma$ associated with the line $\langle x \rangle$. Then the multiplicity decreases, since

$$\operatorname{mult}\langle e_1, \ldots, \hat{e}_i, \ldots, e_k, x\rangle = \alpha_i \operatorname{mult}(\sigma)$$

for $\alpha_i \neq 0$; this formula follows easily from the interpretation of multiplicity as a volume. The argument above can easily be made inductive, proving that the required subdivision $\Sigma'$ exists.

**8.3.** REMARK. The subdivision $\Sigma'$, which gives a resolution of singularities $f\colon X_{\Sigma'} \to X_\Sigma$, can be chosen so that

a) $f$ is an isomorphism over the variety of smooth points of $X_\Sigma$;

b) $f$ is a projective morphism (more precisely, the normalization of the blow-up of some closed **T**-invariant subscheme of $X_\Sigma$).

**8.4.** In the two-dimensional case the regular subdivision of a fan $\Sigma$ can be made canonical; we restrict ourselves to the subdivision of one cone $\sigma$. Let $\Delta \subset \mathbf{Q}^2 = N_\mathbf{Q}$ be the convex hull of the set $(\sigma \cap N) - \{0\}$, and let $x_1, \ldots, x_k$ be the elements of $N$ that lie on the compact faces of $\partial\Delta$. Then the rays $\langle x_1 \rangle, \ldots, \langle x_k \rangle$ give the required subdivision of the cone $\sigma$. That the cones so obtained are basic is due to an elementary fact: if the only integral point of a triangle in the plane are its vertices, then its area is 1/2. The explicit determination of the coordinates of the points $x_1, \ldots, x_k$ is closely connected with expansions in continued fractions.

**8.5. Toric singularities are rational.** Using the results of §7 we prove the following proposition.

**8.5.1.** PROPOSITION. *If $\Sigma'$ is a subdivision of $\Sigma$ and $f\colon X_{\Sigma'} \to X_\Sigma$ is the corresponding morphism* (see 5.5), *then $f_* O_{X_{\Sigma'}} = O_{X_\Sigma}$ and $R^i f_* O_{X_{\Sigma'}} = 0$ for $i > 0$.*

PROOF. Since the question is essentially local, we may assume that $X = X_\Sigma$ is an affine variety, in other words, is of the form $X_{\check{\sigma}}$ for some cone $\sigma$ in $N_\mathbf{Q}$. For each $m \in M$



$$H^i(X_{\Sigma'}, O_{X'})(m) = H^i_{Z_m}(\sigma; K),$$

where

$$Z_m = \{x \in \sigma_\mathbf{R};\ m(x) \geqslant 0\}.$$

If $m \in \check{\sigma}$, then $Z_m = \sigma$ and

$$H^i_{Z_m}(\sigma; K) = H^i(\sigma; K) = \begin{cases} K, & i = 0, \\ 0, & i > 0. \end{cases}$$

But if $m \notin \check{\sigma}$ then $\sigma - Z_m$ is convex and non-empty, so that $H^i_{Z_m}(\sigma; K) = 0$ for all $i$. Finally, we find that $A_{\check{\sigma}} \xrightarrow{\sim} H^0(X_{\Sigma'}, O_{X_{\Sigma'}})$, and $H^i(X_{\Sigma'}, O_{X_{\Sigma'}}) = 0$ for $i > 0$.

**8.5.2.** REMARK. A similar, although more subtle technique can be applied to the study of singularities and cohomological properties of generalized flag varieties $G/B$ and their Schubert subvarieties (see [17], [24], [27]). Flag varieties, like toric varieties, have affine coverings; they also have a lattice of characters and a fan of Weyl chambers.

## §9. The fundamental group

In this section we consider unramified covers and the (algebraic) fundamental group of toric varieties. Throughout we assume that $K$ is *algebraically closed*. First of all, we have the following general fact.

**9.1.** THEOREM. *If $\Sigma$ is a complete fan, then $X_\Sigma$ is simply-connected.*

For $X_\Sigma$ is a complete normal rational variety, so that the assertion follows from [30] (exposé XI, 1.2).

**9.2.** Let us show (at least in characteristic 0) how to obtain this theorem by more "toric" means, and also how to find the fundamental group of an arbitrary toric variety $X_\Sigma$. Let $\Sigma$ be a fan in $N_\mathbf{Q}$; we assume that $|\Sigma|$ is not contained in a proper subspace of $N_\mathbf{Q}$, since otherwise some torus splits off as a direct factor of $X_\Sigma$. Suppose further that $f\colon X' \to X_\Sigma$ is a finite surjective morphism satisfying the following two conditions:

a) $X'$ is normal and connected;

b) $f$ is unramified over the "big" torus $\mathbf{T} \subset X_\Sigma$.

Note that conditions a) and b) hold if $f$ is an etale Galois cover. Now let us restrict $f$ to $\mathbf{T}$, $f\colon f^{-1}(\mathbf{T}) \to \mathbf{T}$. Since in characteristic 0 finite unramified covers of $\mathbf{T}$ are classified by subgroups of finite index in $\pi_1(\mathbf{T}) = \mathbf{Z}^n$, it is not difficult to see that $f^{-1}(\mathbf{T}) = \mathbf{T}'$ is again a torus, and if $M'$ is the character group of $\mathbf{T}'$, then $M$ is a sublattice of finite index in $M'$, and the Galois group of $\mathbf{T}'$ over $\mathbf{T}$ is isomorphic to $M'/M$. In other words,

b′) $f^{-1}(\mathbf{T}) \to \mathbf{T}$ coincides with Spec $K[M'] \to$ Spec $K[M]$.

The subsequent arguments do not use the assumption that the characteristic is 0, but only the properties a) and b′). Let $N' \subset N$ be the inclusion of the dual lattices to $M'$ and $M$. From a) and b′) it follows that the morphism $X' \to X_\Sigma$ is the same as the canonical morphism (see **5.5.2**) $X_{\Sigma,N'} \to X_{\Sigma,N}$. For $X'$ is the normalization of $X_\Sigma$ in the field of rational functions of $\mathbf{T}'$; but $X'_\Sigma = X_{\Sigma,N'}$ is normal and contains $\mathbf{T}'$ as an open piece.

From now on everything is simple. Let us see what condition is imposed on the sublattice $N' \subset N$ by requiring that $X'_\Sigma \to X_\Sigma$ is unramified along $\mathbf{T}_\sigma$, that is, the stratum of $X_\Sigma$ associated with $\sigma \in \Sigma$ (see **5.7**). The character group of $\mathbf{T}_\sigma$ is isomorphic to $M \cap \operatorname{cospan} \check{\sigma}$. Over $\mathbf{T}_\sigma$ there lies the torus $\mathbf{T}'_\sigma$ with the character group $M' \cap \operatorname{cospan} \check{\sigma}$. The condition that $f$ is unramified along $\mathbf{T}_\sigma$ is equivalent to

$$M' \cap (\operatorname{cospan} \check{\sigma})/M \cap \operatorname{cospan} \check{\sigma}) \xrightarrow{\sim} M'/M,$$

or in dual terms

$$N \cap (\sigma - \sigma) = N' \cap (\sigma - \sigma).$$

Now we introduce the following notation: $N_\Sigma \subset N$ is the lattice generated by $\bigcup_{\sigma \in \Sigma} (\sigma \cap N)$. From what we have said above it is clear that $X'_\Sigma \to X_\Sigma$ is unramified if and only if $N_\Sigma \subset N' \subset N$.

**9.3.** PROPOSITION. *Suppose that $K$ is of characteristic* 0 *and that* $|\Sigma|$ *generates* $N_\mathbf{Q}$. *Then* $\pi_1(X_\Sigma) \cong N/N_\Sigma$. *In particular,* $\pi_1(X_\Sigma)$ *is a finite Abelian group.*

As a corollary we get Theorem 9.1, and also the following estimate: if a fan $\Sigma$ contains a $k$-dimensional cone, then $\pi_1(X_\Sigma)$ can be generated by $n-k$ elements (as usual, $n = \dim N_\mathbf{Q}$). Simple examples show that $X_\Sigma$ need not be simply-connected.

**9.4.** Now let us discuss the case when $K$ has positive characteristic $p$. First of all, we cannot expect now to obtain all unramified covers "from the torus". For even the affine line $\mathbf{A}^1$ has many "wild" covers of Artin-Schreier type, given by equations $y^p - y = f(x)$. It is all the more remarkable that when the fan "sticks out in all directions", then the wild effects vanish. We have the following result, which we prove elsewhere:

**9.4.1.** PROPOSITION. *Suppose that $\Sigma$ is not contained in any half-space of $N_\mathbf{Q}$. Let $f\colon X' \to X_\Sigma$ be a connected finite etale cover. Then its restriction to the big torus $f^{-1}(\mathbf{T}) \to \mathbf{T}$ is of the form* $\operatorname{Spec} K[M'] \to \mathbf{T}$, *where $M' \supset M$ and $[M' : M]$ is prime to $p$.*

Arguing as in **9.2** we get the following corollary.

**9.4.2.** COROLLARY. *Suppose that $\Sigma$ is not contained in any half-space. Then $\pi_1(X_\Sigma)$ is isomorphic to the component of $N/N_\Sigma$ prime to $p$.*

## CHAPTER III

## INTERSECTION THEORY

Intersection theory deals with such global objects as the Chow ring, the $K$-functor, and the cohomology ring, which have a multiplication interpreted as the intersection of corresponding cycles. Here we also have a section on the Riemann–Roch theorem, which can be regarded as a comparison between Chow theory and $K$-theory.



### §10. The Chow ring

**10.1.** Chow theory deals with algebraic cycles on an algebraic variety $X$, that is, with integral linear combinations of algebraic subvarieties of $X$. Usually $X$ is assumed to be smooth. If two cycles on $X$ intersect transversally, then it is fairly clear what we should consider as their "intersection"; in the general case we have to shift the cycles about, replacing them by cycles that are equivalent in one sense or another. The simplest equivalence, which is the one we consider henceforth, is *rational* equivalence, in which cycles are allowed to vary in a family parametrized by the projective line $\mathbf{P}^1$. Of course, there remains the question as to whether transversality can always be achieved by replacing a cycle by an equivalent one. In [14] it is shown that this can be done on projective varieties; we shall show below how to do this directly for toric varieties.

Let $A_k(X)$ denote the group of $k$-dimensional cycles on $X$ to within rational equivalence, and let $A_*(X) = \bigoplus_k A_k(X)$. If $f\colon Y \to X$ is a proper morphism of varieties, we have a canonical group homomorphism

$$f_*\colon A_*(Y) \to A_*(X).$$

Later we need the following fact:

**10.2.** LEMMA (see [14]). *If $Y$ is a closed subvariety of $X$, then the sequence*

$$A_*(Y) \to A_*(X) \to A_*(X - Y) \to 0$$

*is exact.*

Let us now go over to toric varieties. Let $\Sigma$ be a fan in $N_\mathbf{Q}$ and $X = X_\Sigma$. We recall that with every cone $\sigma \in \Sigma$ we have associated a closed subvariety $F_\sigma$ in $X$ of codimension $\dim \sigma$ (see **5.7**); let $[F_\sigma] \in A_{n-\dim\sigma}(X)$ denote the class of $F_\sigma$. The importance of cycles of this form is shown by the following proposition.

**10.3.** PROPOSITION. *The cycles of the form $[F_\sigma]$ generate $A_*(X)$.*

PROOF. Using Lemma 10.2 we can easily check that $A_*(\mathbf{T}) \cong \mathbf{Z}$ is generated by the fundamental cycle $\mathbf{T}$. Let $Y = X - \mathbf{T}$. Applying Lemma 10.2 we find that $A_n(X)$ is generated by the fundamental cycle $X = F_{\{0\}}$, and that for $k < n$ the map $A_k(Y) \to A_k(X)$ is surjective. In other words, any cycle on $X$ of dimension less than $n$ can be crammed into the union of the $F_\sigma$ with $\sigma \neq \{0\}$. Since $F_\sigma$ is again a toric variety, an induction on the dimension completes the proof.

We are about to see that on a smooth toric variety $X$ "all cycles are algebraic". To attach a meaning to this statement we could make use of $l$-adic homology. Instead, we suppose that $K = \mathbf{C}$ and use the ordinary homology of the topological space $X(\mathbf{C})$, equipped with the strong topology.

**10.4.** PROPOSITION. *For a complete smooth toric variety $X$ the canonical* *homomorphism (which doubles the degree)*

$$A_*(X) \to H_*(X, \mathbf{Z})$$

*is surjective.*

PROOF. According to Lemma 6.9.2 and 8.3 we can find a regular projective fan $\Sigma'$ that subdivides $\Sigma$; let $X' = X_{\Sigma'}$. From the commutative diagram

$$\begin{array}{ccc} A_*(X') & \to & H_*(X', \mathbf{Z}) \\ \downarrow & & \downarrow \\ A_*(X) & \to & H_*(X, \mathbf{Z}) \end{array}$$

and the fact that $H_*(X', \mathbf{Z}) \to H_*(X, \mathbf{Z})$ is surjective (which is a consequence of Poincaré duality) it is clear that it is enough to prove our proposition for $X'$, that is, to assume that $X$ is projective. But in the projective case, following Ehlers (see [18]), we can display more explicitly a basis of $A_*(X)$ and of $H_*(X, \mathbf{Z})$, and then the proposition follows. Since this basis is interesting for its own sake, we dwell on the projective case a little longer.

