Optimal Algorithms for Curves on Surfaces

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Abstract

We describe an optimal algorithm to decide if one closed curve on a triangulated 2-manifold can be continuously transformed to another, i.e., if they are homotopic. Our algorithm runs in $O(n + k_1 + k_2)$ time and space, where closed curves C_1 and C_2 of lengths k_1 and k_2, resp., on a genus g surface M ($g \neq 2$ if M orientable, and $g \neq 3, 4$ if M is non-orientable) are presented as edge-vertex sequences in a triangulation T of size n of M. This as well implies an optimal algorithm to decide if a closed curve on a surface can be continuously contracted to a point. Except for three low genus cases, our algorithm completes an investigation into the computational complexity of the two classical problems for surfaces posed by the mathematician Max Dehn at the beginning of this century. However, we make novel applications of methods from modern combinatorial group theory for an approach entirely different from previous ones, and much simpler to implement.

1 Introduction

Computational topology is an emerging new subdiscipline of computational geometry. There are many situations in topology when the existence of some structure $\mathcal{S}$ or the decidability of some problem $\mathcal{P}$ has been proved by mathematicians, but the complexity of $\mathcal{S}$ or the efficiency of algorithms to decide $\mathcal{P}$ remains to be thoroughly investigated. Computational topology deals with these algorithmic aspects of topology. A sampling of some relevant recent work may be found in [2, 3, 4, 5, 6, 13, 18].

The topological objects that we consider in this paper are *surfaces* or, equivalently, *2-manifolds*. A 2-manifold is a topological space that at every point locally resembles the Euclidean plane. By a surface or 2-manifold we shall always mean a compact, connected, and boundaryless 2-manifold. Everyday examples of such surfaces include spheres and tori (doughnuts). In fact, any finite object with volume that we care to examine in the three-dimensional world around us is bounded by an *orientable* surface, i.e., a surface with two distinct sides. A well-known exotic example of a surface is the Klein bottle which is non-orientable and cannot be physically realized in three-dimensional space.

Vegter and Yap [18] first examined the computational problems associated with surfaces, in particular with the *combinatorial representation* of surfaces. A combinatorial representation is a representation as a discrete structure, a necessary preliminary in any discrete algorithm for a topological object. An example of a combinatorial representation of a surface M is a *triangulation*, which consists of a decomposition of M into finitely many triangles together with a listing of adjacencies between these triangles.

The particular problems for surfaces that we consider here date back to the beginning of this century when Max Dehn [1] formulated and mathematically solved two now-classical problems, the *contractability* and *transformability* problems, as he termed them (see also Poincaré [12]).

The contractability problem is to decide if a closed curve C on a surface M can be continuously contracted to a point, i.e., if C is *null-homotopic*. Schipper [13] first investigated the complexity of an algorithm for the contractability problem that dynamically maintains a part of the *universal covering space* of M. Subsequently, Dey [3] and Dey and Schipper [5] gave improved implementations of this algorithm. All these use highly complex data structures, and the best result heretofore is in the latter paper, using $O(n + k \log g)$ time, which is suboptimal, and $O(n + k)$ space to decide contractability, if C is of length k on a *genus* g surface with a triangulation of size n.

The second and harder of Dehn's problems, the transformability problem, asks when two closed curves on a surface are continuously transformable into each other, i.e., if they are *homotopic*, denoted $C_1 \simeq C_2$. For example, in Fig. 1(a), $C_1 \simeq C_2$ but $C_1 \not\simeq C_3$. In this paper we describe a time and space optimal algorithm for the transformability problem on a genus g surface M, where $g \neq 2$ if M is orientable, and $g \neq 3, 4$ if M is non-orientable. Given a triangulation T of size n of M and closed curves C_1 and C_2 on M

presented as edge-vertex sequences in T of lengths k_1 and k_2, resp., this algorithm decides if C_1 and C_2 are homotopic in $O(n+k_1+k_2)$ time and space. This immediately implies an optimal algorithm to decide the contractability of C_1 by choosing C_2 to be a point so $k_2 = 0$.

Our approach however abandons universal covering spaces in favor of non-metric combinatorial methods. Specifically, we find a *canonical* representation for closed curves on M as elements of the *fundamental group* $\pi(M)$, observe that deciding the transformability of two curves is equivalent to deciding if their canonical representatives in $\pi(M)$ are *conjugate*, and use methods from combinatorial group theory to efficiently solve this *conjugacy problem* given a special presentation of $\pi(M)$. The resulting algorithm is not only entirely different but much simpler, using data structures no more complex than required to manipulate elementary graphs and stacks.

An interesting interpretation of our result is as a mechanism to decide if one *knot* on a surface can be deformed into another. Given the growing importance of knot theory in physics [8] and molecular biology (in the synthesis of DNA molecules [16]), we hope that results such as this eventually become part of a larger program to investigate these areas from the point of view of computational topology. Such a program could lead to significant practical algorithms.

In Section 2 we discuss some preliminaries. Our algorithm is described in Section 3.

2 Preliminaries

We briefly introduce several terms and notions from combinatorial group theory that we require later. For further discussion of group theory we refer to Rotman [17], and combinatorial group theory, in particular, to Lyndon and Schupp [10]. To save space we skip all preliminary discussion of topology and surfaces, and instead simply refer the reader to Massey [11], Singer & Thorpe [14], and Stillwell [15].

Given a set of symbols X, let X^{-1} denote the set of symbols $\{a^{-1} : a \in X\}$. A *letter* is an element of $X \cup X^{-1}$. A *word* w on X is a finite sequence $a_1 \dots a_k, k \geq 0$, of letters. The *length* of w, denoted $|w|$, is k. An *elementary transformation* of a word w consists of inserting or deleting a subword of the form aa^{-1} or $a^{-1}a$.

