Triangulation and CSG Representation of Polyhedra Through Plane Insertions*

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1 Introduction

The main purpose behind decomposing an object into simpler components is to simplify a problem for complex objects into a number of subproblems dealing with simple objects. In particular, the problem of partitioning a three dimensional polyhedron into simpler parts arises in mesh generation for finite element methods, CAD/CAM applications, computer graphics, motion planning and solid modeling. Though several decomposition problems have been widely researched in two dimensions, very few results exist for their three dimensional counterparts. Two such problems, triangulation and CSG (Constructive Solid Geometry) decomposition of polyhedra are addressed in this paper. We study the complexity of the *plane insertion* paradigm when applied for these two problems. In this paradigm a polyhedron is sliced with planes successively that resolve the reflex edges called *notches* [1] [3]. Chazelle used this paradigm to give an $O(nr^3)$ time and $O(nr^2)$ space algorithm for decomposing a polyhedron with n edges and r notches into convex pieces.

In triangulation we seek for a simplicial decomposition of the given polyhedron that forms a simplicial complex. This means that any two tetrahedra in the triangulation either do not meet or meet along a full facet, or a full edge, or at a vertex. Sometimes this type of simplicial decomposition is called face-to-face triangulation. In three dimensions, there are polyhedra that are not triangulable without additional points called Steiner points. Moreover, as shown by Rupert and Seidel [15], the general problem of determining whether a polyhedron is triangulable without Steiner points or not is NP-hard. Due to these constraints we consider the problem of triangulation with Steiner points. Chazelle showed that $\Omega(r^2)$ convex pieces are necessary for decomposing certain class of polyhedra into convex pieces. This suggests an $\Omega(r^2)$ worst case lower bound on the output size of the triangulations of polyhedra. In [4], Chazelle and Palios have used a technique different from plane insertion to give an $O((n+r^2)\log r)$ time algorithm that produces $O(n+r^2)$ size triangulation for simple polyhedra. These polyhedra are homeomorphic to spheres i.e., they cannot have holes (genus 0) and shells (internal voids) and are manifold [2]. Moreover, this method does not produce a triangulation that is a simplicial complex. However, this method can be modified to produce a simplicial complex with some postprocessing. This tends to produce skinny tetrahedra.

In this paper we study and analyze the plane insertion technique for several reasons. First of all it is simple. It produces face-to-face triangulations of polyhedra with arbitrary genus and shells in a very straight

*Supported in part by ARO Contract DAAG29-85-C0018 under Cornell MSI, NSF grant DMS 88-16286 and ONR contract N00014-88-K-0402.

[1] these are the edges where the internal dihedral angle subtended by two incident facets is greater than 180°

[2] each point on the surface of the polyhedron has an ϵ-neighborhood that is homeomorphic to a 2d-ball or halfball

forward manner. Secondly, it is less susceptible to numerical errors that occurs during computations with finite precision arithmetic. If the planes are chosen from those that subtend the facets adjacent to the reflex edges, one can use the original plane equations while computing the coordinates of the new vertices. This, in effect, reduces the chance of error propagation in numerical computations. Thirdly, plane insertion technique can be extended to higher dimensions in a very natural way. We show that an $O(nr + r^3)$ size triangulation for polyhedra can be computed in $O(nr^2 + r^3 \log r)$ time using plane insertions.

Another motivation to study the plane insertion paradigm is its use in Halfspace CSG (HCSG) representation of polyhedra. In this representation a polyhedron is expressed in terms of the two regularized boolean operations, namely, inetersections and unions, that operate on the closed halfspaces supported by the facets of the polyhedron. Let $N(p)$ represent an ϵ-neighborhood of a point p inside a facet f of a polyhedron S. Let $ri(S)$ represent the relative interior of S. The literal f^+ represents the closed halfspace supported by the facet f such that $N(p) \cap ri(S) \cap f^+$ is nonempty. The literal f^- represents the other closed halfspace. Peterson [14] considered the HCSG representations that use only the closed halfspaces f^+'s. This type of CSG expressions are called Peterson-style CSG formulae. Although it is possible to find such formulae for polygons in 2D, it is not possible to find such formulae for polyhedra in 3D in general [6]. Hence, we allow both closed halfspaces f^+'s and f^-'s in the HCSG representations of polyhedra. In 2D, Dobkin, Guibas, Hershberger, and Snoyeyink [6] gave an $O(n \log n)$ algorithm to compute Peterson-style CSG formulae of size $O(n)$ for polygons with n vertices. They observed that HCSG formulae of size $O(p^3)$ is trivial to obtain for polyhedra with p facets. In [13], Paterson and Yao gave an $O(p^3)$ time algorithm to compute $O(p^2)$ size HCSG formulae for polyhedra with convex facets that have constant size. They used the concept of Binary Space Partition (BSP) trees [10]. We show that the plane insertion paradigm gives an $O(p^{\frac{7}{3}})$ size HCSG formulae for polyhedra that allow nonconvex facets of arbitrary size. This algorithm runs in $O(p^{\frac{10}{3}})$ time. Proving a nontrivial lower bound on the size of HCSG formulae for polyhedra is an open question. We show an $\Omega(p^2)$ lower bound for HCSG formulae that are in Conjunctive Normal Form (CNF) or in Disjunctive Normal Form (DNF).