**10.5.** Let $\Sigma$ be a projective fan, that is, suppose that there exists a function $g: N_{\mathbf{Q}} \to \mathbf{Q}$ that is strictly convex with respect to $\Sigma$. The cones of $\Sigma^{(n)}$ are called *chambers* and their faces of codimension 1 *walls*. The function $g$ allows us to order the chambers of $\Sigma$ in a certain special way, as follows. We choose a point $x_0 \in N_{\mathbf{Q}}$ in general position; then for two chambers $\sigma$ and $\sigma'$ of $\Sigma$ we say that $\sigma' > \sigma$ if $m_{\sigma'}(x_0) > m_\sigma(x_0)$, where $m_\sigma$, $m_{\sigma'} \in M_{\mathbf{Q}}$ are the linear functions that define $g$ on $\sigma$ and $\sigma'$. A wall $\tau$ of a chamber $\sigma$ is said to be *positive* if $\sigma' > \sigma$, where $\sigma'$ is the chamber next to $\sigma$ through $\tau$. We denote by $\gamma(\sigma)$ the intersection of all positive walls of a chamber $\sigma$.

**10.5.1.** LEMMA. *Let $\sigma$ and $\sigma'$ be chambers of $\Sigma$, and suppose that $\sigma' \supset \gamma(\sigma)$. Then $\sigma' \geqslant \sigma$.*

PROOF. Passing to the star of $\gamma(\sigma)$, we may assume that $\gamma(\sigma) = \{0\}$. Then all the walls of $\sigma$ are positive, which is equivalent to $x_0 \in \sigma$. Since $g$ is convex, it then follows that $m_{\sigma'}(x_0) \geqslant m_\sigma(x_0)$ for any chamber $\sigma'$.

**10.6.** PROPOSITION. *For a projective toric variety $X$ the cycles $[F_{\gamma(\sigma)}]$ with $\sigma \in \Sigma^{(n)}$ generate $H_*(X, \mathbf{Z})$.*

PROOF. Let $\mathbf{T}_\tau$ denote the stratum of $X$ associated with $\tau \in \Sigma$ (see 5.7). For a chamber $\sigma$ we set

$$C(\sigma) = \bigcup_{\sigma \supset \tau \supset \gamma(\sigma)} \mathbf{T}_\tau .$$

It is easy to see that $C(\sigma)$ is isomorphic to the affine space $\mathbf{A}^{\operatorname{codim} \sigma}$, and that the closure of $C(\sigma)$ is $F_{\gamma(\sigma)}$. We form the following filtration $\Phi$ of $X$:

$$\Phi(\sigma) = \bigcup_{\sigma \leqslant \sigma'} C(\sigma').$$

**10.6.1.** LEMMA. *The filtration $\Phi$ is closed and exhaustive.*

To check that $\Phi(\sigma)$ is closed it is enough to show that the closure of $C(\sigma)$, that is, $F_{\gamma(\sigma)}$ is contained in $\Phi(\sigma)$. The variety $F_{\gamma(\sigma)}$ consists of the $\mathbf{T}_\tau$ with $\tau \supset \gamma(\sigma)$, and it remains to find for each $\tau$ containing $\gamma(\sigma)$ a chamber $\sigma'$ such that $\sigma' \supset \tau \supset \gamma(\sigma')$ and $\sigma' \geqslant \sigma$. To do this we take $\sigma'$ to be the minimal chamber in the star of $\tau$; then, firstly, $\sigma' \supset \tau \supset \gamma(\sigma')$, and, secondly, $\sigma' \supset \tau \supset \gamma(\sigma)$, from which it follows according to Lemma 10.5.1 that $\sigma' \geqslant \sigma$.



That $\Phi$ is exhaustive is completely obvious, since $C(\sigma) = F_{\{0\}} = X$ if $\sigma$ is the minimal chamber in $\Sigma^{(n)}$.

We return to the proof of Proposition 10.6. Let us show by induction that the homology of $\Phi(\sigma)$ is generated by the cycles $[F_{\gamma(\sigma')}]$ with $\sigma' \geqslant \sigma$. Let $\sigma_0$ be the chamber immediately following $\sigma$ in the order. Everything follows from considering the exact homology sequence of the pair $(\Phi(\sigma), \Phi(\sigma_0))$, which, as is easy to see, is equivalent to the pair $(S^{2 \operatorname{codim} \sigma}$, point). This proves the proposition.

**10.6.2.** REMARK. In fact, the cycles $[F_{\gamma(\sigma)}]$ with $\sigma \in \Sigma^{(n)}$ form a basis of $H_*(X, \mathbf{Z})$. This is easy to obtain if we use the intersection form on $X$ and the dual cell decomposition connected with the reverse order on $\Sigma^{(n)}$. From this we can deduce formulae that connect the Betti numbers of $X$ with the numbers of cones in $\Sigma$ of a given dimension; we obtain these below without assuming $X$ to be projective.

**10.7.** Up to now we have said nothing about intersections; it is time to turn to these. As usual, we set $A^k(X) = A_{n-k}(X)$, and $A^*(X) = \bigoplus_k A^k(X)$. To specify the intersections on $X$ means to equip $A^*(X)$ with the structure of a graded ring.

Among the varieties $F_\sigma$ the most important are the divisors, that is, the $F_\sigma$ with $\sigma \in \Sigma^{(1)}$. We identify temporarily $\Sigma^{(1)}$ with the set of primitive vectors of $N$ lying on the rays $\sigma \in \Sigma^{(1)}$; for such a vector $e \in \Sigma^{(1)}$ we denote by $D(e)$ the divisor $F_{\langle e \rangle}$. To begin with we consider the intersections of such divisors. It is obvious that if $\langle e_1, \ldots, e_k \rangle \in \Sigma^{(k)}$, then the intersection of $D(e_1), \ldots, D(e_k)$ is transversal and equal to $F_{\langle e_1, \ldots, e_k \rangle}$. But if $\langle e_1, \ldots, e_k \rangle$ does not belong to $\Sigma$, then the intersection of $D(e_1), \ldots, D(e_k)$ is empty. Finally, among the divisors $D(e)$ we have the following relations: $\operatorname{div}(x^m) = \sum_e m(e) D(e)$ for each $m \in M$, and hence, $\sum_e m(e)[D(e)] = 0$. Using these observations, let us formally construct a certain ring.

With each $e \in \Sigma^{(1)}$ we associate a variable $U_e$, and let $\mathbf{Z}[U] = \mathbf{Z}[U_e, e \in \Sigma^{(1)}]$ be the polynomial ring in these variables. Next, let $I \subset \mathbf{Z}[U]$ be the ideal generated by the monomials $U_{e_1} \cdot \ldots \cdot U_{e_k}$ such that $\langle e_1, \ldots, e_k \rangle \notin \Sigma$. Finally, let $J \subset \mathbf{Z}[U]$ be the ideal generated by the linear forms $a(m) = \sum_e m(e) U_e$, as $m$ ranges over $M$ (however, it would be enough to take $m$ from a basis of $M$).

**10.7.1.** LEMMA. *The ring $\mathbf{Z}[U]/(I + J)$ is generated by the monomials $U_{e_1} \cdot \ldots \cdot U_{e_k}$, where all the $e_1, \ldots, e_k$ are distinct.*

The proof is by induction on the number of coincidences in $U_{e_1} \cdot \ldots \cdot U_{e_k}$.

Suppose, for example, that $e_1 = e_2$. We choose $m \in M$ such that $m(e_1) = -1$ and $m(e_i) = 0$ for $e_i \neq e_1$. Replacing $U_{e_1}$ by the sum $\sum_{e \neq e_1} m(e)U_e$, which does not involve $e_1, \ldots, e_k$ we reduce the number of coincidences in the terms of the sum $\sum_{e \neq e_1} m(e)U_e U_{e_2} \cdot \ldots \cdot U_{e_k}$. This proves the lemma.

10.7.2. COROLLARY. *The* **Z**-*module* $\mathbf{Z}[U]/(I+J)$ *is finitely generated.*

From the definition of rational equivalence it is clear that by assigning to the monomial $U_{e_1} \cdot \ldots \cdot U_{e_k}$, with $\langle e_1, \ldots, e_k \rangle \in \Sigma^{(k)}$, the cycle $[F_{\langle e_1, \ldots, e_k \rangle}]$ we can obtain a surjective group homomorphism

$$\mathbf{Z}[U]/(I+J) \to A^*(X). \tag{10.7.1}$$

In the projective case, Jurkiewicz [25] has shown that this is an isomorphism. We will show here that this assertion is true for any complete $X_\Sigma$; the isomorphism so obtained defines in $A^*(X)$ a ring structure. To see this we consider also the natural homomorphism

$$A^*(X) \to H^*(X, \mathbf{Z}), \tag{10.7.2}$$

which is defined by the Poincaré duality

$$A^k(X) = A_{n-k}(X) \to H_{2n-2k}(X, \mathbf{Z}) \xrightarrow{\sim} H^{2k}(X, \mathbf{Z})$$

and is surjective according to Proposition 10.4.

10.8. THEOREM. *Let X be a complete smooth toric variety over* **C**. *Then the homomorphisms* (10.7.1) *and* (10.7.2) *are isomorphisms, and the* **Z**-*modules*

$$\mathbf{Z}(U]/(I+J) \xrightarrow{\sim} A^*(X) \xrightarrow{\sim} H^*(X, \mathbf{Z})$$

*are torsion-free. If $a_i = \#(\Sigma^{(i)})$ is the number of i-dimensional cones in $\Sigma$, then the rank of the free* **Z**-*module $A^k(X)$ is equal to*

$$\operatorname{rk} A^k(X) = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} a_{n-i}.$$

PROOF. Let us show, first of all, that the **Z**-module $\mathbf{Z}[U]/(I+J)$ is torsion-free and find its rank. For this purpose we check that for any prime $p$ multiplication by $p$ in $\mathbf{Z}[U]/(I+J)$ is injective. Let $m_1, \ldots, m_n$ be a basis of $M$; it is enough to show that $a(m_1), \ldots, a(m_n), p$ is a regular sequence in $\mathbf{Z}[U]/I$. Since $p$ is obviously not a zero-divisor in $\mathbf{Z}[U]/I$, it remains to check that the sequence $a(m_1), \ldots, a(m_n)$ is regular in $(\mathbf{Z}/p\mathbf{Z})[U]/I$. But according to 3.8, this is a Cohen–Macaulay ring, and the sequence is regular because $(\mathbf{Z}/p\mathbf{Z})[U]/(I+J)$ is finite-dimensional (see Corollary 10.7.2). It also follows from 3.8 that the rank of $\mathbf{Z}[U]/(I+J)$ is $a_n$, the number of chambers of $\Sigma$. On the other hand, the rank of $H^*(X, \mathbf{Z})$ is equal to the Euler characteristic of $X$, which is also equal to $a_n$ (see §11 or §12). It follows from this that the epimorphisms (10.7.1) and (10.7.2) are isomorphisms. The formulae for the ranks of the $A^k(X)$ also follow from 3.8.



10.9. REMARK. Up to now we have assumed that $X$ is a smooth variety. However, the preceding arguments can be generalized with almost no change to the varieties $X_\Sigma$ associated with complete simplicial fans $\Sigma$. The only thing we must do is replace the coefficient ring **Z** by **Q**. There are several reasons for this. The first is Proposition 10.4. The second is that the multiplication table for the cycles $D(e)$ has rational coefficients: if $\sigma = \langle e_1, \ldots, e_k \rangle \in \Sigma^{(k)}$ then

$$D(e_1) \cdot \ldots \cdot D(e_k) = \frac{1}{\operatorname{mult}(\sigma)} [F_\sigma].$$

Finally, Poincaré duality for $X_\Sigma$ holds over **Q** (see §14). Taking account of these remarks we again have isomorphisms

$$\mathbf{Q}[U]/I+J \xrightarrow{\sim} A^*(X)_\mathbf{Q} \xrightarrow{\sim} H^*(X, \mathbf{Q})$$

together with the formulae of Theorem 10.8 for the dimension of $A^k(X)_\mathbf{Q} \xrightarrow{\sim} H^{2k}(X, \mathbf{Q})$, which we will obtain once more in §12.