The *free group* $F(X)$ on the set X is the set $W(X)$ of all words on X modulo the equivalence relation $\sim$, where $w_1 \sim w_2$ if w_2 can be derived from w_1 by a finite sequence of elementary transformations, and with the binary operation that is induced by concatenation. X is called the set of *generators* of $F(X)$. In the following we shall identify a word $w \in W(X)$ with its equivalence class in $F(X)$, and denote the (group) inverse of w by w^{-1}.

A word $w = a_1 \dots a_k \in F(X)$ is *reduced* if it does not contain two successive letters that are inverses of each other; if, in addition, a_k is not the inverse of a_1, then w is *cyclically reduced*. Each element in $F(X)$ has a unique representation as a reduced word. If $w_i, 1 \leq i \leq q$, are words such that in forming the product $z = w_1 \dots w_q$ there is no *cancellation* (i.e., no $w_i, 1 \leq i \leq q-1$, ends in a letter s.t. w_{i+1} begins with the inverse of that letter), then we write $z \equiv w_1 \dots w_q$. A *conjugate* of a word w is a word of the form ywy^{-1}. A subset R of $F(X)$ is called *symmetrized* if all elements of R are cyclically reduced and, for each $r \in R$, all cyclic permutations (i.e., cyclically reduced conjugates) of both r and r^{-1} are also in R.

A group G is said to have *finite presentation* $(X; R)$, where X is a finite set of symbols and $R \subset W(X)$ is also finite, if G isomorphic to the quotient group $F(X)/N$, where N is the smallest normal subgroup of $F(X)$ containing R (i.e., N is the *normal closure* of R in $F(X)$). We say that G is generated by X with the *relations* in R, and write $G = (X; R)$.

For example, $(x; x^3)$ is the 3-element cyclic group, and $(x, y; xyx^{-1}y^{-1})$ is the product of two infinite cyclic groups, i.e, a free abelian group on two generators.

The *word problem* for a group $G = (X; R)$ asks for an algorithm to decide if an element $w \in W(X)$ represents the identity of G. In general, the word problem is unsolvable. The *conjugacy problem*, which is also generally unsolvable, asks to decide if two elements $w_1, w_2 \in W(X)$ represent conjugate elements of G, i.e., if there is $c \in W(X)$ such that $w_1 = cw_2c^{-1}$ holds in G.

Assume now that $G = (X; R)$ is a finite presentation where R is symmetrized. If r_1 and r_2 are distinct elements of R such that $r_1 \equiv bc_1$ and $r_2 \equiv bc_2$, then b is called a *piece* of R (consider the product $r_1^{-1}r_2$ to see that a piece is a subword of an element of R that can be non-trivially cancelled by multiplication with another element of R). R is said to satisfy the *small cancellation* condition $C'(\lambda)$, for the positive real λ, if $r \equiv bc$, where $r \in R$ and b is a piece of R, implies that $|b| < \lambda |r|$. The following consequence of Greendlinger's Lemma for Sixth-Groups ([7], see also [10]) is crucial to an efficient solution of the word problem given certain presentations of fundamental groups of surfaces:

Proposition 1 *If $G = (X; R)$ satisfies $C'(\frac{1}{6})$, then a non-empty reduced word $w \in W(X)$ that represents the identity element of G must contain a subword w' such that there exists a relation $r \in R$ with $r \equiv w'w''$ and $|w'| > \frac{1}{2}|r|$.* ♣

A word $w \in F(X)$ is said to be *R-reduced* if it is reduced and does not contain a subword w' such that there exists a relation $r \in R$ with $r \equiv w'w''$ and $|w'| > \frac{1}{2}|r|$. It is *cyclically R-reduced* if it is cyclically reduced and all its cyclic permutations are R-reduced.

A consequence of another of Greendlinger's results [7] (also [10]) that we shall subsequently use to efficiently solve the conjugacy problem for fundamental groups of surfaces is:

Proposition 2 *If $G = (X; R)$ satisfies $C'(\frac{1}{8})$, then two non-empty cyclically R-reduced words $w_1, w_2 \in W(X)$ represent conjugate elements of G if and only if the equation $w_1^* = hw_2^*h^{-1}$ holds in G, where w_1^**

and w_2^* *are cyclically reduced conjugates of* w_1 *and* w_2, *resp., and* h *is a subword of some relation* $r \in R$. ♣

Note that Greendlinger's original results imply much more than Props. 1 and 2, but these are sufficient for our purposes.

3 The Algorithm

The input to the algorithm includes a triangulation T of size n, consisting of triangles σ_r, $1 \leq r \leq n$, of some surface M, together with cycles C_1 and C_2 on M presented as edge-vertex sequences in T of length k_1 and k_2, respectively. We assume that T is represented by a data structure that allows access to the edges of a triangle, as well as the triangles incident on an edge, in $O(1)$ time. Also, for a reason that will be apparent later, we assume M to be a genus g manifold, where $g \geq 3$ if M is orientable and $g \geq 5$ if M is non-orientable. We shall deal with the remaining cases subsequently.

Say v_1 and v_2 are two vertices on C_1 and C_2, respectively. Find *any* path D from v_1 to v_2, by, say, an $O(n)$-time breadth-first search of the 1-skeleton (i.e., graph) of T, so that D is an edge-vertex sequence of length $O(n)$. It is known (see [14]) that $C_1 \simeq C_2$ *if and only if* C_1 and $D \circ C_2 \circ D^{-1}$ represent conjugate elements in the fundamental group $\pi(M)$, at base-point v_1. Therefore, to avoid clumsy notation later, we shall fudge a little now and assume that we are in fact given C_1 and $D \circ C_2 \circ D^{-1}$ as input, and henceforth denote the latter as C_2. We shall also assume that the edge-vertex sequence representing C_1 is $v_{1,1}e_{1,1}v_{1,2} \ldots e_{1,k_1}v_{1,k_1+1}$ and that representing C_2 is $v_{2,1}e_{2,1}v_{2,2} \ldots e_{2,k_2}v_{2,k_2+1}$, where $v_1 = v_{1,1} = v_{2,1} = v_{1,k_1+1} = v_{2,k_2+1}$. Neither the extra time to find D nor the extra $O(n)$ part in k_2 due to the "hidden" D and D^{-1} affect our future claims on time and space, as they are all of the form $O(n + k_1 + k_2)$.

