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On the number of simplicial complexes in $\mathbb{R}^d$

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Abstract

Using a simplex-crossing counting technique we prove: if the number of non-improperly intersecting simplices with vertices in a set S of n labeled points in $\mathbb{R}^d$ is $O(n^{\lceil d/2\rceil})$, then there are $2^{\Theta(n^{\lceil d/2\rceil})}$ different geometric simplicial complexes with vertices in S. © 1997 Elsevier Science B.V.

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1. Introduction

In this paper we consider the problem of counting the number of combinatorially different geometric simplicial complexes with vertices in a fixed set of n labeled points in $\mathbb{R}^d$, the d-dimensional real space. Geometric simplicial complexes consist of geometric simplices rather than topological simplices. Precise definitions are given in Section 2.

A related problem of counting the number of combinatorially different triangulations with vertices in a fixed labeled point set is considered in [4,8]. Let $t_d(n)$ and $s_d(n)$ denote the maximum number of different topological and geometric triangulations respectively of S^d, the d-dimensional sphere, with n being the number of vertices. Kalai [8] showed that

$$c_1 n^{\lfloor d/2\rfloor} \leqslant \log t_d(n) \leqslant c_2 n^{\lceil d/2\rceil} \log n$$

for some constants c_1, c_2. In [4], Dey showed that $\log s_d(n) = O(n^{\lceil d/2\rceil})$ if at most $O(n^{\lceil d/2\rceil})$ $\lceil d/2\rceil$-simplices can be embedded in $\mathbb{R}^d$ without any crossing. Actually, this upper bound also holds for $\log r_d(n)$, where $r_d(n)$ is the maximum number of geometric triangulations possible with n points

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in $\mathbb{R}^d$. By a geometric triangulation of a point set in $\mathbb{R}^d$ we mean a triangulation of the convex hull of the point set with geometric simplices. The only known lower bound for $\log r_d(n)$ is $\Omega(n)$.

Let $\kappa_d(S)$ denote the number of different geometric simplicial complexes with vertices in a set S of labeled points in $\mathbb{R}^d$, and let

$$\kappa_d(n) = \max_{S\subset\mathbb{R}^d, |S|=n} \kappa_d(S).$$

In contrast to geometric triangulations, it is easy to establish an $\Omega(n^{\lceil d/2\rceil})$ lower bound on the logarithm of $\kappa_d(n)$. However, the upper bound on the number of geometric triangulations does not provide an upper bound on the number of geometric simplicial complexes. This is because, for $d > 2$, not all simplicial complexes in $\mathbb{R}^d$ are extendable to a triangulation of the underlying point set. For example, the boundary complex of the Schönardt polytope [11] is not extendable to a triangulation of the corresponding vertex set.

Previous results on the number of simplicial complexes dealt with all possible simplicial complexes on n vertices in all dimensions. Let $\text{simp}(n)$ denote this number. It follows from the results of [9,10] that $\log \text{simp}(n) = \Theta(\binom{n}{\lfloor n/2\rfloor})$. This paper concentrates on counting the number of simplicial complexes in a fixed dimension $\mathbb{R}^d$. Specifically, we show that $\log \kappa_d(n) = O(n^{\lceil d/2\rceil})$ matching the lower bound if no more than $O(n^{\lceil d/2\rceil})$ simplices can be embedded in $\mathbb{R}^d$ without crossing. In light of the result of Goodman and Pollack [7], this bound for a fixed point set can be extended to cover all point sets of some fixed cardinality. More specifically, they show that there are at most $2^{O(n\log n)}$ combinatorially different configurations of n points in $\mathbb{R}^d$. This result combined with ours shows that there are at most $2^{O(n^{\lceil d/2\rceil}+n\log n)}$ combinatorially different geometric simplicial complexes with n points in $\mathbb{R}^d$ provided at most $O(n^{\lceil d/2\rceil})$ simplices are embeddable in $\mathbb{R}^d$ without crossing.

The rest of the paper is organized as follows. In the next section, we introduce some terminology and present the statement of our main result. In Section 3, we prove a crossing result. Our method is an extension of the method in [5], where it was used to prove a bound on the number of crossings of triangles in $\mathbb{R}^3$. Section 4 generalizes the argument for counting triangulations in [4] to establish the main result. In Section 5 we state some open problems.

2. Definitions and preliminaries

A *d-simplex* σ_T is the convex hull of an affinely independent point set T of size $d+1$. If $V \subseteq T$, then σ_V is a *face* of σ_T. A (*geometric*) *simplicial complex* $\mathcal{K}$ is a finite collection of simplices satisfying the following properties:

(a) if $\sigma_T \in \mathcal{K}$ and $V \subseteq T$, then $\sigma_V \in \mathcal{K}$, and

(b) if $\sigma_V, \sigma_U \in \mathcal{K}$, then $\sigma_V \cap \sigma_U = \sigma_{V\cap U}$.

$\mathcal{K}$ is a *k-complex* if the largest dimension of a simplex in $\mathcal{K}$ is k. For any collection $\mathcal{L}$ of simplices (not necessarily a simplicial complex), we define

$$\mathcal{L}^{(j)} = \{\sigma \in \mathcal{L} \mid \sigma \text{ is a } j\text{-simplex}\}.$$

$\mathcal{L}^{(0)}$ is the set of *vertices* of $\mathcal{L}$.