## §11. The Riemann–Roch theorem

11.1. Let $X$ be a complete variety over a field $K$, and let $\mathfrak{F}$ be a coherent sheaf over $X$. The *Euler–Poincaré characteristic* of $\mathfrak{F}$ is the integer

$$\chi(X, \mathfrak{F}) = \chi(\mathfrak{F}) = \sum_{i \geq 0} (-1)^i \dim_K H^i(X, \mathfrak{F}).$$

The Chow ring $A^*(X)$, which we considered in the last section, is also interesting in that $\chi(X, \mathfrak{F})$ can be expressed in terms of the intersection of algebraic cycles on $X$. This is precisely the content of the Riemann–Roch theorem (see [13] or [32]): *if X is a smooth projective variety, then*

$$\chi(X, \mathfrak{F}) = (\operatorname{ch}(\mathfrak{F}), \operatorname{Td}(X)).$$

Here $\operatorname{ch}(\mathfrak{F})$ and $\operatorname{Td}(X)$ are certain elements of $A^*(X)_\mathbf{Q}$ called, respectively, *the Chern character of* $\mathfrak{F}$ and *the Todd class of X*; and the bracket on the right-hand side denotes the intersection form on $A^*(X)_\mathbf{Q}$, that is, the composite of multiplication in $A^*(X)_\mathbf{Q}$ with the homomorphism $A^*(X)_\mathbf{Q} \to A^*(\text{point})_\mathbf{Q} = \mathbf{Q}$. We apply this theorem to invertible sheaves on a toric variety $X = X_\Sigma$. If for $\mathscr{E} \in \operatorname{Pic} X$ the function $g = \operatorname{ord}(\mathscr{E})$ is convex on $|\Sigma| = N_\mathbf{Q}$, then $H^i(X, \mathscr{E}) = 0$ for $i > 0$, and the Riemann–Roch theorem gives a certain expression for the dimension of $H^0(X, \mathscr{E}) = L(\Delta_g)$, that is, for the number of integral points in the convex polyhedron $\Delta_g \subset M_\mathbf{Q}$.

To begin with, we explain the terms in the Riemann–Roch theorem.

11.2. The Chern character $\operatorname{ch}(\mathscr{E})$ of an invertible sheaf $\mathscr{E}$ is the element of $A^*(X)_\mathbf{Q}$ given by the formula

$$\operatorname{ch}(\mathscr{E}) = e^{[D]} = 1 + [D] + \frac{1}{2!}[D]^2 + \ldots + \frac{1}{n!}[D]^n,$$

where $D$ is a divisor on $X$ such that $\mathscr{E} = O_X(D)$.

11.3. **Chern classes.** Chern classes are needed to define the Todd class. They are given axiomatically by associating with each coherent sheaf $\mathfrak{F}$ on $X$ an element $c(\mathfrak{F}) \in A^*(X)$ so that the following conditions hold:

a) *naturality*: when $f\colon X \to Y$ is a morphism, then

$$c(f^*(\mathfrak{F})) = f^*(c(\mathfrak{F})),$$

b) *multiplicativity*: for each exact sequence of sheaves $0 \to \mathfrak{F}' \to \mathfrak{F} \to \mathfrak{F}'' \to 0$ we have

$$c(\mathfrak{F}) = c(\mathfrak{F}')c(\mathfrak{F}''),$$

c) *normalization*: for a divisor $D$ on $X$ we have

$$c(O(D)) = 1 + [D].$$

The $k$th component of $c(\mathfrak{F})$ is denoted by $c_k(\mathfrak{F})$ and is called the $k$th *Chern class* of $\mathfrak{F}$; $c_0(\mathfrak{F}) = 1$.

Of greatest interest are the Chern classes of the cotangent sheaf $\Omega^1_X$, and also of the sheaves $\Omega^p_X$ and $\check{\Omega}^p_X$, because they are invariantly linked with $X$. The Chern classes of the tangent sheaf $\check{\Omega}^1_X$ are called the *Chern classes of* $X$ and are denoted by $c(X)$. Let us compute $c(X)$ for a complete smooth toric variety $X = X_\Sigma$. Let $e_1, \ldots, e_r$ be all the primitive vectors of $\Sigma^{(1)}$ (see **10.7**), and let $D(e_i)$ be the corresponding divisors on $X$.

**11.4.** PROPOSITION. $c(\Omega^1_X) = \prod_i (1 - D(e_i))$.

PROOF. Let $D$ denote the union of all the $D(e_i)$; then $X - D = \mathbf{T}$. We consider the sheaf $\Omega^1_X(\log D)$ of 1-differentials of $X$ with logarithmic poles along $D$ (see § 15). The sheaf $\Omega^1_X$ is naturally included in $\Omega^1_X(\log D)$, and the Poincaré residue (see § 15) gives an isomorphism

$$\Omega^1_X(\log D)/\Omega^1_X \xrightarrow{\sim} \bigoplus_i O_{D(e_i)}.$$

Using b) we get

$$c(\Omega^1_X) = c(\Omega^1_X(\log D)) \cdot \prod_i c(O_{D(e_i)})^{-1}.$$

Since the sections $\dfrac{dX_1}{X_1}, \ldots, \dfrac{dX_n}{X_n}$ are a basis of $\Omega^1_X(\log D)$, this sheaf is free, and $c(\Omega^1_X(\log D)) = 1$. To find $c(O_{D(e_i)})$ we use the exact sequence

$$0 \to O_X(-D(e_i)) \to O_X \to O_{D(e_i)} \to 0.$$

From this we see that $c(O_{D(e_i)}) = (1 - D(e_i))^{-1}$, as required.

**11.5.** COROLLARY. $c_k(X) = \sum_{\sigma \in \Sigma(k)} [F_\sigma]$.

PROOF. By the preceding proposition, $c(X) = c(\check{\Omega}^1_X) = \prod_i (1 + D(e_i))$, and it remains to expand the product using the multiplication table in **10.7**.

In particular, $c_n(X) = \sum_{\sigma \in \Sigma(n)} [F_\sigma]$ consists of $a_n$ points, where $a_n = \#(\Sigma^{(n)})$ is the number of chambers of $\Sigma$. Since the degree of $c_n(X)$ is equal to the Euler characteristic $E(X)$ of $X$ (see [13], 4.10.1), we obtain the following corollary.



**11.6.** COROLLARY. *For a variety $X = X_\Sigma$ over* $\mathbf{C}$ *the Euler characteristic* $E(X) = \sum_i (-1)^i \dim H^i(X, \mathbf{C})$ *is equal to* $a_n$, *the number of $n$-dimensional cones in* $\Sigma$.

**11.7.** The Todd class is a method of associating with each sheaf $\mathfrak{F}$ a certain element $\mathrm{Td}(\mathfrak{F}) \in A^*(X)_{\mathbf{Q}}$ such that the conditions a) and b) of **11.3** hold, and the normalization c) is replaced by the following condition:

c′) for a divisor $D$ on $X$ we have

$$\mathrm{Td}(O_X(D)) = \frac{D}{1 - e^{-D}} = \Big(\sum_{i \geqslant 0} (-1)^i \frac{D^i}{(i+1)}\Big)^{-1} = 1 + \frac{D}{2} + \frac{D^2}{12} - \frac{D^4}{720} + \ldots .$$

The Todd class can be expressed in terms of the Chern classes:

$$\mathrm{Td}(\mathfrak{F}) = 1 + \frac{c_1(\mathfrak{F})}{2} + \frac{c_2(\mathfrak{F}) + c_1(\mathfrak{F})^2}{12} + \frac{c_1(\mathfrak{F})\, c_2(\mathfrak{F})}{24} + \ldots$$

By the Todd class of a variety $X$ we mean the Todd class of its tangent sheaf, $\mathrm{Td}(X) = \mathrm{Td}(\check{\Omega}^1_X)$.

**11.8.** All this referred to smooth varieties. However, if we are interested only in invertible sheaves on complete toric varieties, we can often reduce everything to the smooth case. Let $\Sigma'$ be a subdivision of $\Sigma$ such that the variety $X' = X_{\Sigma'}$ is smooth and projective. Applying Proposition 8.5.1 to the morphism $f\colon X' \to X$ and to the invertible sheaf $\mathcal{E}$ on $X$ we obtain the formula

$$\chi(X, \mathcal{E}) = \chi(X', f^*(\mathcal{E})) = (\mathrm{ch}(f^*\mathcal{E}), \mathrm{Td}(X')).$$

A consequence of this is the following proposition, which was proved by Snapper and Kleiman for any complete variety.

**11.9.** PROPOSITION. *Let $L_1, \ldots, L_k$ be invertible sheaves on a complete toric variety $X$. Then $\chi(L_1^{\otimes \nu_1} \times \ldots \times L_k^{\otimes \nu_k})$ is a polynomial of degree $\leqslant n = \dim X$ in the (integer) variables $\nu_1, \ldots, \nu_k$.*

PROOF. We may suppose that $X$ is smooth. Let $L_i = O(D_i)$, where the $D_i$ are divisors on $X$. According to the Riemann–Roch formula it is enough to check that $\mathrm{ch}(O(\sum_i \nu_i D_i))$ polynomially depends on $\nu_1, \ldots, \nu_k$ and that its total degree is at most $n$. But according to **11.2**

$$\mathrm{ch}\Big(O\Big(\sum \nu_i D_i\Big)\Big) = 1 + \Big[\sum \nu_i D_i\Big] + \cdots + \frac{1}{n!}\Big[\sum \nu_i D_i\Big]^n,$$

and then everything is obvious.

From the preceding argument it is clear that if $k = n$, then the coefficient of the monomial $\nu_1, \ldots, \nu_k$ in $\chi(O(\sum_i \nu_i D_i))$ is equal to the coefficient of the same monomial in the expression

$$\Big(\mathrm{ch}_n\Big(O\Big(\sum \nu_i D_i\Big)\Big),\ \mathrm{Td}_0(X)\Big) = \frac{1}{n!}\Big[\sum \nu_i D_i\Big]^n$$

(since $\mathrm{Td}_0(X) = 1$), which in turn is equal to the degree of the product $D_1 \cdot \ldots \cdot D_n$, that is, to the so-called *intersection number* of the divisors

$D_1, \ldots, D_n$. Denoting this by $(D_1, \ldots, D_n)$, we have the following corollary:

**11.10.** COROLLARY. *For $n$ divisors $D_1, \ldots, D_n$ on $X_\Sigma$ the intersection number $(D_1, \ldots, D_n)$ is equal to the coefficient of $\nu_1 \ldots \nu_k$ in $\chi(O(\Sigma \nu_i(D_i))$.*

For an arbitrary complete variety the assertion of Corollary 11.10 is taken as the *definition* of $(D_1, \ldots, D_n)$.

**11.11.** COROLLARY. *The self-intersection number $(D^n) = (D, \ldots, D)$ of a divisor $D$ on $X$ is equal to $n!a$, where $a$ is the coefficient of $\nu^n$ in the polynomial $\chi(O(\nu D))$.*

For $(D^n)$ is the coefficient of $\nu_1 \ldots \nu_n$ in

$$\chi(O((\nu_1 + \ldots + \nu_n)D)) = a \cdot (\nu_1 + \ldots + \nu_n)^n + \ldots,$$

that is, $n!\, a$.

**11.12.** Let us apply these corollaries to questions concerning the number of integral points in convex polyhedra. Suppose that $\Delta$ is an *integral* (see **1.4**) polyhedron in $M$, and $\Sigma = \Sigma_\Delta$ the fan in $N_{\mathbf{Q}}$ associated with $\Delta$ (see **5.8**). Since the vertices of $\Delta$ are integral, they define a compatible system $\{m_\sigma\}$ in the sense of **6.2**, and hence also an invertible sheaf $\mathscr{E}$ on $X = X_\Sigma$ (together with a trivialization). The function $\operatorname{ord}(\mathscr{E})$ is convex, therefore, $\chi(\mathscr{E})$ is equal to the dimension of the space $H^0(X, \mathscr{E}) = L(\Delta)$, that is, to the number of integral points in $\Delta$. Since addition of polyhedra corresponds to tensor multiplication of the corresponding invertible sheaves, Corollary 11.10 can be restated as follows:

**11.12.1.** COROLLARY. *The number of integer points of the polyhedron $\Sigma \nu_i \Delta_i$ is a polynomial of degree $\leqslant n$ in $\nu_1, \ldots, \nu_k \geqslant 0$.*

This fact was obtained by different arguments by McMullen [29] and Bernshtein [4].

Corollary 11.11 implies that the self-intersection number $(D^n)$ of the divisor $D$ on $X$ corresponding to $\Delta$ is $n!a$, where $a$ is the coefficient of $\nu^n$ in $l(\nu\Delta)$ $(\nu \geqslant 0)$. As is easy to see, $a$ coincides with $V_n(\Delta)$, the $n$-dimensional volume of $\Delta$, measured with respect to the lattice $M$. So we obtain the following result.