The algorithm consists of two phases: the first phase converts the geometric problem of deciding transformability to the combinatorial one of deciding if two elements in a group are conjugate, while the second phase solves this conjugacy problem. It is in the crucial second phase that we apply our new combinatorial methods.

3.1 Phase 1

This phase consists of two subphases similar to procedures in Dey [3]. However, in order to make this discussion self-contained, we give a brief description.

3.1.1 Subphase 1a

We shall find a *polygonal schema* (see [11, 15, 18]) P representing M such that P has a triangulation T' containing the same number of triangles as T. We shall also find a representations of C_1 and C_2 on P.

The procedure is to construct a sequence of polygons $P_1, \ldots, P_n$ by successively attaching triangles corresponding to triangles in T. Omitting further details of the procedure, we simply observe that on completion we shall have a polygon P with a triangulation T', consisting of the triangles σ'_r, $1 \leq r \leq n$, such that

(a) there is a one-to-one correspondence between the triangles of T and T', together with a vertex-to-vertex identification specified for each pair of corresponding triangles,

(b) each edge e' on $bd(P)$ has a partner edge e'' on $bd(P)$ such that they both correspond to a single edge e in T; further, assuming some arbitrary orientation on the edges of T (say, induced by the data structure representing these edges), we have an orientation on the edges of $bd(P)$ induced by the vertex-to-vertex identification specified in the correspondence between triangles of T and T', and,

(c) we can obtain M by attaching partnered edges of $bd(P)$, taking care to match orientations (as given in (b)) when attaching edges. More precisely, there is a homeomorphism ϕ from M to the quotient space of P modulo the identification of partnered edges.

Thus, appropriately labeling the edges of $bd(P)$, so that partnered edges have the same unsigned symbol and signs represent orientations, P is indeed a polygonal schema for M such that P and M have equal sized triangulation. Say, $bd(P)$ has edges labeled by symbols from the set $\{x_1, x_1^{-1}, \ldots, x_m, x_m^{-1}\}$ such that each unsigned symbol occurs exactly twice.

Next, considering first the cycle C_1, we see that its homeomorphic image by ϕ is an edge-vertex sequence C'_1 that consists of a possibly "disconnected" circular sequence $C'_{1,1}, \ldots, C'_{1,h_1}$, $h_1 \leq k_1$, of *arcs* such that

(a) each arc $C'_{1,j}$ consists of a connected edge-vertex sequence $v_{i_j}e_{i_j}v_{i_j+1} \ldots v_{i'_j}$, where only the first vertex v_{i_j} and last vertex $v_{i'_j}$ of the sequence lie on $bd(P)$, and,

(b) For $1 \leq j \leq h$, the last vertex $v_{i'_j}$ of $C_{1,j}$ and the first vertex $v_{i_{j+1}}$ of $C_{1,j+1}$ on $bd(P)$ are identified by the partnering of oriented edges of $bd(P)$ (of course, "$h_1 + 1 = 1$").

Similar remarks apply to C_2 so that its homeomorphic image by ϕ is an edge-vertex sequence C'_2 that consists of the circular sequence of arcs $C'_{2,1}, \ldots, C'_{2,h_2}$, $h_2 \leq k_2$.

See Fig. 2, forgetting for purpose of convenient illustration, the restriction that genus $g \geq 3$, if M is orientable.

3.1.2 Subphase 1b

The size of the polygonal schema P (i.e., the number of edges on $bd(P) = 2m$), found by Subphase 1a, may be $\Omega(n)$. We shall next find a polygonal schema Q for M which is of minimal size. Such a polygonal schema Q is called a *reduced* polygonal schema for M (see [3, 18]), and, in fact, $bd(Q)$ will have $4g$ or $2g$ edges, according as M is orientable or not.

Denote by G the 1-complex (i.e., graph) formed by taking $bd(P)$ and identifying partnered edges so that orientations match along identified edges. Let Y be a spanning tree of G, and $B = \{b_1, \ldots, b_l\}$ be the set of edges of G *not in* Y. Call the edges of G *in* Y *excess*.

Form the polygonal schema Q as follows: proceed through the sequence of symbols that define P, i.e.,

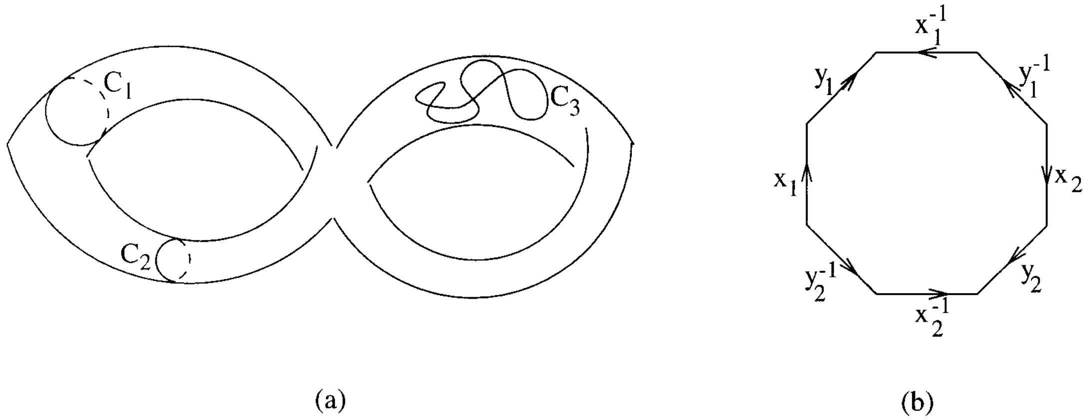

Figure 1: (a) A double torus, an orientable surface of genus 2. (b) A polygonal schema in *canonical* form for the double torus.

the labels of $bd(P)$, deleting those that correspond to an excess edge of G. Recall that each edge of G was formed by identifying two partnered edges, so that each excess edge will result in the deletion of a partnered pair of symbols; call such deleted symbols *excess symbols*. Declare Q to be the polygonal schema defined by the sequence of symbols that remain after deleting excess symbols from P. Clearly, the length of this sequence is $2l$, as 2 symbols remain for each edge in B. Let us write this sequence as $y_1 \dots y_{2l}$, where each unsigned symbol y_i is one of $\{b_1, \dots, b_l\}$, and the sign is assigned according to orientation. See Fig. 2.