Two simplicial complexes $\mathcal{K}_1$, $\mathcal{K}_2$ with vertices in the same labeled point set are *combinatorially different* if and only if there exists a simplex σ_V such that $\sigma_V \in \mathcal{K}_1$ and $\sigma_V \notin \mathcal{K}_2$ or $\sigma_V \notin \mathcal{K}_1$ and

$\sigma_V \in \mathcal{K}_2$. Let $\kappa_d(S)$ denote the number of different geometric simplicial complexes with vertices in a labeled fixed point set $S \subseteq \mathbb{R}^d$, and let

$$\kappa_d(n) = \max_{S \subseteq \mathbb{R}^d, |S|=n} \kappa_d(S).$$

We prove the following theorem.

Theorem 1. $\log \kappa_d(n) = \Theta(n^{\lceil d/2 \rceil})$, *if at most* $\mathrm{O}(n^{\lceil d/2 \rceil})$ *simplices are embedable in* $\mathbb{R}^d$ *without crossing.*

It is easy to see that $\log \kappa_d(n) = \Omega(n^{\lceil d/2 \rceil})$. Let $p(t) = (t, t^2, \ldots, t^d) \in \mathbb{R}^d$ be a point on the moment curve [6]. Let $S = \{p(i) \mid i = 1, \ldots, n\}$ and let $\tau = \lceil d/2 \rceil$. Let $\mathcal{K}$ denote the collection of all simplices σ_T, $T \subseteq S$, $|T| \leqslant \tau$. Then for any two simplices, $\sigma_U, \sigma_V \in \mathcal{K}$, $|U| + |V| \leqslant d+1$. Since S is in general position, σ_U and σ_V are faces of $\sigma_{U \cup V}$. It follows that $\mathcal{K}$ is a simplicial complex. Let $\mathcal{L}$ denote the collection of $(\tau - 1)$-simplices in $\mathcal{K}$. Clearly, the cardinality of $\mathcal{L}$ is $\Theta(n^\tau)$. For every $\mathcal{L}' \subseteq \mathcal{L}$, $(\mathcal{K} - \mathcal{L}) \cup \mathcal{L}'$ is a simplicial complex. This proves the lower bound.

The combinatorial bounds proved in this paper are based on the following proposition.

Conjecture 2. *If* $\mathcal{K}$ *is a simplicial complex embedded in* $\mathbb{R}^d$, *then the total number of simplices in* $\mathcal{K}$ *is* $\mathrm{O}(n^{\lceil d/2 \rceil})$, *where* n *is the number of vertices of* $\mathcal{K}$.

If $\mathcal{K}$ is a d-complex, the conjecture is true as shown in [2]. It is widely believed that the conjecture is true in general. Two simplices σ_U and σ_V have an *improper intersection* if they intersect but the intersection is not $\sigma_{U \cap V}$ (that is, the intersection is not a common face). Conjecture 2 says that the size of a collection of simplices, with vertices from amongst n fixed points in $\mathbb{R}^d$, such that no two simplices in the collection have an improper intersection is $\mathrm{O}(n^\tau)$, where $\tau = \lceil d/2 \rceil$. Two simplices σ_U and σ_V *cross* if they have an improper intersection and $U \cap V = \emptyset$. A collection of simplices is *crossing-free* if no two simplices in the collection cross. An improper intersection is a *non-crossing intersection* if it is not a crossing. To prove Theorem 1, we will need a bound on the size of a collection of crossing-free simplices.

Remark. Since the total number of simplices with vertices in a fixed point set $S \subseteq \mathbb{R}^d$ of size n is $\mathrm{O}(n^{d+1})$, it follows that

$$\kappa_d(S) = \binom{\mathrm{O}(n^{d+1})}{\mathrm{O}(n^\tau)} = 2^{\mathrm{O}(n^\tau \log n)}$$

if Conjecture 2 is true. We aim to strengthen this bound to $2^{\mathrm{O}(n^\tau)}$.

3. A lower bound on the number of crossings

Let $\mathcal{L}$ be some collection of simplices with vertices from a labeled fixed point set $S \subseteq \mathbb{R}^d$ of cardinality n. Further suppose that if $\sigma_T \in \mathcal{L}$ and $V \subseteq T$, then $\sigma_V \in \mathcal{L}$. Let t_k denote the cardinality of $\mathcal{L}^{(k)}$, $0 \leqslant k \leqslant d$. As before, we let $\tau = \lceil d/2 \rceil$. Let $x^{(d)}(n, j, t_j)$ denote the number of crossings of distinct pairs of j-simplices in $\mathcal{L}$. Below, we shall prove a lower bound on $x^{(d)}(n, j, t_j)$ when

$\bigcup_{0 \leqslant k < j} \mathcal{L}^{(k)}$ is a simplicial complex. Note that this requirement and Conjecture 2 imply that $t_k < c_k n^\tau$ for some constants c_k, $0 \leqslant k < j$. We shall need the following lemma which can be found in [4,5].