**11.12.2.** $(D^n) = n! \cdot V_n(\Delta)$.

In general, if the divisors $D_1, \ldots, D_n$ correspond to integral polyhedra $\Delta_1, \ldots, \Delta_n$, then (see also [9])

$$(D_1, \ldots, D_n) = n! \cdot (\text{mixed volume of } \Delta_1, \ldots, \Delta_n).$$

Let us subdivide $\Sigma$ to a regular fan $\Sigma'$. The Todd class $\operatorname{Td}(X_{\Sigma'})$ is a certain combination of the cycles $F_\sigma$ with $\sigma \in \Sigma'$:

$$\operatorname{Td}(X_{\Sigma'}) = \sum_\sigma r_\sigma \cdot [F_\sigma], \ r_\sigma \in \mathbf{Q}.$$

According to the Riemann–Roch formula,

$$\chi(O(D)) = \sum_\sigma \frac{r_\sigma}{(\operatorname{codim} \sigma)!} (D^{\operatorname{codim} \sigma}, [F_\sigma]).$$

If a divisor $D$ corresponds to a polyhedron $\Delta$, then for $\sigma \in \Sigma^{(n-k)}$ the intersection number $(D^k, [F_\sigma])$ is nothing other than $k! V_k(\Gamma_\sigma)$. Here $\Gamma_\sigma$ is the unique $k$-dimensional face of $\Delta$ for which $\sigma \subset \sigma_\Gamma$, and $V_k$ is its $k$-dimensional volume. So we obtain the formula

**11.12.3.** $$l(\Delta) = \sum_\sigma r_\sigma \cdot V_{\operatorname{codim} \sigma}(\Gamma_\sigma).$$

This formula expresses the number of integral points of $\Delta$ in terms of the volumes of its faces. Unfortunately, the numbers $r_\sigma$ are not uniquely determined, and their explicit computation remains an open question (for example, can one say that the $r_\sigma$ depend only on $\sigma$ and not on the fan $\Sigma$?). In the simplest two-dimensional case we obtain for an integral polygon $\Delta$ in the plane the well known and elementary formula

$$l(\Delta) = \text{area}(\Delta) + \tfrac{1}{2}(\text{perimeter}(\Delta)) + 1.$$

Of course, the "length" of each side of $\Delta$ is measured with respect to the induced one-dimensional lattice.

**11.12.4. The inversion formula.** Let $\Delta$ be an $n$-dimensional polyhedron in $M$, and $P(t)$ the polynomial such that $P(\nu) = l(\nu\Delta)$ for $\nu \geqslant 0$ is the number of integral points in $\nu\Delta$. Then $(-1)^n P(-\nu)$ *for $\nu > 0$ is the number of integral points strictly within $\nu\Delta$.*

This is the so-called *inversion formula* (see [19] and [29]). For the proof we again take a regular fan $\Sigma'$ subdividing $\Sigma_\Delta$ and the divisor $D$ on $X_{\Sigma'}$ corresponding to $\Delta$. By Serre duality,

$$(-1)^n P(-\nu) = (-1)^n \chi(O(-\nu D)) = \chi(O(\nu D) \otimes \omega).$$

Now we use the exact sequence (see **6.6**)

$$0 \to \omega_X \to O_X \to O_{D_\infty} \to 0,$$

where $D_\infty = \bigcup_{\sigma \neq \{0\}} F_\sigma$. We obtain $\chi(O(\nu D) \otimes \omega) = \chi(O(\nu D)) - \chi(O(D_\infty) \otimes O(\nu D))$. The first term is the number of integral points in $\nu\Delta$. The second term (for $\nu > 0$) is easily seen to be the number of integral points on the boundary of $\nu\Delta$.

## § 12. Complex cohomology

Here we consider toric varieties over the field of complex numbers $\mathbf{C}$. In this case the set $X(\mathbf{C})$ of complex-valued points of $X$ is naturally equipped with the strong topology, and we can use complex cohomology $H^*(X, \mathbf{C})$, together with the Hodge structure on it. In contrast to § 10, here we only assume that the toric variety $X = X_\Sigma$ is complete.

**12.1.** As in § 7, we can make use of the covering $\{X_{\check\sigma}\}_{\sigma \in \Sigma}$ to compute the cohomology of $X_\Sigma$. Of course, this covering is not acyclic, but this is no disaster, we only have to replace the complex of a covering by the corresponding spectral sequence (see [7], II, 5.4.1). However, it is preferable here to use a somewhat modified spectral sequence, which is more economical and reflects the essence of the matter. This modification is based on using the "simplicial" structure of $\Sigma$. Let us say more about it.

By a contravariant functor on $\Sigma$ we mean a way of associating with each cone $\sigma \in \Sigma$ an object $F(\sigma)$, and with each inclusion $\tau \subset \sigma$ a morphism $\varphi_{\tau,\sigma}: F(\sigma) \to F(\tau)$, such that $\varphi_{\theta,\sigma} = \varphi_{\theta,\tau} \circ \varphi_{\tau,\sigma}$ for $\theta \subset \tau \subset \sigma$. Let $F$ be an additive functor on $\Sigma$; we orient all cones $\sigma \in \Sigma$ arbitrarily and form the complex $C^*(\Sigma, F)$ for which $C^q(\Sigma, F) = \bigoplus_{\sigma \in \Sigma^{(n-q)}} F(\sigma)$, and the differential $d: C^q(\Sigma, F) \to C^{q+1}(\Sigma, F)$ is composed in the usual way from the maps $\pm\varphi_{\tau,\sigma}: F(\sigma) \to F(\tau)$, where $\tau$ ranges over the faces of $\sigma$ of codimension 1, and the sign + or − is chosen according as the orientations of $\tau$ and $\sigma$ agree or disagree. The cohomology of the complex $C^*(\Sigma, F)$ is denoted by $H^*(\Sigma, F)$.

Now let $H^q(\widetilde{\mathbf{C}})$ be the functor on $\Sigma$ that associates with each $\sigma \in \Sigma$ the vector space $H^q(X_{\check\sigma}, \mathbf{C})$.

12.2. THEOREM. *There is a spectral sequence*

$$E_1^{pq} = C^p(\Sigma, H^q(\widetilde{\mathbf{C}})) \Rightarrow H^{p+q}(X, \mathbf{C}).$$

We briefly explain how this spectral sequence is constructed. Unfortunately, I have been unable to copy the construction of the spectral sequence for an open covering, therefore, the first trick consists in replacing an open covering by a closed one. To do this we replace our space $X$ by another topological space $\hat{X}$.

The space $\hat{X}$ consists of pairs $(x, p) \in X \times |\Sigma|$ (here once more $|\Sigma|$ is a subspace of $N_{\mathbf{R}}$ rather than $N_{\mathbf{Q}}$) such that $x \in X_{\check\sigma}$, where $\sigma$ is the smallest cone of $\Sigma$ that contains $p \in |\Sigma|$. In other words, $\hat{X}$ is a fibering over $|\Sigma| = N_{\mathbf{R}}$, and the fibre over a point $p$ lying strictly inside a cone $\sigma$ is the affine toric variety $X_{\check\sigma}$. The projections of $X \times |\Sigma|$ onto its factors give two continuous maps $\rho: \hat{X} \to X$ and $\pi: \hat{X} \to |\Sigma|$. We define $\hat{X}_\sigma$ as $\pi^{-1}(\sigma)$; obviously, $\hat{X}_\sigma$ is a closed subset of $\hat{X}$.

12.2.1. LEMMA. $\rho^*: H^*(X_{\check\sigma}, \mathbf{C}) \to H^*(\hat{X}_\sigma, \mathbf{C})$ *is an isomorphism.*

PROOF. Let $p$ be some point strictly inside $\sigma$; by assigning to a point $x \in X_{\check\sigma}$ the pair $(x, p)$ we define a section $s: X_{\check\sigma} \to \hat{X}_\sigma$ of $\rho: \hat{X}_\sigma \to X_{\check\sigma}$. On the other hand, it is obvious that the embedding $s$ is a deformation retract. (The deformation of $\hat{X}_\sigma$ to $s(X_{\check\sigma})$ proceeds along the rays emanating from $p$.)

12.2.2. COROLLARY. $\rho^*: H^*(X, \mathbf{C}) \to H^*(\hat{X}, \mathbf{C})$ *is an isomorphism.*

For $\rho^*$ effects an isomorphism of the spectral sequences of the coverings $\{X_{\check\sigma}\}$ and $\{\hat{X}_\sigma\}$.

Now replacing $X$ by $\hat{X}$ and $X_{\check\sigma}$ by $\hat{X}_\sigma$, we need only construct the corresponding spectral sequence for $\hat{X}$. For this purpose we consider the functor $\widetilde{\mathbf{C}}$ on $\Sigma$ that associates with each cone $\sigma \in \Sigma$ the sheaf $\mathbf{C}_{\hat{X}_\sigma}$ on $\hat{X}$, the constant sheaf on $\hat{X}_\sigma$ with stalk $\mathbf{C}$ extended by zero to the whole of $\hat{X}$, and



associates with an inclusion $\tau \subset \sigma$ the restriction homomorphism $\mathbf{C}_{\hat{X}_\sigma} \to \mathbf{C}_{\hat{X}_\tau}$. As we have explained in 12.1, there arises a complex $C^*(\Sigma, \widetilde{\mathbf{C}})$ of sheaves on $\hat{X}$.

12.2.3. LEMMA. *The complex $C^*(\Sigma, \widetilde{\mathbf{C}})$ is a resolution of the constant sheaf $\mathbf{C}_{\hat{X}}$.*

PROOF. It is enough to prove the lemma point-by-point. But for each point $\hat{x} \in \hat{X}$ the exactness of the sequence

$$0 \to \mathbf{C}_{\hat{X}} \to C^0(\Sigma, \widetilde{\mathbf{C}}) \to C^1(\Sigma, \widetilde{\mathbf{C}}) \to \dots$$

of sheaves over $\hat{x}$ reduces to the fact that $|\Sigma|$ is a manifold at $\pi(\hat{x})$.

Now the required spectral sequence can be obtained as the spectral sequence of the resolution $C^*(\Sigma, \widetilde{\mathbf{C}})$,

$$E_1^{pq} = H^q(\hat{X}, C^p(\Sigma, \widetilde{\mathbf{C}})) \Rightarrow H^{p+q}(\hat{X}, \mathbf{C}).$$

For $H^q(\hat{X}, C^p(\Sigma, \widetilde{\mathbf{C}})) = C^p(\Sigma, H^q(\widetilde{\mathbf{C}}))$. This proves the theorem.

12.2.4. REMARK. There is an analogous spectral sequence for any sheaf over $X_\Sigma$. The spectral sequence of Theorem 12.2 is interesting for two reasons. Firstly, as we shall soon see, its initial term $E_1^{pq}$ has a very simple structure. Secondly, it degenerates at $E_2$.

12.3. LEMMA. $H^*(X_{\check\sigma}, \mathbf{C}) = \Lambda^*(\text{cospan } \check\sigma) \otimes_{\mathbf{Q}} \mathbf{C}$.

PROOF. We represent $\check\sigma$ as the product of the vector space (cospan $\check\sigma$) and of a cone with vertex. Then everything follows from two obvious assertions:

a) if $\check\sigma$ is a cone with vertex, then $X_{\check\sigma}$ is contractible.

b) for $\mathbf{T} = \text{Spec}\, \mathbf{C}[M]$

$$H^*(\mathbf{T}, \mathbf{C}) = \Lambda^*(M \otimes \mathbf{C}).$$

12.4. LEMMA. $H^q(X, \Omega^p_X) = H^q(\Sigma, H^0(\Omega^p))$.

Here $H^0(\Omega^p)$ on the right-hand side denotes the functor on $\Sigma$ that associates with a cone $\sigma \in \Sigma$ the space $H^0(X_{\check\sigma}, \Omega^p_{X_{\check\sigma}}) = \Omega^p_{A_{\check\sigma}}$. To prove this we have to take the spectral sequence analogous to the one in Theorem 12.2 for the sheaf $\Omega^p_X$ on $X$, and to note that owing to Serre's theorem $H^k(\Omega^p) = 0$ for $k > 0$.

12.4.1. REMARK. The functor $H^0(\Omega^p)$ on $X$ takes values in the category of $M$-graded vector spaces. Since according to Corollary 7.5.1 $H^q(X, \Omega^p_X) = H^q(X, \Omega^p_X)(0)$, we see that *$H^q(X, \Omega^p_X)$ is isomorphic to the $q$th cohomology of the complex $C^*(\Sigma, H^0(\Omega^p)(0))$*, which is the complex of the functor on $\Sigma$ that associates with a cone $\sigma \in \Sigma$ the vector space (see (7.5.1))

$$H^0(X_{\check\sigma}, \Omega^p)(0) = \Omega^p_{A_{\check\sigma}}(0) = \Lambda^p(\text{cospan } \check\sigma) \otimes \mathbf{C}.$$

From this we obtain two results:

a) *the complex $C^*(\Sigma, H^0(\Omega^p)(0))$ can be identified with $C^*(\Sigma, H^p(\widetilde{\mathbf{C}}))$;*

b) $H^q(X, \Omega_X^p)$ *is isomorphic to* $H^q(\Sigma, H^p(\widetilde{\mathbf{C}}))$, the $E_2^{pq}$-term of the spectral sequence in Theorem 12.2.