Now the projection map from G to the quotient space G/Y may be checked to be a homotopy equivalence (G/Y may be thought of as G with the spanning tree Y contracted to a point). Considering G as a subspace of M via the homeomorphism ϕ, this homotopy equivalence extends to the projection from M to M/Y. However, M/Y is homeomorphic to the manifold M' that is represented by the polygonal schema Q. Thus, we have a projection $\psi : M \rightarrow M'$ which is a homotopy equivalence. It follows that, as surfaces, M and M' are homeomorphic, and Q may be considered a polygonal schema for M.

Omitting proof we claim that Q is, in fact, a reduced polygonal schema for M, so that $l = 2g$ if M is orientable, and $l = g$ if M is not orientable; further, $\pi(M) = \pi(M') = (b_1, \dots, b_l; y_1 \dots y_{2l})$.

Since ψ is a homotopy equivalence, deciding the conjugacy of C_1 and C_2 in $\pi(M)$ is equivalent to deciding the conjugacy of $\psi(C_1)$ and $\psi(C_2)$ in $\pi(M')$. Let us now try to compute the element $z_X \in \pi(M')$ that the cycle $\psi(C_X)$ represents, $X = 1, 2$.

First, some observations about the projection ψ. Consider ψ as a map from P to Q, by identifying points of P and Q with the corresponding points in the respective quotient spaces M and M'. We see then that ψ projects $bd(P)$ onto $bd(Q)$, and the interior $int(P)$ of P onto $int(Q)$. It is only $\psi|bd(P)$ that concerns us. Specifically, the behavior of ψ on $bd(P)$ is as follows: ψ projects each edge on $bd(P)$, labeled with a non-excess symbol, onto the corresponding edge on $bd(Q)$; each sequence of edges on $bd(P)$ labeled with excess symbols, that lies between two edges e and e' labeled with non-excess symbols, is projected to the common endpoint between the edges corresponding to e and e' on $bd(Q)$.

Next, consider $\psi(C_1)$: regarding ψ as a map from P to Q, $\psi(C_1)$ is the same as $\psi(C'_1)$. And, from Subphase 1a we have the arc sequence $C'_{1,1}, \dots, C'_{1,h_1}$ of C'. Now, the image $\psi(C'_{1,j}) = \psi(v_{i_j} e_{i_j} v_{i_j+1} \dots v_{i'_j})$ starts at the vertex $\psi(v_{i_j})$ and ends at the vertex $\psi(v_{i'_j})$, both on $bd(Q)$, and no interior vertex of $\psi(C'_j)$ lies on $bd(Q)$. Let the sequence of symbols labeling edges of $bd(Q)$ *clockwise* between $\psi(v_{i_j})$ and $\psi(v_{i'_j})$ be $y_{r_{1,j}}, \dots, y_{s_{1,j}}$ (note that $r_{1,j}$ may be greater than $s_{1,j}$: the list $y_1, \dots, y_{2l}$ is clockwise *circular* around $bd(Q)$). Then, the path $\psi(C'_{1,j})$ is homotopic, with end-points fixed, to the cycle on M' that is represented in $\pi(M')$ by the product $y_{r_{1,j}} \dots y_{s_{1,j}}$. Denote this product by $(r_{1,j}, s_{1,j})$. Completing the traversal of the arc sequence $C'_{1,1}, \dots, C'_{1,h_1}$ of C'_1, we have a representation of the cycle $\psi(C'_1) = \psi(C_1)$ as a product $z_1 = (r_{1,1}, s_{1,1}) \dots (r_{1,h_1}, s_{1,h_1})$, where $h_1 \leq k_1$, in $\pi(M') = (b_1, \dots, b_l; y_1 \dots y_{2l})$. See Fig. 2.

Similarly, we can find a representation of the cycle $\psi(C'_2) = \psi(C_2)$ as a product
$z_2 = (r_{2,1}, s_{2,1}) \dots (r_{2,h_2}, s_{2,h_2})$, where $h_2 \leq k_2$.

Therefore, to solve the original homotopy problem we now have to determine if z_1 and z_2 represent conjugate elements in $\pi(M')$. We solve this combinatorial problem in Phase 2.