Lemma 3. *For $k_1 + k_2 \geqslant d$, let $\Delta_1 \subseteq \mathbb{R}^d$ be a k_1-simplex that improperly intersects a k_2-simplex $\Delta_2 \subseteq \mathbb{R}^d$. Then there exists an ℓ_1-face σ_1 of Δ_1 and an ℓ_2-face σ_2 of Δ_2 such that $\ell_1 + \ell_2 = d$ and σ_1 crosses σ_2.*

Using Lemma 3 and Conjecture 2, we give below a bound on the number of j-simplices in $\mathcal{L}$ if no two j-simplices of $\mathcal{L}$ cross.

Lemma 4. *If Conjecture 2 is true then the following holds. If $t_k < c_k n^\tau$ for some constants c_k, $0 \leqslant k < j$, then there exists a constant c so that if $t_j > cn^\tau$, then there exists a pair of crossing j-simplices in $\mathcal{L}$.*

Proof. Conjecture 2 guarantees a pair of improperly intersecting j-simplices if $t_j > b_1 n^\tau$ for some constant b_1. Suppose that there is no crossing pair amongst the t_j j-simplices in $\mathcal{L}$. The outline of the proof is the following. We shall remove from $\mathcal{L}$ one of the two j-simplices involved in a non-crossing intersection. We show that we remove at most $b_2 n^\tau$ j-simplices by this process, for some constant b_2. At the end, we are left with at least $(c - b_2)n^\tau$ j-simplices such that no two of them have a non-crossing intersection. If $c - b_2 > b_1$, then Conjecture 2 contradicts the supposition that there is no crossing pair of j-simplices.

We remove j-simplices involved in non-crossing intersections according to the following procedure. We let $\mathcal{L}'$ denote the current set of j-simplices; initially, $\mathcal{L}'$ is the same as $\mathcal{L}^{(j)}$, but it changes as we remove j-simplices. We are done when $\mathcal{L}'$ does not contain a pair of improperly intersecting j-simplices. Let $\sigma_U, \sigma_V \in \mathcal{L}'$ be a pair of improperly intersecting j-simplices. By Lemma 3, there exists a face σ_X of σ_U which crosses a face σ_Y of σ_V. Let $I = U \cap V$. Observe that I is non-empty since otherwise σ_U and σ_V cross. Let $I_X = I \cap X$ and $I_Y = I \cap Y$. We have two cases: (a) I_X or I_Y is empty, (b) both I_X and I_Y are non-empty. We remove improperly intersecting pairs in two phases. In phase (i), the pairs that satisfy (a) are removed and in a subsequent phase (ii), the rest of the improperly intersecting pairs (which satisfy (b)) are removed.

Phase (i). Without loss of generality assume that $I_X = \emptyset$. Let

$$\Sigma_V = \{\sigma_Z \in \mathcal{L}' \mid \sigma_Z \text{ is incident to } \sigma_X\}.$$

Phase (ii). Let

$$\Sigma_V = \{\sigma_Z \in \mathcal{L}' \mid \sigma_Z \text{ is incident to } \sigma_X\}.$$

In each phase, we remove all simplices in Σ_V from $\mathcal{L}'$. In phase (i), we charge σ_{Z-V} one unit for the removal of every j-simplex $\sigma_Z \in \Sigma_V$ and in phase (ii), we charge σ_{Z-Y} for the removal of every j-simplex $\sigma_Z \in \Sigma_V$. Below, we show that no simplex is charged more than a constant number of units.

Consider a simplex σ_{Z-V} charged at some step during phase (i). Let $I_1 = Z \cap V$. Then $Z - V = Z - I_1$. In phase (i) we assumed that $I_X = \emptyset$ and so it follows that $X \cap V = \emptyset$ and hence σ_X is a face of σ_{Z-I_1}. We are guaranteed that I_1 is non-empty since otherwise σ_Z and σ_V cross. First, we show that σ_{Z-I_1} is never charged at a later step. Suppose σ_{Z-I_1} is charged at a later step for the removal of some simplex $\sigma_{Z'}$ from $\Sigma_{V'}$. Irrespective of whether this happens in phase (i) or phase (ii), σ_{Z-I_1}

is a face of $\sigma_{Z'}$. But then $\sigma_{Z'}$ would be removed from $\mathcal{L}'$ when Σ_V was processed. So, σ_{Z-I_1} cannot be charged at a later step. This also means that σ_{Z-I_1} was not charged by an earlier step. Finally, it is clear that the step of removing the simplices in Σ_V can charge σ_{Z-I_1} at most $\binom{|V|}{|I_1|} = \binom{j+1}{|I_1|}$ units.