2 Plane Insertions

2.1 Notations

Nonconvexity in a manifold polyhedron S is a result of the presence of notches. All notches of a manifold polyhedron can be removed by repeatedly cutting and splitting it with planes that resolve these notches. If an edge g with f_1, f_2 as its incident facets is a notch, a plane P_g that contains the notch g and subtends an inner-angle greater than $\gamma - 180°$ with both f_1 and f_2, is a valid plane that resolves the notch g. The chosen plane P_g is also called the *notch plane* of g. Clearly, for each notch g, there exist infinite choices for P_g. A notch plane P_g may intersect other notches, thereby producing *subnotches*. See Figure 1.

2.2 Sketch of the Algorithm

Given a polyhedron S with n edges of which r are notches, a notch of S is removed by cutting it with a notch plane. This notch plane intersects possibly other notches to create subnotches of those notches. As the notch elimination process proceeds, the number of polyhedral pieces increases in general. At any generic step of the algorithm all subnotches of a notch, possibly present in different polyhedra, are eliminated by a single notch plane. This process is continued until there is no more notch.

Let S be a polyhedron with a notch g. The intersection of P_g with the polyhedron S is a set of isolated points, segments, and polygons, possibly with holes. We refer to this intersection as *cross-sectional map* and denote it as GP_g. The unique polygon Q_g in GP_g containing the notch on its boundary is called the *cut*. At any generic step of the plane insertion process, let $S_1, S_2, ..., S_k$ be the subpolyhedra containing

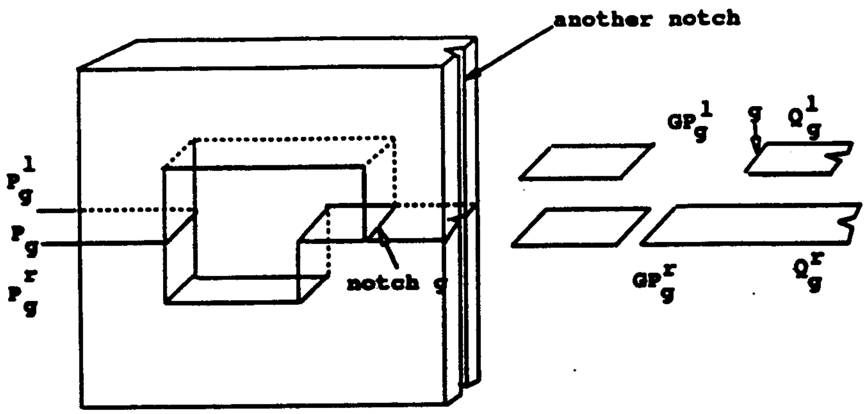

Figure 1: A notch and its notch plane, cross sectional map, cut.

the subnotches $g_1, g_2, ..., g_k$ of a notch g in S. The notch plane P_g passes through $S_1, S_2, ..., S_k$ and possibly other subpolyhedra $S_{k+1}, S_{k+2}, ..., S_t$. We distinguish between two types of slicings with the notch plane P_g. (i) We can split $S_1, S_2, ..., S_t$ along the cross sectional maps $GP_{g_1}, GP_{g_2}, ..., GP_{g_t}$. Effectively, in this case we slice all subpolyhedra completely through which P_g passes. We call each of this type of slicing a *complete cut.* (ii) We can split only $S_1, S_2, ..., S_k$ along only the cuts $Q_{g_1}, Q_{g_2}, ...Q_{g_k}$. We call each of this type of slicing a *bounded cut.* Note that, bounded cuts are sufficient to remove notches and they may not separate the polyhedron into two different pieces. See Figure 1. In this case, two distinct facets corresponding to Q_{g_i}'s are created that overlap geometrically.