**12.5.** THEOREM. *Let* $X = X_\Sigma$, *where* $\Sigma$ *is a complete fan. Then the Hodge–de Rham spectral sequence* (see § 13)

$$E_1^{pq} = H^q(X, \Omega_X^p) \Rightarrow H^{p+q}(X, \mathbf{C})$$

*degenerates at the* $E_1$*-term (that is,* $E_1 = E_\infty$*).*

PROOF. Using Lemma 12.4 we represent the Hodge–de Rham spectral sequence as the spectral sequence of the double complex

$$E_0^{pq} = C^q(\Sigma, H^0(\Omega^p)).$$

This has one combinatorial differential $E_0^{pq} \to E_0^{p,q+1}$ (see **12.1**), and the other differential $E_0^{pq} \to E_0^{p+1,q}$ comes from the exterior derivative $d: \Omega^p \to \Omega^{p+1}$ (see **4.4**). As already mentioned, all the terms carry an $M$-graded structure, and the action of the differentials is compatible with this grading. Thus, the spectral sequence $E$ also splits into a sum of spectral sequences, $E = \bigoplus_{m \in M} E(m)$. It remains to check that each of the $E(m)$ degenerates at the $E_1(m)$-term. We consider separately the cases $m \neq 0$ and $m = 0$.

$m \neq 0$. In this case already $E_1(m) = 0$. For (see **12.4.1**), $E_1^{pq}(m) = H^q(X, \Omega_X^p)(m) = 0$.

$m = 0$. In this case the second differential $E_0^{pq}(0) \to E_0^{p+1,q}(0)$ is zero. For over $m \in M$ the exterior derivative $d: \Omega^p \to \Omega^{p+1}$ acts as multiplication by $m$ (see **4.4**); over $m = 0$ this is zero. This proves the theorem.

This theorem confirms the conjecture in § 13 in the case of toric varieties. What is more important, it implies the following result.

**12.6.** THEOREM. *The spectral sequence of Theorem* 12.2 *degenerates at the* $E_2$-term, that is, $E_2 = E_\infty$.

PROOF. Everything follows from the equalities

$$\dim H^k(X, \mathbf{C}) = \sum_{p+q=k} \dim H^q(X, \Omega_X^p) = \sum_{p+q=k} \dim E_2^{qp}.$$

The first of these follows from Theorem 12.5, and the second from 12.4.1 b).

**12.6.1.** REMARK. There is (or there should be) a deeper reason for the degeneration of the spectral sequence of Theorem 12.2. It consists in the fact that since the open sets $X_{\check{\sigma}}$ that figure in the construction of the spectral sequence of Theorem 12.2 are *algebraic subvarieties* of $X$, *it is a spectral sequence of Hodge structures.* In particular, all the differentials should be morphisms of Hodge structures. On the other hand, the Hodge structure on $H^0(X_{\check{\sigma}}, \mathbf{C})$ is the same as that of some torus (see the proof of Lemma 12.3), and its type is $(p, p)$ (see [15]). Hence it follows that all the differentials $d_i$ for $i > 1$ should change the Hodge type, and thus must be 0. Furthermore, we find that $H^q(\Sigma, H^p(\widetilde{\mathbf{C}}))$ can be identified with the part "of weight $2p$" in $H^{p+q}(X, \mathbf{C})$.

Here are a few consequences of the preceding results. Note, first of all, that



$$\dim \Lambda^p(\operatorname{cospan} \check{\sigma}) = \binom{\dim \operatorname{cospan} \check{\sigma}}{p} = \binom{\operatorname{codim} \sigma}{p}.$$

Hence, for $p > \operatorname{codim} \sigma$ this number is zero, that is, $C^q(\Sigma, H^p(\widetilde{\mathbf{C}})) = 0$ for $p > q$. So we have the following corollary.

**12.7.** COROLLARY. $H^q(X, \Omega_X^p) = 0$ *for* $q < p$.

Furthermore, we see that the "weight" of $H^k(X, \mathbf{C})$ is not greater than $k$, which is as it should be for a complete variety.

In general, if $a_i = \#(\Sigma^{(i)})$ is the number of $i$-dimensional cones of $\Sigma$, then

$$\dim C^q(\Sigma, H^p(\widetilde{\mathbf{C}})) = a_{n-q} \binom{q}{p}. \tag{12.7.1}$$

Let us use this to compute the Euler characteristic of $X$:

$$E(X) = \sum_{p,q} (-1)^{p+q} \dim E_1^{qp} = \sum_{p,q} (-1)^{p+q} a_{n-q} \cdot \binom{q}{p} =$$
$$= \sum_q (-1)^q a_{n-q} \sum_p (-1)^p \binom{q}{p} = \sum_q (-1)^q a_{n-q} \cdot 0^q = a_n.$$

Once more we obtain the result:

**12.8.** COROLLARY. $E(X) = a_n = \#(\Sigma^{(n)})$.

Here are two more facts for arbitrary complete $X_\Sigma$.

**12.9.** PROPOSITION. $H^q(X, O_X) \cong H^q(\Sigma, H^0(\widetilde{\mathbf{C}})) = 0$ *for* $q > 0$.

This has already been proved in Corollary 7.4, and it also follows from the fact that $|\Sigma|$ is a manifold at 0.

**12.10.** PROPOSITION. *For* $p < n$ *we have*

$$H^n(X, \Omega_X^p) \cong H^n(\Sigma, H^p(\widetilde{\mathbf{C}})) = 0.$$

To prove this we have to show that the map $E_1^{n-1,p} \to E_1^{n,p}$ is surjective, that is, the map

$$\bigoplus_{\sigma \in \Sigma^{(1)}} \Lambda^p(\operatorname{cospan} \check{\sigma}) \to \Lambda^p(\operatorname{cospan} \check{0}) = \Lambda^p(M_{\mathbf{Q}}).$$

In the sum on the left-hand side it is enough to take rays $\sigma \in \Sigma^{(1)}$ that generate $N_{\mathbf{Q}}$. After this everything becomes obvious. For let $e_1, \ldots, e_n$ be the dual basis of $M_{\mathbf{Q}}$. Every element of $\Lambda^p(M_{\mathbf{Q}})$ is a sum of expressions of the form $e_{i_1} \wedge \cdots \wedge e_{i_p}$, and since $p < n$, some $i_0 \in [1, n]$ does not occur among $i_1, \ldots, i_p$. But then $e_{i_1} \wedge \cdots \wedge e_{i_p} \in \Lambda^p(\bigoplus_{i \neq i_0} \mathbf{Q} \cdot e_i)$.

Thus, even in the general case in the $E_2^{pq}$-table for the spectral sequence in Theorem 12.2 there are many zeros. If $\Sigma$ is *simplicial*, then only the diagonal terms $E_2^{pp}$ can be non-zero.

**12.11.** PROPOSITION. *Suppose that* $\Sigma$ *is a simplicial fan. Then*

$$H^q(X, \Omega_X^p) \cong H^q(\Sigma, H^p(\widetilde{\mathbf{C}})) = 0 \textit{ for } p \neq q.$$

PROOF. If $q < p$, this follows from Corollary 12.7. If $q > p$, the space $H^q(X, \Omega_X^p)$ is dual to $H^{n-q}(X, \Omega_X^{n-p})$ (see §14), which is zero by Corollary 12.7. Another way of proving the proposition is as follows: the Hodge structure on $H^k(X, \mathbf{C})$ is pure of weight $k$ (see §14), and it remains to use Remark **12.6.1**.

In particular, the odd-dimensional cohomology groups of $X$ are zero, and for the even-dimensional ones we have the resolution

$$0 \to H^{2k}(X, \mathbf{C}) \to C^k(\Sigma, H^k(\widetilde{\mathbf{C}})) \xrightarrow{d} C^{k+1}(\Sigma, H^k(\widetilde{\mathbf{C}})) \to \dots$$

From this we derive formulae for the Betti numbers of $X_\Sigma$ (when $\Sigma$ is a simplicial fan), which we have already met in Theorem 10.8:

$$\dim H^{2k}(X, \mathbf{C}) = a_{n-k}\binom{k}{k} - a_{n-k-1}\binom{k+1}{k} + \dots = \sum_{i=k}^{n}(-1)^{i-k} a_{n-i}\binom{i}{k}.$$

**12.12.** EXAMPLE. As an illustration let us work out an example of a three-dimensional toric variety. For $\Sigma$ we take the fan $\Sigma_\Delta$, where $\Delta \subset \mathbf{Q}^3$ is the octohedron spanned by the vectors $\pm e_i$, where $e_1, e_2, e_3$ is a basis of $\mathbf{Z}^3$. It is easy to see that $X_\Sigma$ is smooth everywhere, except at the six quadratic singularities corresponding to the vertices of $\Delta$.

From the above results it is clear that essentially we have only to deal with the complex $C^*(\Sigma, H^1(\widetilde{\mathbf{C}}))$. More precisely, we compute the kernel of $d\colon C^1(\Sigma, H^1) \to C^2(\Sigma, H^1)$. To do this, we represent the spaces concerned more geometrically on $\Delta$. Now $C^1(\Sigma, H^1)$ consists of cocycles assigning to each edge of $\Delta$ a vector lying on this edge; similarly, $C^2(\Sigma, H^1)$ consists of the vectors lying on the two-dimensional faces of $\Delta$. The differential $d\colon C^1(\Sigma, H^1) \to C^2(\Sigma, H^1)$ takes for each two-dimensional face the sum (respecting the orientation) of the vectors on the edges that bound this face. A direct computation shows that $\operatorname{Ker} d = H^1(\Sigma, H^1(\widetilde{\mathbf{C}}))$ is one-dimensional. From this we obtain

$$-\dim H^2(\Sigma, H^1(\widetilde{\mathbf{C}})) = a_2 - a_1\cdot\binom{2}{1} + a_0\cdot\binom{3}{1} - \dim H^1(\Sigma, H^1(\widetilde{\mathbf{C}})) = 12 - 8\cdot 2 + 1\cdot 3 - 1 = -2,$$

and

$$\dim H^2(\Sigma, H^2(\widetilde{\mathbf{C}})) = a_1 - a_0\cdot\binom{3}{2} = 8 - 1\cdot 3 = 5.$$

Our final table for the Betti numbers $b_i = \dim H^i(X, \mathbf{C})$ is as follows:



| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $b_i$ | 1 | 0 | 1 | 2 | 5 | 0 | 1 |

We note that $H^3(X, \mathbf{C})$ is of weight 2.

# CHAPTER IV
# THE ANALYTIC THEORY

From now we consider almost always varieties over $\mathbf{C}$. For varieties having toroidal singularities we develop a theory analogous to the Hodge–de Rham theory, which allows us to compare complex cohomology with the cohomology of the sheaves of differential forms $\Omega^p$.

## §13. Toroidal varieties

Let $K$ be an algebraically closed field (from **13.3** onwards $K = \mathbf{C}$), and $X$ an algebraic variety over $K$.

**13.1.** DEFINITION. $X$ is said to be (formally) *toroidal at* $x \in X$ if there exists a pair $(M, \sigma)$, where $M$ is a lattice and $\sigma$ a cone in $M$ with vertex, and a formal isomorphism of $(X, x)$ and $(X_\sigma, 0)$. By this we mean an isomorphism of the completions of the corresponding local rings $\hat{O}_{X,x} \xrightarrow{\sim} \hat{O}_{X_\sigma,0}$. The toric variety $X_\sigma$ is called a *local model* for $X$ at $x$.

A variety is said to be *toroidal* if it is toroidal at each of its points. A toroidal variety is naturally stratified into smooth subvarieties according to the types of local models.

**13.2.** EXAMPLE. Any toric variety is toroidal. This is a simple fact, but not a tautology. Since the definition is local, we can assume our toric variety to be affine and isomorphic to $X_\sigma$, and $x$ to lie on the stratum $X_{\sigma_0} \subset X_\sigma$, where $\sigma_0$ is the cospan of $\sigma$. We represent $\sigma$ as $\sigma_0 \times \sigma_1$, where $\sigma_1$ is a cone with vertex. Next, let $\sigma_0'$ be any basic cone in $\sigma_0$ ($\dim \sigma_0' = \dim \sigma_0$), and let $\sigma' = \sigma_0' \times \sigma_1$. Then $X_\sigma = \mathbf{T} \times X_{\sigma_1}$ can be embedded as an open piece in $X_{\sigma'} = \mathbf{A} \times X_{\sigma_1}$. By a shift in $\mathbf{A}$ any point of $\mathbf{T} \subset \mathbf{A}$ can be moved to the origin, which proves that $(X_\sigma, x)$ is toroidal.

The following fact is likewise trivial. Let $X$ be a toroidal variety, and suppose that a divisor $D$ of $X$ intersects all the strata of $X$ transversally. Then $D$ is also toroidal. More precisely, if $x \in D$, then a local model of $(X, x)$ has the form $X_\sigma$, where $\sigma = \theta \times \sigma'$ and $\theta$ is a ray, and then $X_{\sigma'}$ is a local model of $(D, x)$.

In particular, according to Proposition 6.8 the divisor $D_f$ of the zeros of a general Laurent polynomial $f \in L(\Delta)$ is a toroidal variety.

**13.3.** In the definition of a toroidal variety we could require instead of a formal isomorphism the existence of an *analytic* isomorphism between $(X, x)$ and the local model $(X_\sigma, 0)$ (of course, here $K = \mathbf{C}$). However, it follows from Artin's approximation theorem that the two definitions coincide, so that we do not distinguish between them.

