These notes have been prepared for a "DES" set of lectures in Geneva, starting in October 1995. Comments are welcome.

TOPICS ON GEOMETRIC GROUP THEORY
1. INTRODUCTION

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Groups as abstract structures have been recognized progressively during the XIXth century by mathematicians including Gauss (*Disquisitiones arithmeticae* in 1802), Cauchy, Galois, Cayley, Jordan, Sylow, Frobenius, Klein (*Erlangen Programme* in 1872), Lie, Poincaré ... ; see e.g. [Die, chap. III]. Groups are of course sets given with appropriate "multiplications", and they are often given together with actions on interesting geometric objects. But the fact that we want to stress here is that groups are also interesting geometric objects by themselves - a point of view illustrated in the past by Cayley and Dehn (see [ChM, Chap I.5]), and more recently by Gromov (see e.g. [Gro]). More precisely, a finitely generated group can be seen as a metric space (the distance between two points being defined up to "quasi-isometry"), and this gives rise to a very fruitful approach to group theory.

The purpose of these notes is to provide an introduction to this point of view.

Much of what follows is about results of the 40 last years. However, there are related results which are classical; this introduction is about two of them. One goes back to Gauss; the other goes back at least to Polya (1921), and possibly to the first mathematically inclined drunkard.

1.1. THE CIRCLE PROBLEM

Consider a group Γ and a function $\sigma : \Gamma \to \mathbb{R}_+$ which measures in some sense the "size" of elements of Γ.

$$\text{What can be said of} \qquad \sharp\ \{\gamma \in \Gamma \mid \sigma(\gamma) \leq t\} \qquad \text{for large } t \in \mathbb{R}_+ \ ?$$

This theme has many variations. Here we illustrate it with a classical result due to Gauss, for which $\Gamma = \mathbb{Z}^2$ and $\sigma(a,b) = a^2 + b^2$. (See [Cha], Chapter VI, Th. 1; Chandrasekharan quotes Gauss, Werke, (ii), p. 272-5.)

For all $t \geq 0$, set

$$R(t) \;=\; \sharp\ \{\ (a,b) \in \mathbb{Z}^2 \mid a^2 + b^2 \leq t\ \}\,.$$

One has for example

$$R(0) = 1 \qquad R(1) = 5$$
$$R(10) = 37 \qquad R(100) = 317$$
$$R(1000) = 3149 \qquad R(10'000) = 31'417.$$

Theorem 1 (Gauss). *One has $R(t) - \pi t = O(\sqrt{t})$.*

Proof. To each lattice point $(a,b) \in \mathbb{Z}^2$, we associate the unit square of the Euclidean plane with (a,b) as its "south-west corner". If $a^2 + b^2 \leq t$, the square corresponding to (a,b) is inside the disc of radius $\sqrt{t} + \sqrt{2}$; hence

$$R(t) \leq \pi\left(\sqrt{t} + \sqrt{2}\right)^2$$

for all $t \geq 0$. If the square corresponding to (a,b) touches the disc of radius $\sqrt{t} - \sqrt{2}$, then $a^2 + b^2 \leq t$; hence

$$R(t) \geq \pi\left(\sqrt{t} - \sqrt{2}\right)^2$$

for all $t \geq 0$. Thus

$$|R(t) - \pi t| \leq 2\pi\left(1 + \sqrt{2t}\right)$$

for all $t \geq 0$. □

Many mathematicians have worked to improve the error term in Gauss' result. For example, it is known that

$$R(t) - \pi t = O\left(t^\alpha\right)$$

for the following values of α :

$\alpha = \frac{1}{3}$	$\sim 0,3333$	(Sierpinsky, 1906)
$\alpha = \frac{37}{112}$	$\sim 0,3304$	(Van der Corput, 1923)
$\alpha = \frac{15}{46}$	$\sim 0,3261$	(Titchmarch, 1934)
$\alpha = \frac{13}{40} + \epsilon$	$\sim 0,3250$	(Hua, 1941)
$\alpha = \frac{12}{37}$	$\sim 0,3243$	(Chen, 1963)
$\alpha = \frac{35}{108} + \epsilon$	$\sim 0,3241$	(Kolesnik, 1982)
$\alpha = \frac{7}{22} + \epsilon$	$\sim 0,3182$	(Iwaniec and Mozzochi, 1988)

to name but a few (where "$+\epsilon$" means as usual "for all $\epsilon > 0$".) Moreover,

$$\text{if} \qquad R(t) - \pi t = O\left(t^{\alpha}\right) \qquad \text{then} \qquad \alpha \geq \frac{1}{4}$$

(independently due to Landau, 1912, and Hardy, 1915). Apparently, specialists believe that $R(t) - \pi t = O\left(t^{1/4+\epsilon}\right)$ for all $\epsilon > 0$. More on this in the following references.