We obtain the following Lemma from [2].

Lemma 2.1. A manifold polyhedron S having m edges of which r are notches can be partitioned with a notch plane P_g in $O(m + y \log r)$ time and in $O(m)$ space where y is the number of edges on the cross sectional map GP_g.♣

2.3 A 2D Subproblem

Now we focus our attention to a 2D subproblem which is essential for the analysis of bounded cuts. Let L be a set of r lines in 2D that forms an arrangement A. Let E be a set of edges removed from A such that all cells in $A - E$ are convex. Let us denote the new arrangement $A - E$ as A^-. Let C be a set of cells in A^- intersected by a line l. The total number of edges in the cells in C determines the zone complexity $z(l, A^-, r)$ of l in A^-. Of course, the contribution of a line in any single cell is counted only once although it may have several consecutive segments on it in that cell. Let $q(r) = max\{z(l, A^-, r) | \, l$ is any line in any such arrangement $A^-\}$.

Lemma 2.2: $q(r) = O(r^{\frac{3}{2}})$.

Proof: Form a bipartite graph $G = (V_1 \cup V_2, E)$ where each node in V_1 corresponds to a cell in C and each node in V_2 corresponds to a line in L. An edge $e \in E$ connects two vertices $v_1 \in V_1$, $v_2 \in V_2$ if the line corresponding to v_2 contributes an edge to the cell corresponding to v_1. Observe that any four lines in L can contribute simultaneously to at most two cells in C since they are convex. This means that G cannot have $K_{2,5}$ as a subgraph. Thus using the forbidden graph theory [12], G can have at most $O(mr^{\frac{1}{2}} + r)$ edges, where $|C| = m$. Since $|C| \leq r + 1$, we have $q(r) = O(|E|) = O(r^{\frac{3}{2}})$. ♣

Suppose a polyhedron S with n edges and r notches has been sliced with a series of bounded cuts. Let S_1,

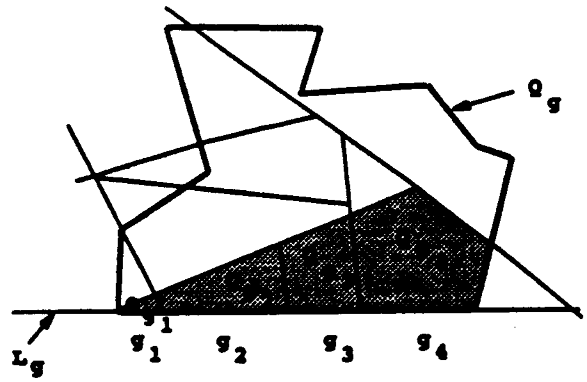

Figure 2: Zones of a line and cuts.

$S_2, ..., S_k$ be the polyhedra in the current decomposition, where each S_i contains a subnotch g_i of a notch g in S. Let x_i be the number of edges on Q_{g_i}.

Lemma 2.3: $x = \Sigma_{i=1}^{k} x_i = O(n + q(r))$.

Proof: Consider the cut Q_g produced by the intersection of S with P_g. The region in Q_g is divided into smaller cells by the segments of *notch lines* produced by the intersection of other notch planes with P_g. We focus on the cells $Q_{g_1}, Q_{g_2}, ..., Q_{g_k}$ adjacent to the subnotches $g_1, g_2, ..., g_k$ of the notch g.

Consider the set of notch line segments separately that divides Q_g. These line segments and the line L_g corresponding to the notch g produce an arrangement T of line segments on the notch plane P_g. Notice that the arrangement T can be thought of as an arrangement A^- for some line arrangement A. The cells adjacent to the line L_g in this arrangement form the *zone* Z_g of L_g. Let the set of vertices and edges of Z_g be denoted as V_g and E_g respectively. At this point we stress the fact that in each single cell of Z_g consecutive segments of a line form a single edge. Actually one can verify that this notion of edges is consistent with our notion of bounded cuts. Overlaying Q_g on T produces $Q_{g_1}, Q_{g_2}, ..., Q_{g_k}$. See Figure 2. These are the cells in $T \cup Q_g$ that are adjacent to the line L_g. Let V'_g and E'_g denote the sets of vertices and edges respectively in $Q_{g_1}, Q_{g_2}, ..., Q_{g_k}$. The vertices in V'_g can be partitioned into three disjoint sets, namely, T_1, T_2, T_3. The set T_1 consists of vertices formed by the intersections of two notch line segments; T_2 consists of vertices of Q_g, and T_3 consists of vertices formed by the intersections of notch line segments with the edges of Q_g. Certainly, $|T_1| = O(q(r))$ since overlaying Q_g on Z_g cannot introduce more vertices in T_1. If Q_g has u' vertices, $|T_2| \leq u'$.

