Lattice Points of Cut Cones

MICHEL DEZA[†] and VIATCHESLAV GRISHUKHIN[‡]

[†]CNRS-LIENS, Ecole Normale Supérieure, Paris

[‡]Central Economic and Mathematical Institute of Russian Academy of Sciences (CEMI RAN), Moscow.

Probab. Comput. 3, No 2, 191–214 (1994, Zbl 824 52015).

Let $\mathbb{R}_+(\mathcal{K}_n), \mathbb{Z}(\mathcal{K}_n), \mathbb{Z}_+(\mathcal{K}_n)$ be, respectively, the cone over $\mathbb{R}$, the lattice and the cone over $\mathbb{Z}$, generated by all cuts of the complete graph on n nodes. For $i \geq 0$, let $A_n^i := \{d \in \mathbb{R}_+(\mathcal{K}_n) \cap \mathbb{Z}(\mathcal{K}_n) : d$ has exactly i realizations in $\mathbb{Z}_+(\mathcal{K}_n)\}$. We show that A_n^i is infinite, except for the undecided case $A_6^0 \neq \emptyset$ and empty A_n^i for $i = 0$, $n \leq 5$ and for $i \geq 2$, $n \leq 3$. The set A_n^1 contains $0, 1, \infty$ nonsimplicial points for $n \leq 4$, $n = 5$, $n \geq 6$, respectively. On the other hand, there exists a finite number $t(n)$ such that $t(n)d \in \mathbb{Z}_+(\mathcal{K}_n)$ for any $d \in A_n^0$; we also estimate such scales for classes of points. We construct families of points of A_n^0 and $\mathbb{Z}_+(\mathcal{K}_n)$, especially on a 0-lifting of a simplicial facet, and points $d \in \mathbb{R}_+(\mathcal{K}_n)$ with $d_{i,n} = t$ for $1 \leq i \leq n-1$.

1. Introduction

In this paper we study integral points of cones. Suppose there is a cone C in $\mathbb{R}^n$ that is generated by its extreme rays $e_1, e_2, ..., e_m$, all $e_i \in \mathbb{Z}^n$.

Let d be a linear combination,

$$d = \sum_{1 \leq i \leq m} \lambda_i e_i. \tag{1}$$

We call the expression a $\mathbb{K}$-realization of d if $\lambda_i \in K$, $1 \leq i \leq m$, and K is either of $\mathbb{R}_+$, $\mathbb{Z}$. $\mathbb{Z}_+$.

If $\lambda_i \geq 0$ for all i, then $d \in C$, and (1) is an R_+-realization of d. If λ_i is an integer for all i, then $d \in L$ where L is a lattice generated by the integral vectors e_i, $1 \leq i \leq m$, and (1) is a $\mathbb{Z}$-realization of d. Obviously $L \subseteq \mathbb{Z}^n$. If $\lambda_i \geq 0$ and is integral for all i, we call the point d an h-point of C. Hence h-points are the points having a Z_+-realization. A

[‡] This work was done during the second author's visit to Laboratoire d'Informatique de l'Ecole Normale Supérieure, Paris

point $d \in C \cap L$ is called a quasi-h-point if it is not an h-point. In other words, d is a quasi-h-point if it has $\mathbb{R}_+$- and $\mathbb{Z}$-realizations but no $\mathbb{Z}_+$-realization.

We consider cut cones, *i.e.* those where e_i are cut vectors. Let $\mathscr{K}_n$ be the set of all nonzero cut vectors of a complete graph on n vertices. Then $\mathbb{R}_+(\mathscr{K}_n)$ is the cut cone. The members of the cut cone $\mathbb{R}_+(\mathscr{K}_n)$ are exactly semimetrics, which are isometrically embedded into some l_1-space, *i.e.* into $\mathbb{R}^n$ with the metric $|x-y|_{l_1}$. Between them, the members of *integer* cut cone $\mathbb{Z}_+(\mathscr{K}_n)$ are exactly semimetric subspaces of some hypercube $\{0,1\}_m$ equipped with the Hamming metric. In particular, the graphic metric $d(G)$ belongs to $\mathbb{Z}_+(\mathscr{K}_n)$, $(1/2)\mathbb{Z}_+(\mathscr{K}_n)$ if and only if G is an isometric subgraph of a cube or of a halved cube, respectively. The above equivalences explain the interest of the cut cones, such as $\mathbb{R}_+(\mathscr{K}_n)$ and $\mathbb{Z}_+(\mathscr{K}_n)$. See [12] for a detailed survey of applications of cut polyhedra. As examples, we recall applications for binary addressing in telecomunication networks, the max-cut problem in Combinatorial Optimization, and the feasibility of multicommodity flows. More specifically, the integer cut cone $\mathbb{Z}_+(\mathscr{K}_n)$ provides some tools for Design Theory (see, for example, [9] and Section 8 below) and for the large subject of embedding graphs in hypercubes.

In fact, those problems are related to feasibility problems of the integer program

$$\{A\lambda = d,\ \lambda \in \mathbb{Z}^m_+\}, \tag{2}$$

where A is the $n \times m$ matrix whose columns are the vectors e_i.

In this paper we attack the integer programming aspects of the cut cones, the main general problem of which is to give a criterion of membership in $\mathbb{Z}_+(\mathscr{K})$, $\mathscr{K} \subseteq \mathscr{K}_n$, for metrics of given class. Examples of possible approaches to it are as follows.

1 Criteria in terms of inequalities and comparisions, as in [3]: $\mathscr{K} = \mathscr{K}_n$, $(n \le 5)$; [10], [13]: $\mathscr{K}$ is a simplex, *i.e.* cuts of $\mathscr{K}$ are linearly independent, $\mathscr{K} = Odd\mathscr{K}_n$.

2 Criteria in terms of enumeration, as in [1] for (1,2)-valued d, or in [15] for $d = d(G)$, where G is a distance-regular graph.

3 A polynomial criterion as in [14] for graphic $d = d(G)$ and other of d.

But in this paper we use other concepts (quasi-h-points and scales), which come from the basic concept of the Hilbert base; see Sections 3 and 4, and 8 and 9 below, respectively.

Finally, we also address adjacent problems on cut lattices (characterization and some arithmetic properties), and on the number of representations of a metric in $\mathbb{Z}_+(\mathscr{K}_n)$.

2. Definitions and notation

Set $V_n = \{1, ..., n\}$, $E_n = \{(i,j) : 1 \le i < j \le n\}$, then $K_n = (V_n, E_n)$ denotes the complete graph on n points. Denote by $P_{(i_1, i_2, ..., i_k)} = P_k$ the path in K_n going through the vertices $i_1, i_2, ..., i_k$.

For $S \subseteq V_n$, $\delta(S) \subseteq E_n$ denote the *cut* defined by S, with $(i,j) \in \delta(S)$ if and only if $|S \cap \{i,j\}| = 1$. Since $\delta(S) = \delta(V_n - S)$, we take S such that $n \notin S$. The incidence vector of the cut $\delta(S)$ is called a *cut vector* and, by abuse of language, is also denoted by $\delta(S)$. Besides, $\delta(S)$ determines a distance function (in fact, a semimetric) $d_{\delta(S)}$ on points of V_n as follows: $d_{\delta(S)}(i,j) = 1$ if $(i,j) \in \delta(S)$, otherwise the distance between i and j is equal to 0. For the sake of simplicity, we set $\delta(\{i,j,k,...\}) = \delta(i,j,k,...)$.

We use $\mathscr{K}_n$ to denote the family of all nonzero cuts $\delta(S)$, $S \subseteq V_n$. For any family $\mathscr{K} \subseteq \mathscr{K}_n$, define the cone $C(\mathscr{K}) := \mathbb{R}_+(\mathscr{K})$ as the conic hull of cuts in $\mathscr{K}$. So, by definition, $\mathscr{K}$ is the set of extreme rays of the cone $C(\mathscr{K})$. The cone $C(\mathscr{K})$ lies in the space $\mathbb{R}(\mathscr{K})$ spanned by the set $\mathscr{K}$. We set $C_n := C(\mathscr{K}_n)$.

So, each point $d \in C(\mathscr{K})$ has a representation $d = \sum_{\delta(S)\in\mathscr{K}} \lambda_S \delta(S)$. Since $\lambda_S \geq 0$, the representation is called the $\mathbb{R}_+$*-realization* of d. The number $\sum_{\delta(S)\in\mathscr{K}} \lambda_S$ is called *the size* of the $\mathbb{R}_+$-realization.

The lattice $L(\mathscr{K}) := \mathbb{Z}(\mathscr{K})$ is the set of all integral linear combinations of cuts in $\mathscr{K}$. Let $L_n = L(\mathscr{K}_n)$. The lattice L_n is easily characterized: $d \in L_n$ if and only if d satisfies the following *condition of evenness*

$$d_{ij} + d_{ik} + d_{jk} \equiv 0 \pmod 2), \text{ for all } 1 \leq i < j < k \leq n. \tag{3}$$

So, $2\mathbb{Z}^{n(n-1)/2} \subset L_n \subset \mathbb{Z}^{n(n-1)/2}$.

The points of $L(\mathscr{K})$ with nonnegative coefficients, *i.e.*, the points of $\mathbb{Z}_+(\mathscr{K})$, are called *h-points*. We denote the set of h-points of the cone $C(\mathscr{K})$ by $hC(\mathscr{K})$. For $d \in \mathbb{Z}_+(\mathscr{K})$, any decomposition of d as a nonnegative integer sum of cuts is called a $\mathbb{Z}_+$*-realization* of d. An h-point of C_n is (seen as a semimetric) exactly isometrically *embeddable into a hypercube* (or *h-embeddable*) semimetric. This explains the name of an h-point.

For $d \in C_n$, define

$$s(d) := \text{ minimum size of } \mathbb{R}_+\text{-realizations of } d,$$

$$z(d) := \text{ minimum size of } \mathbb{Z}_+\text{-realizations of } d \text{ if any.}$$

Let $d(G)$ be the shortest path metric of a graph G. We set

$$z_n^t := z(2td(K_n)).$$

For this special case, $G = K_n$, $s(d) = s(2td(K_n))$ is equal to $a_n^t := \frac{tn(n-1)}{\lfloor n/2\rfloor \lceil n/2\rceil}$.

A point $d \in C(\mathscr{K})$ is called a *quasi-h-point* of $C(\mathscr{K})$ if d belongs to $L(\mathscr{K})$ but has no $\mathbb{Z}_+$-realization. We set

$$A(\mathscr{K}) := C(\mathscr{K}) \cap L(\mathscr{K}) - \mathbb{Z}_+(\mathscr{K}).$$

Recall (see [18]) that a *Hilbert basis* is a set of vectors $e_1, ..., e_k$ with the property that each vector lying in both the lattice and the cone generated by $e_1, ..., e_k$ is a nonnegative integral combination of these vectors. $A(\mathscr{K}) = \emptyset$ would mean that $\mathscr{K}$ is a Hilbert basis of $C(\mathscr{K})$. Actually, $\mathscr{K}$ would be the minimal Hilbert basis of $C(\mathscr{K})$ if it is a Hilbert basis, since $\mathscr{K}$ is the set of extreme rays of $C(\mathscr{K})$ (see [4]).

Define

$$A^i(\mathscr{K}) := \{d \in C(\mathscr{K}) \cap L(\mathscr{K}) : d \text{ has exactly } i\ \mathbb{Z}_+\text{-realizations}\},$$

$$A_n^i := A^i(\mathscr{K}_n).$$

So, the above defined set $A(\mathscr{K})$ is $A^0(\mathscr{K})$. Define

$$\eta^i(d) := \min\{t \in \mathbb{Z}_+ : td \text{ has} > i\ \mathbb{Z}_+\text{-realizations}\}$$

$$= \min\{t \in \mathbb{Z}_+ : td \notin A^k(\mathscr{K}) \text{ for all } 0 \leq k \leq i\}.$$

A cone $C = \mathbb{R}_+(\mathcal{K})$ is said to be *simplicial* if the set $\mathcal{K}$ is linearly independent; a point $d \in C$ is said to be *simplicial* if d lies on a simplicial face of C, *i.e.*, if d admits a unique $\mathbb{R}_+$-realization.

