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ON THE LARGEST NONTRIVIAL POLE OF THE DISTRIBUTION $|f|^s$

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1. Introduction

Let $f \in \mathbb{R}[x_1, ..., x_n]$ be a polynomial which is non degenerate (over $\mathbb{R}$) with respect to its Newton polyhedron $\Gamma(f)$ at the origin (see [AVG] and [DS1,1.1]). Assume also that $f(0) = 0$ and that 0 is a critical point of f. Fix $\eta \in (\mathbb{N} \setminus \{0\})^n$ and let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function with *compact support contained in a sufficiently small neighbourhood of* 0. We are interested in the integral

$$Z(s) = \int_{\mathbb{R}^n} |f(x)|^s x^{\eta-1} \varphi(x) dx,$$

for $s \in \mathbb{C}, Re(s) \geq 0$, where $x^{\eta-1} = x_1^{\eta_1-1} x_2^{\eta_2-1} ... x_n^{\eta_n-1}$ with $\eta = (\eta_1, ..., \eta_n)$. It is well-known that the function $s \mapsto Z(s)$ has an analytic continuation to a meromorphic function on $\mathbb{C}$ which we denote again by $Z(s)$.
Put $s_0 = \frac{-1}{t_0}$ where $t_0 \in \mathbb{R}$ is the smallest value of t such that $t\eta \in \Gamma(f)$. Denote by τ_0 the intersection of all facets of $\Gamma(f)$ which contain $t_0\eta$, and let ρ_0 be the codimension of τ_0 in $\mathbb{R}^n$. We will always suppose that $s_0 \notin \mathbb{Z}$.
It is well-known [V2, 1.4] that all poles of $Z(s)$ are real and $\leq s_0$, except possibly some poles which are integers. (These exceptions do not "contribute" to the asymptotic expansion of $\int_{\mathbb{R}^n} \varphi(x) e^{2\pi i \tau f(x)} x^{\eta-1} dx$ for $\tau \to +\infty$ cf. [V2, 0.4], and we consider them as "trivial"). Moreover if $Z(s)$ has a pole at s_0 then its multiplicity is $\leq \rho_0$, see [V2, 1.4] and [DS1, 1.3].
One expects that "usually" s_0 is a pole of $Z(s)$ with multiplicity ρ_0 for suitable φ, but there are however exceptions as is shown in [DS2, § 6.2]. It is an open problem to determine these exceptional cases.
Instead of working with $Z(s)$ we will often consider the integral

$$I(s) = \int_{\mathbb{R}_+^n} |f(x)|^s x^{\eta-1} \varphi(x) dx$$

for $s \in \mathbb{C}, Re(s) \geq 0$, where $\mathbb{R}_+ = \{t \in \mathbb{R} | t \geq 0\}$; Z(s) and I(s) being related as explained in [DS1.1.16]. The function $s \mapsto I(s)$ has an analytic continuation to a meromorphic function on $\mathbb{C}$ which we denote again by $I(s)$. Similarly as for $Z(s)$, if $I(s)$ has a pole at s_0 then its multiplicity is $\leq \rho_0$.

The principal result of this paper is a formula (Theorem 2.1) for $\lim_{s \to s_0} (s - s_0)^{\rho_0} I(s)$. As a consequence of this formula and [DS2, §6.2] we obtain in §5 the following result which was conjectured in [DS2, Conjecture 3] :

Theorem 1.1 *Suppose that the face τ_0 is unstable. If $Z(s)$ has a pole at s_0 then its multiplicity is $< \rho_0$.*

As in [DS2, §1] we call a face τ of $\Gamma(f)$ *unstable* if there exists an index j $(1 \leq j \leq n)$ such that the following two conditions are satisfied :

(i) $\tau \subset \{(\alpha_1, ..., \alpha_n) \in \mathbb{R}^n | 0 \leq \alpha_j \leq 1\}$ and $\tau \not\subseteq \{(\alpha_1, ..., \alpha_n) \in \mathbb{R}^n | \alpha_j = 0\}$,

and

(ii) for each compact face σ of $\Gamma(f)$ contained in $\tau \cap \{(\alpha_1, ..., \alpha_n) \in \mathbb{R}^n | \alpha_j = 1\}$, the polynomial f_σ does not vanish on $(\mathbb{R} \setminus \{0\})^n$, where f_σ is defined as follows :

For any face σ of $\Gamma(f)$ we put $f_\sigma := \sum_{\alpha \in \sigma \cap \mathbb{N}^n} a_\alpha x^\alpha$, where $f(x) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha$.
We tried for a long time to prove Theorem 1.1 by using only the methods of [DS2], but we never succeeded in this way.
The authors of the present paper first proved Theorem 2.1 by using methods of [DS1] and [S]. But here Theorem 2.1 is proved by using toroidal resolution of singularities and ideas of Langlands [La]. Some more details can be found in [L].