S.W. Graham and G. Kolesnik: *Van der Corput's Method of Exponential Sums*, Cambridge Univ. Press, 1991.

E. Grosswald: *Representation of integers by sums of squares*, Springer, 1985; see pages 20-22.

G.H. Hardy and E.M. Wright: *An Introduction to the Theory of Numbers, 5^{th} ed*, Oxford Univ. Press 1979; see page 272.

I.A. Ivic: *The Riemann Zeta Function*, J. Wiley, 1985; see in particular pages 372, 375, 384.

B. Lichtin: *Geometric Features of Lattice Point Problems*, in *Singularity Theory*, D.T. Lê, K. Saito and B. Teissier eds, World Scientific, 1995; see pages 370-443; this has an exposition of $R(t) - \pi t = O\left(t^{1/3+\epsilon}\right)$.

W. Sierpinski: *Elementary Theory of Numbers, 2^{nd} ed.*, North Holland, 1988; see pages 383 ff.

1.2. Polya's recurrence theorem

Consider the *simple random walk* on the lattice $\mathbb{Z}$ of integers. In loose terms, a walker is at the origin at time 0 and moves one step left or one step right, equiprobably, after each unit of time.

The question of *recurrence* for $\mathbb{Z}$ is to know whether the walker has 100 % chance to visit again the origin infinitely many times. (In the original paper [Pol] of 1921, Polya's question was slightly different; see Section 5.3 in [DoS].)

The answer is yes. Indeed (we follow Section 7.2 of [DoS]), the number of all paths of length $2n$ is 2^{2n}; among these, the number of those ending in the origin is $\binom{2n}{n}$ because they involve a choice of n steps right among their $2n$ steps. Hence the probability of being at the origin after $2n$ steps is

$$u_{2n} = \frac{1}{2^{2n}}\binom{2n}{n}.$$

(Observe that $u_{2n+1} = 0$.) Using Stirling's formula $k! \sim k^k e^{-k}\sqrt{2\pi k}$, one has

$$u_{2n} \sim \frac{1}{2^{2n}} \frac{(2n)^{2n} e^{-2n}\sqrt{2\pi 2n}}{n^{2n} e^{-2n} 2\pi n} = \frac{1}{\sqrt{\pi n}}.$$

Hence

$$\sum_{k=1}^{\infty} u_k = \sum_{n=1}^{\infty} u_{2n} \sim \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}} = \infty$$

and the simple random walk on $\mathbb{Z}$ just considered is recurrent.

Consider the similar problem on the lattice $\mathbb{Z}^2$ of the Euclidean plane. The walk is again recurrent.

Let us show this. The number of all paths of length $2n$ is now 4^{2n}. Among these, the number of paths that return to the origin after k steps north, k steps south, $n-k$ steps east and $n-k$ steps west is

$$\begin{pmatrix} & & 2n & \\ k & k & n-k & n-k \end{pmatrix} = \frac{(2n)!}{k!\, k!\, (n-k)!\, (n-k)!}\,.$$

Hence the probability of being at the origin after $2n$ steps is

$$u_{2n} = \frac{1}{4^{2n}} \sum_{k=0}^{n} \frac{(2n)!}{k!\, k!\, (n-k)!\, (n-k)!} = \frac{1}{4^{2n}} \frac{(2n)!}{n!\, n!} \sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k}\,.$$

Now, given a box of size $2n$ with n black balls and n white balls, the number of n-balls subsets can be computed in two ways, so that

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k}\,.$$

Thus

$$(*) \qquad u_{2n} = \frac{1}{4^{2n}} \frac{(2n)!}{n!\, n!} \binom{2n}{n} = \left\{ \frac{1}{2^{2n}} \binom{2n}{n} \right\}^2\,.$$

Using Stirling's formula, one has

$$u_{2n} \sim \frac{1}{\pi n} \qquad \text{and} \qquad \sum_{k=1}^{\infty} u_k = \sum_{n=1}^{\infty} u_{2n} \sim \sum_{n=1}^{\infty} \frac{1}{\pi n} = \infty$$

as above, so that the simple random walk on $\mathbb{Z}^2$ is indeed recurrent.

The situation is different in one more dimension: the simple random walk on $\mathbb{Z}^3$ is *transient* (this means precisely "non recurrent").