Let $\dim \mathcal{K}$ be the dimension of the space spanned by $\mathcal{K}$. Call $e(\mathcal{K}) := |\mathcal{K}| - \dim \mathcal{K}$, the *excess* of $\mathcal{K}$. Set

$$\mathcal{K}_n^l = \{\delta(S) \in \mathcal{K}_n : |S| = l \text{ or } n - |S| = l\}.$$

For even n we also set

$$Even\mathcal{K}_n = \{\delta(S) \in \mathcal{K}_n : |S|, n - |S| \equiv 0 \ (\text{mod } 2)\},$$

$$Odd\mathcal{K}_n = \{\delta(S) \in \mathcal{K}_n : |S|, n - |S| \equiv 1 \ (\text{mod } 2)\}.$$

For a subset $T \subseteq V_n$ denote

$$EvenT\mathcal{K}_n = \{\delta(S) \in \mathcal{K}_n : |S \cap T| \equiv 0 \ (\text{mod } 2)\},$$

$$OddT\mathcal{K}_n = \{\delta(S) \in \mathcal{K}_n : |S \cap T| \equiv 1 \ (\text{mod } 2)\}.$$

So $Even\mathcal{K}_n = EvenT\mathcal{K}_n$, $Odd\mathcal{K}_n = OddT\mathcal{K}_n$ for $T = V_n$, n even.

Remark that $\mathcal{K}_{2m}^m = \{\delta(S) \in \mathcal{K}_{2m}^m : 1 \notin S\} = \{\delta(S) \in \mathcal{K}_{2m}^m : 1 \in S\}$.

Denote by $\mathcal{K}_n^{i,j}, \mathcal{K}_n^{\neq i}, \mathcal{K}_n^{\not\equiv i (\text{mod } a)}$ the families of $\delta(S) \in \mathcal{K}_n$ with $|S| \in \{i, j, n-i, n-j\}$, $|S| \notin \{i, n-i\}$, $\min\{|S|, n-|S|\} \not\equiv i (\text{mod } a)$, respectively.

We write C_b^a for $C(\mathcal{K}_b^a)$, where a and b are indices or sets of indices.

3. Families of cuts $\mathcal{K}$ with $A(\mathcal{K}) = \emptyset$

Of course $A(\mathcal{K}) = \emptyset$ if $e(\mathcal{K}) = 0$, i.e. if the cone $C(\mathcal{K})$ is simplicial. It is easy to see that $C(\mathcal{K}_n^l)$ is simplicial if and only if either $l = 1$, or $l = 2$, or $(l, n) = (3, 6)$. Also $e(\mathcal{K}_3) = 0$, what is a special case of the formula

$$e(\mathcal{K}_n) = 2^{n-1} - 1 - \binom{n}{2}.$$

Some examples of $\mathcal{K}$ with a positive excess but with $A(\mathcal{K}) = \emptyset$ are:

(a) $\mathcal{K}_4$, $\mathcal{K}_5$ with excess 1 and 5, respectively. The first proof was given in [3]; for details of the proof see [10], where, for any $d \in C_n \cap L_n$, $n = 4, 5$, the explicit $\mathbb{Z}_+$-realization of d is given.

(b) $Odd\mathcal{K}_6$ with the excess 1. For the proof see [10].

(c) (See the case $n = 5$ of Theorem 6.2 below.) The family of cuts (with excess 5) on a facet of $C(\mathcal{K}_6)$ that is a 0-lifting of a simplicial pentagonal facet of $C(\mathcal{K}_5)$.

But $\mathcal{K}_n^{1,2}$ with excess n has $A(\mathcal{K}) \neq \emptyset$ for $n \geq 6$. Below we give some examples of $\mathcal{K}$ with $A(\mathcal{K}) \neq \emptyset$, which are, in a way, close to the above examples of $\mathcal{K}$ with $A(\mathcal{K}) = \emptyset$.

We denote by $Q(b)$ the linear form $\sum_{1 \leq i < j \leq n} b_i b_j x_{ij}$ for $b \in \mathbb{Z}^n$. If $\sum_{i=1}^n b_i = 1$, the inequality $Q(b) \leq 0$ is called a *hypermetric inequality*. We call $d \in \mathbb{R}^{n(n-1)/2}$ a *hypermetric* if it satisfies all the hypermetric inequalities. We denote the hypermetric inequality by $Hyp_n(b)$. It is easy to verify that $\delta(S)$ satisfies all hypermetric inequalities. Moreover, for large classes of parameters b (see [4], [6]) $Hyp_n(b)$ is a facet of $C(\mathcal{K}_n)$. The only known case when a hypermetric face is simplicial is (up to permutation) $Hyp_n(1^2, -1^{n-3}, n-4)$, $n \geq 3$, and (its 'switching' in terms of [6]) $Hyp_n(-1, 1^{n-2}, -(n-4))$. Call the facet $Hyp_n(1^2, -1^{n-3}, n-4)$ the *main n-facet*. Call the facet $Hyp_n(1^2, 0^k, -1^{n-k-3}, n-k-4)$ the *k-fold 0-lifting* of the main (n-k)-facet. It is a facet of $C(\mathcal{K}_n)$, because every k-fold 0-lifting of a facet of C_{n-k} is a facet of C_n (see [4]). We call 1-fold 0-lifting simply 0-lifting. We list, up to a permutation, all facets of $C(\mathcal{K}_n)$ for $3 \leq n \leq 6$:

- The unique type of facets of $C(\mathcal{K}_3)$ is the main 3-facet (triangle inequality);
- The unique type of facets of $C(\mathcal{K}_4)$ is the main 4-facet (which is the 0-lifting $Hyp_4(-1, 1^2, 0)$ of a main 3-facet);
- All facets of $C(\mathcal{K}_5)$ are 2-fold 0-liftings of a main 3-facet (i.e. 0-lifting of a main 4-facet), and the main 5-facet $Hyp_5(1^3, -1^2)$, called the *pentagonal* facet;
- All facets of $C(\mathcal{K}_6)$ are: 2-fold 0-liftings of a main 4-facet, 0-lifting of a main 5-facet, the main 6-facet $Hyp_6(2, 1, 1, -1^3)$ and its 'switching' $Hyp_6(-2, -1, 1^4)$.

Lemma 3.1. *If $\mathcal{K}$ is a family of cuts $\delta(S)$, $|S| \leq (n/2)$, lying on a face F of C_n, the family*

$$\mathcal{K}' = \mathcal{K} \cup \{\delta(\{n+1\})\} \cup \{\delta(S \cup \{n+1\}) : \delta(S) \in \mathcal{K}\}$$

is the family of cuts lying on a 0-lifting of the face F. If, for the above $\mathcal{K}$, $C(\mathcal{K})$ is a simplicial facet of C_n, we obtain, for $n \geq 4$,

$$e(\mathcal{K}') = n(n-3)/2.$$

Proof. If $C(\mathcal{K})$ is a simplicial facet of C_n, then dim $\mathcal{K} = |\mathcal{K}| = \binom{n}{2} - 1$. Obviously, $|\mathcal{K}'| = 2|\mathcal{K}| + 1$. Since $\mathcal{K}'$ is a simplicial facet of C_{n+1}, we have, dim $\mathcal{K}' = \binom{n+1}{2} - 1$ also. Hence

$$\begin{aligned} e(\mathcal{K}') &= |\mathcal{K}'| - \dim \mathcal{K}' \\ &= (2|\mathcal{K}| + 1) - \dim \mathcal{K}' \\ &= 2(\binom{n}{2} - 1) + 1 - (\binom{n+1}{2} - 1) \\ &= n(n-3)/2. \end{aligned}$$

□

Recall that $A(\mathcal{K}) = \emptyset$ for $\mathcal{K} = \mathcal{K}_5, \mathcal{K}_6^1, \mathcal{K}_6^2, \mathcal{K}_6^3, \mathcal{K}_6^{1,3} = Odd\mathcal{K}_6$, and for the family of any (except triangle) facet of $\mathcal{K}_6$, since $\mathcal{K}_6^i$ is simplicial for $i = 1, 2, 3$, and $\mathcal{K}_5$, $Odd\mathcal{K}_6$ are examples given at the beginning of this section.

4. Antipodal extension

A fruitful method of obtaining quasi-h-points is the *antipodal extension operation* at the point n. For $d \in \mathbb{R}^{n(n-1)/2}$ we define $ant_\alpha d \in \mathbb{R}^{n(n+1)/2}$ by

$$\begin{aligned} (ant_\alpha d)_{ij} &= d_{ij} \text{ for } 1 \leq i < j \leq n, \\ (ant_\alpha d)_{n,n+1} &= \alpha, \\ (ant_\alpha d)_{j,n+1} &= \alpha - d_{jn} \text{ for } 1 \leq j \leq n-1. \end{aligned}$$

For $\mathscr{K} \subseteq \mathscr{K}_n$, define

$$ant\mathscr{K} = \{ant_1\delta(S) : \delta(S) \in \mathscr{K}\} \cup \{\delta(n+1)\}.$$

Note that

$$ant_1\delta(S) = \delta(S) \text{ if } n \in S, \text{ and } ant_1\delta(S) = \delta(S \cup \{n+1\}) \text{ if } n \notin S.$$

Hence

$$ant\mathscr{K} = \{\delta(S) : \delta(S) \in \mathscr{K}, n \in S\} \cup \{\delta(S \cup \{n+1\}) : \delta(S) \in \mathscr{K}, n \notin S\}.$$

Observe that if $d \in C(\mathscr{K})$ and $d = \sum_{\delta(S)\in\mathscr{K}} \lambda_S \delta(S)$, then

$$\begin{aligned} ant_\alpha d &= \sum_{\delta(S)\in\mathscr{K}} \lambda_S ant_\alpha \delta(S) + \alpha(1 - \sum_S \lambda_S)\delta(n+1) \\ &= \sum_{\delta(S)\in\mathscr{K}} \lambda_S ant_1 \delta(S) + (\alpha - \sum_S \lambda_S)\delta(n+1). \end{aligned} \quad (4)$$

Also, if

$$ant_\alpha d = \sum_{\delta(S)\in\mathscr{K}} \lambda_S ant_1\delta(S) + \lambda_0\delta(n+1),$$

then $\alpha = \sum_S \lambda_S + \lambda_0$, and $d = \sum_{\delta(S)\in\mathscr{K}} \lambda_S\delta(S)$ is the projection of $ant_\alpha(d)$ on $\mathbb{R}^{n(n-1)/2}$.

So $ant_\alpha d \in \mathbb{R}(ant\mathscr{K})$ if and only if $d \in \mathbb{R}(\mathscr{K})$.

Note that the cone $\mathbb{R}(ant\mathscr{K})$ is the intersection of the triangle facets $Hyp_{n+1}(1^2, -1_j, 0^{n-2})$, where $b_n = b_{n+1} = 1$, $b_j = -1$ and $b_i = 0$ for $i \neq j$, $1 \leq i \leq n-1$.