2. Statement of the principal result

Let $F_1, ..., F_r$ be the facets of $\Gamma(f)$ that contain $t_0 \eta$. Let ξ_{F_i} be the vector, with components relative prime in $\mathbb{N}$, orthogonal to F_i, and let N_{F_i} be $\min\{\langle x, \xi_{F_i} \rangle | x \in \Gamma(f)\}$.
Put $\overset{o}{\tau}_0 = \sum_{i=1}^{r} \mathbb{R}_+ \xi_{F_i}$.
After a permutation of the coordinates we may assume that the standard basis $e_1, ..., e_n$ of $\mathbb{R}^n$ satisfies $\mathbb{R}^n = \tilde{\tau}_0^0 + \sum_{i=\rho_0+1}^{n} \mathbb{R} e_i$ and $e_{m+1}, ... e_n$ are those among $e_1, ..., e_n$ which are parallel to τ_0, where $\tilde{\tau}_0^0$ is the vectorspace spanned by $\overset{o}{\tau}_0$.
Let K be $conv\{0, \frac{\xi_{F_1}}{N_{F_1}}, ..., \frac{\xi_{F_r}}{N_{F_r}}, e_{\rho_0+1}, ... e_n\}$, where *conv* indicates the convex hull. We denote by Vol(K) the volume of K.

Theorem 2.1. *With the above notation and assumptions, we have that*

$$\lim_{s \to s_0} (s - s_0)^{\rho_0} \int_{\mathbb{R}_+^n} |f(x)|^s x^{\eta - 1} \varphi(x) dx \tag{2.1.1}$$

equals

$$n! \ Vol\ (K)\ PV \int_{\mathbb{R}_+^{n-\rho_0}} |f_{\tau_0}(1, ..., 1, y_{\rho_0+1}, ..., y_n)|^{s_0} \varphi(0, ..., 0, y_{m+1}, ..., y_n) \prod_{j=\rho_0+1}^{n} y_j^{\eta_j - 1} dy_{\rho_0+1} \wedge ... \wedge dy_n. \tag{2.1.2}$$

Here the Principal Value Integral PV $\int_{\mathbb{R}_+^{n-\rho_0}} \ldots$ is by definition the value at $(s_0, 0)$ of the meromorphic continuation to $\mathbb{C}^2$ of the function

$$(2.1.3)\quad \begin{aligned} I(s,\ell) := \int_{\mathbb{R}_+^{n-\rho_0}} & |f_{\tau_0}(1,\ldots,1,y_{\rho_0+1},\ldots,y_n)|^s \varphi(0,\ldots,0,y_{m+1},\ldots,y_n) \\ & \prod_{j=\rho_0+1}^{n} y_j^{\eta_j - 1} \prod_{j=\rho_0+1}^{m} (y_j^2+1)^{-\ell} dy_{\rho_0+1} \wedge \ldots \wedge dy_n, \end{aligned}$$

defined for $Re(s) > 0$ and $\frac{Re(\ell)}{Re(s)}$ sufficiently big. This meromorphic continuation to $\mathbb{C}^2$ exists and is indeed holomorphic at $(s_0, 0)$. Moreover if $s_0 > -1$, then the integral in (2.1.2) converges absolutely and equals its principal value (i.e. the value at $(s_0, 0)$ of the meromorphic continuation of $I(s,\ell)$).

Theorems 1.1 and 2.1 remain valid with $|f|$ replaced by $f_+ := \max(f, 0)$ and f_{τ_0} by $(f_{\tau_0})_+$. Indeed the proofs remain the same. If τ_0 is simplicial and if each term in f_{τ_0} corresponds to a vertex of τ_0, then we moreover obtained, by using Theorem 2.1, an explicit formula for $\lim_{s \to s_0} (s - s_0)^{\rho_0} Z(s)$ in terms of special values of the gamma function (see [L].)