3.2 Phase 2

We have from Phase 1 words $z_1 = (r_{1,1}, s_{1,1}) \dots (r_{1,h_1}, s_{1,h_1})$ and $z_2 = (r_{2,1}, s_{2,1}) \dots (r_{2,h_2}, s_{2,h_2})$ in $F(b_1 \dots, b_l)$, and must

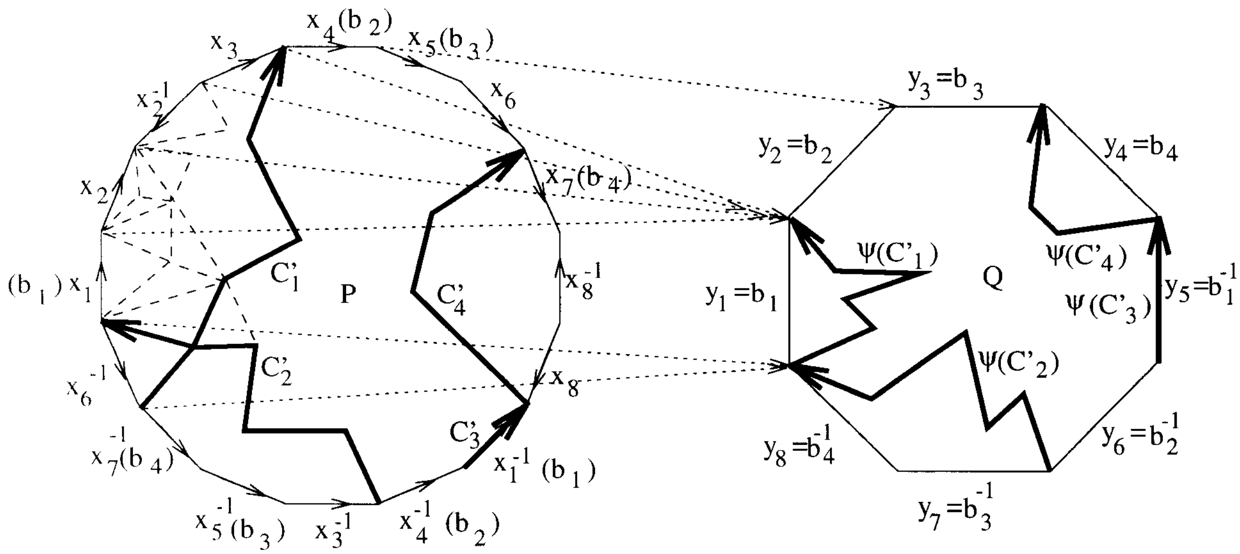

Figure 2: A polygonal schema P for a double torus and its reduction Q, ***not*** in canonical form: (i) The broken lines show part of the triangulation T' of P, (ii) The bold lines in P and Q show cycles C' and $\psi(C')$, respectively, (iii) A symbol b_i in parenthesis indicates that the corresponding edge of P is associated with the edge b_i of the spanning tree G, (iv) The dotted lines indicate some vertex mappings by ψ, (v) $\psi(C') = (1,1)(7,8)(6,4)(5,3)$.

decide if they represent conjugate elements in the group $\pi(M') = (b_1, \ldots, b_l; y_1 \ldots y_{2l})$.

First, we require certain notations and preprocessing. If $w_1, w_2 \in F(b_1, \ldots, b_l)$ are equal in $F(b_1, \ldots, b_l)$, i.e. as strings, denote this as $w_1 = w_2$; while, if w_1 and w_2 represent equal elements of $\pi(M')$, i.e. they project to equal elements in $(b_1, \ldots, b_l; y_1 \ldots y_{2l})$, denote this as $w_1 \approx w_2$. Of course, $w_1 = w_2 \Rightarrow w_1 \approx w_2$.

Denote by $\overline{(u,v)}$ the product given by the subword $y_u^{-1} \ldots y_v^{-1}$ of the circular sequence of symbols $y_{2l}^{-1} \ldots y_1^{-1}$ (with y_{2l}^{-1} following y_1^{-1}: imagine traversing Q *counterclockwise*). If either (r,s) or $\overline{(u,v)}$ consists of a single letter, write it as that letter (e.g., (r,r) is written as y_r, $\overline{(u,u)}$ as y_u^{-1}, and we assume this is always done when required). Let $|(r,s)|$ and $|\overline{(u,v)}|$ denote the length of the product that each represents. A consequence of the representation $\pi(M') = (b_1, \ldots, b_l; y_1 \ldots y_{2l})$ is:

Proposition 3 $(r,s) \approx \overline{(r-1, s+1)}$, *where* $|\overline{(r-1,s+1)}| = 2l - |(r,s)|$, *and* $(r,s)^{-1} = \overline{(s+1, r-1)} = \overline{(s,r)}$. *A similar statement holds for* $\overline{(u,v)}$. ♣

Preprocess z_X, $X = 1,2$, as follows:

For each term $(r_{X,j}, s_{X,j})$ of z,

(a) if $1 \leq |(r_{X,j}, s_{X,j})| \leq l$, leave it unchanged (if $|(r_{X,j}, s_{X,j})| = 1$ it is written as a single letter, of course),

(b) if $l < |(r_{X,j}, s_{X,j})| < 2l - 1$, replace it by $\overline{(r_{X,j} - 1, s_{X,j} + 1)}$,

(c) if $|(r_{X,j}, s_{X,j})| = 2l - 1$, replace it by the single letter $y_{r_{X,j}-1}^{-1}$, and,

(d) if $|(r_{X,j}, s_{X,j})| = 2l$, delete it.

This $O(h_X)$-time preprocessing gives, for $X = 1,2$, a product $z'_X = c_{X,1} \ldots c_{X,h'_X}$, s.t. $h'_X \leq h_X$, and $z'_X \approx z_X$ by Prop. 3, of terms either of the form (r,s), of length $\leq l$, or of the forms $\overline{(u,v)}$ and y_u, of length $< l$. Call such terms *rectified terms*, and such a product a *rectified product*, and denote the number of terms as the *height* of the product.

Let $y = y_1 \ldots y_{2l}$ and let R denote the set of $4l$ relations, each of length $2l$, consisting of y, y^{-1}, and all their cyclic permutations. It may be seen that $\pi(M') = (b_1, \ldots, b_l; R)$, where R is now a finite symmetrized set of relations (in fact, each $y' \in R$ is a conjugate of either y or y^{-1}, so $y = 1 \Leftrightarrow y' = 1$).

