GRÖBNER BASES OF CERTAIN DETERMINANTAL IDEALS

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Abstract. We prove that the ideal of the locus of points $(x_1, \dots, x_n)$ in $\mathbb{C}^n$ for which the $n \times k$ matrix $(x_i^{\mu_j})$ has rank smaller than k is generated by the determinants of the $k \times k$ minors of this matrix, if $\mu_1 < \dots < \mu_k$ and $\mu_1 = 0$ or 1. Moreover, these generators form a universal Gröbner basis. Special cases of our result apply for ideals arising in the study of chromatic numbers of graphs or identities of matrices, and ideals of truncations of hyperplane arrangements related to pseudo-reflection groups.

1. Introduction

We work over an algebraically closed field F of characteristic zero. Let $\mu = (\mu_1, \dots, \mu_k)$ be a strictly increasing sequence of non-negative integers with $k \geq 2$, that is, $0 \leq \mu_1 < \dots < \mu_k$. Consider the map

$$\phi_{n,\mu} : \mathbb{A}^n \to \mathbb{A}^{nk}$$
$$(x_1, \dots, x_n) \mapsto (x_i^{\mu_j})$$

from the n dimensional affine space to the nk dimensional affine space, where $n \geq k \geq 2$, which maps a point $(x_1, \dots, x_n)$ to the $n \times k$ matrix whose (i, j) entry is $x_i^{\mu_j}$. We study

$$X_{n,\mu} = \{x \in \mathbb{A}^n | \mathrm{rk}(\phi_{n,\mu}(x)) \leq k - 1\}$$

the locus of points in $\mathbb{A}^n$ whose image under $\phi_{n,\mu}$ has rank smaller than k. One can also think of $X_{n,\mu}$ as the set of points $x \in \mathbb{A}^n$ with the property that there exists a polynomial (depending on x) $f(t) = a_1 t^{\mu_1} + \dots + a_k t^{\mu_k} \in F[t]$ such that all the coordinates of x are roots of f.

The aim of the present paper is to give generators (a Gröbner basis) of the ideal $I(X_{n,\mu})$ of polynomials in $F[t_1, \dots, t_n]$ vanishing on $X_{n,\mu}$. We put

$$\delta_\mu(t_1, \dots, t_k) = det((t_i^{\mu_j})_{1 \leq i,j \leq k})$$

Supported by OTKA no. F023436 and the Bolyai Research Fellowship. .

1991 *Mathematics Subject Classification.* Primary 13C40, Secondary 13P10, 14M12.

Key words and phrases. Determinantal ideal, Gröbner basis, degree, Hilbert polynomial, subspace arrangement, reflection group.

so δ_μ is the determinant of the $k \times k$ matrix whose (i,j) entry is $t_i^{\mu_j}$. Note that the integers $\lambda_i = \mu_i - i + 1$ form a non-decreasing sequence of non-negative integers $\lambda = (\lambda_1, \dots, \lambda_k)$, and the *Schur function* s_λ is usually defined by the equality

$$\delta_\mu(t_1, \dots, t_k) = s_\lambda(t_1, \dots, t_k) \prod_{1 \le i < j \le k} (t_j - t_i)$$

(see for example [Ma]). Since the points $x \in X_{n,\mu}$ are characterized by the property that all the $k \times k$ minors of $\phi_{n,\mu}(x)$ are degenerate, the polynomials $\delta_\mu(t_{i_1}, \dots, t_{i_k})$ with $1 \le i_1 < \dots < i_k \le n$ define $X_{n,\mu}$ set theoretically. We shall show that they generate the ideal $I(X_{n,\mu})$, unless δ_μ has multiple factors. So our first question is that when the polynomial δ_μ has multiple factors. This is answered by the following elementary lemma.

Lemma 1.1. *The polynomial $\delta_\mu(t_1, \dots, t_k)$ has multiple factors in $F[t_1, \dots, t_k]$ if and only if $\mu_1 \ge 2$.*

Proof. The equality

$$\delta_\mu(t_1, \dots, t_k) = (t_1 \cdots t_k)^{\mu_1} det((t_i^{\mu_j - \mu_1})_{1 \le i,j \le k})$$

shows that if δ_μ has no multiple factors, then $\mu_1 \le 1$.