3. Toric manifolds

Let L be a lattice in $\mathbb{R}^n$, for example $\mathbb{Z}^n$. A cone Δ in $\mathbb{R}^n$ is called L-simple if it is generated by a set of vectors which are part of a basis for L. Let F be a fan (see [AVG, p. 192- 193]) consisting of L-simple cones in $\mathbb{R}^n$ (i.e. a L-simple fan). To the pair (L, F) one associates in a canonical way a real analytic manifold $X_{L,F}$ (called the toric manifold associated to L, F) see [AVG, p. 193-196]. Each n-dimensional cone $\Delta \in F$ yields an open subset $U_{L,F,\Delta}$ of $X_{L,F}$ which is a copy of $\mathbb{R}^n$ (called a standard chart[1]), and each ordered basis $\{\xi_1, \ldots, \xi_n\}$ of Δ yields affine coordinates $(y_1, \ldots, y_n)$ on $U_{L,F,\Delta}$ (called the standard coordinates associated to the basis $\{\xi_1, \ldots, \xi_n\}$). A fan F_1 is finer than a fan F_2 (notation $F_1 < F_2$), if each cone of F_1 is contained in a cone of F_2. To fans $F < F'$ and lattices $L \subset L'$ in $\mathbb{R}^n$ one associates in a canonical way an analytic map $X_{L,F} \to X_{L',F'}$, (see [AVG, p. 197] when $L = L'$). Even when L is not contained in L', there is a natural map $\pi : X_{L,F}(\mathbb{R}_+) \to X_{L',F'}(\mathbb{R}_+)$ which is given on corresponding charts by monomials with nonnegative real exponents. (With $X_{L,F}(\mathbb{R}_+)$ we mean the set of points on $X_{L,F}$ which have nonnegative standard coordinates). More precisely let $\Delta \in F$, $\Delta' \in F'$, be n-dimensional with $\Delta \subset \Delta'$ and let $\{\xi_1, \ldots, \xi_n\}$, resp. $\{\xi'_1, \ldots, \xi'_n\}$ be ordered sets of generators for Δ, resp. Δ', that are part of a basis for L resp. L'. Then the restriction of the natural map π to $U_{L,F,\Delta}$ takes values in $U_{L',F',\Delta'}$ and is given in the standard coordinates (associated to $\{\xi_1, \ldots, \xi_n\}$, resp. $\{\xi'_1, \ldots, \xi'_n\}$) by $y'_j = \prod_{i=1}^n y_i^{c_{ij}}$ for $j = 1, \ldots, n$, where c_{ij} is given by $\xi_i = \sum_{j=1}^n c_{ij} \xi'_j$.

[1] The standard charts cover the manifold $X_{L,F}$

4. Proof of Theorem 2.1

We assume that $\overset{o}{\tau}_0$ is $\mathbb{Z}^n$-simple. The general case is left to the reader and is obtained by making a sum over the cones in a subdivision of $\overset{o}{\tau}_0$ in $\mathbb{Z}^n$-simple cones. For ease of notation we also suppose that $\eta = (1, 1, ..., 1)$.

Let $L_1 = \mathbb{Z}^n$ and F_1 be a L_1-simple fan subordinated (in the sense of [AVG, p. 199]) to the Newtonpolyhedron $\Gamma(f)$ of f at 0. Then the natural map $\pi_1 : X_{L_1,F_1} \to \mathbb{R}^n$ is an embedded resolution of singularities of f in a neighbourhood of the origin in $\mathbb{R}^n$ [AVG, p. 201 Théorème 2].

Varchenko [V2] has studied the meromorphic continuation of $\int_{\mathbb{R}^n} |f|^s \varphi \cdot x^{\eta - 1} dx$ by using the resolution π_1, pulling back the integral by π_1. We assume the reader is familiar with this work.

Next we define the closed submanifold Y of X_{L_1,F_1} (with codimension ρ_0), by requiring for every n-dimensional $\Delta \in F_1$ that

$$U_{L_1,F_1,\Delta} \cap Y = \phi, \text{ if } \overset{o}{\tau}_0 \not\subseteq \Delta$$

$$U_{L_1,F_1,\Delta} \cap Y = \text{ locus } (y_1 = y_2 = ... = y_{\rho_0} = 0), \text{ if } \overset{o}{\tau}_0 \subset \Delta$$