A toroidal variety is normal and Cohen–Macaulay. On it we can define sheaves of differential forms $\Omega_X^p$ and an exterior derivative $d\colon \Omega_X^p \to \Omega_X^{p+1}$, as in §4. This gives rise to the *algebraic de Rham complex*

$$\Omega_X^\cdot = \{\Omega_X^0 \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \dots\}.$$

Note that these are coherent sheaves on $X$ for the Zariski topology.

From now on we assume that $K = \mathbf{C}$. Let $X^{\mathrm{an}}$ denote the analytic space associated with a variety $X$. Then $O_X^{\mathrm{an}}$ is the sheaf of germs of holomorphic functions on $X^{\mathrm{an}}$; for a coherent sheaf $\mathcal{F}$ on $X$ we denote by $\mathcal{F}^{\mathrm{an}} = \mathcal{F} \otimes_{O_X} O_X^{\mathrm{an}}$ the analytic version of $\mathcal{F}$. Extending the $d$ to the analytic version of $\Omega_X^p$ in the natural way, we obtain the *analytic de Rham complex*

$$\Omega_X^{\cdot,\mathrm{an}} = \{\Omega_X^{0,\mathrm{an}} \xrightarrow{d} \Omega_X^{1,\mathrm{an}} \xrightarrow{d} \dots\}.$$

Note that these are now sheaves relative to the strong topology.

13.4. PROPOSITION. *The complex $\Omega_X^{\cdot,\mathrm{an}}$ is a resolution of the constant sheaf $\mathbf{C}_X$ on $X^{\mathrm{an}}$.*

PROOF. That the complex $\mathbf{C} \to \Omega_X^{\cdot,\mathrm{an}}$ is exact can be verified locally, and by going to a local model we may assume that $(X, x) = (X_\sigma, 0)$. We consider the map of $A$-modules $h\colon \Omega_A^{p+1} \to \Omega_A^p$ (where $A = \mathbf{C}[\sigma \cap M]$), that was introduced in the proof of Lemma 4.5. Taking the tensor product with $O_{X,0}^{\mathrm{an}}$ over $A$ we get a homomorphism of $(O_{X,0}^{\mathrm{an}})$-modules $h\colon \Omega_{X,0}^{p+1,\mathrm{an}} \to \Omega_{X,0}^{p,\mathrm{an}}$. Now we consider the action of the operator $d\circ h + h\circ d$. An element of $\Omega_{X,0}^{p,\mathrm{an}}$ is a convergent series $\sum_{m \in \sigma \cap M} \omega_m x^m$, where $\omega_m \in \Lambda^p(V_{\Gamma(m)}) \subset \Lambda^p(M \otimes \mathbf{C})$, and $d\circ h + h\circ d$ takes it into the series $\sum_m \lambda(m)\omega_m x^m$. For $p > 0$ this transformation has an inverse, which takes the series $\Sigma\omega_m x^m$ into the series $\Sigma \frac{1}{\lambda(m)} \omega_m x^m$, which obviously converges. This shows that the complex is acyclic in its positive terms. The fact that the kernel of $d\colon O_{X,0}^{\mathrm{an}} \to \Omega_{X,0}^{1,\mathrm{an}}$ is just $\mathbf{C}$ is obvious.

13.5. COROLLARY. *For a toroidal variety $X$ there is a Hodge-de Rham spectral sequence*

$$E_1^{pq} = H^q(X^{\mathrm{an}}, \Omega_X^{p,\mathrm{an}}) \Rightarrow H^{p+q}(X^{\mathrm{an}}, \mathbf{C}).$$

Supposing $X$ to be complete and using the results of *GAGA*, this spectral sequence can be rewritten as

$$E_1^{pq} = H^q(X, \Omega_X^p) \Rightarrow H^{p+q}(X^{\mathrm{an}}, \mathbf{C}). \tag{13.5.1}$$



Thus, the problem of computing the cohomology of a toroidal variety becomes almost algebraic. The word "almost" could be removed if the following were proved:

13.5.1. CONJECTURE. *For a complete toroidal algebraic variety $X$ the spectral sequence* (13.5.1) *degenerates at the $E_1$-term and converges to the Hodge filtration on $H^k(X, \mathbf{C})$.*

As Steenbrink has shown (see also the next section), this conjecture holds for quasi-smooth varieties; it also holds for toric varieties (Theorem 12.5). In §15 we will construct a spectral sequence that generalizes (13.5.1) to the case of non-complete toroidal varieties.

13.6. COROLLARY. *If $X$ is an affine toroidal variety, then $H^k(X, \mathbf{C}) = 0$ for $k > \dim X$.*

For in this case $X^{\mathrm{an}}$ is a Stein space, and $H^q(X^{\mathrm{an}}, \Omega_X^{p,\mathrm{an}}) = 0$ for $q > 0$.

## §14. Quasi-smooth varieties

14.1. A toroidal variety $X$ is said to be *quasi-smooth*[1] if all the local models $X_\sigma$ are associated with simplicial cones $\sigma$. A smooth variety is, of course, quasi-smooth.

Let $X$ be an $n$-dimensional quasi-smooth variety: using Corollary 4.9 we see that for $k > 0$

$$\mathcal{E}xt^k_{O_X}(\Omega_X^p, \Omega_X^n) = 0.$$

Hence and from Proposition 4.7 it follows that

$$\mathrm{Ext}^k_{O_X}(X;\ \Omega_X^p, \Omega_X^n) = H^k(X, \mathcal{H}om(\Omega^p, \Omega^n)) = H^k(X, \Omega_X^{n-p}).$$

If we assume, in addition, that $X$ is projective (or would completeness suffice?) we deduce from Serre–Grothendieck duality that the pairing

$$H^q(X, \Omega_X^p) \times H^{n-q}(X, \Omega_X^{n-p}) \to H^n(X, \Omega_X^n) = K$$

is non-degenerate.

14.2. PROPOSITION. *Let $X$ be a projective quasi-smooth variety, and $\rho\colon \overline{X} \to X$ a resolution of singularities. Then the homomorphism*

$$\rho^*\colon H^k(X, \Omega_X^p) \to H^k(\overline{X}, \Omega_{\overline{X}}^p)$$

*is injective.*

PROOF. We use the commutative diagram

$$\begin{array}{ccc} H^q(\overline{X}, \Omega_{\overline{X}}^p) \times H^{n-q}(\overline{X}, \Omega_{\overline{X}}^{n-p}) & \to & H^n(\overline{X}, \Omega_{\overline{X}}^n) \\ \uparrow{\scriptstyle \rho^*\times\rho^*} & & \wr\uparrow{\scriptstyle \rho^*} \\ H^q(X, \Omega^p) \times H^{n-q}(X, \Omega_X^{n-p}) & \to & H^n(X, \Omega_X^n) \end{array}$$

and the fact that the lower pairing is non-degenerate.

14.3. THEOREM (Steenbrink [31]). *Let $X$ be a projective quasi-smooth*

[1] A closely related notion, that of $V$-manifolds, was introduced by Baily [36], [37].

*variety over* **C**. *Then the Hodge–de Rham spectral sequence* (13.5.1)

$$E_1^{pq}=H^q(X,\ \Omega_X^p)\Rightarrow H^{p+q}(X,\ \mathbf{C})$$

*degenerates at the* $E_1$*-term and converges to the Hodge filtration on* $H^k(X, \mathbf{C})$.

PROOF. Let $p: \overline{X}\to X$ be a resolution of singularities; we consider the morphism of spectral sequences

$$\begin{array}{ccc}\overline{E}_1^{pq}=H^q(\overline{X},\ \Omega_{\overline{X}}^p) & \Rightarrow & H^{p+q}(\overline{X},\ \mathbf{C})\\ \uparrow p^* & & \uparrow p^*\\ E_1^{pq}=H^q(X,\ \Omega_X^p) & \Rightarrow & H^{p+q}(X,\ \mathbf{C}).\end{array}$$

According to classical Hodge theory (see [8]), the assertions of the theorem hold for the upper spectral sequence; in particular, $\overline{E}_1=\overline{E}_2=\ldots=\overline{E}_\infty$, and all the differentials $\overline{d}_i$ are zero for $i\geqslant 1$. Let us show by induction that $E_i$ for $i\geqslant 1$ maps injectively to $\overline{E}_i$. For $i=1$ this follows from Proposition 14.2; let us go from $i$ to $i+1$. Since $E_i\subset\overline{E}_i$ and $\overline{d}_i=0$, we have $d_i=0$, so that $E_{i+1}$ is equal to $E_i$ and is again a subspace of $\overline{E}_{i+1}$.

We have, thus, shown that $E_1=E_\infty$ and is included in $\overline{E}_\infty=\overline{E}_1$. From this it follows that $H^k(X, \mathbf{C})$ is included in $H^k(\overline{X}, \mathbf{C})$, and the limit filtration $'F$ on $H^k(X, \mathbf{C})$ is induced by the limit filtration $'\overline{F}$ on $H^k(\overline{X}, \mathbf{C})$.

Finally, because the Hodge filtration is functorial (see [15]), the Hodge filtration $F$ on $H^k(X, \mathbf{C})$ is induced by the Hodge filtration $\overline{F}$ on $H^k(\overline{X}, \mathbf{C})$. It remains to make use of the already mentioned fact that $\overline{F}={}'\overline{F}$. The theorem is now proved.

14.4. COROLLARY. *For a projective quasi-smooth variety* $X$ *the Hodge structure on* $H^k(X, \mathbf{C})$ *is pure of weight* $k$, *and the* $H^{pq}(X)$ *are isomorphic to* $H^q(X, \Omega^p)$.

For $H^k(X, \mathbf{C})$ is a substructure of the pure structure on $H^k(\overline{X}, \mathbf{C})$, as is clear from the proof of the theorem.

The purity of the cohomology of a quasi-smooth $X$ also follows from the fact that Poincaré duality holds for the complex cohomology of $X$. This duality is, in turn, a consequence of the fact that a quasi-smooth variety is a *rational homology manifold* (see [15]).

## §15. Differential forms with logarithmic poles

Up to now we have dealt with "regular" differential forms. However, in the study of the cohomology of "open" varieties differential forms with so-called logarithmic poles are useful. We begin with the simplest case of a smooth variety.

15.1. Let $X$ be a smooth variety (over **C**), and $D$ a smooth subvariety of codimension 1. Let $z_1, \ldots, z_n$ be local coordinates at a point $x\in X$, and let $z_n=0$ be the local equation of $D$ at this point. A 1-form $\omega$ on $X-D$ is said to have a *logarithmic pole along* $D$ *at* $x$ if $\omega$ can be expressed in a neighbourhood of $x$ as

$$\omega=f_1(z)\,dz_1+\ldots+f_{n-1}(z)\,dz_{n-1}+f_n(z)\,\frac{dz_n}{z_n},$$

where $f_1, \ldots, f_n$ are regular functions in a neighbourhood of $x$ in $X$. It is easy to check that this definition does not depend on the choice of local coordinates. Considering the germs of such forms we obtain the sheaf $\Omega_X^1(\log D)$ of germs of differential 1-forms on $X$ with logarithmic poles along $D$. Locally this sheaf is generated by the forms $dz_1, \ldots, dz_{n-1}, dz_n/z_n$, so that it is a locally free $O_X$-module of rank $n=\dim X$.

For any $p\geqslant 0$ we set

$$\Omega_X^p(\log D)=\Lambda^p(\Omega_X^1(\log D));$$

this is again a locally free sheaf containing $\Omega_X^p$.

The role played by forms with logarithmic poles is explained by the fact that locally they represent the cohomology of $X-D$ near $D$. For around a point $x\in D$ the manifold $X-D$ has, from the homological point of view, the structure of a circle $S$, and the cycle $S$ is caught by the form $d(\log z_n)=dz_n/z_n$

$$\int_S dz_n/z_n=2\pi\sqrt{-1}.$$

15.2. We now turn to the more general case. Namely, we suppose that $X$ is merely a *normal* variety, and that the divisor $D\subset X$ is merely *smooth at its generic point.* We consider an open subvariety $U\subset X$ such that a) $U$ is smooth, b) $D_U=D\cap U$ is a smooth divisor on $U$, and c) $X-U$ is of codimension greater than 1 in $X$. Let $j: U\to X$ be the inclusion. We set

$$\Omega_X^p(\log D)=j_*(\Omega_U^p(\log D_U))$$

and call this the *sheaf of germs of p-differentials on X with logarithmic poles along D.* It can be verified that the definition is independent of the choice of $U$.

15.3. In the classical case, $X$ is a smooth variety and $D$ is a divisor with normal crossings. If $z_1, \ldots, z_n$ are local coordinates and $D$ is given by an equation $z_{k+1}\cdot\ldots\cdot z_n=0$, then $\Omega_X^1(\log D)$ is generated by the forms $dz_1, \ldots, dz_k, dz_{k+1}/z_{k+1}, \ldots, dz_n/z_n$ and is again locally free, and $\Omega_X^p(\log D)=\Lambda^p(\Omega^1(\log D))$. The connection of such sheaves with the cohomology of $X-D$ is established by the following theorem (Deligne [8]): *there is a spectral sequence*

$$E_1^{pq}=H^q(X,\ \Omega_X^p(\log D))\Rightarrow H^{p+q}(X-D,\ \mathbf{C}),$$

*which degenerates at the* $E_1$*-term and converges to the Hodge filtration on* $H^k(X-D, \mathbf{C})$.