To prove the converse direction we may assume that $\mu_1 = 0$, since in this case none of $t_1, \dots, t_k$ is a divisor of δ_μ. We apply induction on k. The case $k = 2$ is obvious, because then we have $\delta_\mu(t_1, t_2) = t_2^{\mu_2} - t_1^{\mu_2}$, and this polynomial has no multiple factors in characteristic zero. Assume now that $k \ge 3$, and

$$\delta_\mu(t_1, \dots, t_k) = f^2 \cdot g$$

for some $f, g \in F[t_1, \dots, t_k]$, where f is non-constant. Since δ_μ is homogeneous, f and g are also homogeneous. Expand f and g according to the powers of t_k :

$$f = \sum_{i=0}^{r} a_i(t_1, \dots, t_{k-1}) t_k^i \quad \text{and} \quad g = \sum_{j=0}^{s} b_j(t_1, \dots, t_{k-1}) t_k^j,$$

where a_r and b_s are non-zero. On expanding the determinant of $(t_i^{\mu_j})$ according to its last row, we get

$$\delta_\mu = (-1)^{k+1}(t_1 \cdots t_{k-1})^{\mu_2} \delta_{\mu'}(t_1, \dots, t_{k-1}) + \dots + t_k^{\mu_k} \delta_{\mu''}(t_1, \dots, t_{k-1}),$$

where $\mu' = (0, \mu_3 - \mu_2, \dots, \mu_k - \mu_2)$ and $\mu'' = (\mu_1, \dots, \mu_{k-1})$. It follows that $\delta_{\mu''} = a_r^2 b_s$ and $(-1)^{k+1}(t_1 \cdots t_{k-1})^{\mu_2} \delta_{\mu'} = a_0^2 b_0$, and by the induction hypothesis a_r must be a non-zero constant and a_0^2 is a divisor of $(t_1 \cdots t_{k-1})^{\mu_2}$.

On the other hand, we may assume that f is fixed up to sign by the symmetric group $\mathrm{Sym}(k)$ acting on $F[t_1, \dots, t_k]$ via permutation of the variables. Indeed, take an irreducible factor q of f, then q^2 divides δ_μ. Now let $q = q_1, \dots, q_m$ be the pairwise different images of q under the action of $\mathrm{Sym}(k)$, up to scalar multiplication. Since $\pi(\delta_\mu) = \mathrm{sign}(\pi)\delta_\mu$

holds for any $\pi \in \mathrm{Sym}(k)$, we have that $\pi(q)^2$ divides δ_μ. The polynomials $q_1, \dots, q_m$ are pairwise non-associate irreducibles, hence $q_1^2 \cdots q_m^2$ is a divisor of δ_μ. So $f = q_1 \cdots q_m$ has the required properties.

Summarizing, the leading term of f as a polynomial of t_k is ct_k^r, where $r \geq 1$ and c is a non-zero constant. The symmetry of f implies that

$$f = c(t_1^r + \cdots + t_k^r) + \text{ other terms},$$

hence

$$a_0(t_1, \dots, t_{k-1}) = c(t_1^r + \cdots + t_{k-1}^r) + \text{ other terms},$$

which contradicts the fact that a_0^2 is a divisor of $(t_1 \cdots t_{k-1})^{\mu_2}$. □

Now we state the main result of the paper.

Theorem 1.2. *Let $n \geq k \geq 2$, and let $\mu = (\mu_1, \dots, \mu_k)$ be a strictly increasing sequence of non-negative integers with $\mu_1 \leq 1$. Then the ideal of $X_{n,\mu}$ is generated by $\delta_\mu(t_{i_1}, \dots, t_{i_k})$ with $1 \leq i_1 < \cdots < i_k \leq n$. Moreover, this generating set is a universal Gröbner basis of $I(X_{n,\mu})$.*