where $(y_1, ..., y_n)$ are the standard coordinates associated to an ordered basis $\{\xi_1, ..., \xi_n\}$ of Δ with $\xi_1, ..., \xi_{\rho_0} \in \overset{o}{\tau}_0$. It is easy to verify (and well-known in the theory of toric varieties [Da, 5.7] and [F, 3.1]) that $Y = X_{L_2,F_2}$ where the lattice L_2 and the fan F_2 in $\mathbb{R}^{n-\rho_0}$ are constructed as follows : Let $\tilde{F}_1$ be the set consisting of all $\Delta \in F_1$ which contain $\overset{o}{\tau}_0$. Then the lattice L_2 and the fan F_2 are obtained by projecting L_1 and $\tilde{F}_1$ parallel to $\tilde{\tau}_0^0$ onto $\mathbb{R}e_{\rho_0+1} + ... + \mathbb{R}e_n = \mathbb{R}^{n-\rho_0}$. Note that the cones of F_2 are L_2- simple.

Put $L_3 = \mathbb{Z}e_{\rho_0+1} + ... + \mathbb{Z}e_n \subset \mathbb{R}^{n-\rho_0}$ and let F_3 be the fan in $\mathbb{R}^{n-\rho_0}$ consisting of all octants (i.e. all the connected components of $(\mathbb{R} \setminus \{0\})^{n-\rho_0}$). Then $X_{L_3,F_3} = (\mathbb{P}^1_{\mathbb{R}})^{n-\rho_0}$, where $\mathbb{P}^1_{\mathbb{R}}$ denotes the real projective line.
By refining the fan F_1 we may suppose that $F_2 < F_3$. Then there is a natural map

$$\pi_2 : Y(\mathbb{R}_+) = X_{L_2,F_2}(\mathbb{R}_+) \to X_{L_3,F_3}(\mathbb{R}_+) = (\mathbb{P}^1_{\mathbb{R}_+})^{n-\rho_0},$$

as explained in 3. (Here $\mathbb{P}^1_{\mathbb{R}_+} = \mathbb{P}^1_{\mathbb{R}} \setminus \{$ the negative real numbers $\}$.)

We are going to study the meromorphic continuation of the integral $I(s, \ell)$ in (2.1.3) by pulling it back through π_2 to an integral on $Y(\mathbb{R}_+)$.
Let γ on $(\mathbb{P}^1_{\mathbb{R}})^{n-\rho_0}$ be given by

$$\gamma := |f_{\tau_0}(1, ..., 1, z_{\rho_0+1}, ..., z_n)|^{s_0} \varphi(0, ..., 0, z_{m+1}, ..., z_n) |dz_{\rho_0+1} \wedge ... \wedge dz_n|,$$

where $z_{\rho_0+1}, ..., z_n$ are the standard affine coordinates on $\mathbb{R}^{n-\rho_0}$, and put

$$h_1 := |f_{\tau_0}(1, ..., 1, z_{\rho_0+1}, ..., z_n)| \text{ and } h_2 := \prod_{j=\rho_0+1}^{m} (z_j^2 + 1)^{-1}.$$

Note that $I(s,\ell) = \int_{\mathbb{R}_+^{n-\rho_0}} |h_1|^{s-s_0}|h_2|^{\ell}\gamma = \int_{Y(\mathbb{R}_+)} |h_1 \circ \pi_2|^{s-s_0}|h_2 \circ \pi_2|^{\ell}\pi_2^*(\gamma)$.

Let $\Delta \in \hat{F}_1$ be n-dimensional and generated by $\xi_1, ..., \xi_n$ with $\xi_1, ..., \xi_{\rho_0} \in \overset{0}{\tau}_0$.
Put $N_i = \min\{\langle x, \xi_i\rangle | x \in \Gamma(f)\}$ and $\nu_i =$ sum of the coordinates $\xi_{i,j}$ of ξ_i. It is a straightforward excersise to verify that on $Y \cap U_{L_1,F_1,\Delta}$ we have

$$(*) \qquad (n!\ \mathrm{Vol}\ (K)\prod_{i=1}^{\rho_0} N_i)\pi_2^*(\gamma) = \left.\frac{|\prod_{i=1}^{\rho_0} y_i|\pi_1^*(\varphi|f|^{s_0}|dx|)}{|dy_1 \wedge ... \wedge dy_{\rho_0}|}\right|_{y_1=y_2=...=y_{\rho_0}=0}$$

where $(y_1, ..., y_n)$ are the standard coordinates associated to $\{\xi_1, ..., \xi_n\}$. (Note that $N_i s_0 + \nu_i = 0$ for $i = 1, ..., \rho_0$.)