Proposition 4.1. (Proposition 2.6 of [8])

(i) *$ant_\alpha d \in L_{n+1}$ if and only if $d \in L_n$ and $\alpha \in \mathbb{Z}$,*

(ii) *$ant_\alpha d \in C_{n+1}$ if and only if $d \in C_n$ and $\alpha \geq s(d)$,*

(iii) *$ant_\alpha d \in hC_{n+1}$ if and only if $d \in hC_n$ and $\alpha \geq z(d)$,*

(iv) *$ant_\alpha d$ is a simplicial point of C_{n+1} if and only if d is a simplicial point of C_n and $\alpha \geq s(d)$.* □

Clearly, $s(ant_\alpha d) = \alpha$ if $ant_\alpha d \in C_{n+1}$ and $z(ant_\alpha d) = \alpha$ if $ant_\alpha d \in hC_{n+1}$. Also, $ant_\alpha d \in A_n^i$ for $i > 0$ if and only if $d \in A_n^i$, $\alpha \in \mathbb{Z}_+$, $\alpha \geq z(d)$.

Proposition 4.1 obviously implies the following important corollary.

Corollary 4.2. *Let $d \in hC_n$, and α be an integer such that $s(d) \leq \alpha < z(d)$. Then $ant_\alpha d \in A(ant\mathscr{K}_n) \subset A_{n+1}^0$, i.e. $ant_\alpha d$ is a quasi-h-point in C_{n+1}.*

5. Spherical t-extension and gate extension

Let $d \in C_{n+1}$. We write $d = (d^0, d^1)$, where

$$d^0 = \{d_{ij} : 1 \leq i < j \leq n\}, \quad d^1 = \{d_{i,n+1} : 1 \leq i \leq n\}.$$

A point $d \in C_{n+1}$ is called the *spherical t-extension*, or simply *t-extension*, of the point $d^0 \in C_n$ if $d = (d^0, d^1)$ and $d^1_{i,n+1} = t$ for all $i \in V_n$. We denote the spherical t-extension of d^0 by $ext_t d^0$.

Let j_n be the n-vector whose components are all equal to 1. Then for the t-extension (d^0, d^1), we have $d^1 = tj_n$.

Proposition 5.1. *$ext_t d$ is a hypermetric if and only if*

(i) *d is a hypermetric,*

(ii) $t \geq (\sum b_i b_j d_{ij})/\Sigma(\Sigma - 1)$

for all integers $b_1, ..., b_n$ with $\Sigma := \sum_1^n b_i > 1$ and g.c.d. $b_i = 1$.

Proof. If $ext_t d$ is hypermetric, then $\sum b_i b_j (ext_t d)_{ij} \leq 0$ for any $b_1, ..., b_n$, $b_{n+1} \in \mathbb{Z}_+$ with $\sum b_i = 1$, *i.e.*

$$\sum_{1 \leq i < j \leq n} b_i b_j d_{ij} + \sum_{1 \leq i \leq n} b_i b_{n+1} t \leq 0.$$

Since $b_{n+1} = 1 - \Sigma$, the second term is equal to $-t\Sigma(\Sigma - 1)$. We obtain (i) if $b_{n+1} = 0$ or 1; otherwise $\Sigma(\Sigma - 1) \neq 0$, and we get (ii). □

Corollary 5.2. *$ext_t d$* is a semimetric if and only if d is a semimetric and $t \geq (1/2)\max_{(ij)} d_{ij}$.

In fact, apply (ii) above to the case $b_i = b_j = 1$, $b_{n+1} = -1$ and $b_k = 0$ for other b's.

As with Proposition 5.1, one can check that $ant_t d$ is a hypermetric (a semimetric) if and only if d is a hypermetric (a semimetric, respectively) and

$$t \geq (\sum_{1 \leq i < j \leq n} b_i b_j d_{ij})/\Sigma(\Sigma - 1) + \sum_1^n b_i d_{in}/\Sigma$$

for any integers $b_1, ..., b_n$ with $\Sigma := \sum_1^n b_i > 1$ and g.c.d. $b_i = 1$

$$(t \geq \frac{1}{2}\max_{1 \leq i < j \leq n-1}(d_{ij} + d_{in} + d_{jn}), \text{respectively}).$$

There is another operation, similar to antipodal extension operation. We call it the *gate extension operation* at the point n (called the *gate*). For $d \in \mathbb{R}^{n(n-1)/2}$, define $gat_\alpha d \in \mathbb{R}^{n(n-1)/2}$ by

$$(gat_\alpha d)_{ij} = d_{ij} \text{ for } 1 \leq i < j \leq n,$$
$$(gat_\alpha d)_{n,n+1} = \alpha,$$
$$(gat_\alpha d)_{i,n+1} = \alpha + d_{in} \text{ for } 1 \leq i \leq n - 1.$$

The following identity shows that $gat_\alpha d$ is, in a sense, a complement of $ant_\alpha d$:

$$ant_\alpha d + gat_{2t-\alpha} d = 2ext_t d. \tag{5}$$

Recall that we take S in $\delta(S)$ such that $n \notin S$. Hence, for $\mathcal{K} \subseteq \mathcal{K}_n$, we have

$$gat\ \mathcal{K} = \mathcal{K} \cup \{\delta(n+1)\}.$$

Actually, $ant\ \mathcal{K}_n = OddT\mathcal{K}_{n+1}$, $gat\ \mathcal{K}_n = \{\delta(n+1)\} \cup EvenT\mathcal{K}_{n+1}$, for $T = \{n, n+1\}$.

Note that the cone $\mathbb{R}_+(gat\ \mathscr{K})$ is the intersection of the triangle facets $Hyp_{n+1}(1_i, 0^{n-2}, -1, 1_{n+1})$, where $b_i = b_{n+1} = 1,\ b_n = -1,\ b_j = 0$ for $j \neq i,\ 1 \leq j \leq n-1$.

It is clear that any $\mathbb{R}_+$-realization of $gat_\alpha d$ (if it belongs to C_{n+1}) has the form $\sum_S \lambda_S \delta(S) + \alpha\delta(n+1)$ where $n+1 \notin S$, and where the above realization is any $\mathbb{R}_+$-realization of d. So, $gat_\alpha d \in L_{n+1}(C_{n+1}, hC_{n+1}, A^i_{n+1}$, respectively) if and only if $d \in L_n(C_n, hC_n, A^i_n$, respectively) and $\alpha \in \mathbb{Z}(\mathbb{R}_+, \mathbb{Z}_+, \mathbb{Z}$, respectively).

Also, $gat_\alpha d$ is a hypermetric (a metric) if and only if $\alpha \in \mathbb{R}_+$ and d is a hypermetric (a metric, respectively).

Hence if $\alpha \in \mathbb{Z}_+$, we have

$$gat_\alpha d \in A^i_{n+1} \Longleftrightarrow d \in A^i_n. \tag{6}$$

In particular, $gat_\alpha d$ is a quasi-h-point if and only if d is.

The following facts are obvious.

1 If d_i is the t_i-extension of d^0_i, $i = 1, 2$, then $d_1 + d_2$ is the $(t_1 + t_2)$-extension of $d^0_1 + d^0_2$.

2 If d^0 lies in a facet of the cut cone, the t-extension of d^0 lies in the 0-lifting of the facet.

We call a point $d \in C_n$ *even* if all distances d_{ij} are even.

Let $d = \sum_S \lambda_S \delta(S)$ be a $\mathbb{Z}_+$-realization of an h-point d. We call the realization *(0,1)-realization* (*$2\mathbb{Z}_+$-realization*) if all λ_S are equal to 0 or 1 (are even, respectively). We have

Fact. *Let d be an h-point. Then $d = d_1 + d_2$, where d_1 has a (0,1)-realization, and d_2 has a $2Z_+$-realization.*

Obviously, if d has a $2\mathbb{Z}_+$-realization, d is even. But if d is even, it can have no $2\mathbb{Z}_+$-realizations.

The following Proposition is an analog of Proposition 4.1.

Proposition 5.3.

(i) *$ext_t d \in L_{n+1}$ if and only if $d \in 2Z^{n(n-1)/2}$ and $t \in Z$,*

(ii) *$ext_t d \in C_{n+1}$ if $d \in C_n$ and $2t \geq s(d)$,*

(iii) *suppose that d has $2Z_+$-realizations, and let $z_{even}(d)$ denote their minimal size; then $ext_t d \in hC_{n+1}$ if $d \in hC_n$ and $2t \geq z_{even}(d)$.*

Proof. (i) is implied by the trivial equality $d_{i,n+1} + d_{j,n+1} + d_{ij} = 2t + d_{ij},\ 1 \leq i < j \leq n$.

From (5) we have $ext_t d = (1/2)(ant_\alpha d + gat_{2t-\alpha} d)$. Taking $\alpha = s(d)$ and applying (ii) of Proposition 4.1 we get (ii).

Taking $\alpha = z_{even}(d)$, applying (iii) of Proposition 4.1 and using $ant_{++z_{even}(d)}, gat_{2t-z_{even}(d)} d \in 2\mathbb{Z}_+(\mathscr{K}_{n+1})$, we get (iii). □

Define $ext^m_t d = ext_t(ext^{m-1}_t d)$, where $ext^1_t d = ext_t d$.

Proposition 5.4. *If $2t \geq s(d)$, then $ext^m_t d \in C_{n+m}$ for any $m \in Z_+$, and*

$$\max(s(ext^{m-1}_t d),\ 2t - \frac{t}{\lceil m/2 \rceil}) \leq s(ext^m_t d) \leq 2t - 2^{-m}(2t - s(d)).$$

Proof. From Proposition 5.3(ii) we get

$$s(ext_t d) \leq \frac{1}{2} s(ant_{s(d)}d + gat_{2t-s(d)}d) = t + \frac{1}{2}s(d) \leq 2t.$$

By induction on m, we obtain $ext_t^m d \in C_{n+m}$ for all $m \in \mathbb{Z}_+$, and the upper bound for $s(ext_t^m d)$.

The lower bound is implied by the fact that the restriction of $ext_t^m d$ on m extension points is $td(K_m)$. Since $s(td(K_m)) = (1/2)a_m^t$ (see Section 2), we have

$$s(ext_t^m) \geq s(td(K_m)) = \frac{1}{2}\frac{tm(m-1)}{\lfloor m/2 \rfloor \lceil m/2 \rceil} = 2t - \frac{t}{\lceil m/2 \rceil}.$$

□

Remark. So, if $s(d) \leq 2t$, then $\lim_{m\to\infty} s(ext_t^m d) = 2t$.

Probably, there exist $m_0 = m_0(t, d)$ such that $s(ext_t^m d) = 2t$ for $m \geq m_0$.

We conjecture that $ext_t^m d \notin C_{n+m}$ for $m > m_1$ if $s(d) > 2t$. For example, if $t = 1$ and $d = d(G)$ ($d(G)$ is the shortest path metric of the graph G), then it can be proved that $m_1 = 2$.

If the conjecture is true,

$$s(d) = 2\min\{t : ext_t^m d \in C_{n+m} \text{ for all } m \in Z_+\}.$$

Recall, that Proposition 4.1(ii) implies

$$s(d) = \min\{\alpha : ant_\alpha d \in C_{n+1}\}.$$

In terms of $ext_n^m d$ we also have analogs of (i) and (iii) of Proposition 4.1.

Proposition 5.5.

(i) *$ext_t^m d \in L_{n+m}$ for all $m \in \mathbb{Z}_+$ if and only if $d \in 2Z^{n(n-1)/2}$ and t is even.*

(ii) *$ext_t^m d \in hC_{n+m}$ for all $m \in \mathbb{Z}_+$ if and only if t is an even positive integer, and $ext_{t/2}^1 d \in hC_{n+1}$.*

Proof. The evenness of t follows from $ext_t^3 d \in L_{n+3}$. So, (i) is implied by Proposition 5.3(i).