Now consider a simplex σ_{Z-Y} charged at some step during phase (ii). Because of the above argument, σ_{Z-Y} was not charged during phase (i). Let $I_2 = Z \cap Y$. I_2 must be non-empty since otherwise σ_Z and σ_V intersect improperly and $I_Y = I_2 = \emptyset$ satisfies condition (a) implying that σ_V should have been removed from Σ_Z during phase (i). Since σ_X and σ_Y cross, we have $X \cap Y = \emptyset$ and so σ_X is a face of $\sigma_{Z-Y} = \sigma_{Z-I_2}$. Now σ_{Z-I_2} cannot be charged at a later step for the removal of some simplex $\sigma_{Z'}$ from $\Sigma_{V'}$ because if σ_{Z-I_2} is a face of $\sigma_{Z'}$, then $\sigma_{Z'}$ would be removed from $\mathcal{L}'$ when Σ_V was processed. This also means that σ_{Z-I_2} was not charged by an earlier step in phase (ii). As before, the step of removing the simplices in Σ_V can charge σ_{Z-I_2} at most $\binom{|V|}{|I_2|} = \binom{j+1}{|I_2|}$ units.

Let $b_3 = \sum_{0 \leqslant k < j} c_k$. Since the size of $\bigcup_{0 \leqslant k < j} \mathcal{L}^{(k)}$ is at most $b_3 n^\tau$, it follows that the total number of j-simplices removed is at most $b_2 n^\tau$, where $b_2 = 2^{j+1} b_3$. □

Now, we are ready to prove a lower bound on $x^{(d)}(n, j, t_j)$.

Lemma 5. *Let $j \geqslant \tau$. If for some constants c, c_{j-1}, $t_{j-1} < c_{j-1} n^\tau$ and there exists a pair of crossing j-simplices whenever $t_j > cn^\tau$, then there exist constants c', h so that*

$$x^{(d)}(n, j, t_j) \geqslant c' \binom{n}{2j+2} \left(t_j \Big/ \binom{n}{j+1} \right)^{1+\gamma_j},$$

when $t_j > hn^\tau$. Here

$$\gamma_j = \frac{j+1}{j+1-\tau} > 1.$$

Proof. Since we are interested in a lower bound, we can assume that $\mathcal{L}^{(j)}$ realizes the lower bound for $x^{(d)}(n, j, t_j)$. Let **bound** denote the term

$$c' \binom{n}{2j+2} \left(t_j \Big/ \binom{n}{j+1} \right)^{1+\gamma_j}.$$

We shall proceed by induction on $t = t_j$. We choose $h = c + 1$. We have at least $t - cn^\tau \geqslant n^\tau$ crossings since there is a crossing for every j-simplex above cn^τ.

First, we dispense with the case where n is no greater than the constant $n_0 = 2j + 2$. In this case, t is also a constant, and we can make **bound** $\leqslant 1$ by simply choosing a sufficiently small c'. Thus, the lower bound holds in this case since as we saw above, we have at least n^τ crossings. For the rest of the induction step, we will assume that $n > n_0$. We have two cases.

Case 1 (base case). $hn^\tau \leqslant t \leqslant (h + c_{j-1}) n^\tau$.

Since we have at least n^τ crossings, it suffices to show that **bound** $\leqslant n^\tau$. Since $n > 2j + 2$, $\binom{n}{j+1} \geqslant b_1 n^{j+1}$ for some constant b_1. Since $t \leqslant (h + c_{j-1}) n^\tau$, we have

$$\textbf{bound} \leqslant c' \frac{n^{2j+2} (h + c_{j-1})^{1+\gamma_j} n^{\tau(\gamma_j+1)}}{b_1^{(1+\gamma_j)} n^{(j+1)(\gamma_j+1)}} = b_2 n^\tau,$$

where $b_2 = c'(h + c_{j-1})^{1+\gamma_j} / b_1^{(1+\gamma_j)}$ is a constant. $b_2 < 1$ if c' is small enough.

Case 2 (induction step). $t > (h + c_{j-1})n^{\tau}$.

Let $T(w)$ denote the set of j-simplices in $\mathcal{L}^{(j)}$ that are not incident to a vertex $w \in \mathcal{L}^{(0)}$ and let $t(w) = |T(w)|$. Now $t(w) \geqslant t - t_{j-1} > hn^{\tau}$. For every pair of crossing j-simplices Δ_1 and Δ_2, we count all vertices except those incident to Δ_1 and Δ_2. Alternatively, this count can be obtained by summing up all crossings between j-simplices in $T(w)$ for each vertex w. Thus, we have

$$(n-2j-2)x^{(d)}(n,j,t) = \sum_{w\in\mathcal{L}^{(0)}} x^{(d)}\big(n-1,j,t(w)\big)$$
$$\geqslant c'\binom{n-1}{2j+2}\left(\sum_{w\in\mathcal{L}^{(0)}} t(w)^{1+\gamma_j} \Big/ \binom{n-1}{j+1}^{1+\gamma_j}\right)$$

by induction, since $t(w) > hn^{\tau}$. Now $\sum_{w\in\mathcal{L}^{(0)}} t(w) = (n-j-1)t$. Thus

$$\sum_{w\in\mathcal{L}^{(0)}} t(w)^{1+\gamma_j} \geqslant n\left(\frac{(n-j-1)t}{n}\right)^{1+\gamma_j}.$$

This implies that

$$x^{(d)}(n,j,t) \geqslant c'\frac{n}{n-2j-2}\binom{n-1}{2j+2}\left(\frac{(n-j-1)t}{n\binom{n-1}{j+1}}\right)^{1+\gamma_j}$$
$$\geqslant c'\binom{n}{2j+2}\left(\frac{t}{\binom{n}{j+1}}\right)^{1+\gamma_j}. \qquad \square$$

By the pigeon-hole principle, it follows that there is at least one j-simplex in $\mathcal{L}^{(j)}$ that crosses at least $x^{(d)}(n,j,t_j)/t_j$ other j-simplices. Hence we have the following lemma.