Formula $(*)$ is really the key of the proof of the Theorem. It relates PV $\int_{\mathbb{R}_+^{n-\rho_0}} \gamma$ to a principal value integral on $Y(\mathbb{R}_+)$ of the right side of $(*)$. But Langlands' work [La] implies that a *differently defined* principal value integral on $Y(\mathbb{R}_+)$ of the right side of $(*)$ equals the limit in (2.1.1). So to prove Theorem 2.1 it suffices to show that the two definitions of the PV coincide, which is not difficult. However we prefer to give a selfcontained proof of Theorem 2.1, without using Langlands' theory.

From [V1, p.260] it follows that at each point $P \in Y(\mathbb{R}_+) \cap U_{L_1,F_1,\Delta}$ which is contained in a sufficiently small neighbourhood of $\pi_1^{-1}(0)$, there exist local coordinates $y'_1, ..., y'_n$ on $U_{L_1,F_1,\Delta}$ centered at P such that *locally* at P we have :

(i) $y'_i = y_i$ for $i = 1, ..., \rho_0$ and for any i in $\{\rho_0 + 1, \ldots, n\}$ with $y_i(P) = 0$; thus Y is given by $y'_1 = ... = y'_{\rho_0} = 0$ and the positivity of all standard coordinates on $U_{L_1,F_1,\Delta}$ is equivalent to the positivity of these y'_i for which $y_i(P) = 0$.
(ii) $\pi_1^*(|f|^s|dx|) = |v_1|^s|v_2|\prod_{i=1,...,n}|y'_i|^{N'_i s+\nu'_i-1}|dy'_1 \wedge ... \wedge dy'_n|$,
where v_1 and v_2 are nonvanishing analytic functions, $(N'_i, \nu'_i) = (N_i, \nu_i)$ for any i with $y_i(P) = 0$ and $(N'_i, \nu'_i) \in \{(1,1), (0,1)\}$ if $y_i(P) \neq 0$.
(iii) $\pi_2^*(\gamma) = (n!Vol(K)\prod_{i=1}^{\rho_0} N_i)^{-1}\prod_{i=\rho_0+1}^{n}|y'_i|^{N'_i s_0+\nu'_i-1}\times$
$(|v_1|^{s_0}|v_2|(\varphi \circ \pi_1))\Big|_{y'_1=...=y'_{\rho_0}=0}|dy'_{\rho_0+1} \wedge ... \wedge dy'_n|$.
This follows from $(*)$ and (ii), and holds for any C^∞-function φ on $\mathbb{R}^n$.
(iv) $|h_1 \circ \pi_2| = |u|\prod_{i=\rho_0+1}^{n}|y'_i|^{a_i}$, $|h_2 \circ \pi_2| = |w|\prod_{i=\rho_0+1}^{n}|y'_i|^{b_i}$,
where $a_i, b_i \in \mathbb{Q}$ and u, w are nonvanishing functions with u analytic and with w analytic in $y_i'^{c_i}$ for suitable $c_i \in \mathbb{Q}, c_i > 0, i = \rho_0 + 1, ..., n$. This follows easily from (iii) and the nature of π_2. Moreover one can take $c_i = 1$ when $y_i(P) \neq 0$.

Note that the exponents $N'_i s_0 + \nu'_i - 1$ for $i = \rho_0 + 1, ..., n$ are among the numbers

$$(**) \qquad s_0 \notin \mathbb{Z},\ 0,\ N_j s_0 + \nu_j - 1 > -1 \text{ for } j = \rho_0 + 1, ..., n,$$

because $N'_i s_0 + \nu'_i = N_i s_0 + \nu_i > 0$ when $y_i(P) = 0, i > \rho_0$.