A crucial consequence of Q being a reduced polygonal schema, we omit all proofs, is:

Proposition 4 *Considering y to be circular sequence of symbols of length $2l$, a given pair of adjacent symbols $y_i y_{i+1}$ at position i in y cannot occur at any other position in y, and the pair $y_{i+1}^{-1} y_i^{-1}$ cannot occur at all in y (i.e., $y_i y_{i+1}$ cannot occur at all in y^{-1}).* ♣

which allows us to employ Greendlinger's powerful results:

Proposition 5 *It follows from Prop. 4 that the set of relations R satisfies the small cancellation condition $C'(\frac{1}{4g-1})$ if M' is orientable (when $2l = 4g$), and $C'(\frac{1}{2g-1})$ if M' is non-orientable (when $2l = 2g$). Therefore, with our restriction that $g \geq 3$ if M is orientable, and $g \geq 5$ if M is non-orientable, R always satisfies $C'(\frac{1}{8})$, and so also $C'(\frac{1}{6})$.*

It follows then from Prop. 1 that, if a nonempty $w \in F(b_1, \ldots, b_l)$ is represented as a rectified product, then, either

1. *$w \not\approx 1$, or*
2. *w can be shortened to a rectified product $\overline{w}$ of lesser height by concatenating two adjacent terms of w and replacing them, possibly applying Prop. 3, with one term, so that $w \approx \overline{w}$.* ♣

Comment: It is precisely because Q is in reduced rather than *canonical* form (see [18] and Fig. 1(b)) that we need Greendlinger's results to prove that *Dehn-type algorithms* (see [10]) apply to the word and conjugacy problems for $\pi(M')$.

The preceding proposition suggests a procedure to find a *canonical* representation of each $w \in F(b_1, \ldots, b_l)$ as an equal word in $\pi(M')$. The idea is to use a stack S to successively insert terms from w, in the process combining adjacent terms when possible according to clause 2 of Prop. 5. Thus the contents of S are always rectified terms. We first define two functions *push*(c) and *apply*(c), where c is a rectified term.

Define *push*(c) to be the operation of pushing c into S. There are 12 cases to consider for the function *apply*(c), given by the 3 possibilities for c (the forms (r,s), $\overline{(u,v)}$, or y_u) and 4 for S (it is empty, or its top term is of the form (r,s), $\overline{(u,v)}$, or y_u). In the 3 cases that S is empty, define *apply*(c) = *push*(c). When S is not empty, we shall define *apply*(c) only for the 3 cases that arise given $c = (r,s)$. The other 6 cases may be similarly defined. Let the top element of S be d. The 3 cases we consider are:

(a) $d = (u,v)$:

function *apply*(c) (* given $c = (r,s)$ and $d = (u,v)$ *);
begin
pop S;
(* labels are for referencing in a subsequent discussion *)
A: if $r = v + 1$ then
  begin
  if $|(u,s)| \leq l$ then *push*(u,s)
  else if $l < |(u,s)| < 2l$ then *apply*$\overline{(u-1, s+1)}$
    else (* $|(u,s)| = 2l$, so $c = d^{-1}$ *) *do nothing*
  end
B: else if $y_v = y_r^{-1}$ then *push*$(u, v-1)$, *push*$(r+1, s)$
C: else *push*(u,v), *push*(r,s)
end

(b) $d = \overline{(u,v)}$:

function *apply*(c) (* given $c = (r,s)$ and $d = \overline{(u,v)}$ *);
begin
pop S;
if $y_v^{-1} = y_{r-1}$ then
  begin
  if $|(r-1,s)| \leq l$ then *push*$\overline{(u, v+1)}$, *push*$(r-1, s)$
  else (* $|(r-1,s)| = l+1$ *) *push*$\overline{(u, v+1)}$,
  push$\overline{(r-2, s+1)}$
  end
else if $y_v^{-1} = y_r^{-1}$ then
  begin
  if u appears before s going clockwise from r then
  apply$(u+1, s)$
  else if u appears after s going clockwise from r then
    apply$\overline{(u, s+1)}$
    else (* $u = s$, and $c = d^{-1}$ *) *do nothing*
  end
  else *push*$\overline{(u,v)}$, *push*(r,s)
end

(c) $d = y_u$:

function *apply*(c) (* given $c = (r,s)$ and $d = y_u$ *);
begin
pop S;
if $y_u = y_{r-1}$ then
  begin
  if $|(r-1,s)| \leq l$ then *push*$(r-1, s)$
  else (* $|(r-1,s)| = l+1$ *) *push*$\overline{(r-2, s+1)}$
  end
else if $y_u = y_r^{-1}$ then *apply*$(r+1, s)$
  else *push* y_u, *push*(r,s)
end

Our procedure to find a canonical form for $w \in F(b_1, \ldots, b_l)$ is:

function *canonical* (*input*: word w given as a rectified product $c_1 \ldots c_{h'}$);
begin
$S := \emptyset$;
for $j := 1$ to h' do *apply* c_j;
return $\overline{w}$, the word contained as a rectified product in S reading the terms from bottom to top
end

The following efficiently solves the word problem for $\pi(M')$:

Proposition 6 *For a rectified product $w = c_1 \ldots c_{h'}$, canonical(w) is R-reduced. Therefore, by Prop. 1, $w \approx 1$ if and only if canonical$(w) = 1$, the empty word.* ♣

Call a rectified product $w = c_1 \ldots c_{h'}$ *stable* if *canonical*$(w) = w$. The following two lemmas should be intuitively clear, and we omit proofs.