Recall the result of [5] that $t\sum_1^n \delta(i)$ is the unique $\mathbb{Z}_+$-realization of $td(K_n)$ for even t and $m \geq (t^2/4) + (t/2) + 3$. Using this fact, we get that any $\mathbb{Z}_+$-realization of $ext_t^m d$ contains $t/2$ cuts $\delta(i)$ for some i if m is large enough. □

6. Quasi-h-points of 0-lifting of the main facet

Consider the main facet

$$F_0(n) = Hyp_n(1^2, -1^{n-3}, n-4) = Hyp_n(b^0),$$

where $b_1^0 = b_2^0 = 1$, $b_i^0 = -1$, $3 \leq i \leq n-1$, $b_n^0 = n-4$. The cut vectors $\delta(S)$ lying in the facet are defined by equations $b(S) \equiv \sum_{i\in S} b_i = 0$ or 1. We take S not containing n. Then $S \in \mathscr{S}$, where

$$\mathscr{S} = \{\{1\}, \{2\}, \{1i\}, \{2i\}\{12i\}\ (3 \le i \le n-1), \{12ij\}\ (3 \le i < j \le n-1)\}.$$

We set

$$m = |\mathscr{S}| = \frac{n(n-1)}{2} - 1.$$

Every n-facet contains at least m cut vectors. Since the main n-facet contains exactly m cuts, it is simplicial.

The 0-lifting of the main facet is the facet

$$F(n) = Hyp_{n+1}(1^2, -1^{n-3}, n-4, 0).$$

Besides the above cuts $\delta(S), S \in \mathscr{S}$, it contains, according to Lemma 3.1, only the cuts $\delta(S \cup \{n+1\}), S \in \mathscr{S}$, and $\delta(n+1)$.

Note that $A(\mathscr{K}) = \emptyset$ for the main n-facet (as for any simplicial $C(\mathscr{K})$).

Now we consider even points having no $2\mathbb{Z}_+$-realization. The simplest such points are points having a (0,1)-realization. We call these points *even (0,1)-points.*

Let $d^0 \in F_0(n)$ be an even h-point, and let $\sum_{S \in \mathscr{S}_0} \lambda_S \delta(S)$ be one of its $\mathbb{Z}_+$-realizations. Consider a minimal set of comparisions mod 2 that λ_S's have to satisfy. The comparisions are implied by the conditions $d_{ij} \equiv 0$ for all pairs (ij). Since $d^0 \in L_n$, we have $d_{ij} \equiv d_{ik} + d_{jk}$ (mod 2) for all ordered triples (ijk). Hence independent comparisions are implied by the comparisions $d_{in} \equiv 0$ (mod 2), $1 \le i \le n-1$. The comparisions are as follows. (For the sake of simplicity, we set $\lambda_{\{ij\ldots\}} = \lambda_{ij\ldots}$ and omit the indication (mod 2)).

$$\lambda_{1i} + \lambda_{2i} + \lambda_{12i} + \sum_{3 \le j \le n-1, j \ne i} \lambda_{12ij} \equiv 0,\ 3 \le i \le n-1,$$

$$\lambda_1 + \sum_{3 \le i \le n-1} (\lambda_{1i} + \lambda_{12i}) + \sum_{3 \le i < j \le n-1} \lambda_{12ij} \equiv 0, \tag{7}$$

$$\lambda_2 + \sum_{3 \le i \le n-1} (\lambda_{2i} + \lambda_{12i}) + \sum_{3 \le i < j \le n-1} \lambda_{12ij} \equiv 0.$$

The system of comparisions (7) has $n-1$ equations with $m = n(n-1)/2 - 1$ unknowns. Hence the number of (0,1)-solutions distinct from the trivial zero solution is equal to $2^{m-(n-1)} - 1 = 2^{\binom{n-1}{2}-1} - 1$.

This shows that all points of $F_0(3)$ have $2\mathbb{Z}_+$-realizations. The only even (0,1)-points of $F_0(4)$ are 2 points $2d(K_3)$ with $d_{13} = 0$ or $d_{23} = 0$, and the point $2d(K_4 - P_{(1,2)})$. There are 31 even (0,1)-points in $F_0(5)$.

Since there are exponentially many even (0,1)-points in $F_0(n)$, we consider points of the following type and call them *special.*

For these points the coefficients λ_S are

$$\lambda_1 = a_1,\ \lambda_2 = a_2,\ \lambda_{1i} = b_1,\ \lambda_{2i} = b_2,\ \lambda_{12i} = c_1,\ 3 \le i \le n-1,$$

$$\lambda_{12ij} = c_2,\ 3 \le i < j \le n-1.$$

Here $a_i, b_i, c_i,\ i = 1, 2$, are equal to 0 or 1.

If we set

$$k = n - 3, \; l = \frac{n(n-1)}{2},$$

then for the special points, (7) takes the form

$$b_1 + b_2 + c_1 + (k-1)c_2 \equiv 0,$$

$$a_1 + k(b_1 + c_1) + \frac{k(k-1)}{2} c_2 \equiv 0,$$

$$a_2 + k(b_2 + c_1) + \frac{k(k-1)}{2} c_2 \equiv 0.$$

Since we have 3 equations for 6 variables, we can express 3 variables a_1, a_2, c_1 through the other 3 variables b_1, b_2, c_2.

There are 4 families of the solutions of the system depending on the value of k (mod 4). The solutions are as follows (undefined equivalences are taken by (mod 2)).

$$k \equiv 0 \text{ (mod 4)}, \; a_1 = a_2 = 0, \; c_1 \equiv b_1 + b_2 + c_2,$$

$$k \equiv 1 \text{ (mod 4)}, \; a_1 = b_2, \; a_2 = b_1, c_1 \equiv b_1 + b_2, \; c_2 \text{ arbitrary},$$

$$k \equiv 2 \text{ (mod 2)}, \; a_1 = a_2 = c_2, \; c_1 \equiv b_1 + b_2 + c_2,$$

$$k \equiv 3 \text{ (mod 4)}, \; a_1 \equiv b_2 + c_2, \; a_2 \equiv b_1 + c_2, \; c_1 \equiv b_1 + b_2.$$

In each case we obtain 7 nontrivial special even (0,1)-points.

Turning our attention to the definition of $\mathcal{S}$, for $a = 0, \pm$, we denote by λ^a_{ik}, λ^a_k the k-vectors with the components λ^a_{ij}, $3 \le j \le n-1$, $i = 1, 2$, λ^a_{12j}, $3 \le j \le n-1$, respectively. Similarly, λ^a_l is the l-vector with the components λ^a_{12ij}, $3 \le i < j \le n-1$.

In this notation a special point d^0 has a (0,1)-realization λ^0 such that $\lambda^0_i = a_i$, $\lambda^0_{ik} = b_i j_k$, $i = 1, 2$, $\lambda^0_k = c_1 j_k$ and $\lambda^0_l = c_2 j_l$.

Recall that special points are simplicial. Therefore their size is equal to $\sum_{S \in \mathcal{S}} \lambda_S$. We show below that the t-extension of 2 special points with $(a_1, a_2, b_1, b_2, c_1, c_2) = (1, 1, 0, 0, 0, 1)$ and $(0, 1, 0, 1, 1, 1)$ are quasi-h-points for $n \equiv 2$ (mod 4).

For $n = 6$ the points d^0 are $d(K_6 - P_3)$ and $ant_{10}(ext_4 d(K_4))$. Another example of $d \in A^0_7$ is $ant_6(ext_3 d(K_5)) = d^{5,3}$ in terms of Corollary 6.6 below.

Proposition 6.1. *Let d^0 be one of the 7 special points of the main facet $F_0(n)$. Let t be a positive integer such that $t \ge (1/2) \sum_{S \in \mathcal{S}} \lambda^0_S$. Then the t-extension of d^0 is an h-point if $n \not\equiv 2$ (mod 4), and if $n \equiv 2$ (mod 4), then there is a point d^0 such that its t-extension is a quasi-h-point, namely the point with $(a_1, a_2, b_1, b_2, c_1, c_2) = (1, 1, 0, 0, 0, 1)$.*

Proof. Recall that we can take $\mathcal{S}$ such that $n \notin S$ for all $S \in \mathcal{S}$.

We apply equation (2) to the t-extension d. In this case the matrix A takes the form

$$A = \begin{pmatrix} B & B & 0 \\ D & \overline{D} & j_n \end{pmatrix}$$

Here the first m columns correspond to sets $S \in \mathcal{S}$, the next m columns correspond to sets $S \cup \{n+1\}$, $S \in \mathcal{S}$, and the last $(2m+1)$th column corresponds to $\{n+1\}$. The size of the matrix B is $\binom{n}{2} \times m$, and $D, \overline{D}$ are $n \times m$ matrices such that $D + \overline{D} = J$, where J is the matrix all of whose elements are equal to 1. Each column of the matrix J is the vector j_n consisting of n 1's. In this notation, we can write J as the direct product $J = j_n \times j_m^T$. Hence for any m-vector a we have $Ja = (j_m, a)j_n$.

The rows of D and $\overline{D}$ are indexed by pairs $(i, n+1)$, $1 \le i \le n$. The S-column of the matrix D is the (0,1)-indicator vector of the set S. Since $n \notin S$ for all $S \in \mathcal{S}$, the last row of D consists of 0's only.

We look for solutions of the system (2) for this matrix A such that λ is a nonnegative integral (2m+1)-vector. We set

$$\mu_S = \lambda_{S\cup\{n+1\}}, \ S \in \mathcal{S}, \ \gamma = \lambda_{\{n+1\}}.$$

Then the system (2) takes the form

$$B(\lambda + \mu) = d^0,$$

$$D(\lambda - \mu) + (\gamma + (j_m, \mu))j_n = d^1.$$

Now, if we set $\lambda^+ = \lambda + \mu$, $\lambda^- = \lambda - \mu$, $\gamma_1 = \gamma + (j_m, \mu)$, and recall that $d^1 = tj_n$, we obtain the equations

$$B\lambda^+ = d^0,$$

$$D\lambda^- + \gamma_1 j_n = tj_n. \tag{8}$$

Recall that the last row of D is the 0-row. Hence the last equation of the system (8) gives $\gamma_1 = t$, and the equation (8) takes the form

$$D\lambda^- = 0.$$

A solution $(\lambda^+, \lambda^-, \gamma_1)$ is feasible if the vector (λ, μ, γ) is nonnegative. Since

$$\lambda = \frac{1}{2}(\lambda^+ + \lambda^-), \ \mu = \frac{1}{2}(\lambda^+ - \lambda^-), \text{ and } \gamma = t - (j_m, \mu),$$

a solution $(\lambda^+, \lambda^-, \gamma_1)$ is feasible if

$$\lambda^+ \ge 0, \ |\lambda^-| \le \lambda^+, \text{ and } t \ge (j_m, \mu). \tag{9}$$

Since the main facet $F_0(n)$ is simplicial, the system $B\lambda^+ = d^0$ has the full rank m such that $\lambda^+ = \lambda^0$ is the unique solution.

We try to find an integral solution for λ^-. By (9), we have that $|\lambda^-| \le \lambda^0$. This implies that $\lambda_S^- \neq 0$ only for sets S where $\lambda_S^0 \neq 0$. Since λ^0 is a (0,1)-vector, an integral λ_S^- takes the value 0 and ± 1 only.