In order to explain the structure of the proof we introduce some terminology. Recall that for any homogeneous ideal I the factor algebra $F[t_1, \dots, t_n]/I$ is naturally graded, and there exists a polynomial $h_I(t) \in \mathbb{Q}[t]$ called the *Hilbert polynomial of I* such that the dimension of the degree d homogeneous component of $F[t_1, \dots, t_n]/I$ is $h_I(d)$ for sufficiently large d. The leading term of $h_I(t)$ is $\frac{e(I)}{m!}t^m$, where $e(I)$ is a positive integer, called the *degree of I*, and $m = \dim(I) - 1$. Now let M be a monomial ideal, that is, an ideal generated by monomials. Obviously, for any monomial ideal M' containing M we have that $\dim(M') \leq \dim(M)$, and in the case of equality $e(M') \leq e(M)$. We say that M is *critical*, if for any monomial ideal M' strictly containing M we have that $\dim(M') < \dim(M)$ or $e(M') < e(M)$. The strategy of the proof of Theorem 1.2 is based on the fact that the initial monomial ideal of $I(X_{n,\mu})$ turns out to be critical. To verify the conjecture that a given critical monomial ideal M generated by the initial monomials of certain polynomials vanishing on $X_{n,\mu}$ is indeed the initial ideal of $I(X_{n,\mu})$ it is sufficient to show that the dimension and the degree of M is the same as for $X_{n,\mu}$. In Section 2 we derive the necessary information on the geometry of $X_{n,\mu}$. Section 3 is devoted to the combinatorial study of the ideals generated by the initial monomials of $\delta_\mu(t_{i_1}, \dots, t_{i_k})$, which we need to finish the proof of Theorem 1.2.

An interesting special case is $\mu = (0, 1, \dots, k)$, when $X_{n,(0,1,\dots,k)}$ is the subspace arrangement consisting of points which have at most k different coordinates. The Kleitman-Lovász theorem [L] asserts that $\prod_{1 \leq r < s \leq k+1}(t_{j_r} - t_{j_s})$ (with $1 \leq j_1 < \cdots < j_{k+1} \leq n$) generate the ideal in question. The additional fact that the given generators form a universal Gröbner basis is due to de Loera [Lo]. The motivation of [L] comes from graph theory. An n-variable polynomial p_G is associated with any graph G having n nodes, and it is observed that the chromatic number of G is greater than k if and only if p_G is contained in $I(X_{n,(0,1,\dots,k)})$. So the result of [Lo] gives some information on graph colorings.

The description of $I(X_{n,\mu})$ when $\mu = (0, 1, \dots, k)$ or $\mu = (0, 1, \dots, k-2, k)$ has consequences also in the theory of polynomial identities and weak identities of matrices, as it is shown in [D].

In Section 4 we draw attention to other special cases of our result, which deal with subspace arrangements related to complex pseudo-reflection groups. More precisely, let G be a finite complex pseudo-reflection group, and consider the set of reflecting hyperplanes belonging to G. Take the union of k dimensional subspaces which are intersections of reflecting hyperplanes. The vanishing ideal of this subspace arrangement is described by Theorem 1.2 for certain pseudo-reflection groups, including the Weyl groups of type A_n, B_n, and D_n.

We finish the introduction with some comments on the result. $X_{n,\mu}$ is the set of points in the n dimensional affine space whose any k coordinates satisfy the alternating relation $\delta_\mu = 0$. Now let $l(t_1,\dots,t_k)$ be any homogeneous alternating polynomial, that is,

$$l(t_1,\dots,t_k) = s(t_1,\dots,t_k) \prod_{1\le i<j\le k} (t_i - t_j),$$

where $s(t_1,\dots,t_k)$ is a symmetric polynomial. Assume that $l(t_1,\dots,t_k)$ has no multiple factors in $F[t_1,\dots,t_k]$. The following related problem is studied in [D]. We put

$$Z_{n,l} = \{x \in \mathbb{A}^n | x \text{ has at most k different coordinates and } l(x_{i_1},\dots,x_{i_k}) = 0 \text{ for any } 1 \le i_1 < \dots < i_k \le n\},$$

and let denote $\delta(t_1,\dots,t_{k+1}) = \prod_{1\le r<s\le k+1}(t_s - t_r)$. We show in [D] that

$$I(Z_{n,l}) = \langle l(t_{i_1},\dots,t_{i_k}),\ \delta(t_1,t_{j_1},\dots,t_{j_k}) | 1 \le i_1 < \dots < i_k \le n,\ 2 \le j_1 < \dots < j_k \le n\rangle,$$

and this generating set is a Gröbner basis with respect to the lexicographic monomial order induced by $t_1 \prec \dots \prec t_n$. The motivation to study $I(Z_{n,l})$ in [D] came from the theory of polynomial identities of matrices, therefore the condition "x has at most k different coordinates" was natural. However, one can also ask whether the ideal of

$$X_{n,l} = \{x \in \mathbb{A}^n | l(x_{i_1},\dots,x_{i_k}) = 0 \text{ for any } 1 \le i_1 < \dots < i_k \le n\}$$

is generated by the polynomials $l(t_{i_1},\dots,t_{i_k})$ (with $1 \le i_1 < \dots < i_k \le n$). The answer to this question is no in general, as the following example shows, so it has some interest that we have affirmative answer in the determinantal case.