15.4. Later we shall be interested in the toroidal case. Generalizing

Definition 13.1 we say that a pair $(X, D)$ (where $X$ is a variety and $D$ a divisor on $X$) is *toroidal* if for each point $x \in X$ we can find a local toric model $X_\sigma$ such that $D$ goes over into a **T**-invariant divisor $D_\sigma$ on $X_\sigma$. Such a divisor $D_\sigma$ is a union of subvarieties $X_\theta \subset X_\sigma$, where $\theta$ ranges over certain faces of $\sigma$ of codimension 1.

To give an idea of the local structure of the sheaves $\Omega_X^p(\log D)$ in the toroidal case, we devote some time to the study of the corresponding toric case.

Thus, let $\sigma$ be an $n$-dimensional cone in an $n$-dimensional lattice $M$; let $I$ be a set of faces of $\sigma$ of codimension 1. In §4 above we associated with each face $\tau$ the space $V_\tau = (\tau - \tau) \underset{\mathbf{Q}}{\otimes} K$. Now we set for each face $\theta$ of codimension 1

$$V_\theta(\log) = \begin{cases} V & \text{if } \theta \in I, \\ V_\theta & \text{if } \theta \notin I. \end{cases}$$

By analogy with the module $\Omega_A^p$ (see **4.2**) we introduce the $M$-graded $A$-module

$$\Omega_A^p(\log) = \bigoplus_{m \in \sigma \cap M} \Omega_A^p(\log)(m),$$

setting for each $m \in \sigma \cap M$

$$\Omega_A^p(\log)(m) = \Lambda^p\Big(\bigcap_{\theta \ni m} V_\theta(\log)\Big).$$

**15.5.** PROPOSITION. *Let $X = X_\sigma$, $D = \bigcup_{\theta \in I} X_\theta$. Then the sheaf of $O_X$-modules $\Omega_X^p(\log D)$ is associated with the A-module $\Omega_A^p(\log)$.*

The proof is completely analogous to that of Proposition 4.3.

The derivatives $d\colon \Omega^p \to \Omega^{p+1}$ extend to derivatives $d\colon \Omega^p(\log D) \to \Omega^{p+1}(\log D)$, which under the identifications of Proposition 15.5 transform into $M$-homogeneous derivatives, which over $m \in M$ are constructed like the exterior product with $m \otimes 1 \in V$.

**15.6.** The sheaves $\Omega_X^p(\log D)$ have the so-called *weight* filtration $W$. We explain this in the toric case, where it turns into an $M$-graded filtration

$$0 \subset W_0\Omega_A^p(\log) \subset W_1\Omega_A^p(\log) \subset \ldots \subset W_p\Omega_A^p(\log) = \Omega_A^p(\log).$$

On a homogeneous component over $m \in M$ this is given by the formula

$$(W_k\Omega_A^p(\log))(m) = \Omega_A^{p-k}(m) \wedge \Omega_A^k(\log)(m).$$

In particular, for the quotients we have

$$(W_k/W_{k-1})(m) = \Lambda^{p-k}(V_{\Gamma(m)}) \otimes \Lambda^k(V_{\Gamma(m)}(\log)/V_{\Gamma(m)}).$$

**15.7. The Poincaré residue.** In the case of a *simplicial* cone $\sigma$ these quotients have an interesting interpretation. Suppose that $\sigma$ is given by $n$ linear inequalities $\lambda_i \geqslant 0$, for $i = 1, \ldots, n$, where $\lambda_i\colon M \to \mathbf{Z}$ are linear functions. The faces of $\sigma$ correspond to subsets of $\{1, \ldots, n\}$. Let $I = \{r+1, \ldots, n\}$, and let the corresponding divisor be $D = D_{r+1} \cup \ldots \cup D_n$. Finally, let $\sigma_0$ be the face of $\sigma$ corresponding to $I$.

In this situation we define the *Poincaré residue isomorphism*

$$W_k\Omega_A^p(\log)/W_{k-1}\Omega_A^p(\log) \xrightarrow{\sim} \bigoplus_\tau \Omega_{A_\tau}^{p-k},$$

where the summation on the right is over the faces $\tau$ of codimension $k$ that contain $\sigma_0$. This isomorphism is $M$-homogeneous, and it is enough to specify it over each $m \in M$. Let $I(m) = \{i \mid \lambda_i(m) = 0\}$. Then on the left-hand side we have the space

$$\Lambda^{p-k}(V_{I(m)}) \otimes \Lambda^k(V_{I(m)}(\log)/V_{I(m)}).$$

where $V_J$ denotes the intersection of the kernels of $\lambda_i \otimes 1\colon V \to K$, $i \in J$. The space $V_{I(m)}(\log)$ is the same as $V_{I(m)-I}$, and the functions $\lambda_i \otimes 1$ with $i \in I \cap I(m)$ give an isomorphism

$$V_{I(m)}(\log)/V_{I(m)} \xrightarrow{\sim} K^{I \cap I(m)}$$

Thus, the space $\Lambda^k(V_{I(m)}(\log)/V_{I(m)})$ has a canonical basis in correspondence with $k$-element subsets of $I \cap I(m)$, that is, with faces of $\sigma$ of codimension $k$ containing both $\sigma_0$ and $m$. We claim that if $\tau$ is such a face, then $\Omega_{A_\tau}^{p-k}(m) = \Lambda^{p-k}(V_{I(m)})$. For $\Omega_{A_\tau}^{p-k}(m)$ is the $(p-k)$th exterior power of the subspace of $V_\tau = V_{I_\tau}$ cut out by the equations $\lambda_i \otimes 1 = 0$, $i \in I(m) - I_\tau$, that is, just $V_{I(m)}$. If a face $\tau$ of codimension $k$ contains $\sigma_0$ but not $m$, then $\Omega_{A_\tau}^{p-k}(m) = 0$.

It is easy to globalize the Poincaré residue isomorphisms. Let $X$ be a quasi-smooth variety, and suppose that the divisor $D$ consists of quasi-smooth components $D_1, \ldots, D_N$ that intersect quasi-transversally. Then we have isomorphisms

$$W_k\Omega_X^p(\log D)/W_{k-1}\Omega_X^p(\log D) \xrightarrow{\sim} \bigoplus_{0 \leq i_1 < \ldots < i_k \leq N} \Omega_{D_{i_1 \ldots i_k}}^{p-k},$$

where $D_{i_1 \ldots i_k} = D_{i_1} \cap \ldots \cap D_{i_k}$.

**15.8.** Let us now show how to apply differentials with logarithmic poles to the cohomology of open toric varieties. Let $X$ be a *complete* variety over **C**, and $D$ a *Cartier divisor* on $X$ (that is, $D$ can locally be defined by one equation). Suppose that the pair $(X, D)$ is *toroidal.* Then the following theorem holds.

**15.9.** THEOREM. *In the notation and under the hypotheses of* **15.8** *there exists a spectral sequence*

$$E_1^{pq} = H^q(X, \Omega_X^p(\log D)) \Rightarrow H^{p+q}(X - D, \mathbf{C}).$$

Of course, this is the spectral sequence of the complex

$$\Omega_X^\cdot(\log D) = \{\Omega_X^0(\log D) \xrightarrow{d} \Omega_X^1(\log D) \xrightarrow{d} \ldots\}.$$

Supposedly, this degenerates at the $E_1$-term and converge to the Hodge filtration on $H^k(X - D, \mathbf{C})$ (Conjecture 13.5.1). When $X$ is quasi-smooth, this is actually so (see [31]), and the weight filtration $W$ on the complex $\Omega_X^{\boldsymbol{\cdot}}(\log D)$ induces the weight filtration of the Hodge structure on $H^k(X-D, \mathbf{C})$.

Leaving aside the necessary formal incantations on the hypercohomology of complexes (in the spirit of [8]), the content of the proof of Theorem 15.9 reduces to the following. We consider the sheaf morphism

$$\varphi: \mathcal{H}^k(\Omega_X^{\boldsymbol{\cdot}}(\log D)^{\mathrm{an}}) \to R^k j_*(\mathbf{C}_{X-D})$$

(here $\mathcal{H}$ denotes the cohomology sheaf of a complex, and $j$ the embedding of $X-D$ in $X$), that takes a closed $k$-form over an open $W \subset X$ into the de Rham cohomology class on $W-D$ defined by it. To prove Theorem 15.9 we have to establish the following assertion, which generalizes Proposition 13.4.

**15.10.** LEMMA. *$\varphi$ is an isomorphism of sheaves.*

PROOF OF THE LEMMA. Since the assertion is local, we can check it point by point. Going over to a local model, we may assume that $X = X_\sigma$ and that $D$ is given by an equation $x^{m_0}$, with $m_0 \in \sigma \cap M$. Let 0 be the "vertex" of $X$; we have to prove that

$$\mathcal{H}^k(\Omega_X^{\boldsymbol{\cdot}}(\log D)^{\mathrm{an}})_0 \to R^k j_*(\mathbf{C}_{X-D})_0$$

is an isomorphism.

THE RIGHT-HAND SIDE. By definition,

$$R^k j_*(\mathbf{C}_{X-D})_0 = \varprojlim_W H^k(W-D, \mathbf{C}),$$

where $W$ ranges over a basis of the neighbourhoods of 0 in $X$. We specify such a basis explicitly. For this purpose we introduce a function $\rho: X(\mathbf{C}) \to \mathbf{R}$ measuring the "distance from 0".

We fix a linear function $\lambda: M \to \mathbf{Z}$ such that $\lambda(\sigma) \geqslant 0$ and $\lambda^{-1}(0) \cap \sigma = \{0\}$. We recall (see 2.3) that a $\mathbf{C}$-valued point $x \in X(\mathbf{C})$ is a homomorphism of semigroups $x: \sigma \cap M \to \mathbf{C}$. We set

$$\rho(x) = \max_{m \neq 0}\{|x(m)|^{\frac{1}{\lambda(m)}}\}.$$

Here $m$ ranges over the non-zero elements of $\sigma \cap M$ (or just over a set of generators of this semigroup). Now $\rho$ is continuous, and $\rho(x) = 0$ if and only if $x = 0$. Therefore, the sets $W_\varepsilon = \rho^{-1}([0, \varepsilon])$ for $\varepsilon > 0$ form a basis of the neighbourhoods of 0.

The group $\mathbf{R}_+^*$ of positive real numbers acts on $X(\mathbf{C})$ by the formula: for $r > 0$ and $x \in X(\mathbf{C})$

$$(r \cdot x)(m) = r^{\lambda(m)} x(m).$$

This action preserves the strata of $X$ and, in particular, the divisor $D$. Since $\rho(r \cdot x) = r\rho(x)$, we see that all the sets $W_\varepsilon - D$ are homotopy-equivalent to $X-D$. Hence,



$$\varprojlim_{\varepsilon > 0} H^k(W_\varepsilon - D, \mathbf{C}) = H^k(X-D, \mathbf{C}).$$

We know the cohomology of $X_\sigma - D = X_{\sigma - \langle m_0 \rangle}$ from Lemma 12.3, and finally,

$$R^k j_*(\mathbf{C}_{X-D})_0 = \Lambda^k(\mathrm{cospan}(\sigma - \langle m_0 \rangle)) \otimes \mathbf{C}.$$

THE LEFT-HAND SIDE. This is the cohomology of the complex of $O_{X,0}^{\mathrm{an}}$-modules $\Omega_X^{\boldsymbol{\cdot}}(\log D)_0^{\mathrm{an}}$. We argue as in the proof of Proposition 13.4. Again we use the homomorphism $h: \Omega_X^{p+1}(\log D)_0^{\mathrm{an}} \to \Omega_X^p(\log D)_0^{\mathrm{an}}$. Over $m \in M$ the operator $d \circ h + h \circ d$ acts as multiplication by $\lambda(m)$. For a non-zero $m \in \sigma \cap M$ this number $\lambda(m)$ is invertible, so that we can only have cohomology over $m = 0$, and this reduces to the cohomology of the complex $\Omega_A^{\boldsymbol{\cdot}}(\log D)(0)$ with the zero differential. We deduce from this that $\mathcal{H}^k(\Omega_X^{\boldsymbol{\cdot}}(\log D)^{\mathrm{an}})_0 = \Omega_A^k(\log)(0)$ is isomorphic to the $k$th exterior power of $\bigcap_{\theta \ni 0} V_\theta(\log) = \bigcap_{\theta \ni m_0} V_\theta$. It remains to remark that $\bigcap_{\theta \ni m_0} (\theta - \theta)$ is the same as the space $\mathrm{cospan}(\sigma - \langle m_0 \rangle)$, and this proves the theorem.