To count the number of vertices in T_3, we first assume that Q_g does not have any hole. Consider an edge e in E_g that contributes one or more edge segments to E'_g as a result of intersections with Q_g. If e contributes y edges to Z'_g, it hides at least $y - 2$ vertices of Q_g from L_g. Further, each vertex of Q_g is hidden from L_g by a unique segment that is closest to L_g. Thus, $|T_3| \leq |T_2| + 2|T_1| = O(u' + q(r))$.

In the case when Q_g has holes, we can prove $|T_3| = O(u' + q(r))$ by creating a polygon without hole from Q_g. See [2] for details.

Putting all these together, we have $|V'_g| = |T_1| + |T_2| + |T_3| = O(u' + q(r)) = O(n + q(r))$. Since $Q_{g_1}, Q_{g_2}, ..., Q_{g_k}$ form a planar graph, we have $x = |E'_g| = O(|V'_g|) = O(n + q(r))$.♣

2.4 Complexity of Bounded Cuts

As observed earlier, all notches of a polyhedron S can be eliminated with a series of bounded cuts. Actually, if S has r notches, bounded cuts with r notch planes are sufficient to remove all notches. All polyhedra in the final decomposition after all these bounded cuts are thus convex. The Lemma below estimates the number of edges in the final decomposition.

Lemma 2.4: The total number of edges in the final decomposition of the polyhedron S with r bounded cuts is $O(nr + rq(r))$.

Proof: Edges in the final decomposition consist of newly generated edges by the bounded cuts, and the edges of S that are not intersected by any notch plane. By Lemma 2.3, a single bounded cut generates $O(n + q(r))$ edges. Thus, r bounded cuts generate $O(nr + rq(r))$ new edges. Hence, the total number of edges in the final decomposition is $O(n + nr + rq(r)) = O(nr + rq(r))$.♣

Lemma 2.4 gives an upper bound on the total number of edges present in all subpolyhedra encountered at any generic step of the plane insertion process. However, for our analysis we also need to estimate the number of edges present on the cross sectional maps produced by a notch plane. Note that this number is not same as the number of edges in all cuts produced by the notch plane. Let $S_1, S_2, ..., S_k$ be the polyhedra in the current decomposition, where each S_i contains a subnotch g_i of a notch g. Let y_i be the total number of edges on the cross sectional map in S_i.

Lemma 2.5: $y = \sum_{i=1}^{k} y_i = O(n + r^2)$.

Proof: Consider the cells in $\cup_{i=1}^{k} GP_{g_i}$ created by the notch line segments and the edges of GP_g on P_g. The vertices on $\cup_{i=1}^{k} GP_{g_i}$ can be partitioned into three sets, viz., T_1, T_2 and T_3. The set T_1 consists of vertices that are created by intersections two notch lines. The set T_2 consists of vertices of GP_g and the set T_3 consists of vertices that are created by intersections of edges of GP_g and notch lines . Since there are at most r notch lines , $|T_1| = O(r^2)$. Certainly, $|T_2| = O(n)$. By a Lemma in [2], each notch line can intersect GP_g in at most $O(r)$ segments since GP_g has at most r polygons containing no more than r reflex vertices all together. This gives $|T_3| = O(r^2)$. Thus,

$$\begin{aligned} y = \sum_{i=1}^{k} y_i &= |T_1| + |T_2| + |T_3| \\ &= O(n + r^2). \clubsuit \end{aligned}$$

Theorem 2.1: A manifold polyhedron S, possibly with holes and shells and having r notches and n edges can be decomposed into $O(r^2)$ convex polyhedra in $O(nr^2 + r^{\frac{7}{2}})$ time and $O(nr + r^{\frac{5}{2}})$ space.