As before, we have

$$\begin{aligned} u_{2n} &= \frac{1}{6^{2n}} \sum_{\substack{j,k \geq 0 \\ j+k \leq n}} \frac{(2n)!}{j!\, j!\, k!\, k!\, (n-j-k)!\, (n-j-k)!} \\ &= \frac{1}{2^{2n}} \binom{2n}{n} \sum_{\substack{j,k \geq 0 \\ j+k \leq n}} \left(\frac{1}{3^n} \frac{n!}{j!\, k!\, (n-j-k)!} \right)^2\,. \end{aligned}$$

Now. for each $j, k \geq 0$ with $j + k \leq n$, one has

$$(**) \qquad \frac{n!}{j!\ k!\ (n-j-k)!} \leq \frac{n!}{\left[\frac{n}{3}\right]!\ \left[\frac{n}{3}\right]!\ \left[\frac{n}{3}\right]!}$$

(Exercise 3). Consequently

$$u_{2n} \leq \frac{1}{2^{2n}} \binom{2n}{n} \left(\frac{1}{3^n} \frac{n!}{\left(\left[\frac{n}{3}\right]!\right)^3} \right) \sum_{\substack{j,k \geq 0 \\ j+k \leq n}} \frac{1}{3^n} \frac{n!}{j!\ k!\ (n-j-k)!} .$$

As the last sum is 1, one has

$$u_{2n} \leq \frac{1}{2^{2n}} \binom{2n}{n} \left(\frac{1}{3^n} \frac{n!}{\left(\left[\frac{n}{3}\right]!\right)^3} \right) .$$

Stirling's fomula shows that

$$\frac{1}{2^{2n}} \binom{2n}{n} \frac{1}{3^n} \frac{n!}{\left(\left[\frac{n}{3}\right]!\right)^3} = \frac{1}{2^{2n} 3^n} \frac{(2n)!}{n!\ \left(\left[\frac{n}{3}\right]!\right)^3} \sim \frac{\sqrt{2}}{\left(\sqrt{\frac{2\pi}{3}}\right)^3 n^{\frac{3}{2}}}$$

and one has finally

$$\sum_{k=1}^{\infty} u_k = \sum_{n=1}^{\infty} u_{2n} \leq K \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} < \infty$$

for an appropriate constant K.

Theorem 2 (Polya, 1921). *The simple random walk on* $\mathbb{Z}^d$ *is*

$$\begin{cases} \textit{recurrent if } d = 1 \textit{ or } d = 2 \\ \textit{transient if } d \geq 3. \end{cases}$$

Proof. The proof above for $d = 3$ carries over (with minor changes only) to the case $d \geq 3$. □

Feller adds that, in dimension 3, the probability of return to the initial position is about 0.35, and the expected number of returns is consequently

$$0.65 \sum_{k \geq 1} k(0.35)^k = 0.65 \frac{0.35}{(1-0.35)^2} = \frac{0.35}{0.65} = 0.53$$

(Section 7 of Chapter 14 in [Fel]).

The following result, considerably deeper, is due to Varopoulos (see [VSC], in particular the end of Chapter VI).

Theorem 3 (Varopoulos, 1980 's). *Let Γ be a finitely generated group and let*

$$p \; : \; \Gamma \longrightarrow [0,1]$$

be a symmetric probability measure on Γ with finite support which generates Γ. If the random walk defined by Γ and p is recurrent, then

either Γ is a finite group,
or Γ has a subgroup of finite index isomorphic to $\mathbb{Z}$,
or Γ has a subgroup of finite index isomorphic to $\mathbb{Z}^2$.

EXERCISES AND COMPLEMENTS

(1) Consider analogous problems to the circle problem in dimensions $d \geq 3$, and with ellipsoids instead of with balls.
[Hint: see e.g. Lichtin's paper quoted at the end of 1.1.]

(2) Explain Formula $(*)$ above, showing that the probability u_{2n} for $\mathbb{Z}^2$ is the square of the probability u_{2n} for $\mathbb{Z}$.
[Hint: see if necessary [DoS], Section 7.6.]

(3) For positive integers a, b such that $a < b$, check that $a!b! \geq (a+1)!(b-1)!$. Deduce from this Equality $(**)$ used above, shortly before Theorem 2.

(4) Check the details of the proof of Theorem 2 for $d \geq 3$.

(5) A walker on $\mathbb{N}$ is in 0 at time 0 and in 1 at time 1. If he is in k at time $n \geq 1$, then either $k = 0$ and he stays there at time $n+1$, or $k \geq 1$ and he moves equiprobably one step left or one step right. Denote by P_k^n the probability for the walker to be in k at time n, and set $P_k(z) = \sum_{n=0}^{\infty} P_k^n z^n$ (a formal power series, which is called the *generating function* of the sequence $(P_k^n)_{n \geq 0}$).

(i) Compute P_k^n for $n \leq 10$.