## APPENDIX 1

## DEPTH AND LOCAL COHOMOLOGY

This question is also treated in [11] and [22]. Let $A$ be a local Neotherian ring with maximal ideal $\mathfrak{m}$, and $F$ a Noetherian $A$-module. A sequence $a_1, \ldots, a_n$ of elements of $A$ is said to be *$F$-regular* if $a_i$ for each $i$ from 1 to $n$ is not a zero-divisor in the $A$-module $F/(a_1, \ldots, a_{i-1})F$. For the connection between regularity of a sequence and acyclicity of the Koszul complex, see [11].

The length of a maximal $F$-regular sequence is called the *depth* of $F$ and is denoted by $\mathrm{prof}(F)$. We always have $\mathrm{prof}(F) \leqslant \dim(F)$; if this equality holds, then $F$ is said to be a *Cohen–Macaulay module.* A ring $A$ is said to be a *Cohen–Macaulay ring* if it is a Cohen–Macaulay module over itself. If $A$ is an $n$-dimensional Cohen–Macaulay ring, then a sequence $a_1, \ldots, a_n$ is regular if and only if the ideal $(a_1, \ldots, a_n)$ has finite codimension in $A$.

Let $H_{\mathfrak{m}}^0(F)$ denote the submodule of $F$ consisting of the elements of $F$ that are killed by some power of $\mathfrak{m}$. This $H_{\mathfrak{m}}^0$ is a left-exact additive functor on the category of $A$-modules; the $q$th right derived functor $H_{\mathfrak{m}}^q$ associated with $H_{\mathfrak{m}}^0$ is called the *$q$th local cohomology functor.*

PROPOSITION ([22]). *For a natural number $n$ the following are equivalent:*

a) $\mathrm{prof}(F) > n$;

b) $H_{\mathfrak{m}}^i(F) = 0$ *for all* $i \leqslant n$.

COROLLARY. *Let $0 \to F \to G \to H \to 0$ be an exact sequence of $A$-modules, with* $\mathrm{prof}\, G = n$ *and* $\mathrm{prof}\, H = n - 1$. *Then* $\mathrm{prof}\, F = n$.

The preceding proposition is applied to the local cohomology long exact sequence.

## APPENDIX 2

## THE EXTERIOR ALGEBRA

This question is treated in more detail in [5], Ch. III, §§ 5 and 8.

Let $K$ be a commutative ring with 1, and let $V$ be a $K$-module. Let $p \geqslant 1$ be an integer; the *pth exterior power* of $V$ is the quotient module of the $p$-fold tensor product $V \otimes \ldots \otimes V = V^{\otimes p}$ by the submodule $N$ generated by primitive tensors $x_1 \otimes \ldots \otimes x_p$ in which at least two terms $x_i$ and $x_j$ are equal. The $p$th exterior power of $V$ is written $\Lambda^p(V)$; we set $\Lambda^0(V) = K$.

The direct sum $\Lambda^*(V) = \bigoplus_{p \geqslant 0} \Lambda^p(V)$ is called the *exterior algebra* of $V$; its multiplication is given by the *exterior product* $(x, y) \mapsto x \wedge y$. The exterior product is skew-symmetric, that is, if $x \in \Lambda^p(V)$ and $y \in \Lambda^q(V)$, then $x \wedge y = (-1)^{pq} y \wedge x$.

If $V_1$ and $V_2$ are $K$-modules, then we have a canonical isomorphism of graded skew-symmetric $K$-algebras.

$$\Lambda^*(V_1 \oplus V_2) = \Lambda^*(V_1) \underset{K}{\otimes} \Lambda^*(V_2).$$

From this we can deduce by induction that if $V$ is a free $K$-module with a basis $e_1, \ldots, e_n$ then $\Lambda^p(V)$ has a basis consisting of the expressions $e_{i_1} \wedge \ldots \wedge e_{i_p}$ with $1 \leqslant i_1 < \ldots < i_p \leqslant n$.

The (left) multiplication by an element $v \in V$ defines a module homomorphism $v \wedge : \Lambda^p(V) \to \Lambda^{p+1}(V)$, which we denote by ${}_vE$. Since $v \wedge v = 0$, we obtain a complex of $K$-modules

$$0 \to \Lambda^0(V) \xrightarrow{v\wedge} \Lambda^1(V) \xrightarrow{v\wedge} \Lambda^2(V) \xrightarrow{v\wedge} \ldots$$

Conversely, if we take a linear map $\lambda$: $V \to K$, then there is a unique way of extending $\lambda$ to a $K$-linear derivation of the algebra $\Lambda^*(V)$ that decreases degrees by 1. This is called the *right internal multiplication* by $\lambda$ and is denoted by $\llcorner \lambda$ or $I_\lambda$. In particular, for $x \in \Lambda^p(V)$ we have

$$(x \wedge y) \llcorner \lambda = (x \llcorner \lambda) \wedge y + (-1)^p x \wedge (y \llcorner \lambda). \tag{1}$$

If $e_1, \ldots, e_n$ is a basis of $V$, then $\llcorner \lambda$ can be given by the formula

$$(e_{i_1} \wedge \ldots \wedge e_{i_p}) \llcorner \lambda = \sum_{h=1}^{p} (-1)^h \lambda(e_{i_h}) e_{i_1} \wedge \ldots \wedge \widehat{e_{i_h}} \wedge \ldots \wedge e_{i_p}.$$

It is easy to check that $I_\lambda \circ I_\lambda = 0$, and again we get a complex

$$0 \leftarrow \Lambda^0(V) \xleftarrow{\lambda} \Lambda^1(V) \xleftarrow{\llcorner\lambda} \Lambda^2(V) \xleftarrow{\llcorner\lambda} \ldots,$$

which is known as the *Koszul complex* of the sequence $\lambda(e_1), \ldots, \lambda(e_n)$ of elements of $K$. The relation (1) for $x \in V = \Lambda^1(V)$ can be rewritten in the form

$$I_\lambda \circ {}_xE + {}_xE \circ I_\lambda = \lambda(x) \tag{2}$$

and can be interpreted as a homotopy. Thus, if $a_1, \ldots, a_n$ generate the unit ideal of $K$, that is, if $a_1 f_1 + \ldots + a_n f_n = 1$ for some $f_i \in K$, then by taking for $x$ the vector $(f_1, \ldots, f_n) \in K^n$ we find that the Koszul complex for $a_1, \ldots, a_n$ is homotopic to 0, and hence is acyclic.



Now suppose that $K$ is a field. If $W$ is a subspace of $V$, then there arises an increasing filtration on $\Lambda^p(V)$,

$$0 \subset W_0 \subset W_1 \subset \ldots \subset W_p = \Lambda^p(V),$$

where $W_k = (\Lambda^{p-k}(W)) \wedge (\Lambda^k(V))$. For the quotients we have isomorphisms

$$W_k/W_{k-1} \simeq \Lambda^{p-k}(W) \otimes \Lambda^k(V/W). \tag{3}$$

Finally, if $W_1, \ldots, W_n$ are subspaces of $V$, then

$$\Lambda^p(\cap W_i) \simeq \cap \Lambda^p(W_i). \tag{4}$$

## APPENDIX 3

## DIFFERENTIALS

1. Let $K$ be a commutative ring with $1, \Lambda^*$ a graded skew-symmetric $K$-algebra, and $M^*$ a graded $\Lambda^*$-module. A $K$-linear map of degree $\nu$

$$D: \Lambda^* \to M^*$$

is called a *K-derivation* if for $\lambda \in \Lambda^p$ and $\mu \in \Lambda^*$

$$D(\lambda\mu) = D(\lambda)\mu + (-1)^{p\nu} \lambda \cdot D(\mu). \tag{1}$$

2. Let $A$ be a commutative $K$-algebra, with the multiplication $\mu$: $A \underset{K}{\otimes} A \to A$, so that $\mu(a \otimes b) = ab$. Let $I$ denote the kernel of $\mu$. The $A$-module $I/I^2$ is called the module of *K-differentials* of $A$ and is denoted by $\Omega^1_{A/K}$. The name is explained by the fact that the map

$$d: A \to \Omega^1_{A/K}, \tag{2}$$

defined by $d(a) = (a \otimes 1 - 1 \otimes a) \bmod I^2$ is a $K$-derivation (and is universal is a certain sense). Note that $\Omega^1_{A/K}$ is generated as $A$-module by the set $d(A)$.

For example, if $A = K[T_1, \ldots, T_n]$ is the polynomial ring in $T_1, \ldots, T_n$, then $\Omega^1_{A/K}$ is the free $A$–module with the basis $dT_1, \ldots, dT_n$.

3. We write $\Omega^p_{A/K} = \Lambda^p(\Omega^1_{A/K})$.

PROPOSITION. *There exists one and only one K-derivation of degree* 1 *of the algebra* $\Omega^* = \Lambda^*(\Omega^1_{A/K})$

$$d: \Omega^p \to \Omega^{p+1},$$

*for which* $d \circ d = 0$ *and which for* $p = 0$ *coincides with* (2).

PROOF. UNIQUENESS: Since $d$ is to be a derivation, it suffices to verify uniqueness for $p = 1$. The $A$-module $\Omega^1$ is generated by $d(A)$, so that it is enough to check that for $a, b \in A$ the expression $d(a \cdot db)$ is uniquely determined. But

$$d(a \cdot db) = da \wedge db + a \cdot d(db) = da \wedge db \tag{3}$$

by the condition that $d \circ d = 0$.

EXISTENCE: We define $d: \Omega^1 \to \Omega^2$ by (3). More precisely, we consider first the $K$-linear homomorphism

$$d': A \underset{K}{\otimes} A \xrightarrow{d \otimes d} \Omega^1 \underset{K}{\otimes} \Omega^1 \to \Lambda^2(\Omega^1) = \Omega^2.$$

Since

$$d'((a \otimes 1 - 1 \otimes a)(b \otimes 1 - 1 \otimes b)) =$$
$$= d'(ab \otimes 1 - a \otimes b - b \otimes a + 1 \otimes ab) = -da \wedge db - db \otimes da = 0,$$

$d'$ vanishes on $I^2$ and hence defines a $K$-linear map $d: \Omega^1 \to \Omega^2$.

For $p \geqslant 2$ we define $d: \Omega^p \to \Omega^{p+1}$ by means of (1). To see that $d$ is well-defined we have to check that $d$ vanishes on primitive tensors of the form $\ldots \otimes \omega \otimes \ldots \otimes \omega \otimes \ldots$ with $\omega \in \Omega^1$. We leave the details to the reader.

**4.** If $A$ is a *graded* $K$-algebra, then it is easy to check that the constructions of **2.** and **3.** above are compatible with the grading. Thus, the multiplication $\mu: A \underset{K}{\otimes} A \to A$ is homogeneous of degree 0, $I = \operatorname{Ker} \mu$ is a homogeneous ideal in $A \underset{K}{\otimes} A$, $I/I^2 = \Omega^1_{A/K}$ is a graded $A$-module, and $d: A \to \Omega^1$, together with the remaining $d: \Omega^p \to \Omega^{p+1}$, are homogeneous homomorphisms of degree 0.

**5.** $\Omega^p_{A/K}$ is called the *module of p-differentials*, and *d the exterior derivative*. These formations are functorial. In particular, if $S$ is a multiplicative set in $A$, then there are canonical isomorphisms (see [10])

$$\Omega^p_{A[S^{-1}]/K} \simeq \Omega^p_{A/K} \otimes_K A[S^{-1}].$$

This allows us to glue the modules $\Omega^p$ together into a sheaf on an arbitrary $K$-scheme $X$; the resulting sheaves of $O_X$-modules are denoted by $\Omega^p_{X/K}$ and are called the *sheaves of p-differentials of X over K*. The derivatives $d$ can also be glued together into $K$-linear sheaf morphisms

$$d: \Omega^p_{X/K} \to \Omega^{p+1}_{X/K}.$$

For a *smooth* $K$-scheme $X$ the sheaves of $O_X$-modules $\Omega^p_{X/K}$ are *locally free*.

**6.** REMARK. The sheaves of differentials with which we work in the main text differ from those defined here, although the notation is the same. In the smooth case the two definitions agree.

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# LATTICES OF VARIETIES OF LINEAR ALGEBRAS

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Contents



## Introduction

Let $k$ be a fixed commutative and associative ring with unit. *A linear $k$-algebra* is a $k$-module $A$ in which a bilinear multiplication $xy \in A$ is given, where $x, y \in A$. It is clear from this definition that the multiplication in $A$ can be fairly arbitrary, and the study and classification of arbitrary linear algebras is, therefore, quite an impossible task. For this reason we are forced to assume that the multiplication $xy$ satisfies some identities or laws, for example, the associative law $(xy)z = x(yz)$ for all $x, y, z \in A$, or the commutative law $xy = yx$ for all $x, y \in A$, etc. In this way we arrive at the concept of a *variety of algebras* – the class of all linear $k$-algebras in which a fixed collection of identities holds. By Birkhoff's theorem ([35], Ch. VI, [53]), varieties are precisely classes that are closed with respect to subalgebras, direct products, and factor algebras.

We now recall the concept of a lattice. A *lattice* is a partially ordered set in