We write the matrix $(D, j_n) \equiv D_n$ explicitly:

$$D_n = \begin{pmatrix} 1 & 0 & j_k^T & 0 & j_k^T & j_l^T & 1 \\ 0 & 1 & 0 & j_k^T & j_k^T & j_l^T & 1 \\ 0 & 0 & I_k & I_k & I_k & G_k & j_k \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The first, the second and the last rows of the matrix D_n are indexed by the pairs $(1, n+1), (2, n+1)$ and $(n, n+1)$, respectively. The third row consists of matrices with k rows corresponding to the pairs $(i, n+1)$ with $3 \le i \le n-1$. The columns of D_n are indexed by sets $S \in \mathscr{S}_0 \cup \{n+1\}$ in the sequence $\{1\}, \{2\}, \{1i\}, \{2i\}, \{12i\}$, $3 \le i \le n-1$, $\{12ij\}$, $3 \le i < j \le n-1$, $\{n+1\}$. I_k is the $k \times k$ unit matrix, and G_k is the $k \times l$ incidence matrix of the complete graph K_k. G_k contains exactly two 1's in each column, *i.e.* $j_k^T G_k = 2j_l^T$. The matrix $D_{n'}$ is an obvious submatrix of D_n, for $n' < n$.

In the above notation, the equation $D\lambda^- = 0$ takes the form

$$\lambda_i^- + j_k^T(\lambda_{ik}^- + \lambda_k^-) + j_l^T \lambda_l^- = 0, \ i = 1, 2,$$

$$\lambda_{1k}^- + \lambda_{2k}^- + \lambda_k^- + G_k \lambda_l^- = 0.$$

Since $j_k^T G_k = 2j_l^T$, the last equality implies that

$$j_k^T(\lambda_{1k}^- + \lambda_{2k}^- + \lambda_k^-) + 2j_l^T \lambda_l^- = 0.$$

Hence the above system implies

$$\lambda_1^- + \lambda_2^- + j_k^T \lambda_k^- = 0.$$

Recall that we look for a $(0, \pm 1)$-solution. Note that if $\lambda_S^+ = 1$ and $\lambda_S^- = 0$, then $\lambda_S = \mu_S = 1/2$ is nonintegral. Hence we shall look for a solution such that $\lambda_S^- = \pm\lambda_S^0$. So, such a solution is nonzero where λ_S^0 is nonzero.

The main part of the above equations is contained in the term $G_k \lambda_l^-$. We can treat the (± 1)-variables $(\lambda^-)_{ij} \equiv \lambda_{12ij}^-$ as labels of edges of the complete graph K_n. Now the problem is reduced to finding such a labelling of edges of K_n that the sum of labels of edges incident to a given vertex is equal to a prescribed value, usually equal to 0 or ± 1. The existence of such a solution depends on a possibility of factorization of K_n into circuits and 1-factors.

Corresponding facts can be found in [16, Theorems 9.6 and 9.7].

A tedious inspection shows that a feasible labelling exists for each of the 7 special points if $n \not\equiv 2 \pmod 4$ (*i.e.* if $k \not\equiv 3 \pmod 4$), and for 5 special points if $n \equiv 2 \pmod 4$. For the other point with $(a_1, a_2, b_1, b_2, c_1, c_2) = (1, 1, 0, 0, 0, 1)$ there is no feasible solution, *i.e.* there are S such that $\lambda_S^- = 0 \neq \pm\lambda_S^0$.

Now the assertion of the proposition follows. □

In the table below, t-extensions of some special points are given explicitly. The last column of the table gives a point of A_{4m-1}^0 for any $m \ge 2$.

$n \pmod 4) \equiv$	3	0	1	2
d_{12}	$n-3$	0	$n-1$	2
d_{1i} $(3 \le i \le n-1)$	$\binom{n-4}{2}+1$	$\binom{n-4}{2}$	$\binom{n-4}{2}+2$	$\binom{n-4}{2}+1$
d_{2i} $(3 \le i \le n-1)$	$\binom{n-3}{2}$	$\binom{n-4}{2}$	$\binom{n-3}{2}+1$	$\binom{n-4}{2}+1$
$d_{ij} (i \neq j)$ $(3 \le i,j \le n-1)$	$2(n-4)$	$2(n-5)$	$2(n-4)$	$2(n-5)$
d_{1n}	$\binom{n-3}{2}$	$\binom{n-3}{2}$	$\binom{n-3}{2}+1$	$\binom{n-3}{2}+1$
d_{2n}	$\binom{n-2}{2}$	$\binom{n-3}{2}$	$\binom{n-2}{2}+1$	$\binom{n-3}{2}+1$
d_{in} $(3 \le i \le n-1)$	$n-3$	$n-4$	$n-3$	$n-4$
$d_{in+1} (i \neq n+1)$	$\binom{n-2}{2}/2$	$\binom{n-3}{2}/2$	$(\binom{n-2}{2}+3)/2$	$(\binom{n-3}{2}+3)/2$

Remarks.

(a) For the smallest possible $n \equiv 2 \pmod 4$, and $n \ge 6$, (*i.e.*, for $n = 6$) distance d is the 3-extension of $d_6 = 2d(K_6 - P_{(1,6,2)})$, corresponding to the special point (1,1,0,0,0,1). On the other hand, the 3-extension of $2d(K_5 - P_{(1,2,5)})$ by the point 6 is an h-point.
For $n \equiv 0$ and $n \equiv 3 \pmod 4$ this d is an antipodal extension at the point 2, *i.e.*, $d_{in} + d_{2i} = d_{2n}$ for all i.

(b) If we consider λ_l^0 such that $\lambda_{12ij}^0 = 0$ or 1, the problem is reduced to a factorization of the graph whose edges are pairs (ij) such that $\lambda_{12ij}^0 \neq 0$.

(c) In fact, we can take t slightly smaller. By (9), we must have $t \ge (j_m, \mu)$. Let r be the number of $S \in \mathscr{S}_0$ such that $\lambda_S = 1$. Then $(j_m, \mu) \le (1/2)(\sum_{S\in\mathscr{S}_0} \lambda_S^0 - r)$.

Proposition 6.2. *Let $\mathscr{K}$ be the family of cuts lying on the 0-lifting $F(n)$ of the main facet $F_0(n)$. Then $A(\mathscr{K}) = \emptyset$ if and only if $n \le 5$.*

Proof. By Lemma 6.1, $F(6)$ has quasi-h-points, and (6) implies that quasi-h-points exist in all $F(n)$ for $n > 6$. We prove that there is no quasi-h-point on $F(n)$ for $n \le 5$.

We use the above notation and the equations $B(\lambda + \mu) = d^0$, $D_n(\lambda - \mu) + \gamma_1 j_n = d^1$. The first equation has the unique solution $\lambda + \mu = \lambda^0$. Hence $2D_n\lambda - D_n\lambda^0 + \gamma_1 j_n = d^1$, where $\gamma_1 = \gamma + (j_{m_0}, \lambda^0) - (j_{m_0}, \lambda)$. The last row gives $\gamma_1 = d_{n,n+1}$. Hence the ith row of the equation with D_n takes the form

$$(D_n\lambda)_i = \frac{1}{2}((D_n\lambda^0)_i + d_{i,n+1} - d_{n,n+1}).$$

It can be shown that the condition of evenness (3) implies that the right-hand side is an integer for $n \le 5$. Moreover, for $n \le 5$, the matrix D_n is unimodular, *i.e.*, $|\det D'| \le 1$ for each $n \times n$ submatrix D' of D_n. Therefore any solution λ is an integer. This implies that μ and $\gamma = d_{n,n+1} - (j_{m_0}, \mu)$ are integers, too.

So, all points $d \in L_{n+1} \cap F(n)$ have a $\mathbb{Z}_+$-realization (λ, μ, γ) for $n \le 5$. □

We now give some other examples of $\mathbb{Z}_+$-realizations of t-extensions of even h-points.

Using the fact that $\sum_{i\in V_n}\delta(i)$ is the unique $\mathbb{Z}_+$-realization of $2d(K_n)$ for $n\neq 4$, (see [5]), we obtain the following lemma.

Lemma 6.3. *The only* $\mathbb{Z}_+$*-realizations of* $ext_t(2d(K_n))$, $n\geq 5$, $t\in\mathbb{Z}_+$, *are*

(1) $$\sum_{i\in V_n}\delta(i)+(t-1)\delta(n+1) \text{ for } t\geq 1,$$

(1′) $$\sum_{i\in V_n}\delta(i,n+1)+(t-n+1)\delta(n+1) \text{ for } t\geq n-1.$$

Proof. Note that $d^0=2d(K_n)$ is an even (0,1)-point of C_n. The coefficients of its (0,1)-realization λ^0 are as follows: $\lambda^0_S=1$ if $S=\{i\}$, $1\leq i\leq n-1$, or $S=V_{n-1}$, and $\lambda^0_S=0$ for other S. (Recall that we use S such that $n\notin S$.) Since it is a unique $\mathbb{Z}_+$-realization of d^0, the equation $B\lambda^+=d^0$ has the unique integral solution $\lambda^+=\lambda^0$.

The submatrix of D consisting of columns corresponding to S with $\lambda^+_S\neq 0$, and without the last zero row, has the form $D=(I_{n-1},j_{n-1})$. Hence the unique (±1)-solutions of the equation $D\lambda^-=0$ are as follows:

(1) $\lambda^-_i=1$, $1\leq i\leq n-1$, $\lambda^-_{V_{n-1}}=-1$, and
(2) $\lambda^-_i=-1$, $1\leq i\leq n-1$, $\lambda^-_{V_{n-1}}=1$.

Since $(j_m,\mu)=1$ in the first case, and $(j_m,\mu)=n-1$, in the second, we have $\gamma=t-1$, and $\gamma=t-n+1$, respectively. These solutions give the above $\mathbb{Z}_+$-realizations (1) and (1′). □

If we define $d^{n,t}=ant_{2t}ext_t(2d(K_{n-1}))$, we obtain

$$d^{n,t}_{ij}=2,\ 1\leq i<j\leq n-1,\ d_{i,n}=d_{i,n+1}=t,\ 1\leq i\leq n,\ d_{n,n+1}=2t.$$

If we apply (4) to (1) and (1′) of Lemma 6.3 (where n is interchanged with $n-1$), we obtain (2), and (2) with n and $n+1$ interchanged, of Lemma 6.4 below. Summing these two expressions, we obtain the symmetric expression (3) of that lemma.

Lemma 6.4. *For* $d^{n,t}$ *the following holds*

(2) $$d^{n,t}=\sum_{i\in V_{n-1}}\delta(i,n+1)+(t-1)\delta(n)+(t-n+2)\delta(n+1),$$

(3) $$2d^{n,t}=\sum_{i\in V_{n-1}}(\delta(i,n)+\delta(i,n+1))+(2t-n+1)(\delta(n)+\delta(n+1)).$$

Lemma 6.5. *For* $n\geq 6$, $d^{n,t}$ *is h-embeddable if and only if* $t\geq n-2$. *Moreover, for* $t\geq n-2$, *the only* $\mathbb{Z}_+$*-realizations are (2) and its image under the transposition (n,n+1).*

Proof. In fact, if we use Lemma 6.4, the restrictions of an h-embedding of $d^{n,t}$ onto $V_{n+1}-\{n\}$ and V_n has to be of the form (1) and (1′) or (1′) and (1). □

The realizations (2) and (3) of Lemma 6.4 imply

Corollary 6.6. *$d^{n,t}$ is a quasi-h-point of C_n and $(antC_n) \cap C^{1,2}_{n+1}$ having the scale 2 if $\lceil (n-1)/2 \rceil \le t \le n-3$, $n \ge 5$.*

In fact, for $n = 7$ we only have to prove that $2d(K_6 - P_{(5,6)})$ is a quasi-h-point of scale 2, and this will be done in Section 7. For $n \ge 8$ we use (2), (3) and Lemma 6.4.

Remark. $d^{n-1,2} = 2d(K_n - P_2)$ and it is a quasi-h-point for $n \ge 6$. Its scale lies in the segment $[\lceil n/4 \rceil, n/2)$. $d^{n-1,2} \in Z(ant\mathscr{K}_{n-1} \cap \mathscr{K}^{1,2}_n)$ (see Remark (c) following Lemma 7.1 below) for $n \ge 6$, but $d^{n-1,2} \in R_+(ant\mathscr{K}_{n-1} \cap \mathscr{K}^{1,2}_n)$ only for $n = 6$.