Lemma 6. *Let Conjecture 2 hold and let $j \geqslant \tau$. Then there exists a j-simplex in $\mathcal{L}$ that crosses at least $h_j t^{\gamma_j}/n^{(\gamma_j-1)(j+1)}$ other j-simplices of $\mathcal{L}$ for some constant $h_j > 0$, when $t = t_j > hn^{\tau}$, where h is the constant in Lemma 5.*

4. Counting the number of simplicial complexes

Let $S \subseteq \mathbb{R}^d$ be a labeled fixed point set of cardinality n. Let $\mathcal{F}(j)$ denote the set of all simplicial j-complexes with vertices in S. Let $\Delta(j)$ denote the set of all j-simplices with vertices in S. For a simplicial complex $\mathcal{K} \in \mathcal{F}(j-1)$ and a collection of j-simplices $T \subseteq \Delta(j)$, define

$$\mathcal{L}(j,T,\mathcal{K}) = \{\mathcal{K}\cup T' \mid T' \subseteq T,\ \mathcal{K}\cup T' \text{ is a simplicial complex}\}.$$

Thus $\mathcal{L}(j,T,\mathcal{K})$ is the collection of simplicial j-complexes $\mathcal{K}'$ such that j-simplices of $\mathcal{K}'$ come from T and the k-dimensional simplices of $\mathcal{K}'$ are the same as those in $\mathcal{K}$, $0 \leqslant k < j$. Define

$$F(j,t,\mathcal{K}) = \max_{T\subseteq\Delta(j),|T|=t} |\mathcal{L}(j,T,\mathcal{K})|$$

and

$$F(j,t) = \sum_{\mathcal{K}\in\mathcal{F}(j-1)} F(j,t,\mathcal{K}).$$

Observe that $F(j,\binom{n}{j+1}) = |\mathcal{F}(j)|$. We shall show that $F(j,t) = 2^{O(n^\tau)}$ for $0 \leqslant j \leqslant d$ if Conjecture 2 holds. Since

$$\kappa_d(S) = \sum_{0\leqslant j\leqslant d} F\left(j, \binom{n}{j+1}\right),$$

it follows that $\kappa_d(S) = 2^{O(n^\tau)}$. Thus to establish Theorem 1, we only need to prove the following lemma.

Lemma 7. $F(j,t) = 2^{O(n^\tau)}$ *if Conjecture 2 holds.*

Proof. We shall use induction, both on j and t. We shall show that for every $\mathcal{K} \in \mathcal{F}(j-1)$, $F(j,t,\mathcal{K}) = 2^{O(n^\tau)}$. This implies that there exists a constant $c > 0$ such that $F(j,t,\mathcal{K}) \leqslant 2^{cn^\tau}$. We shall inductively assume that

$$|\mathcal{F}(j-1)| \leqslant 2^{n^\tau+(j-1)cn^\tau},$$

whence it follows that

$$F(j,t) = \sum_{\mathcal{K}\in\mathcal{F}(j-1)} F(j,t,\mathcal{K}) \leqslant 2^{n^\tau+(j-1)cn^\tau}2^{cn^\tau} = 2^{n^\tau+jcn^\tau},$$

and so

$$|\mathcal{F}(j)| \leqslant 2^{n^\tau+jcn^\tau} = 2^{O(n^\tau)}.$$

For $j < \tau$, the number of j-simplices with vertices in S is bounded by $n^{j+1} \leqslant n^\tau$. Thus $|\Delta(j)|$ is bounded by $O(n^\tau)$, and so the size of the power set of $\Delta(j)$ is at most $2^{O(n^\tau)}$. This implies that $F(j,t,\mathcal{K}) = 2^{O(n^\tau)}$ for any complex $\mathcal{K} \in \mathcal{F}(j-1)$. Because of the above argument and since $|\mathcal{F}(0)| = 2^n$, it follows that the inductive hypothesis holds for $j < \tau$. In the following, we consider the case when $j \geqslant \tau$, and induct on t.