Proof: Since any notch can have at most $r - 1$ subnotches during the notch elimination process, there can be at most $r - 1$ subpolyhedra involved per bounded cut. Since each subpolyhedron is split into at most two pieces, each bounded cut produces at most $O(r)$ new polyhedra. Thus r bounded cuts produce at most $O(r^2)$ convex pieces in the final decomposition.

At a generic instance of the algorithm let $S_1, S_2, ..., S_k$ be k distinct (nonconvex) polyhedra in the current decomposition, where each S_i contains a subnotch g_i of a notch g that is going to be removed. Let S_i have m_i edges of which r_i are notches. Let y_i be the number of edges on GP_{g_i} of S_i and $y = \sum_{i=1}^{k} y_i$.

Applying Lemma 2.1, removal of a notch g can be carried out in $O(\sum_{i=1}^{k}(m_i + y_i \log r_i)$ time. By Lemma 2.4 $\sum_{i=1}^{k} m_i = O(nr + rq(r))$, and by Lemma 2.5 $y = O(n + r^2)$; we have $O(\sum_{i=1}^{k}(m_i + y_i \log r_i) = O(nr + rq(r) + r^2 \log r)$. Thus to remove r notches we need $O(nr^2 + r^2 q(r) + r^3 \log r)$ time. Using the bound on $q(r)$ in Lemma 2.2, we get an $O(nr^2 + r^{\frac{7}{2}})$ time bound. By Lemma 2.4 the space complexity is $O(nr + rq(r)) = (nr + r^{\frac{5}{2}})$.♣

Improvement of Complexity: Recently, Hershberger and Snoyeyink [11] proved that $q(r) = O(r^{\frac{4}{3}})$. Applying this result the plane insertions when used as bounded cuts decompose a nonconvex polyhedron into convex pieces in $O(nr^2 + r^{\frac{10}{3}})$ time and in $O(nr + r^{\frac{7}{3}})$ space.

3 Triangulation

We observe that triangulating each convex piece as produced by bounded cuts does not yield a triangulation of the original polyhedron S. Two facets created corresponding to the cut Q_g may be decomposed differently later by other notch planes. Thus, the triangulation of the portions where these facets touch each other may not match. This produces a triangulation of S that is not a simplicial complex. We can overcome this problem if we use complete cuts. For such plane insertions, we cannot use Lemma 2.2 to determine the space complexity since the new edges created by complete cuts are not restricted to the regions adjacent to the notch g. In fact, in this case, we have to consider all the edges inside and on $\bigcup_{i=1}^{t} GP_{g_i}$ where P_g passes through $S_1, S_2, ..., S_t$. The natural expectation is that the complete cuts increase the time and space complexity considerably. In the following two Lemmas we show that the time and space complexities do not change much due to complete cuts.

Lemma 3.1: If a polyhedron S is decomposed by complete cuts, the number of edges in the final decomposition is $O(nr + r^3)$.

Proof: By a similar argument of Lemma 2.5, the number of edges on and inside $\bigcup_{i=1}^{t} GP_{g_i}$ is only $O(n+r^2)$. This implies that one complete cut generates $O(n+r^2)$ new edges. Thus, r complete cuts produce $O(nr+r^3)$ new edges.♣

If we use the similar analysis of Theorem 2.1 for complete cuts, we get $O(\sum_{i=1}^{k} m_i) = O(nr + r^3)$ by Lemma 3.1. This gives a straightforward $O(nr^2 + r^4)$ time complexity for decompositions with complete cuts. However, the following Lemma prevents this increase in time complexity.

Lemma 3.2: If a polyhedron S is decomposed by complete cuts, the total number of edges in subpolyhedra $S_1, S_2, ..., S_t$ through which a complete cut passes is only $O(nr)$.

Proof: Consider the complete cut corresponding to the plane P_g. let R be the set of planes used before P_g for other complete cuts. The planes in $R \cup P_g$ form an arrangement A of planes in three dimensions. The cells adjacent to the plane P_g in A constitute the zone Z_g of P_g. By well known zone theorem [8], the number of edges in Z_g is $O(q^2)$ if there are q planes in the arrangement. Let A' be the new arrangement obtained by superimposing the boundary facets of S on Z_g. Consider the cells adjacent to P_g that constitute the zone Z'_g in A'. Subpolyhedra through which P_g passes consist of cells that are members of Z'_g. Thus, the number of edges in Z'_g gives an upper bound on the number of edges of subpolyhedra through which P_g passes. To count the number of edges in Z'_g, we carefully analyze the effect of superimposing p boundary facets of S on Z_g.