The cone $(antC_{n-1}) \cap C^{1,2}_n$ has excess 1. It has $2n-2$ cuts $\delta(i, n-1), \delta(i, n), \delta(n-1), \delta(n)$, for $i \in V_{n-2}$, its dimension is $2n-3$, and there is the following unique linear dependency:

$$\sum_{i \in V_{n-2}} \delta(i, n-1) + (n+4)\delta(n) = \sum_{i \in V_{n-2}} \delta(i, n) + (n-4)\delta(n-1).$$

The two sides of this equation differ only by the transposition $(n-1, n)$.

The number of quasi-h-points in $(antC_{n-1}) \cap C^{1,2}_n$ is 0 for $n = 5$ (since it is so for the larger cone C_5) and $\ge n - 2 - \lceil n/2 \rceil = \lfloor n/2 \rfloor - 2$, which is implied by Corollary 6.6. Perhaps, it is exactly 1 for $n = 6, 7$.

7. Cones on 6 points

Consider the following cones generated by cut vectors on 6 points:

$$C_6,\ C^1_6,\ C^2_6 = EvenC_6,\ C^3_6,\ C^{1,2}_6,\ C^{1,3}_6 = OddC_6,\ C^{2,3}_6,\ antC_5.$$

Recall (see Section 3) that the facets of C_6 are, up to permutations of V_6, as follows:

(a) 3-fold 0-lifting of the main 3-facet, 3-gonal facet $Hyp_6(1^2, -1, 0^3)$,

(b) 0-lifting of the main 5-facet, 5-gonal facet $Hyp_6(1^3, -1^2, 0)$,

(c) the main 6-facet and its 'switching' (7-gonal simplicial facets) $Hyp_6(2, 1^2, -1^3)$ and $Hyp_6(-2, -1, 1^4)$.

Let

$$d_6 := 2d(K_6 - P_{(5,6)}).$$

Recall that (up to permutations) d_6 is the only known quasi-h-point of C_6.

The following lemma is useful for what follows. It can be checked by inspection. Recall that $V_n = \{1, 2, ..., n\}$.

Lemma 7.1.

(1) *All $\mathbb{Z}_+$-realizations of $2d_6$ are*

(1*a*)
$$2d_6 = \sum_{i \in V_4} (\delta(i,5) + \delta(i,6)) \in \mathbb{Z}_+(\mathscr{K}^2_6) = \mathbb{Z}_+(Even\mathscr{K}_6),$$

(1*b*)
$$2d_6 = (\delta(5) + \delta(6)) + \sum_{i \in V_3} (\delta(i,4,5) + \delta(i,4,6)) \in$$
$$\mathbb{Z}_+(\mathscr{K}^{1,3}_6) = \mathbb{Z}_+(Odd\mathscr{K}_6),$$

(1c) $$2d_6 = \delta(5)+\delta(j,5)+\sum_{i\in V_4-\{j\}}(\delta(i,j,6)+\delta(i,6)) \text{ for } j \in V_4.$$

(2) *Some representations of* $d_6 = 2d(K_6 - P_{(5,6)})$ *in* L_6 *are*

(2a) $$d_6 = \delta(5)+\sum_{i\in V_4}\delta(i,6)-\delta(6) \in L_6^{1,2},$$

(2b) $$d_6 = 2\delta(5)+2\delta(6)+\sum_{i\in V_4}\delta(i)-\delta(5,6) \in L_6^{1,2},$$

(2c) $$d_6 = \sum_{i\in V_4}\delta(V_4-\{i\})-\delta(5,6)-\sum_{i\in V_4}(\delta(i,i+1,6)-\delta(i,i+1)) \in L_6^{2,3}.$$

Here $i+1$ *is taken by mod 4.*

Remarks.

(a) The projection of 2(a) onto $V_6-\{1\}$ gives the $\mathbb{Z}_+$-realization $2d(K_5 - P_{(5,6)}) = \delta(5)+\sum_{i=2,3,4}\delta(i,6)$; it and its permutation by the transposition (5,6) are the only $\mathbb{Z}_+$-realizations of the above h-point.

(b) 'Small' pertubations of d_6 do not produce other quasi-h-points. For example, one can check that

$$d_6+\delta(1,2) = \delta(1)+\delta(2)+\delta(6)+\delta(1,2,5)+\delta(3,5)+\delta(4,5);$$

it and its permutation by the transposition (5,6) are the only $\mathbb{Z}_+$-realizations of this h-point.

(c) Actually, 2(a) is the case $n=5, \alpha=4$ of

$$\begin{aligned} ant_\alpha(2d(K_n)) &= \delta(n)+\sum_{i\in V_{n-1}}\delta(i,n+1)-(n-\alpha)\delta(n+1) \\ &= \sum_{i\in V_{n+1}}\delta(\{i\})+(\frac{\alpha}{2}-1)(\delta(\{n\})+\delta(\{n+1\})-\delta(\{n,n+1\})). \end{aligned}$$

(d) One can check that $L_n^{\neq 1} \subset L_n$ strictly, and $2\mathbb{Z}^{15} \subset L_6^{\neq 1}$ strictly. Note that $L_6^{2,3} = L_6^{\neq 1}$. On the other hand, $L_n^{i,j} = L_n$ if and only if $(i,j) = (1,2)$.

(e) By 1(a) and 1(b) of Lemma 7.1 we have

$$2d_6 \in hC_6^2 \text{ and } 2d_6 \in hC_6^{1,3},$$

$$\text{but } 2d_6 \notin L_6^2 \cup L_6^{1,3} = L(Even\mathcal{K}_6) \cup L(Odd\mathcal{K}_6).$$

We call a subcone of C_n a *cut subcone* if its extreme rays are cuts.

Lemma 7.2. *Let* $d \in A(\mathcal{K})$ *and let* $\mathcal{K}(d)$ *be the set of cuts of a minimal cut subcone of* C_n *containing* d. *Then*

(i) $d \in A(\mathcal{K}')$ *for any* $\mathcal{K}'$ *such that* $\mathcal{K}(d) \subseteq \mathcal{K}' \subseteq \mathcal{K}$,

(ii) $e(\mathcal{K}') = 1$ *implies* $\mathcal{K}' = \mathcal{K}(d)$.

Proof. In fact, $d \notin \mathbb{Z}_+(\mathcal{K}(d))$ implies $d \notin \mathbb{Z}_+(\mathcal{K}')$, and $d \in \mathbb{Z}(\mathcal{K}(d)) \cap C(\mathcal{K}(d))$ implies $d \in \mathbb{Z}(\mathcal{K}') \cap C(\mathcal{K}')$, and (i) follows. If $e(\mathcal{K}') = 1$, any proper cut subcone of $C(\mathcal{K})$ is simplicial and has no quasi-h-points. □

Now we remark that the cone $C_6^{1,2} \cap ant\ C_5$ has excess 1, since it has dimension 9 and contains 10 cuts $\delta(5), \delta(6), \delta(i,5), \delta(i,6)$, $1 \le i \le 4$, with the unique linear dependency

$$\sum_{i \in V_n} (\delta(i,5) - \delta(i,6)) = 2(\delta(5) - \delta(6)).$$

Proposition 7.3. *$d_6 = 2d(K_6 - P_2) \in A(\mathcal{K}_6)$ and it is a quasi-h-point of the following proper subcones of C_6: $C_6^{1,2}$, $C_6^{2,3}$, ant C_5, the triangle facet $Hyp(1^2, -1, 0^3)$ and $C_6^{1,2} \cap ant\ C_5$ (which is a minimal cut subcone of C_6 containing d).*

Proof. The point d_6, is the antipodal extension $ant_4(d_5)$ of the point $d_5 := 2d(K_5)$. The minimum size of $\mathbb{Z}_+$-realizations of d_5 is equal to $z(d_5) = z_5^1 = 5$, since its only $\mathbb{Z}_+$-realization is the following decomposition $2d(K_5) = \sum_{i=1}^5 \delta(i)$.

The minimum size of $\mathbb{R}_+$-realizations of d_5 is $s(d_5) = a_5^1 = 10/3$, which is given by the $\mathbb{R}_+$-realization $d_5 = (1/3)/\sum_{1 \le i < j \le 5} \delta(ij)$.

Since $10/3 < 4 < 5$, we deduce that $d_6 = 2d(K_6 - P_{\{5,6\}}) \notin \mathbb{Z}_+(C_6)$.

But $d_6 \in C_6 \cap L_6$, from (1) and (2) of Lemma 7.1. So, $d_6 \in A_6^0$. Now, from 1(a) and (2) of the same lemma, we have $d_6 \in C(\mathcal{K}_6^{1,2} \cap ant\ \mathcal{K}_5) \cap L(\mathcal{K}_6^{1,2} \cap ant\ \mathcal{K}_5)$, and so, using (ii) of Lemma 7.2, we get that $\mathcal{K}_6^{1,2} \cap ant\ \mathcal{K}_5$ is a minimal subcone $\mathcal{K}(d)$.

Using (i) of Lemma 7.2, and the fact that *ant* C_5 is the intersection of some triangular facets, we get the assertion of Proposition 7.3 for $C_6^{1,2}$, *ant* C_5 and the triangle facet. Finaly, 1(a) and 2(c) of Lemma 7.1 imply that $d_6 \in A(\mathcal{K}_6^{2,3})$. □

Remarks.

(a) On the other hand, the following subcones $C(\mathcal{K})$ of C_6 have $A(\mathcal{K}) = \emptyset$: 5 simplicial cones C_6^i, $i = 1,2,3$, both 7-gonal facets, and nonsimplicial cones: C_5, $C_6^{1,3} = OddC_6$, and 5-gonal facet.

(b) Nonsimplicial cones $C_6, C_6^{1,2}, C_6^{2,3}, C_6^{1,3}$, $C_5, ant\ C_5$, $Hyp_6(1^2, -1, 0^3)$, $Hyp_6(1^3, -1^2, 0)$ have excess 16, 6, 10, 1, 5, 5, 9, 5, respectively. The cones $C_6, C_6^{1,2}, C_6^{2,3}, C^{1,3}, C_5$ have, respectively, 210, 495, 780, 60, 40 facets and the facets are partitioned, respectively, into 4, 5, 8, 1, 2 classes of equivalent facets up to permutations.

8. Scales

In this section we consider the scale $\eta^0(ant_\alpha 2d(K_n))$, which is, by Proposition 4.1(iii), equal to $\min\{t \in \mathbb{Z}_+ : \alpha t \ge z_n^t\}$, especially for two extreme cases $\alpha = 4$ and $\alpha = n - 1$. The number t below is always a positive integer.

Denote by $H(4t)$ a Hadamard matrix of order $4t$, and by $PG(2,t)$ a projective plane of order t.

It is proved in [5] that $t \sum_1^n \delta(\{i\})$ is the unique $\mathbb{Z}_+$-realization of $2td(K_n)$ if $n \ge t^2 + t + 3$, and that for $n = t^2 + t + 2$, $2td(K_n)$ has other $\mathbb{Z}_+$-realizations if and only if there exists a $PG(2, t)$. Below, in $(iv_1) - (iv_3)$ of Theorem 8.1, we reformulate this result in terms of A_n^1, $\eta^1(2d(K_n))$, z_n^t, using the following trivial relations

$$\eta^1(2d(K_n)) \geq t + 1 \Leftrightarrow 2td(K_n) \in A_n^1 \Leftrightarrow z_n^t = nt \Leftrightarrow$$

$$\Leftrightarrow t \sum_1^n \delta(\{i\}) \text{ is the unique } \mathbb{Z}_+\text{-realization of } 2td(K_n).$$

(iii_2) of Theorem 8.1 follows from a result of Ryser (reformulated in terms of z_n^t in [9, Theorem 4.6(1)]) that $z_n^t \geq n - 1$ with equality if and only if $n = 4t$ and there exists an $H(4t)$.