First, we dispense with the case $j = d$. Let $\mathcal{K} \in \mathcal{F}(d-1)$. Let $\Gamma \subseteq \Delta(d)$ be a collection of d-simplices so that for every $\sigma \in \Gamma$, $\mathcal{K} \cup \{\sigma\}$ is a d-complex. We claim that a $(d-1)$-simplex σ_U of $\mathcal{K}$ can be incident to at most 2 d-simplices in Γ. Suppose not and let $\sigma_{U\cup\{p_1\}}$, $\sigma_{U\cup\{p_2\}}$ and $\sigma_{U\cup\{p_3\}}$ be three d-simplices of Γ incident to σ_U. At least two points from p_1, p_2 and p_3, say p_1 and p_2, lie on the same side of the hyperplane $\mathrm{aff}(U)$. But then $\sigma_{U\cup\{p_2\}}$ improperly intersects some $(d-1)$-face σ' of $\sigma_{U\cup\{p_1\}}$, or vice-versa. Without loss of generality, assume that $\sigma_{U\cup\{p_2\}}$ improperly intersects some $(d-1)$-face σ' of $\sigma_{U\cup\{p_1\}}$. Since $\sigma' \in \mathcal{K}$, it follows that $\mathcal{K} \cup \{\sigma_{U\cup\{p_2\}}\}$ is not a simplicial complex, contradicting the assumption that $\sigma_{U\cup\{p_2\}} \in \Gamma$. Thus at most 2 d-simplices of Γ can be incident to a $(d-1)$-simplex of $\mathcal{K}$. Since Conjecture 2 implies that the number of $(d-1)$-simplices in $\mathcal{K}$ is $O(n^\tau)$, it follows that the size of Γ is $O(n^\tau)$. For any t, $F(j,t,\mathcal{K}) \leqslant |\mathcal{L}(j,\Gamma,\mathcal{K})|$ and so we have $F(j,t,\mathcal{K}) = 2^{O(n^\tau)}$. In the following, we only consider the case when $\tau \leqslant j < d$.

Let $\lambda_j > 2$ be a large enough constant (to be determined later) so that $\lambda_j > h$ and $h_j > \gamma_j/\lambda_j$, where h is the constant in Lemma 5 and h_j is the constant in Lemma 6. Recall $\gamma_j = (j+1)/(j+1-\tau) > 1$.

Fix a complex $\mathcal{K} \in \mathcal{F}(j-1)$, and consider a set $T \subseteq \Delta(j)$ of size t that realizes the maximum $F(j,t,\mathcal{K})$. When $t \leqslant \lambda_j n^\tau$, the number of subsets T' of T is bounded by $2^{\mathrm{O}(n^\tau)}$. The bound on $F(j,t,\mathcal{K})$ follows.

Let $t > \lambda_j n^\tau$. We show that $F(j,t,\mathcal{K}) \leqslant C^{n^\tau} f(j,t)$, where

$$C = (2\lambda_j)^{\lambda_j + 1/\lambda_j^{\gamma_j - 2}} \quad \text{and} \quad f(j,t) = \left(\frac{t}{n^\tau}\right)^{-\lambda_j n^{\tau\gamma_j}/t^{\gamma_j - 1}}.$$

Certain useful properties of $f(j,t)$ are discussed in Appendix A. In particular, property (P1) states that $f(j,t) \leqslant 1$ for $\lambda_j n^\tau \leqslant t \leqslant \binom{n}{j+1}$, implying that $F(j,t,\mathcal{K}) = 2^{\mathrm{O}(n^\tau)}$. We divide the proof into two cases.

Case 1 (base case). $\lambda_j n^\tau < t \leqslant 2\lambda_j n^\tau$.

Recall $\lambda_j > 2$. Since the number of subsets of T is at most $2^{2\lambda_j n^\tau}$, we have

$$\begin{aligned} F(j,t,\mathcal{K}) &\leqslant 2^{2\lambda_j n^\tau} \leqslant (2\lambda_j)^{\lambda_j n^\tau} (t/n^\tau)^{\lambda_j n^{\tau\gamma_j}/t^{\gamma_j-1}} f(j,t) \\ &\leqslant (2\lambda_j)^{\lambda_j n^\tau} (2\lambda_j)^{\lambda_j n^{\tau\gamma_j}/(\lambda_j^{\gamma_j-1} n^{\tau(\gamma_j-1)})} f(j,t) = C^{n^\tau} f(j,t). \end{aligned}$$

Case 2 (induction step). $t > 2\lambda_j n^\tau$.

Since $\mathcal{K}$ is a simplicial complex, by Conjecture 2 the number of k-simplices in $\mathcal{K}$ is $\mathrm{O}(n^\tau)$ for $0 \leqslant k < j$. So Lemma 6 applies with $\mathcal{L} = \mathcal{K} \cup T$. Let σ be the j-simplex in T that crosses at least

$$h_j \frac{t^{\gamma_j}}{n^{(\gamma_j-1)(j+1)}} \geqslant \frac{\gamma_j t^{\gamma_j}}{\lambda_j n^{\tau\gamma_j}}$$

other j-simplices of T. We get the following recurrence:

$$F(j,t,\mathcal{K}) \leqslant F(j,t-1,\mathcal{K}) + F\left(j, t - \frac{\gamma_j t^{\gamma_j}}{\lambda_j n^{\tau\gamma_j}}, \mathcal{K}\right).$$

Let $\rho = t/n^\tau$. Then $2\lambda_j < \rho < n^{j+1-\tau}$.