Let $n = 3$ and let $l = (t_1 - t_2)(t_1^3 + t_2^3)$. We claim that $X_{3,l}$ does not contain a point with pairwise different coordinates. Indeed, assume in the contrary that (x_1, x_2, x_3) is such a point, its coordinates are necessarily non-zero. Then $\frac{x_2}{x_1}$, $\frac{x_3}{x_1}$, $\frac{x_3}{x_2}$ must all be roots of the polynomial $1 + t^3$. The roots of $t^3 + 1$ are ρ, ρ^5 and -1, where ρ is a primitive 6th root of unity. But the ratio of any two element of $\{\rho, \rho^5, -1\}$ is not contained in this set, a contradiction. Therefore, the polynomial $(t_1 - t_2)(t_1 - t_3)(t_2 - t_3)$ lies in $I(X_{3,l})$, but it is of degree 3, hence is clearly not contained in $(l(t_1,t_2), l(t_1,t_3), l(t_2,t_3))$.

On the other hand, $\prod_{1\le r<s\le k+1}(t_{j_r} - t_{j_s})$ is not always contained in $I(X_{n,l})$. For example, $X_{3,t_1^3 - t_2^3}$ contains the point $(1, \rho, \rho^2)$, where ρ is a primitive third root of unity.

Another possibility to generalize Theorem 1.2 is to consider $X_{n,\mu,m}$, the locus of points in $\mathbb{A}^n$ whose image under $\phi_{n,\mu}$ has rank smaller than m. (When $m = k$, the algebraic set

$X_{n,\mu,k}$ coincides with $X_{n,\mu}$.) Though the $m \times m$ minors of $\phi_{n,\mu}(t_1,\dots,t_n)$ define $X_{n,\mu,m}$ set theoretically, they do not generate its ideal in general. For example, take $\mu = (0,1,3,6)$, $m = 3$, and $n = 3$. The 3×3 minors of $\phi_{3,\mu}(t_1,t_2,t_3)$ have no multiple factors in this case, however, they do not generate a radical ideal. Indeed, we claim that $(t_1t_2 + t_1t_3 + t_2t_3)(t_1-t_2)(t_1-t_3)(t_2-t_3)$ vanishes on $X_{3,(0,1,3,6),3}$. Since this polynomial is not divisible by $(t_1+t_2+t_3)(t_1-t_2)(t_1-t_3)(t_2-t_3)$, and all the 3×3 minors not from the first three columns are of degree greater than 5, this polynomial is not contained in the ideal generated by the 3×3 minors of $\phi_{3,\mu}(t_1,t_2,t_3)$. To show the claim take a point x on $X_{3,(0,1,3,6),3}$ with three different coordinates. Then the first two columns of $\phi_{3,(0,1,3,6)}(x_1,x_2,x_3)$ are linearly independent, so their span contains the third and fourth columns. Therefore there are $a,b,c,d \in \mathbb{C}$ such that $x_i^3 = ax_i + b$, and $x_i^6 = cx_i + d$ for $i = 1,2,3$, hence $(ax_i+b)^2 = cx_i + d$, implying that $a = 0$. So x_1, x_2, x_3 are the third roots of b, and the claim follows. Similarly, the 3×3 minors of $\phi_{n,(0,1,3,6)}(t_1,\dots,t_n)$ do not generate a radical ideal for any $n \geq 3$.

The author thanks Endre Szabó for helpful discussions.