Hence we see that the integrand of $I(s,\ell) = \int_{Y(\mathbb{R}_+)} |h_1 \circ \pi_2|^{s-s_0}|h_2 \circ \pi_2|^{\ell}\pi_2^*(\gamma)$ *locally* looks like the integrand in the integral $J(k,\ell)$ in Lemma 4.1 below, with k replaced by $s - s_0$, v by v_1, θ by $|v_2|(\varphi \circ \pi_1)$ and (N_i, ν_i) by (N'_i, ν'_i). Because $I(s,\ell)$ converges absolutely for any compactly supported C^∞-function φ on $\mathbb{R}^n$, whenever $Re(s) > 0$ and

$\frac{Re(\ell)}{Re(s)}$ is sufficiently big, we see that $b_i \geq 0, a_i \geq 0$ if $b_i = 0$ and $N_i' s_0 + \nu_i' > 0$ if $a_i = b_i = 0$, for all $i = \rho_0 + 1, \ldots, n$. Thus by using a suitable partition of unity[2] on X_{L_1, F_1} (and the properness of π_1) we obtain by Lemma 4.1 below that (2.1.1) equals (2.1.2), and that the meromorphic continuation of $I(s, \ell)$ is analytic in $(s_0, 0)$. Finally the last assertion of the Theorem follows from $(**)$ which implies that $\int_{\mathbb{R}_+^{n-\rho_0}} \gamma = \int_{Y(\mathbb{R}_+)} \pi_2^*(\gamma)$ converges when $s_0 > -1$. $\square$

Lemma 4.1. *Let $N_i, \nu_i \in \mathbb{R}, N_i \geq 0, \nu_i > 0$, for $i = 1, ..., n$. Let $s_0 \in \mathbb{R}, s_0 < 0$. Suppose that $N_i s_0 + \nu_i = 0$ for $i = 1, ..., \rho_0 \leq n$ and that $N_i s_0 + \nu_i \notin -\mathbb{N}$ for $i > \rho_0$. Let θ be a C^∞ function on $\mathbb{R}^n$ with compact support, and v an analytic nonvanishing function on a neighbourhood of the support of θ. Then*

(i) the meromorhpic continuation of

$$(s - s_0)^{\rho_0} \int_{\mathbb{R}_+^n} \theta |v|^s (\prod_{i=1}^n y_i^{N_i s + \nu_i - 1}) \qquad dy_1 \wedge \ldots \wedge dy_n$$

is holomorphic in s_0 with value say A.

(ii) Moreover let $a_i, b_i \in \mathbb{R}$ for $i = \rho_0 + 1, ..., n$ and let u, w be real valued functions of $y_{\rho_0+1}, \ldots, y_n \in \mathbb{R}$ which do not vanish and which are analytic in $|y_i|^{c_i}$ for suitable $c_i \in \mathbb{Q}, c_i > 0$ for $i = \rho_0 + 1, \ldots, n$, on a neighbourhood of the support of θ. Consider the integral

$$J(k, \ell) := \int_{\mathbb{R}_+^{n-\rho_0}} (\theta |v|^{s_0})\Big|_{y_1 = \ldots = y_{\rho_0} = 0} (\prod_{i=\rho_0+1}^n y_i^{N_i s_0 + \nu_i - 1 + a_i k + b_i \ell}) |u|^k |w|^\ell dy_{\rho_0+1} \wedge \ldots \wedge dy_n.$$

Suppose that $b_i \geq 0, a_i \geq 0$ if $b_i = 0$ and $N_i s_0 + \nu_i > 0$ if $a_i = b_i = 0$, for all $i = \rho_0 + 1, \ldots, n$. Assume that $N_i s_0 + \nu_i > 0$ whenever $c_i \notin \mathbb{N}$. Then for $Re(k)$ and $\frac{Re(\ell)}{Re(k)}$ sufficiently big, the integral $J(k, \ell)$ converges absolutely to an analytic function which has a meromorphic continuation to $\mathbb{C}^2$. Moreover this meromorphic continuation is holomorphic at (0,0) with value $A \prod_{i=1}^{\rho_0} N_i$.

Proof. Consider the integral

$$G(s, k, \ell) := (s - s_0)^{\rho_0} \int_{\mathbb{R}_+^n} \theta |v|^s (\prod_{i=1}^{\rho_0} y_i^{N_i s + \nu_i - 1})(\prod_{i=\rho_0+1}^n y_i^{N_i s + \nu_i - 1 + a_i k + b_i \ell}) |u|^k |w|^\ell dy_1 \wedge \cdots \wedge dy_n.$$

It is clear that this integral converges absolutely to an analytic function G on the open connected set

$$D_0 := \{(s, k, \ell) \in \mathbb{C}^3 | Re(s) > s_0, Re(N_i s + \nu_i + a_i k + b_i \ell) > 0 \text{ for } i = \rho_0 + 1, \ldots, n\} \neq \emptyset,$$