(ii) Check that one has

$$\begin{aligned} P_0(z) &= 1 + z\Big[P_0(z) - 1\Big] + \frac{z}{2} P_1(z) \; , \\ P_1(z) &= z\Big[1 + \frac{1}{2} P_2(z)\Big] \; , \\ P_k(z) &= z\Big[\frac{1}{2} P_{k-1}(z) + \frac{1}{2} P_{k+1}(z)\Big] \qquad \text{for all} \qquad k \geq 2 \; . \end{aligned}$$

(iii) Deduce from (ii) that

$$P_k(z) = 2\left(\frac{1 - \sqrt{1 - z^2}}{z}\right)^k$$

for all $k \geq 1$ and that

$$P_0(z) \;=\; 1 \;+\; \frac{1-\sqrt{1-z^2}}{1-z} \; .$$

Check that the first terms of the Taylor developement of this series at the origin fit with the figures found in (i).

(iv) For any convergent power series around the origin $f(z) = \sum_{n\geq 0} f_n z^n$, check that

$$(1-z)f'(z) \;-\; f(z) \;=\; \sum_{n\geq 1} n(f_n - f_{n-1})z^{n-1}.$$

In particular, for $f = P_0$, one has

$$\begin{aligned}(1-z)P_0'(z) \;-\; P_0(z) \;&=\; \frac{z}{\sqrt{1-z^2}} \;-\; 1 \;=\; \sum_{n\geq 1} n(P_0^n - P_0^{n-1})z^{n-1} \\ &=\; -1 + z + \frac{1}{2}z^3 + \frac{1{\cdot}3}{2{\cdot}4}z^5 + \frac{1{\cdot}3{\cdot}5}{2{\cdot}4{\cdot}6}z^7 + \ldots + \frac{1}{2^{2k}}\binom{2k}{k}z^{2k+1} + \ldots \; .\end{aligned}$$

Check that these values of $n(P_0^n - P_0^{n-1})$ confirm the values found in (i).

(v) Show that the *expected duration*

$$\sum_{n=1}^{\infty} n\left(P_0^n - P_0^{n-1}\right)$$

of return to the origin is infinite.

[Hint. One may use the following consequence of Abel's theorem, from the theory of functions of one complex variable: if $f(z) = \sum_{n\geq 0} f_n z^n$ is a holomorphic function in the open unit disc and if the series is convergent at the point 1, then

$$\lim_{r \overset{<}{\to} 1} f(r) \;=\; \sum_{n\geq 0} f_n \;.]$$

(vi) For a direct estimate of the expected duration of return to the origin, see also Section 3 of Chapter 14 in [Fel].

(6) There has been a lot of work done on problems of self-avoiding walks. Consider an integer $d \geq 2$ and the lattice $\mathbb{Z}^d$. A *self-avoiding walk* of length N in $\mathbb{Z}^d$ is a sequence $\big(\omega(0), \omega(1), \ldots, \omega(N)\big)$ of points in $\mathbb{Z}^d$ such that $\omega(j) \neq \omega(k)$ for $j, k \in \{0, 1, \ldots, N\}, j \neq k$ and such that $\omega(j)$ is a neighbour of $\omega(j-1)$ for $j \in \{1, \ldots, N\}$. Let $c_{d,N}$ denote the number of these with $\omega(0)$ the origin and let

$$\left\langle |\omega(N)|^2 \right\rangle \;=\; \frac{1}{c_{d,N}} \sum_{\substack{\omega \text{ of length } N \\ \omega(0)=0}} \|\omega(N)\|^2$$

denote the *mean-square displacement* (where $||\omega(N)||$ denotes a Euclidean norm). It is an easy exercise to show that

$$d \leq \limsup_{N\to\infty} (c_{d,N})^{\frac{1}{N}} \leq 2d-1$$

for all $d \geq 2$.

Two basic problems are to understand $c_{d,N}$ and $\langle|\omega(N)|^2\rangle$ for large N. One of the first result in this area is that the limit

$$\mu_d = \lim_{N\to\infty} (c_{d,N})^{\frac{1}{N}}$$

exists (Hammersley and Morton, 1954); it is called the *connective constant* for dimension d. The exact value of μ_d is unknown, even if $d = 2$ or $d = 3$. For the mean-square displacement, let us mention the *conjecture* according to which

$$\lim_{N\to\infty} \frac{\langle|\omega(N)|^2\rangle}{N^{2\nu}}$$

exists and is neither 0 nor ∞ for the value $\nu = \frac{3}{4}$. There are conjectural asymptoptic behaviours for large N of the form

$$c_{d,N} \sim A\mu^N N^{\gamma-1}$$
$$\langle|\omega(N)|^2\rangle \sim DN^{2\nu}$$

where A, D, μ, γ, ν are constants depending on the dimension d. Many of these constants have conjectural values which are not rigorously demonstrated.

Research in this field is strongly motivated by various applications, such as models for linear polymer molecules in chemical physics, spin models for ferromagnetism in statistical physics, or percolation theory. For a recent state of the art, see [MaS].

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