2. On the geometry of $X_{n,\mu}$

In this section we analyze a bit the geometry of $X_{n,\mu}$. We assume here that the base field is $F = \mathbb{C}$, the field of complex numbers. We pass to the $n-1$ dimensional projective space $\mathbb{P}^{n-1}$, whose points are the lines in $\mathbb{A}^n$ containing the origin. A point of $\mathbb{P}^{n-1}$ is usually written as $(x_1 : \cdots : x_n)$, by which we mean the line spanned by the non-zero $(x_1,\dots,x_n) \in \mathbb{A}^n$. The coordinate ring $F[t_1,\dots,t_n]$ of $\mathbb{A}^n$ becomes the *homogeneous coordinate ring* of $\mathbb{P}^{n-1}$. Obviously, $X_{n,\mu}$ is a cone over the origin, hence the one dimensional subspaces contained in $X_{n,\mu}$ form a projective algebraic subset of $\mathbb{P}^{n-1}$, we denote it by $Y_{n,\mu}$. The ideal $I(Y_{n,\mu})$ of polynomials in $F[t_1,\dots,t_n]$ vanishing on $Y_{n,\mu}$ is the same as $I(X_{n,\mu})$.

Throughout this section we use the following notation:

$$\pi : \mathbb{P}^{n-1} \setminus \{p\} \to \mathbb{P}^{n-2}$$
$$(x_1 : \cdots : x_n) \mapsto (x_1 : \cdots : x_{n-1})$$

is the projection from $p = (0 : \cdots : 0 : 1) \in \mathbb{P}^{n-1}$.

First we show that $Y_{n,\mu}$ is unmixed of dimension $k-2$. For the proof we need to introduce a slightly more general class of varieties. As earlier, let $n, k \geq 2$ be integers, and $\mu = (\mu_1,\dots,\mu_k)$ with $0 \leq \mu_1 < \cdots < \mu_k$. For any positive integer m less than or equal to $n+1$ and k we put

$$Y_{n,\mu,m} = \{(x_1 : \cdots : x_n) | \mathrm{rk}((x_i^{\mu_j})_{i=1,\dots,n}^{j=1,\dots,k}) \leq m-1\}.$$

Note that in the special case $m = k$ we get $Y_{n,\mu,k} = Y_{n,\mu}$.

Proposition 2.1. *Any irreducible component of $Y_{n,\mu}$ is of dimension $k-2$.*

Proof. In the special case $m = 2$ it is easy to see that $Y_{n,\mu,2}$ is a finite subset of $\mathbb{P}^{n-1}$.

When $n = m-1$, then obviously $Y_{m-1,\mu,m} = \mathbb{P}^{m-2}$.

Now assume that $n \geq m \geq 3$, and by induction on m and n we show that $\dim(Y_{n,\mu,m}) \leq m-2$. It is clear that $\pi(Y_{n,\mu,m} \setminus \{p\})$ is contained in $Y_{n-1,\mu,m}$. By the induction hypothesis

$\dim(Y_{n-1,\mu,m-1}) \leq m-3$. Since adding one row to a matrix increases its rank at most by one, the whole $\pi^{-1}(Y_{n-1,\mu,m-1}) \cup \{p\}$ belongs to $Y_{n,\mu,m}$, and all of its components have dimension bounded by $(m-3)+1 = m-2$. Now take an irreducible component Y of $Y_{n,\mu,m}$ such that $\pi(Y)$ is not contained in $Y_{n-1,\mu,m-1}$. Denote by Z the Zariski closure of $\pi(Y)$ in $Y_{n-1,\mu,m}$. Then Z is an irreducible projective variety, and $Z \setminus Y_{n-1,\mu,m-1}$ is a Zariski dense open subset in it. Consider the restriction

$$\mathrm{res}_Y \pi : Y \setminus \{p\} \to Z$$

of the regular map π. For any $z \in Z \setminus Y_{n-1,\mu,m-1}$ there exists $1 \leq i_1 < \cdots < i_{m-1} \leq n-1$ and $1 \leq j_1 < \cdots < j_m \leq k$ such that some $m-1 \times m-1$ minor of the $m-1 \times m$ matrix $(z_{i_r}^{j_s})$ is non-degenerate. Then if $(z_1 : \cdots : z_{n-1} : z_n)$ is contained in $(\mathrm{res}_Y \pi)^{-1}(z)$, then z_n must be a root of the non-zero polynomial $\delta_{(\mu_{j_1},\ldots,\mu_{j_m})}(z_{i_1},\ldots,z_{i_{m-1}},t) \in \mathbb{C}[t]$. So Z contains a Zariski dense open subset over which $\mathrm{res}_Y(\pi)$ has finite fibers, implying that $\dim(Y) = \dim(Z)$ (see for example [M, 3.13]). By induction on n we know that $m-2 \geq \dim(Y_{n-1,\mu,m}) \geq \dim(Z)$.