Lemma 1 *For a rectified product w, if $canonical(w) = c_1 \dots c_{h''}$, then each of the rectified products $c_1 \dots c_j$, $j \leq h''$, is stable.* ♣

Lemma 2 *If $w = c_1 \dots c_{h''}$ is a stable product, then so is $w^{-1} = c_{h''}^{-1} \dots c_1^{-1}$ (here $(r,s)^{-1} = \overline{(s,r)}$, $\overline{(u,v)}^{-1} = (v,u)$). In other words, $canonical(w^{-1}) = (canonical(w))^{-1}$.*

Further, if $w = c_1 \dots c_{h''}$ is a cyclically reduced stable product, then so is every cyclic permutation of $c_1 \dots c_{h''}$. ♣

The next lemma requires careful analysis of the function *apply*:

Lemma 3 *If $w = c_1 \dots c_{h''}$ is a stable product (of height h'') and c is a rectified term, then, either*

1. *$height(canonical(wc)) = h'' + 1$, when $canonical(wc) = wc$, or*
2. *$height(canonical(wc)) = h''$, when $canonical(wc) = c_1 \dots c_{h''-1}\, c'$, where $c' \approx c_{h''}\, c$, or*
3. *$height(canonical(wc)) = h'' - 1$, when $c = c_{h''}^{-1}$ and $canonical(wc) = c_1 \dots c_{h''-1}$.*

A similar statement holds for $canonical(cw)$.

Proof. Cases 1 and 3 are straightforward to verify as in these cases *apply* does not recur. For case 2 we have to check the various possibilities in the function *apply*. For example, in case (a) of $apply(c)$, where $d = c_{h''}$, the only *possibility* for S to decrease, other than when $c = d^{-1}$, is when $r = v + 1$ and $l < |\overline{(u,s)}| < 2l$, in which case the recursive call is to $apply\overline{(u-1, s+1)}$ on stack $S - d$. Now, *suppose* the top of $S - d$ is of the form $d' = \overline{(u', v')}$, when we are again in a case *analogous* to (a), in that the top of the stack and the element *applyed* are of the same form. However, in this situation, the only clause of the conditional statement that is applicable must be that labeled C (given labeling similar to (a) in this analogous case) leaving the height of S at h''. For if not, if either clauses labeled A or B is applicable, that would imply that d' is of such a value that $w = c_1 \dots d'd$ cannot be stable, contradicting an initial assumption. This, of course, has to be carefully checked and we omit details.

We also omit discussing the other cases.

The second statement of the lemma follows from the first and Lemma 2. ♣

Our main claim is the following:

Proposition 7 *The word $canonical(w)$ is indeed a canonical form for words of $F(b_1, \dots, b_l)$ that represent the same element of $\pi(M')$. Precisely, $w_1 \approx w_2$ if and only if $canonical(w_1) = canonical(w_2)$.*

Proof. Examining function *canonical* it is easily checked that, for any w, $canonical(w) \approx w$. It follows that $canonical(w_1) = canonical(w_2) \Rightarrow w_1 \approx w_2$.

Conversely, suppose that $w_1 \approx w_2$, and let

$$canonical(w_1) = c_{1,1} \dots c_{1,h_1''} \text{ and}$$
$$canonical(w_2) = c_{2,1} \dots c_{2,h_2''}.$$

We shall prove the equality $canonical(w_1) = canonical(w_2)$ by induction on $\min(h_1'', h_2'')$. If $\min(h_1'', h_2'') = 0$, the equality follows from Prop. 6. Assume inductively that the equality is true if $\min(h_1'', h_2'') \leq N$, for some $N \geq 0$, and consider the case when, in fact, $\min(h_1'', h_2'') = N+1$, where, w.l.o.g, we suppose that $h_1'' = N + 1$.

Since, $w_1 \approx w_2$ and $w_X \approx canonical(w_X)$, $X = 1, 2$, we have

$$canonical(w_1) \approx canonical(w_2) \Rightarrow$$
$$c_{1,1} \dots c_{1,h_1''} \approx c_{2,1} \dots c_{2,h_2''} \Rightarrow$$
$$c_{1,1} \dots c_{1,h_1''-1} \approx c_{2,1} \dots c_{2,h_2''}(c_{1,h_1''}{}^{-1}).$$

Now, by Lemma 1, the product on the LHS of the last equation is stable, i.e.,

$$canonical(c_{1,1} \dots c_{1,h_1''-1}) = c_{1,1} \dots c_{1,h_1''-1}.$$

Therefore, by the inductive hypothesis we must have

$$c_{1,1} \dots c_{1,h_1''-1} = canonical(c_{2,1} \dots c_{2,h_2''}(c_{1,h_1''}{}^{-1})).$$

For the preceding equation to hold we must further have, by Lemma 3, that

$$c_{1,h_1''} = c_{2,h_2''} \text{ and } c_{1,1} \dots c_{1,h_1''-1} = c_{2,1} \dots c_{2,h_2''-1},$$

which implies that,

$$c_{1,1} \dots c_{1,h_1''} = c_{2,1} \dots c_{2,h_2''}, \text{ and, indeed,}$$

$canonical(w_1) = canonical(w_2)$. ♣

We return to the problem of deciding if $z_X' = c_{X,1} \dots c_{X,h_X'}$, $X = 1, 2$, represent conjugate elements of the group $\pi(M') = (b_1, \dots, b_l; y_1 \dots y_{2l})$.

Compute $canonical(z_X') = c_{X,1}^c \dots c_{X,h_X''}^c$, say, $X = 1, 2$. Since $z_X' \approx canonical(z_X')$, the problem of deciding the conjugacy of z_X', $X = 1, 2$, in $\pi(M')$, is equivalent to that of deciding the conjugacy of $canonical(z_X')$, $X = 1, 2$, in $\pi(M')$. We may assume $canonical(z_X') \neq 1$, $X = 1, 2$, as otherwise the problem is trivial. Both $c_{X,1}^c \dots c_{X,h_X''}^c$, $X = 1, 2$, and each of their cyclic permutations are R-reduced, and we may further assume that they are cyclically reduced (for, if not, they may be rendered so by simple linear-time preprocessing). So, assume $c_{X,1}^c \dots c_{X,h_X''}^c$, $X = 1, 2$, are non-empty cyclically R-reduced words, and, w.l.o.g, assume that $h_1'' \leq h_2''$.