Theorem 8.1.

(i_1) *$ant_\beta 2td(K_n) \in C_{n+1}$ if and only if $\beta \geq \frac{tn(n-1)}{\lfloor n/2 \rfloor \lceil n/2 \rceil}$;*
(i_2) *$ant_\beta 2td(K_n) \in A^0$ if and only if $\frac{tn(n-1)}{\lfloor n/2 \rfloor \lceil n/2 \rceil} \leq \beta < z_n^t$, $\beta \in \mathbb{Z}_+$;*
(i_3) *$ant_\beta 2td(K_n) \in hC_{n+1}$ if and only if $\beta \geq z_n^t$, $\beta \in \mathbb{Z}_+$;*
(i_4) *$ant_\alpha 2d(K_n) \in C_{n+1} \cap L_{n+1}$ if and only if $\frac{n(n-1)}{\lfloor n/2 \rfloor \lceil n/2 \rceil} \leq \alpha$, $\alpha \in \mathbb{Z}_+$.*

Moreover, if $d = ant_\alpha 2d(K_n) \in C_{n+1} \cap L_{n+1}$, then
(ii_1) *either $n = 3, d \in A_3^1$, is simplicial, $d = ant_3 2d(K_4)$ (so $\eta^i(d) = 1$ for $i \geq 0$),*
or $d \in A_n^1$, d is not simplicial, $\alpha \geq n \geq 4$ (so $\eta^0(d) = 1$), or $d \in A_n^0$ (so $\eta^0(d) \geq 2$),
(ii_2) *$\eta^0(d) = \min\{t : z_n^t \leq \alpha t\}$.*
(iii_1) *$\eta^0(ant_4 2d(K_n)) = \eta^0(2d(K_{n+1} - P_{(1,2)})) = \eta^0(2d(K_{n\times 2}))$;*
(iii_2) *$\lceil n/4 \rceil \leq \eta^0(ant_4 2d(K_n)) \leq \min\{t \in Z_+ : n \leq 4t$ and there exists a $H(4t)\} < n/2$;*
(iii_3) *For $n = 4t, 4t - 1$, we have $\eta^0(ant_4 2d(K_n)) = \lceil n/4 \rceil = t$ if and only if there exists an $H(4t)$;*
(iv_1) *$\eta^0(ant_{n-1} 2d(K_n)) = \eta^1(2d(K_n)) \leq \min\{n - 3, \eta^1(2d(K_{n+1}))\}$;*
(iv_2)

$$\left\lceil (1/2)(\sqrt{4n-7} - 1) \right\rceil = \min\{t \in Z_+ : n \leq t^2 + t + 2\}$$
$$\leq \eta^0(ant_{n-1} 2d(K_n))$$
$$\leq \min\{t \in Z_+ : n \leq t^2 + t + 2 \text{ and there exists a } PG(2, t)\};$$

(iv_3) *For $n = t^2 + t + 2$, we have $\eta^0(ant_{n-1} 2d(K_n)) = \left\lceil (1/2)(\sqrt{4n-7} - 1) \right\rceil = t$ if and only if there exists a $PG(2, t)$.*

Remarks.

(a) For $i \geq 0$, we have $\eta^{i+1}(2d(K_4)) = i + 1$, but $\eta^i(ant_3(2d(K_4))) = 1$, since $ant_3(2d(K_4))$ is a simplicial point. For $i \geq 0$ and $n \geq 5$, we have $\eta^{i+1}(2d(K_n)) \leq \eta^i(ant_{n-1}(2d(K_n)))$ with equality for $i = 0$ and for some pair (i, n) with $i \geq 1$. Propositions 5.9–5.11 of [9] imply that

$$\eta^{i+1}(2d(K_5)) = \eta^i(ant_4(2d(K_5))) = 2 \text{ for } i = 0, 1;$$

$$\eta^3(2d(K_5)) = \eta^2(ant_4(2d(K_5))) = \eta^4(2d(K_5)) = 3;$$

$$\eta^5(2d(K_5)) = \eta^4(ant_4(2d(K_5))) = \eta^3(ant_4(2d(K_5))) = 4.$$

(b) Using the well-known fact that $H(4t)$ exists for $t \le 106$, we obtain

$$\eta^0(ant_4(2d(K_n))) = \eta^0(2d(K_{n+1} - P_2)) = \eta^0(2d(K_{n\times 2})) = \lceil n/4 \rceil \text{ for } n \in [4, 424];$$

(c) Using the well-known fact that $PG(2,t)$, $t \le 11$, exists if and only if $t \ne 6, 10$, we get for $a_n = \eta^0(ant_{n-1}(2d(K_n))) = \eta^1(2d(K_n))$, that $6 \le a_n \le 7$ for $33 \le n \le 43$, $10 \le a_n \le 11$ for $93 \le n \le 111$, and $a_n = \lceil (1/2)(\sqrt{4n-7} - 1) \rceil$ for all other $n \in [4, 134]$.

(d) (iii), (iv) of Theorem 8.1 imply that

$$\eta^0(d(K_{2t\times 2})) \ge 2t \text{ with equality if and only if there exists } H(4t),$$

$$\eta^1(d(K_{t^2+t+2})) \ge 2t \text{ with equality if and only if there exists } PG(2,t).$$

Note also that $a_n \le n - 3$ with equality if and only if $n = 4, 5$.

Proof of (iv_1). For $n \ge 4$ we have

$$\left\lceil \frac{1}{2}(\sqrt{4n-7} - 1) \right\rceil \le \eta^1(2d(K_n)) = \eta^0(ant_{n-1}(2d(K_n))) \le n - 3.$$

In fact, we have

$$\eta^1(2d(K_n)) = \min\{t \in Z_+ : z_n^t < nt\},$$

$$\eta^0(ant_N(2d(K_n))) = \min\{t \in Z_+ : z_n^t \le Nt\},$$

since $2td(K_n)$ has the following $\mathbb{Z}_+$-realization $t\sum_1^n \delta(\{i\})$ of maximal size nt, and since $t(ant_N(2d(K_n))) \in hC_{n+1}$ if and only if $2td(K_n)$ admits a $\mathbb{Z}_+$-realization of size at most Nt. Denote

$$p = \eta^1(2td(K_n)), \quad q = \eta^0(ant_{n-1}(2d(K_n))).$$

Then $p \le q$, because $z_n^q \le (n-1)q$ implies $z_n^q \le nq$. Also, $q \le n-3$, because $2(n-3)d(K_n)$ has the $\mathbb{Z}_+$-realization $\sum_1^{n-1}((n-4)\delta(\{i\}) + \delta(\{i,n\}))$ of size $(n-3)(n-1)$. On the other hand, $p \ge q$, because $z_n^p < np$ implies $z_n^p \le np - (n-3)$, which is proved in [9, Proposition 5.3]. So $z_n^p \le np - q \le np - p$. We have $p \ge \lceil (1/2)(\sqrt{4n-7} - 1) \rceil$, because otherwise $n \ge p^2 + p + 3$, and using [5], $2td(K_n)$ has exactly one $\mathbb{Z}_+$-realization, in contradiction with the definition of p. □

Theorem 8.2. *Let* $\eta_n^i = \eta^i(2d(K_n))$. *Then*

(i) $\eta_n^0 < \infty$ *for* $d \in L_n \cap C_n$,

(ii) $\eta_n^{i-1} | \eta_n^i$ *for* $i \ge 1$, *and* $\eta_{n-1}^i | \eta_n^i$ *for* $n \ge 5$,

(iii) $\eta^i(ad) = \lceil \eta^i(d)/a \rceil$ *for* $d \in C_n \cup L_n$, $i \ge 0$, $a \in \mathbb{Z}_+$.

Proof. (i) Define

$$Y = L_n \cap C_n \cap \{\sum \lambda_S \delta(S) : 0 \le \lambda_S \le 1\}.$$

Clearly, Y is finite, and one can find $\lambda \in \mathbb{Z}_+$ such that λd is an h-point for every $d \in Y$.

Let $d \in L_n \cap C_n$ have an $\mathbb{R}_+$-realization $d = \sum \mu_S \delta(S)$. Clearly the coefficients μ_S are rational numbers. We have $d = d_1 + d_2$, where $d_1 = \sum \lfloor \mu_S \rfloor \delta(S)$, and $d_2 = \sum(\mu_S - \lfloor \mu_S \rfloor)\delta(S)$. By the construction, d_1 is an h-point. Since $d_2 = d - d_1$ and $d \in L_n \cap C_n$, $d_1 \in$

$L_n \cap C_n$, we obtain $d_2 \in Y$. Hence there is λ such that $\lambda d_2 \in hC_n$, and we obtain that $\lambda d = \lambda d_1 + \lambda d_2$ is an h-point, too.
(ii) Obvious.
(iii) Take $\lambda = \eta^i(ad)$, that is $\lambda(ad)$ has at least $i+1$ $\mathbb{Z}_+$-realizations. Hence $\lambda a \geq \eta^i(d)$ implies $\lambda \geq \lceil \eta^i(d)/a \rceil$, that is, $\eta^i(ad) \geq \lceil \eta^i(d)/a \rceil$.

Now, take $\lambda = \lceil \eta^i(d)/a \rceil$. So, $\lambda - 1 < \eta^i(d)/a \leq \lambda \Rightarrow (\lambda - 1)a < \eta^i(d) \leq \lambda a$. Hence λad has at least $i+1$ $\mathbb{Z}_+$-realizations, implying that $\lambda \geq \eta^i(ad)$, and so $\lceil \eta^i(d)/a \rceil \geq \eta^i(ad)$. □

Remarks.

(a) $\eta_4^i = \eta^i(2d(K_4)) = i$ for $i \geq 1$; $\eta_n^0 = 1$ if and only if $n = 4, 5$.

(b) For $d \notin L_n$ and $\lambda \in \mathbb{Z}_+$, we have $\lambda d \in L_n$ implies that λ is even (because $(\lambda d_{ij} + \lambda d_{ik} + \lambda d_{jk})/2 = \lambda(d_{ij} + d_{ik} + d_{jk})/2$). Hence, for $d \in \mathbb{Z}^{\binom{n}{2}} - A_n^0$, we have either $d \notin L_n$ (so $\eta^0(d)$ is even), or $\eta^0(d) = 1$ (*i.e.* $d \in hC_n$). Since $d(G) \notin A_n^0$ for any connected graph G on n vertices (see [14]), we have either $\eta^0(d(G)) = 1$ or $\eta^0(d(G))$ is even. But, for example, $\eta^0(2d(K_{10} - P_2)) = \eta^0(2d(K_{9\times 2})) = 3$.

It will be interesting to see whether η_n^0 and $\max\{\eta^0(d) : d \in A_n^0\}$ are bounded from above by *const* $\times\, n$.

The best-known lower bound for the last number is $\eta^0(d(K_n - P_2))$, which belongs to the interval $[2\lceil (n-1)/4 \rceil, n-2]$.

It is proved in [19] that for a graphic metric $d = d(G)$, we have

(i) $\eta^0(d) \leq n - 2$ if $d(G) \in C_n$,

(ii) $\eta^0(d) \in \{1, 2\}$, that is, G is an isometric subgraph of a hypercube or a halved cube if $d(G)$ is simplicial.

9. h-points

Recall that any point of $\mathbb{Z}_+(\mathscr{K}_n) = hC_n$ is called an h-point.