Let f_i be a facet of S that contributes to the boundaries of some cells in Z'_g. Consider the lines of intersections between f_i and the facets of Z'_g. These lines together with the line segments supporting the edges of f_i form an arrangement of line segments on the plane supporting f_i. Let B_i denote the facets in this arrangement that are inside f_i. Further, let B'_i denote the set of facets in B_i that are adjacent to line segments supporting the edges of f_i; B''_i denote the rest of the facets in B_i. See Figure 3. In the following, by $V(F)$ and $E(F)$ we denote the number of vertices and edges respectively in a set of facets F.

Let F_i denote the set of facets in B''_i that do not have any edge formed by the intersection of P_g with f_i; F'_i denote the rest of the facets in B''_i. The facets in F_i are created by slicing the cells in Z_g completely by f_i such that f_i does not intersect P_g inside those cells. Consider the facets in F_i separately. They do not intersect P_g or have edges inside cells of Z_g. For each vertex of Z_g, there is at most one facet in F_i that hides the vertex from P_g and is closest to P_g. Therefore the number of facets in F_i that

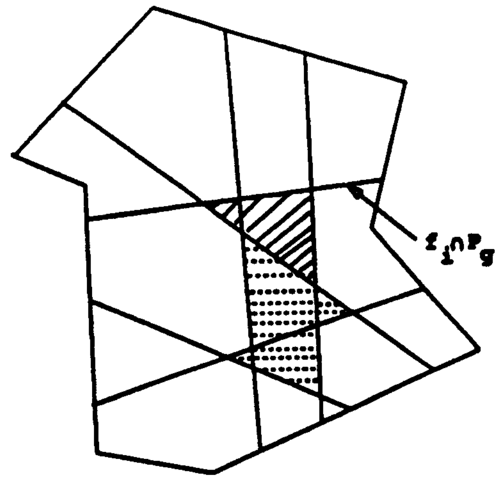

Figure 3: The facets in F_i are hatched with dotted lines; facets in F_i' are hatched with solid lines; facets in B_i' are not hatched.

contribute to Z_g' is $O(q^2)$. Each edge of Z_g is cut by at most one of the facets in F_i, and the vertices of these facets are produced only by intersections of edges in Z_g with them. Therefore the total complexity of these facets is $\sum_{i=1}^{p} E(F_i) = O(q^2)$. All other facets in B_i'' (if any) are adjacent to the line of intersection of f_i with P_g. Thus, the facets in F_i' are members of the zone of this line in an arrangement of $O(q)$ lines. Since there can be at most p lines of intersection between the planes supporting the facets of S and P_g we get $\sum_{i=1}^{p} E(F_i') = O(pq)$ by applying the zone theorem of line arrangement. This gives $\sum_{i=1}^{p} E(B_i'') = \sum_{i=1}^{p} E(F_i) + E(F_i') = O(pq + q^2)$.

To estimate the number of edges in the facets of B_i', consider the arrangement of q lines that represent the intersections between the plane of f_i and the planes in R. Now look at the restriction of this arrangement to the face f_i. The complexity of the zone of an edge in this arrangement is $O(q+|f_i|)$ (cf. the Combination Lemma of [7]). Summing over all facets f_i gives a bound of $O(nq + n) = O(nq)$ on $E(B_i')$. Combining all these, we get that the number of new edges contributed to Z_g' as a result of superimposing p facets of S on Z_g is only $O(pq + nq + q^2) = O(nr)$ since $q = O(r)$, $p = O(n)$. This immediately implies that Z_g' has at most $O(r^2 + nr) = O(nr)$ edges. Thus, the total number of edges in subpolyhedra $S_1, S_2, ..., S_t$ through which the plane P_g passes is at most $O(nr)$. ♣

Theorem 3.1: A manifold polyhedron S with arbitrary genus and having n edges of which r are reflex can be triangulated with complete cuts in $O(nr^2 + r^3 \log r)$ time and $O(nr + r^3)$ space.