We have shown that any irreducible component of $Y_{n,\mu,m}$ has dimension not exceeding $m-2$. On the other hand, a result from [Mac] shows that any irreducible component of $Y_{n,\mu}$ has dimension not smaller than $k-2$ (see also [E, Exercise 10.9]). □

Remark. $Y_{n,\mu,m}$ need not be unmixed in general. For example, $Y_{3,(0,1,3,4),3}$ is the union of $Y_{3,(0,1,2)}$ (which is the union of the three lines $V(t_i - t_j)$, $1 \leq i < j \leq 3$) and the two points $\{(1:\rho:\rho^2),(1:\rho^2:\rho)\}$, where ρ is a primitive third root of unity.

It is clear that π maps $Y_{n,\mu} \setminus \{p\}$ into $Y_{n-1,\mu}$. Denote by

$$\psi = \mathrm{res}_{Y_{n,\mu}} \pi : Y_{n,\mu} \setminus \{p\} \to Y_{n-1,\mu}$$

the restriction of π to $Y_{n,\mu}$.

Proposition 2.2. *Assume that $n > k \geq 3$ and $\mu_1 \leq 1$. Then there exists a Zariski dense open subset U of $Y_{n-1,\mu}$ such that for any $y \in U$ the fiber $\psi^{-1}(y)$ consists of μ_k different points.*

Proof. Let denote $K = F[t_1,\ldots,t_{k-1}]$. Consider $f(t) = \delta_\mu(t_1,\ldots,t_{k-1},t)$ as a polynomial of t over the ring K. It is of degree μ_k, and its leading coefficient is $\delta_{\mu'}(t_1,\ldots,t_{k-1})$, where μ' is the sequence obtained by removing μ_k from μ. Consider the discriminant $D_\mu(t_1,\ldots,t_{k-1})$ of $f(t)$, so D_μ is an element of K. We claim that D_μ is non-zero. Indeed, suppose in the contrary that $D_\mu = 0$. It means that $f(t)$ has a multiple root in the algebraic closure of the quotient field L of K. Therefore $f(t)$ has multiple factors in $L(t)$, and it follows easily that $f(t)$ has multiple factors in $K[t]$, contradicting to Lemma 1.1.

Take an irreducible component Y of $Y_{n-1,\mu}$, and assume that for some $1 \leq i_1 < \cdots < i_{k-1} \leq n-1$, the polynomial $\delta_{\mu'}(t_{i_1},\ldots,t_{i_{k-1}})$ does not vanish identically on Y. Then the image of Y under the projection

$$\tau : Y \setminus V(t_{i_1},\ldots,t_{i_{k-1}}) \to \mathbb{P}^{k-2}$$
$$(x_1 : \cdots : x_{n-1}) \mapsto (x_{i_1} : \cdots : x_{i_{k-1}})$$

is not contained in $V(\delta_{\mu'})$. Since all the coordinates of a point in the fiber of τ over any $(x_1 : \cdots : x_{k-1})$ contained in the set $\tau(Y)\setminus V(\delta_{\mu'})$ must be roots of the non-zero polynomial $\delta_\mu(x_{i_1},\ldots,x_{i_{k-1}},t)$, the map $\tau : Y \setminus V(t_{i_1},\ldots,t_{i_{k-1}}) \to \overline{\tau(Y)}$ has finite fibers over a Zariski dense open subset of the irreducible variety $\overline{\tau(Y)}$. It follows that $\dim(\overline{\tau(Y)}) = \dim(Y) = k-2$, so τ is a dominating morphism into $\mathbb{P}^{k-2}$. Hence by the previous paragraph $D_\mu(t_{i_1},\ldots,t_{i_{k-1}})$ is not identically zero on Y.

Now remove from $Y_{n-1,\mu}$ the $k-3$ dimensional algebraic subset $Y_{n-1,\mu'}$, and from all irreducible components Y of $Y_{n-1,\mu}$ remove the proper subvarieties $Y\cap V(\delta_{\mu'}(t_{i_1},\ldots,t_{i_{k-1}}))$ and $Y\cap V(D_\mu(t_{i_1},\ldots,t_{i_{k-1}}))$ for all $1\le i_1<\cdots<i_{k-1}\le n-1$ such that $\delta_{\mu'}(t_{i_1},\ldots,t_{i_{k-1}})$ is not identically zero on Y. In this way we get a Zariski dense open subset U of $Y_{n-1,\mu}$, and we claim that this U satisfies the conditions of the proposition.