[2] Note that $\lim_{s \to s_0} (s - s_0)^{\rho_0} \int_{X_{L_1, F_1}(\mathbb{R}_+)} \pi_1^*(|f|^s |dx|)\theta = 0$ whenever θ is a C^∞-function with compact support disjoint with Y.

because $N_i s_0 + \nu_i = 0$ for $i = 1, \ldots, \rho_0$. There exists ε in $\mathbb{R}, \varepsilon > 0$, such that G has a continuation to an analytic function, again denoted by G, on the open connected set

$$D := \{(s,k,\ell) \in \mathbb{C}^3 | Re(s) > s_0 - \varepsilon, Re(N_i s + \nu_i + a_i k + b_i \ell) > 0 \text{ for } i = \rho_0 + 1, \ldots, n\} \supset D_0.$$

This follows from integration by parts with respect to the variables $y_1, \ldots, y_{\rho_0}$, to raise the exponents of these variables. Moreover the function G on D has a meromorphic continuation $[G]_{ac}$ to $\mathbb{C}^3$. Indeed this follows again by partial integration when all c_i are integral and one reduces to this case by a change of variables $y_i = {y'_i}^d$ with d$\in \mathbb{N}$. Moreover $[G]_{ac}$ is holomorphic at $(s_0, 0, 0)$ because it follows from $N_i s_0 + \nu_i \notin -\mathbb{N}$, for $i > \rho_0$, that integration by parts with respect to the variables y_i, for which $c_i \in \mathbb{N}$, raises the exponent of y_i without introducing a pole at $(s_0, 0, 0)$. (Note that we avoid integration by parts with respect to the variables y_i for which $c_i \notin \mathbb{N}$. An integration by parts with respect to one of these variables could cause problems and is not needed because we assume $N_i s_0 + \nu_i > 0$ for these i, which implies that the exponent of such y_i has not to be raised.)

We recall the following principle which follows easily from the basic properties of meromorphic functions in several variables [GF]. Let G be a holomorphic function on a nonempty open connected subset D of $\mathbb{C}^n$ which has a meromorphic continuation $[G]_{ac}$ to $\mathbb{C}^n$. Let L be an affine subspace of $\mathbb{C}^n$ with $L \cap D \neq \emptyset$. Then the restriction $G_{|L\cap D}$ of G to $L \cap D$ has a unique meromorphic continuation $[G_{|L\cap D}]_{ac}$ to L and $[G_{|L\cap D}]_{ac}$ is holomorphic at P with value $[G]_{ac}(P)$ at each point P$\in L$ where $[G]_{ac}$ is holomorphic.

By applying this principle with $L = \{(s,k,l) \in \mathbb{C}^3 | k = l = 0\}$ and $P = (s_0, 0, 0)$, we see that assertion (i) of lemma 4.1 is true with A=$[G]_{ac}((s_0, 0, 0))$.

Because of the assumption on a_i, b_i, there moreover exist N,M in $\mathbb{N}$ such that $\{s_0\} \times W \subset D$, where

$$W := \{(k,\ell) \in \mathbb{C}^2 | Re(k) > N, \frac{Re(\ell)}{Re(k)} > M\}.$$

The principle above with $L = \{s_0\} \times \mathbb{C}^2$ and $P = (s_0, 0, 0)$ yields that $G_{|\{s_0\}\times W}$ has a meromorphic continuation to $L = \{s_0\} \times \mathbb{C}^2$ which is holomorphic at $(s_0, 0, 0)$ with value $[G]_{ac}((s_0, 0, 0)) = A$. Thus to prove assertion (ii) of lemma 4.1, it suffices to prove that $J_{|W}$ equals $(\prod_{i=1}^{\rho_0} N_i) G_{|\{s_0\}\times W}$. But since $N_i s + \nu_i = N_i(s - s_0)$ for $i = 1, \ldots, \rho_0$, this follows easily from the well-known formula

$$\lim_{s \to s_0} (s - s_0)^{\rho_0} \int_{[0,1]^{\rho_0}} \psi(s, y_1, \ldots, y_{\rho_0}) \prod_{i=1}^{\rho_0} y_i^{N_i(s-s_0)-1} dy_1 \wedge \cdots \wedge dy_{\rho_0} = \frac{\psi(s_0, 0, \ldots, 0)}{\prod_{i=1}^{\rho_0} N_i},$$

which holds for any continuous function ψ on $\mathbb{R} \times [0,1]^{\rho_0}$. □