Proof: We proceed as in the proof of Theorem 2.1. We get $\sum_{i=1}^{k} m_i = O(nr)$ using Lemma 3.2. This gives an $O(nr^2 + r^3 \log r)$ time bound for convex decomposition through complete cuts. Lemma 3.1 gives $O(nr + r^3)$ space complexity. Each convex piece can be triangulated in a straightforward way by triangulating its facets and joining all triangles thus produced to a point inside the convex piece. However, we need to ensure that all pairs of facets that overlap completely on one another have same triangulation. Since the facets in each such pair have same topological structure and have the same geometric location, any deterministic algorithm that triangulates a facet can be made to produce same triangulations for both facets. This triangulation phase does not increase the time and space complexity.♣

4 CSG Representation

The plane insertions through bounded cuts as described in section 3 can easily be extented to give HCSG representation of polyhedra from their boundary representations.

For each notch g in S, if the plane supporting one of the facets adjacent to g is chosen as the notch plane for g, all facets of the convex pieces in the final decomposition lie only on the supporting planes of the facets of S. Further, each convex piece can be expressed as the intersection of closed halfspaces supporting its facets. Finally, S can be represented as the union of the expressions obtained for each convex piece. This gives a HCSG formula for S. The number of literals in this formula is equal to the number of facets present in the convex pieces.

Theorem 4.1: For any manifold polyhedron, an HCSG representation of size $O(pl + l^{\frac{7}{3}})$ can be computed in $O(pl^2 + l^{\frac{10}{3}})$ time, where p is the number of facets in S of which l are adjacent to notches.

Proof: The total number of edges in the final decomposition through bounded cuts is $O(nr + r^{\frac{7}{3}})$ (using improved bound of [11] on $q(r)$). Certainly, $r = O(l)$, and since S is a manifold polyhedron $n = O(p)$. Thus, the total number of facets in the convex pieces of final decomposition is $O(pl + l^{\frac{7}{3}})$ which determines the size of the HCSG representation of S. The time complexity for this HCSG computation is same as that of computing the convex decomposition of S through bounded cuts. Expressed in terms of p and l this complexity is $O(pl^2 + l^{\frac{10}{3}})$.♣

4.1 Lower Bound

Let $(\alpha_{11} o_{11} \alpha_{12} ...) r_1 (\alpha_{21} o_{21} \alpha_{22} ...) r_2 ... (\alpha_{k1} o_{k1} ...)$ be an HCSG formula for a polyhedron where o_{ij}'s and r_i's denote the operators intersections ($\cap$) or unions ($\cup$), and α_{ij}'s denote the literals corresponding to the halfspaces. In case where $o_{ij} = \cap$ and $r_i = \cup$ for all i, j, we say that the given HCSG is in Disjunctive Normal Form (DNF). On the other hand, if $o_{ij} = \cup$ and $r_i = \cap$ for all i, j, we say that the given formula is in Conjunctive Normal Form (CNF).

Lemma 4.1: There exists a class of polyhedra for which any DNF HCSG formula has a size of $\Omega(p^2)$, where p is the number of facets in them.

Proof: Consider the polyhedron S as constructed by Chazelle in [3] to prove a lower bound on the number of convex pieces needed to decompose a non-convex polyhedron. See Figure 4. The notches of this polyhedron form two sets of line segments, each lying on the surface of a hyperbolic paraboloid which have a small distance of ϵ between them. Let Σ denote the region between these two hyperbolic paraboloid surfaces each containing r notches. Assuming unit distances between consecutive notches, the volume of Σ is $\Omega(\epsilon r^2)$. Chazelle showed that a single convex polyhedron whose volume lies inside S can occupy only $O(\epsilon)$ volume in Σ, thus requiring $\Omega(r^2)$ convex pieces to cover Σ. Let $C = C_1 \cup C_2 \cup \ ... \ \cup C_k$ be a DNF HCSG formula for S where each C_i represents the maximal collection of literals along with only intersection operators in between them. Each C_i represents a closed convex polyhedron S_i that lies inside S. The convex polyhedra corresponding to $C_i, i = 1, ..., k$ cover the polyhedron S and hence Σ. Thus k must be $\Omega(r^2)$ giving an $\Omega(r^2)$ lower bound on the size of C. The worst-case lower bound of $\Omega(p^2)$ follows immediately from the fact that S can be made to have $r = \Omega(p)$.♣

Lemma 4.2: There exists a class of polyhedra for which any CNF HCSG formula has a size of $\Omega(p^2)$ where p is the number of facets in them.