Indeed, take any $y=(x_1:\cdots:x_{n-1})\in U$. It is not contained in $Y_{n-1,\mu'}$, hence there exists some $1\le j_1<\cdots<j_{k-1}\le n-1$ with $\delta_{\mu'}(x_{j_1},\ldots,x_{j_{k-1}})\neq 0$. Hence the $j_1,\ldots,j_{k-1}$ rows of $\phi_{n-1,\mu}(y)$ are linearly independent, and all the other rows of $\phi_{n-1,\mu}(y)$ are contained in their span. So $x=(y:x_n)$ is contained in $Y_{n,\mu}$ if and only if the last row of $\phi_{n,\mu}(x)$ is contained in the span of its $j_1,\ldots,j_{k-1}$ rows, that is, if x_n is a root of $\delta_\mu(x_{j_1},\ldots,x_{j_{k-1}},t)$. But the discriminant of this degree μ_k polynomial is non-zero by the construction of U, hence it has μ_k different roots, which yield the μ_k points in the fiber of ψ over y. □

Let Y be an irreducible projective subvariety of $\mathbb{P}^n$ of dimension r, let x be a point in $\mathbb{P}^n$, and let L be an $n-r$ dimensional linear subspace of $\mathbb{P}^n$ intersecting Y in finitely many points, y is one of them. We refer to [M] for the definitions of the following basic notions of algebraic geometry: the *degree* $\deg(Y)$ of Y, the *multiplicity* $\mathrm{mult}_x(Y)$ of the point x on Y, and the *intersection multiplicity* $i(y;Y\cap L)$ of Y and L in the point y. If Y is not necessarily irreducible, but unmixed, that is, its irreducible components $Y_1,\ldots,Y_m$ all have dimension r, then one can define $\deg(Y)=\sum_{j=1}^m \deg(Y_j)$, $\mathrm{mult}_x(Y)=\sum_{j=1}^m \mathrm{mult}_x(Y_j)$, $i(y;Y\cap L)=\sum_{j=1}^m i(y;Y_j\cap L)$. Recall that $\dim(Y)=\dim(I(Y))-1$ and $\deg(Y)=e(I(Y))$. These definitions apply for $Y_{n,\mu}$, and we have the following recursive formula for its degree.

Proposition 2.3. *Assume that $n>k\ge 3$ and $\mu_1\le 1$. Then we have the formula*

$$deg(Y_{n,\mu}) = mult_p(Y_{n,\mu}) + \mu_k deg(Y_{n-1,\mu}).$$

Proof. Let M be an $n-k$ dimensional linear subvariety of $\mathbb{P}^{n-2}$, and denote by $L=\pi^{-1}(M)\cup\{p\}$ its preimage under the projection π, so L is an $n-k+1$ dimensional linear subspace of $\mathbb{P}^{n-1}$ containing p. It is well known (see for example [M, Chapter 5.]), that if M is sufficiently general, then the following conditions hold:

(1) M intersects $Y_{n-1,\mu}$ in $\deg(Y_{n-1,\mu})$ points, and all of them are contained in the Zariski dense open subset U taken from Proposition 2.2;
(2) $i(p;Y_{n,\mu}\cap L)=\mathrm{mult}_p(Y_{n,\mu})$;
(3) L intersects $Y_{n,\mu}$ transversally in finitely many points besides p, that is, $i(x;Y_{n,\mu}\cap L)=1$ for any $x\neq p$ from the finite set $L\cap Y_{n,\mu}$.

Assume that M and L satisfy the above conditions. Then by [M, 5.3] and the choice of M we have

$$\deg(Y_{n,\mu}) = \sum_{x\in Y_{n,\mu}\cap L} i(x;Y_{n,\mu}\cap L) = i(p;Y_{n,\mu}\cap L) + \sum_{x\in Y_{n,\mu}\cap L\setminus\{p\}} 1$$
$$= \mathrm{mult}_p(Y_{n,\mu}) + |Y_{n,\mu}\cap L\setminus\{p\}|.$$

Clearly, we have $Y_{n,\mu}\cap L\setminus\{p\} = \psi^{-1}(M\cap Y_{n-1,\mu})$, and by Proposition 2.2 and the choice of M this set contains $\mu_k \deg(Y_{n-1,\mu})$ points. □