$$\begin{aligned} t - \frac{\gamma_j t^{\gamma_j}}{\lambda_j n^{\tau\gamma_j}} &= \rho n^\tau - \frac{\gamma_j \rho^{\gamma_j} n^{\tau\gamma_j}}{\lambda_j n^{\tau\gamma_j}} = \rho n^\tau \left(1 - (\gamma_j/\lambda_j)\rho^{\tau/(j+1-\tau)}/n^\tau\right) \\ &> \rho n^\tau (1 - \gamma_j/\lambda_j) \quad \text{because } \rho < n^{j+1-\tau} \\ &> \lambda_j n^\tau \quad \text{if } \lambda_j > 2\gamma_j. \end{aligned}$$

So we can apply the inductive assumption and get

$$\begin{aligned} F(j,t,\mathcal{K}) &\leqslant F(j,t-1,\mathcal{K}) + F\left(j, t - \frac{\gamma_j t^{\gamma_j}}{\lambda_j n^{\tau\gamma_j}}, \mathcal{K}\right) \\ &< C^{n^\tau} f(j,t-1) + C^{n^\tau} f\left(j, t - \frac{\gamma_j t^{\gamma_j}}{\lambda_j n^{\tau\gamma_j}}\right) \\ &< C^{n^\tau} f(j,t) \quad \text{by property (P5) of } f(j,t). \end{aligned}$$

We note that property (P5) applies only when ρ is larger than the maximum of $(2\lambda_j')^{1/\gamma_j}$ and $\mathrm{e}^{1/(\gamma_j-1)} + 1/n^\tau$. We can coerce ρ to be always larger than this maximum by choosing a sufficiently large λ_j since $\rho > 2\lambda_j$.

To wrap up the proof, we simply choose λ_j large enough so that it satisfies the requirements of the above proof and so that the properties of $f(j,t)$ discussed in Appendix A hold. □

5. Concluding remarks

We have derived an asymptotically tight upper bound on $\log \kappa_d(n)$ based on a conjecture that any simplicial complex embeddable in $\mathbb{R}^d$ without improper intersection contains at most $O(n^{\lceil d/2 \rceil})$ simplices. A natural question that arises is whether these bounds extend to topological simplicial complexes. Let Δ_{n-1} denote a geometric $(n-1)$-simplex and let $\mathcal{L}$ be the collection of all j-faces of Δ_{n-1}, $0 \leqslant j < d+1$. Let $\mathcal{L}'$ be a subcomplex of $\mathcal{L}$ and let $g : \bigcup_{\sigma \in \mathcal{L}'} \sigma \to \mathbb{R}^d$ be an embedding. Then $\mathcal{K} = \{g(\sigma) \mid \sigma \in \mathcal{L}'\}$ is a *topological simplicial complex* in $\mathbb{R}^d$. The vertex set of $\mathcal{K}$ is $g(\mathcal{L}'^{(0)})$. Although we assumed a linear embedding of simplices, our result is valid for any fixed map $g' : \bigcup_{\sigma \in \mathcal{L}} \sigma \to \mathbb{R}^d$ such that g' restricted to each $\sigma \in \mathcal{L}$ is an embedding. However, our counting method fails when several such maps are considered. Hence, the result does not immediately extend to topological simplicial complexes since it is possible to embed the simplices of $\mathcal{L}$ in $\mathbb{R}^d$ in more than one way.

Related to determining the number of geometric simplicial complexes is the question of determining the number of geometric triangulations, $r_d(n)$, on n vertices in $\mathbb{R}^d$. Clearly, the upper bound on $\log \kappa_d(n)$ holds for $\log r_d(n)$, see also [4]. As mentioned in Section 1, the lower bound on $\log r_d(n)$ is $\Omega(n)$. Reducing the huge gap between the upper and lower bounds on $\log r_d(n)$ remains a challenge to date.

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Appendix A. Properties of $f(j,t) = (t/n^{\tau})^{-\lambda_j n^{\tau\gamma_j}/t^{\gamma_j - 1}}$

We shall assume that $t \leqslant n^{j+1}$.

$$\text{(P1)} \quad f(j,t) \leqslant 1 \quad \text{for } n^{\tau} \leqslant t \leqslant \binom{n}{j+1}.$$

This is easy to see since $t/n^{\tau} \geqslant 1$, and the exponent is negative.

$$\text{(P2)} \quad f'(j,t) = \frac{\partial}{\partial t} f(j,t) > \lambda_j \frac{n^{\tau\gamma_j}}{t^{\gamma_j}} f(j,t) \quad \text{if } t > \mathrm{e}^{2/(\gamma_j - 1)} n^{\tau}.$$

Again, this is straightforward since

$$f'(j,t) = f(j,t) \lambda_j \frac{n^{\tau\gamma_j}}{t^{\gamma_j}} \left((\gamma_j - 1) \ln \left(\frac{t}{n^{\tau}} \right) - 1 \right).$$

This implies that $f(j,t)$ is a monotonically increasing function of t when $t > \mathrm{e}^{1/(\gamma_j-1)} n^{\tau}$.

$$\text{(P3)}\quad f(j,t-1) < f(j,t)\frac{t^{\gamma_j}}{t^{\gamma_j}+\lambda_j n^{\tau\gamma_j}} \quad \text{if } t > \mathrm{e}^{2/(\gamma_j-1)} n^{\tau}+1.$$

By the mean value theorem, $f(j,t)-f(j,t-1) = f'(j,t')$ for some $t-1 \leqslant t' \leqslant t$. By property (P2),

$$f(j,t)-f(j,t-1) > \lambda_j \frac{n^{\tau\gamma_j}}{t^{\gamma_j}} f(j,t-1).$$

(P3) follows.

$$\text{(P4)}\quad f\left(j, t-\frac{\gamma_j t^{\gamma_j}}{\lambda_j n^{\tau\gamma_j}}\right) \leqslant \lambda'_j \frac{n^{\tau\gamma_j}}{t^{\gamma_j}} f(j,t),$$

where $\lambda'_j = (4^{\gamma_j})^{2^{\tau}}$ is a constant, provided $\lambda_j > 2\gamma_j$ and $\tau \leqslant j < d$.