Proof: Consider a polyhedron S_0 constructed as follows. Let S_1 be the unbounded polyhedron obtained by taking the closure of the complement of the Chazelle's Polyhedron. The unbounded polyhedron S_1 has an internal void whose boundary is same as that of Chazelle's polyhedron. Let S_2 be a cube, large enoug' to contain the internal void of S_1 inside. Let $S_0 = cl(S_1 \cap S_2)$. The polyhedron S_0 is a closed polyhedron.

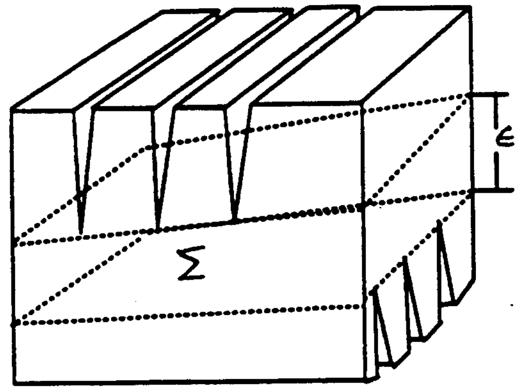

Figure 4: Chazelle's Polyhedron.

Its outer boundary consists of six facets of the cube S_1, and its inner boundary consists of the boundary of Chazelle's polyhedron. Let $C = C_1 \cap C_2 \cap ... \cap C_k$ be a CNF HCSG formula for S_0 where each C_i represents the maximal collection of literals along with only union operators. Let $C_i = H_{i1} \cup H_{i2} ... \cup H_{il}$ where H_{ij}'s represent closed halfspaces. Let $\overline{H}^* = cl(\overline{H})$ where $\overline{H}$ represents the complement of H. Let $\overline{C_i}^* = \overline{H_{i1}}^* \cap \overline{H_{i2}}^* ... \cap \overline{H_{il}}^*$. The HCSG formula $\overline{C}^* = \overline{C_1}^* \cup \overline{C_2}^* \cup ... \cup \overline{C_k}^*$ is a DNF HCSG formula that represents two disjoint polyhedra, the Chazelle's polyhedron and the unbounded polyhedron $cl(\overline{S_2})$ corresponding to the complement of S_2. Each $\overline{C_i}^*$ represents a convex polyhedron that lies completely either inside the Chazelle's polyhedron or inside the unbounded polyhedron $cl(\overline{S_2})$. Since the portion denoted by Σ in the Chazelle's polyhedron is covered by convex polyhedra that lie inside it, k must be $\Omega(r^2)$. Making $r = \Omega(p)$, we can have $k = \Omega(p^2)$. ♣

In Lemma 4.1 we proved that Chazelle's polyhedron has $\Omega(p^2)$ DNF HCSG formula. However, one can verify that this polyhedron has $O(p)$ CNF HCSG formula. Similarly, the polyhedron S_0 in Lemma 4.2 has $\Omega(p^2)$ CNF HCSG formula and $O(p)$ DNF HCSG formula. However, it is not difficult to show that there is a polyhedron for which any CNF or DNF HCSG formula has size $\Omega(p^2)$.

Theorem 4.2: There exists a class of polyhedra for which any CNF or DNF HCSG formula has a size of $\Omega(p^2)$ where p is the number of facets in them.

Proof: Consider a polyhedron that is formed by gluing Chazelle's polyhedron with the polyhedron S_0 along one of the six facets of the cube. From the proof of Lemma 4.1 and 4.2, it is clear that any CNF or DNF HCSG formula for this polyhedron has $\Omega(p^2)$ size.

5 Conclusions

It is often desirable to produce well shaped tetrahedra in a triangulation of polyhedral domains. In [5], we have given an algorithm to produce guaranteed quality (with respect to shape) triangulation of a convex polyhedron. To triangulate each convex piece produced by complete cuts, we can use this algorithm if we are concerned with the shape of the tetrahedra. However, this method has the limitation that the convex polyhedra produced through the convex decomposition phase may be very bad in shape. An algorithm that achieves guaranteed quality triangulations directly on nonconvex polyhedra is more practical.

Reducing the time and space complexities for triangulation and reducing the gap between upper and lower bounds of HCSG representations of polyhedra with arbitrary genus remain as challenging questions. Currently, research is going on to settle these questions.

Acknowledgements: The author is thankful to Prof. C. Bajaj and Prof. K. Sugihara of Purdue University for their many valuable comments on this research. A preliminary version of this paper appeared in *Seventh Annual Symposium on Computational Geometry (ACM).*

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