Since R satisfies $C'(\frac{1}{8})$ (see Prop. 5) we have, by Prop. 2, that $c_{X,1}^c \dots c_{X,h_X''}^c$, $X = 1, 2$, represent conjugate elements of $\pi(M')$ if and only if there exists c such that

$$(c_{2,1}^c \dots c_{2,h_2''}^c)^* \approx c\, (c_{1,1}^c \dots c_{1,h_1''}^c)^*\, c^{-1}, \qquad (1)$$

where $(c^c_{X,1} \cdots c^c_{X,h''_X})^*$ is a cyclic permutation of $c^c_{X,1} \cdots c^c_{X,h''_X}$, $X = 1, 2$, and c is a subword of some relation $y' \in R$. By Prop. 3 we may assume that $c = (r, s)$, and, say, equation (1), in fact, holds with

$$(c^c_{X,1} \cdots c^c_{X,h''_X})^* = c^c_{X,i_X} \cdots c^c_{X,h''_X} c^c_{X,1} \cdots c^c_{X,i_X-1},$$

where

$$1 \leq i_X \leq h''_X, X = 1, 2.$$

Thus equation (1) becomes

$$c^c_{2,i_2} \cdots c^c_{2,h''_2} c^c_{2,1} \cdots c^c_{2,i_2-1} \approx c\, c^c_{1,i_1} \cdots c^c_{1,h''_1} c^c_{1,1} \cdots c^c_{1,i_1-1}\, c^{-1}. \quad (2)$$

By Prop. 7, equation (2) holds if and only if

$$canonical(c^c_{2,i_2} \cdots c^c_{2,h''_2} c^c_{2,1} \cdots c^c_{2,i_2-1}) = canonical(c\, c^c_{1,i_1} \cdots c^c_{1,h''_1} c^c_{1,1} \cdots c^c_{1,i_1-1}\, c^{-1}). \quad (3)$$

Since $c^c_{X,1} \cdots c^c_{X,h''_X}$, $X = 1, 2$, is stable it may be deduced from Lemmas 2 and 3 that

$$canonical(c^c_{2,i_2} \cdots c^c_{2,h''_2} c^c_{2,1} \cdots c^c_{2,i_2-1}) = c^c_{2,i_2} \cdots c^c_{2,h''_2} c^c_{2,1} \cdots c^c_{2,i_2-1}, \quad (4)$$

and

$$canonical(c\, c^c_{1,i_1} \cdots c^c_{1,h''_1} c^c_{1,1} \cdots c^c_{1,i_1-1}\, c^{-1}) = c'\, c^c_{1,i_1+1} \cdots c^c_{1,h''_1} c^c_{1,1} \cdots c^c_{1,i_1-2}\, c'', \quad (5)$$

where, either $c' = c\, c^c_{1,i_1}$, the concatenation of two rectified terms, or c' is one rectified term s.t. $c' \approx c\, c^c_{1,i_1}$; and a similar statement relates c'' and $c^c_{1,i_1-1}\, c^{-1}$.

With equations (4) and (5), rewrite (3) as

$$c^c_{2,i_2} \cdots c^c_{2,h''_2} c^c_{2,1} \cdots c^c_{2,i_2-1} = c'\, c^c_{1,i_1+1} \cdots c^c_{1,h''_1} c^c_{1,1} \cdots c^c_{1,i_1-2}\, c''. \quad (6)$$

Examining equation (6), we see that the string $c^c_{1,1} \cdots c^c_{1,h''_1}$, with *some* two adjacent terms deleted (possibly the first and the last), must occur as a substring of the *circular* string $c^c_{2,1} \cdots c^c_{2,h''_2}$ (with the first term following the last). Whether this is true may be checked with a linear-time Knuth-Morris-Pratt matching algorithm [9]. If it is indeed true then we can further determine in constant time if there does exist $c = (r, s)$ s.t. equation (6), in fact, holds.

This completes our description of the procedure to decide if z'_X, $X = 1, 2$, represent conjugate elements of $\pi(M')$.

Analysis: It is not hard to verify that Subphases 1a and 1b each complete in $O(n + k_1 + k_2)$ time and space. As for Phase 2, it is sufficient to analyze function *canonical* observing that each iteration of the *for* loop

(a) adds at most one term to the stack, and,
(b) costs $O(1+$ number of terms it deletes from the stack).

It follows that the total cost of *canonical* is $O(h')$. It may then be verified that Phase 2 completes in $O(n + k_1 + k_2)$ time and space.

Remarks: When M is orientable with genus $g = 0$ (sphere), 1 (torus) and when M is non-orientable with genus $g = 1$ (klein bottle), the transformability problem can be easily solved in optimal time and space using the method described in [1]. The only exceptional cases that remain are $g = 2$ (double torus) when M is orientable, and $g = 3, 4$ when M is non-orientable. Unfortunately, our main algebraic tool Prop. 2 fails in these cases. However, the contractibility problem for these cases can be solved in $O(n + (k_1 + k_2) \log g))$ time and $O(n + k_1 + k_2)$ space using the algorithm of [5], which is optimal since g is bounded.

We summarize in the following:

Theorem 1 *Given a triangulation T of size n of a genus g surface M,*
(i) it can be decided if a closed curve C presented as edge-vertex sequences of length k in T is contratible in optimal $O(n + k)$ time and space, and
(ii) it can be decided if two such closed curves C_1 and C_2 of lengths k_1 and k_2, resp., are homotopic in optimal $O(n + k_1 + k_2)$ time and space except for cases, where M is genus 2 orientable surface, or genus 3 or genus 4 non-orientable surface. ♣

Acknowledgement

We thank Tarun K. Mukherjee, Jadavpur University, Calcutta, for pointing out a valuable reference. The first author acknowledges the support of the NSF grant CCR-93-21799 while he was in Dept. of Comp. Sci., Indiana-Purdue University, Indianapolis, IN 46202, USA.

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