Let

$$\mu = \frac{\gamma_j t^{\gamma_j-1}}{\lambda_j n^{\tau\gamma_j}}.$$

Since

$$\frac{t^{\gamma_j-1}}{n^{\tau\gamma_j}} = \left(\frac{t}{n^{j+1}}\right)^{\tau/(j+1-\tau)} \leqslant 1$$

and $\lambda_j > 2\gamma_j$, we have $0 < \mu < 1/2$. As a result, $1+\mu \leqslant 1/(1-\mu)^{\gamma_j-1}$. This is easy to see when $\gamma_j \geqslant 2$ since we have $1/(1-\mu)^{\gamma_j-2} \geqslant 1 \geqslant 1-\mu^2$. The only case when $\gamma_j < 2$ is when the dimension d is even and $j = d$. But this is precluded since $j < d$.

Observe that $f(j,t) = (t/n^{\tau})^{-\gamma_j/\mu}$. Also,

$$f\big(j, t(1-\mu)\big) = f\left(j, t-\frac{\gamma_j t^{\gamma_j}}{\lambda_j n^{\tau\gamma_j}}\right) = a(j,t,\mu)b(j,t,\mu),$$

where

$$a(j,t,\mu) = \left(\frac{t}{n^{\tau}}\right)^{-\gamma_j/(\mu(1-\mu)^{\gamma_j-1})} \quad \text{and} \quad b(j,t,\mu) = \left((1-\mu)^{-1/\mu}\right)^{\gamma_j/(1-\mu)^{\gamma_j-1}}.$$

Now,

$$\begin{aligned} a(j,t,\mu) &= (t/n^{\tau})^{-\gamma_j/(\mu(1-\mu)^{\gamma_j-1})} = f(j,t)^{1/(1-\mu)^{\gamma_j-1}} \\ &\leqslant f(j,t)^{1+\mu} \quad \text{since} \quad 1+\mu \leqslant 1/(1-\mu)^{\gamma_j-1} \quad \text{and} \quad f(j,t) \leqslant 1 \\ &= f(j,t) n^{\tau\gamma_j}/t^{\gamma_j}. \end{aligned}$$

We claim that

$$g(\mu) = (1-\mu)^{-1/\mu} \leqslant 4 \quad \text{for } 0 < \mu < 1/2.$$

To see this, note that by Taylor series expansion

$$\ln g(\mu) = \sum_{i \geqslant 0} \frac{\mu^i}{i+1},$$

which is an increasing function of $\mu > 0$. Since e^x is an increasing function of x, it follows that $g(\mu) \leqslant g(1/2) = 4$ for $0 < \mu < 1/2$. Now,

$$\begin{aligned} b(j,t,\mu) &= \left((1-\mu)^{-1/\mu}\right)^{\gamma_j/(1-\mu)^{\gamma_j - 1}} \\ &\leqslant 4^{\gamma_j/(1-\mu)^{\gamma_j-1}} \quad \text{since} \quad 4 \geqslant (1-\mu)^{-1/\mu} \\ &< \left(4^{\gamma_j}\right)^{2^\tau} \quad \text{since} \quad 0 < \mu < 1/2 \quad \text{and} \quad \gamma_j - 1 \leqslant \tau. \end{aligned}$$

(P5) $$f(j,t-1) + f\left(j, t - \frac{\gamma_j t^{\gamma_j}}{\lambda_j n^{\tau\gamma_j}}\right) < f(j,t)$$

when $\tau \leqslant j < d$,

$$t > \max\left\{e^{2/(\gamma_j-1)} n^\tau + 1, (2\lambda'_j)^{1/\gamma_j} n^\tau\right\}$$

and $\lambda_j > 2\lambda'_j$, where λ'_j is the constant in (P4).

Because of properties (P3) and (P4), it suffices to show that

$$\frac{t^{\gamma_j}}{t^{\gamma_j} + \lambda_j n^{\tau\gamma_j}} + \lambda'_j \frac{n^{\tau\gamma_j}}{t^{\gamma_j}} \leqslant 1,$$

which is equivalent to showing that

$$\left(\frac{t}{n^\tau}\right) \geqslant \left(\frac{\lambda_j \lambda'_j}{\lambda_j - \lambda'_j}\right)^{1/\gamma_j}.$$

The inequality holds because the term on the right is bounded by $(2\lambda'_j)^{1/\gamma_j}$.

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