5. Proof of Theorem 1.1

Applying Theorem 2.1 to both f and f_{τ_0} we see that $\lim_{s \to s_0} (s - s_0)^{\rho_0} Z(s)$ and

$$\lim_{s \to s_0} (s - s_0)^{\rho_0} \int_{\mathbb{R}^n} |f_{\tau_0}(x)|^s x^{\eta - 1} \varphi(x) dx \tag{5.1}$$

are equal up to a strictly positive factor (which is a quotient of volumes). Hence it suffices to prove that the limit in (5.1) is zero, i.e. to prove that Theorem 1.1 holds for f replaced by f_{τ_0}. Since all vertices of $\Gamma(f_{\tau_0})$ are contained in τ_0, this can be done by using material from [DS2] as follows:

Proof for f replaced by f_{τ_0}. We assume that τ_0 is unstable relatively to the index j=n. For any vector u in $\mathbb{R}^n_+$, we denote by F(u) the set of all x in $\Gamma(f_{\tau_0})$ where $\langle x, u\rangle$ is minimal. Let H_0 be $\{x \in \mathbb{R}^n | x_n = 0\}$ and H_1 be $\{x \in \mathbb{R}^n | x_n = 1\}$. By using the material of section 4 in [DS2], it suffices to prove that there exists a decomposition of $\mathbb{R}^n_+$ in cones C_i spanned by $\{u_1^{(i)}, ..., u_{n-1}^{(i)}, e_n\}$ such that for every i

(1) $\cap_{j=1}^{n-1} F(u_j^{(i)}) \neq \emptyset$,

(2) at most ρ_0 -1 of the $u_j^{(i)}$ are contained in $\overset{o}{\tau}_0$,

(3) for every subset J of {1,...,n-1 } the face $\tau = \cap_{j\in J} F(u_j^{(i)})$ satisfies

  (a) if $\tau \cap H_0 = \emptyset$, then $\tau \cap H_1 \neq \emptyset$,

  (b) if $\tau \cap H_0 = \emptyset$ and if $\tau \cap H_1$ is compact, then $f_{(\tau\cap H_1)}$ does not vanish on $(R \setminus \{0\})^n$.

To prove the existence of such a decomposition, we will construct one. We consider the set of cones $\{p^0 \cap H_0 | \text{p vertex of } \Gamma(f_{\tau_0})\}$ where $p^0 := \{u \in \Gamma(f_{\tau_0}) | F(u) \ni p\}$. We refine this decomposition of $\mathbb{R}^n_+ \cap H_0$ by dividing every cone in simplicial subcones, to obtain a decomposition $(\tilde{C}_i)_{i\in I}$. We claim that the decomposition of $\mathbb{R}^n_+$ consisting of the cones $C_i := conv(\tilde{C}_i, e_n)$ for i in I, satisfies conditions (1),(2) and (3).
Condition (1) is satisfied since the cones $\tilde{C}_i$ are subordinated to $\Gamma(f_{\tau_0})$.Since τ_0 is unstable relatively to x_n, we have that $\dim(\overset{o}{\tau}_0 \cap H_0) < \dim(\overset{o}{\tau}_0) = \rho_0$ which implies (2). For an arbitrary i $\in$ I and J subset of $\{1, ..., n-1\}$, let τ be $\cap_{j\in J} F(u_j^{(i)})$. Since τ is a nonempty face of $\Gamma(f_{\tau_0})$ by (i), it contains at least one vertex of $\Gamma(f_{\tau_0})$, cf. [R, 18.5.3]. Since each vertex of $\Gamma(f_{\tau_0})$ is contained in τ_0, we conclude that τ contains at least one vertex of τ_0. Since τ_0 is unstable relatively to x_n, all vertices of τ_0 are contained in $H_0 \cup H_1$. Let $\tau \cap H_0 = \emptyset$, then $\tau \cap H_1 \neq \emptyset$ which proofs (3) (a). Note that $\tau \cap H_1$ is a face of $\Gamma(f_{\tau_0})$. Suppose moreover that $\tau \cap H_1$ is compact, then $\tau \cap H_1 = conv\{p_1, ..., p_r\}$ where the p_i are vertices of $\Gamma(f_{\tau_0})$, cf. [R, 18.5.1]. Since each vertex of $\Gamma(f_{\tau_0})$ is contained in τ_0, we conclude that $\tau \cap H_1 \subset \tau_0$. Assertion (3) then follows from the unstability of τ_0. □

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