A point d is called *k-gonal*, if it satisfies all hypermetric inequalities $Hyp_n(b)$ with $\sum_1^n |b_i| = k$.

The following cases are examples of when the conditions $d \in L_n$ and hypermetricity of d imply that d is an h-point.

(a) [14], [17]: If $d = d(G)$ and G is bipartite, then 5-gonality of d implies that $d \in hC_n$;

(b) [1]: If $\{d_{ij}\} \in \{1, 2\}$, $1 \leq i < j \leq n$, then $d \in L_n$ and 5-gonality of d imply that $d \in hC_n$ (actually, $d = d(K_{1,n-1})$, $d(K_{2,2})$ or $2d(K_n)$ in this case);

(c) [2]: If $n \geq 9$ and $\{d_{ij}\} \in \{1, 2, 3\}$, $1 \leq i < j \leq n$, then $d \in L_n$ and ≤ 11-gonality of d imply that $d \in hC_n$.

So, the cases (a), (b), (c) are among known cases when the problem of testing membership of d in hC_n can be solved by a polynomial time algorithm. The polynomial testing holds for any $d = d(G)$ (see [19]) and for 'generalized bipartite' metrics (see [7] which generalizes the cases (b) and (c) above).

Cases (a), (b) and (c) imply (i), (ii) and (iii), respectively, of

Corollary 9.1. *If $d \in A_n^0$, then none of the following hold*

(i) $d = d(G)$ *for a bipartite graph G,*

(ii) $\{d_{ij}\} \in \{1,2\}$, $1 \le i < j \le n$,

(iii) $\{d_{ij}\} \in \{1,2,3\}$, $1 \le i < j \le n$, *if* $n \ge 9$.

A point $d \in \mathbb{Z}_+(\mathcal{K}_n) = hC_n$ is called *rigid* if d admits a unique $\mathbb{Z}_+$-realization. In other words, d is rigid if and only if $d \in A_n^1$. Clearly, if $d \in hC_n$ is simplicial, d is rigid. Rigid nonsimplicial points are more interesting. Hence we define the set

$$\tilde{A}_n^1 := \{d \in A_n^1 : d \text{ is not simplicial}\},$$

and call its points *h-rigid*.

Theorem 9.2.

(i) $A_n^0 = \emptyset$ for $n \le 5$, $2d(K_6 - P_2) \in A_6^0$, $|A_n^0| = \infty$ for $n \ge 7$,

(ii) $\tilde{A}_n^1 = \emptyset$ for $n \le 4$, $\tilde{A}_5^1 = \{2d(K_5)\}$, $|\tilde{A}_n^1| = \infty$ for $n \ge 6$,

(iii) for $i \ge 2$, $A_n^i = \emptyset$ if $n \le 3$, $|A_n^i| = \infty$ if $n \ge 4$.

Proof. (i) and (ii) The first equalities in (i) and (ii) are implied by results in [3]. The inclusion in (i) is implied by [1]. The second equality in (ii) is proved in [13]. We have $|A_n^0| = \infty$ for $n \ge 7$, because $A_6^0 \ne \emptyset$ and $|A_{n+1}^i| = \infty$ whenever $A_n^i \ne \emptyset$ from (6).

We prove the third equality of (ii): $|\tilde{A}_n^1| = \infty$ for $n \ge 6$. The equality is implied by the fact that $ant_\alpha(2d(K_n)) \in \tilde{A}_{n+1}^1$ for any $n \ge 5$, $\alpha \in \mathbb{Z}_+$, $\alpha \ge n$. We prove the inclusion.

Recall that $2td(K_n)$ has the unique $\mathbb{Z}_+$-realization of size tn if $n \ge t^2 + t + 3$. (See [5] or the beginning of Section 8). For $t = 1$ we obtain the equality $z(2d(K_n)) = n$ for $n \ge 5$ Using the fact that $2d(K_n)$ is not simplicial for $n \ge 4$, and (iv) of Proposition 4.1 we obtain the required inclusion.

(iii) Since C_3 is simplicial, $A_3^i = \emptyset$ for $i \ge 2$. Consider now $n = 4$. We show that $A_4^i = \{2(i-1)d(K_4) + d : d \text{ is a simplicial h-point of } C_4\}$. This follows from the fact that the only linear dependency on cuts of C_4 is, up to a multiple, $\delta(1) + \delta(2) + \delta(3) + \delta(4) = \delta(1,4) + \delta(2,4) + \delta(3,4)$.

So, $|A_4^i| = \infty$, because there are an infinity of simplicial points, *e.g.*, $\lambda d(K_{2,2})$ for $\lambda \in \mathbb{Z}_+$. Finally we use (6). □

Some questions.

(a) Is it true that all 10 permutations of $d_6 = 2d(K_6 - P_2)$ are only quasi-h-points of C_6? If yes, these 10 points and 31 nonzero cuts from $\mathcal{K}_6$ form a Hilbert basis of C_6.

(b) Does there exist a ray $\{\lambda d : \lambda \in \mathbb{R}_+\} \subset C_n$ containing an infinite set of quasi-h-points? Recall that we got in Section 6 examples of rays $\{d^0 + td^1 : t \ge 0\}$ containing infinitely many quasi-h-points.

Lemma 9.3. *Let $d \in A_n^0$, and let $d = ant_\alpha d'$ where $d' \notin A_{n-1}^0$. Then d' is an h-point and* $z(d') \ge \lceil s(d') \rceil + 1$.

Proof. In fact, $d \in C_n \cap L_n$, so $d' \in C_{n-1} \cap L_{n-1}$. But $d' \notin A_{n-1}^0$, so d' is an h-point of C_{n-1}. Hence by Proposition 4.1(ii), $\alpha \in \mathbb{Z}_+$, $s(d') \le \alpha < z(d')$.

Note that for $n \geq 5$ we have $2d(K_{n\times 2}) \in A^0_{2n}$, $2d(K_{n\times 2}) = ant_4 d'$, where $d' \in A^0_{2n-1}$ and $d' = ant_4 d''$ for $d'' \in A^0_{2n-2}$, and so on.

So, d' is neither a simplicial point nor an antipodal extension (*i.e.* $d' \notin \mathbb{R}_+(ant\mathscr{K}_{n-2})$), nor $d' \in \mathbb{Z}_+(\mathscr{K}^m_{n-1})$, $m = \lfloor (n-1)/2 \rfloor$, because in each of these 3 cases we have for an h-point d', $z(d') = s(d')$; this also implies that by Proposition 4.1(iv), d itself is not simplicial. □

The following proposition makes plausible the fact that the metric $d_6 = 2d(K_6 - P_2)$ is the unique (up to permutations) quasi-h-point of C_6.

Proposition 9.4. *Let* $d \in A^0_6$, $d = ant_\alpha d'$ *and* $d \neq d_6$. *Then*

(a) *both* d *and* d' *are not simplicial;*

(b) $d' \notin \mathbb{R}_+(ant\mathscr{K}_4)$, $d' \notin \mathbb{Z}_+(\mathscr{K}^2_5)$;

(c) $d' \neq \lambda d(G)$ *for any* $\lambda \in \mathbb{Z}_+$ *and any graph* G *on 5 vertices;*

(d) d' *has at least two* $\mathbb{Z}_+$*-realizations.*

Proof. Since $A^0_5 = \emptyset$ by [3], we can apply Proposition 9.3, and (a) and (b) follow. One can see by inspection, that among all 21 connected graphs on 5 vertices, the only graphs G with nonsimplicial $d(G) \in C_6$ are the following 3 graphs: K_5, $K_5 - P_2$, and $K_4.K_2 = K_4$ with an additional vertex adjacent to a vertex of K_4. For these graphs, $\lambda d(G)$ is an h-point if and only if $\lambda \in 2\mathbb{Z}_+$.

Since $2d(K_5 - P_2) = ant_4(2d(K_4))$, then, according to (b), $d' \neq \lambda d(K_5 - P_2)$.

Since for any $\lambda \in \mathbb{Z}_+$ we have $z(2\lambda d(K_4.K_2)) = 5\lambda = s(2\lambda d(K_4.K_2))$, and (by Proposition 9.3) $s(d') < z(d')$, then $d' \neq \lambda d(K_4.K_2)$.

There remains the case $d' = \lambda d(K_5)$. We have $s(d') = \lambda 5/3$, $z(d') = 5$ for $\lambda = 2$ and $z(d') = s(d')$ for $\lambda \in 2\mathbb{Z}_+$, $\lambda > 2$. (See [9, Proposition 5.11]). So $s(d') \leq \alpha < z(d')$ implies $\lambda = 2$, $\alpha = 4$, *i.e.*, exactly the case $d = ant_4(2d(K_5))$. This proves (c).

Finally, (d) follows from the fact (see [13]) that $2d(K_5)$ is the unique nonsimplicial h-point of C_5 with unique $\mathbb{Z}_+$-realization. □

References

[1] Assouad, P. and Deza, M. (1980) Espaces métriques plongebles dans un hypercube: aspects combinatoires *Annals of Discrete Math.* **8** 197–210.

[2] Avis, D. (1990) On the complexity of isometric embedding in the hypercube: In Algorithms. *Springer-Verlag Lecture Notes in Computer science*, **450** 348–357.

[3] Deza, M. (1960) On the Hamming geometry of unitary cubes. *Doklady Academii Nauk SSSR* **134**, 1037–1040 (in Russian) *Soviet Physics Doklady* (English translation) **5** 940–943.

[4] Deza, M. (1973) Matrices de formes quadratiques non négatives pour des arguments binaires. *C. R. Acad. Sci. Paris* **277** 873–875.

[5] Deza, M. (1973) Une proprieté éxtremal des plans projectifs finis dans une classe de codes equidistants. *Discrete Mathematics* **6** 343–352.

[6] Deza, M. and Laurent, M. (1992) Facets for the cut cone I, II. *Mathematical Programming* **52** 121–161, 162–188.

[7] Deza, M. and Laurent, M. (1991) Isometric hypercube embedding of generalized bipartite metrics, Research report 91706-OR, Institut für Discrete Mathematik, Universität Bonn.

[8] Deza, M. and Laurent, M. (1992) Extension operations for cuts. *Discrete Mathematics* **106-107** 163–179.

[9] Deza, M. and Laurent, M (1992) Variety of hypercube embeddings of the equidistant metric and designs. *Journal of Combinatorics, Information and System sciences* (to appear).

[10] Deza, M. and Laurent, M. (1993) The cut cone: simplicial faces and linear dependencies. *Bulletin of the Institute of Math. Academia Sinica* **21** 143–182.

[11] Deza, M., Laurent, M. and Poljak, S. (1992) The cut cone III: on the role of triangle facets. *Graphs and Combinatorics* **8** 125–142.

[12] Deza, M. and Laurent, M. (1992) Applications of cut polyhedra, Research report LIENS 92-18, ENS. *J. of Computational and Applied Math* (to appear).

[13] Deza, M. and Singhi, N. M. (1988) Rigid pentagons in hypercubes. *Graphs and Combinatorics* **4** 31–42.

[14] Djokovic, D.Z. (1973) Distance preserving subgraphs of hypercubes. *Journal of Combinatorial Theory* **B14** 263–267.

[15] Koolen, J. (1990) On metric properties of regular graphs, Master's thesis, Eindhoven University of Technology.

[16] Harary, F. (1969) *Graph Theory*, Addison-Wesley.

[17] Roth, R. L. and Winkler, P. M. (1986) Collapse of the metric hierarchy for bipartite graphs. *European Journal of Combinatorics* **7** 371–375.

[18] Schrijver, A. (1986) *Theory of linear and integer programming*, Wiley.

[19] Shpectorov, S. V. (1993) On scale embeddings of graphs into hypercubes. *European Journal of Combinatorics* **14**.