Now we shall determine the tangent cone of $Y_{n,\mu}$ at p. Consider the affine space $\mathbb{A}^{n-1} = \mathbb{P}^{n-1}\setminus V(t_n)$ with affine coordinates $(\frac{t_1}{t_n},\dots,\frac{t_{n-1}}{t_n}) = (u_1,\dots,u_{n-1})$. The point p corresponds to the origin, and denote by X the affine part $Y_{n,\mu}\cap\mathbb{A}^{n-1}$ of $Y_{n,\mu}$. Any non-zero polynomial $f\in F[u_1,\dots,u_{n-1}]$ can be decomposed as a sum $f = f_s + f_{s+1} + \cdots + f_m$, where f_i is homogeneous of degree i, and f_s is non-zero. The homogeneous component f_s is called the *leading term of f at 0,* and is denoted by $\mathrm{lead}_0(f)$. Recall that the *tangent cone* TC_0X at 0 is the common zero locus of the leading terms of the elements in the ideal of X, so $TC_0X = V(\mathrm{lead}_0(f)|f\in I(X))$ (see for example [M, 5.10]). The multiplicity $\mathrm{mult}_0(X) = \mathrm{mult}_p(Y_{n,\mu})$ equals the degree of $\langle \mathrm{lead}_0(f)|f\in I(X)\rangle$.

Proposition 2.4. *The tangent cone of X at 0 is the algebraic set $X_{n-1,\mu'}$, where μ' is the sequence $(\mu_1,\dots,\mu_{k-1})$.*

Proof. Denote $\hat{\mu}_i$ the sequence of $k-1$ increasing integers obtained by removing the ith member of μ, for example, $\hat{\mu}_k = \mu'$. For any $1\le i_1 < \cdots < i_{k-1}\le n-1$ the polynomial $\delta_\mu(t_{i_1},\dots,t_{i_{k-1}},t_n)$ is contained in $I(Y_{n,\mu})$, and on passing to $\mathbb{A}^{n-1} = \mathbb{P}^{n-1}\setminus V(t_n)$ it corresponds to

$$\delta_{\hat{\mu}_k}(u_{i_1},\dots,u_{i_{k-1}}) - \delta_{\hat{\mu}_{k-1}}(u_{i_1},\dots,u_{i_{k-1}}) \pm\cdots + (-1)^{k+1}\delta_{\hat{\mu}_1}(u_{i_1},\dots,u_{i_{k-1}}),$$

as one can see it from the expansion of the determinant of $(t_i^{\mu_j})_{1\le i,j\le k}$ according to its last row. The leading terms $\delta_{\mu'}(u_{i_1},\dots,u_{i_{k-1}})$ (with $1\le i_1<\cdots<i_{k-1}\le n-1$) are contained in $I(TC_0X)$, and their common zero locus is $X_{n-1,\mu'}$ by definition, implying that $X_{n-1,\mu'}$ contains TC_0X.

Now we turn to the proof of the reverse inclusion. Let $x = (x_1,\dots,x_{n-1})\in X_{n-1,\mu'}$ be a non-zero point. We claim that the line $\{\lambda x|\lambda\in\mathbb{C}\}$ belongs to TC_0X. We use the following geometric characterization of the tangent cone [M, 5.7 b)]: the projectivized tangent cone $\mathbb{P}TC_0X$ is the set of limiting positions of lines $\overline{0q}$ as $q\in X\setminus\{0\}$ approaches 0.

Clearly, some $(z_1,\dots,z_{n-1})$ belongs to X if and only if $(z_1,\dots,z_{n-1},1)$ belongs to $X_{n,\mu}$. If the rank of the matrix $(x_i^{\mu_j})_{i=1,\dots,n-1}^{j=1,\dots,k}$ is at most $k-2$, then the whole line $\overline{0x}$ lies on X, so it belongs to TC_0X.

From now on we suppose that $\mathrm{rk}((x_i^{\mu_j})_{i=1,\dots,n-1}^{j=1,\dots,k}) = k-1$, say the first $k-1$ rows of this matrix are linearly independent. So in the $(k-1)\times k$ matrix $(x_i^{\mu_j})_{i=1,\dots,k-1}^{j=1,\dots,k}$ the $(k-1)\times(k-1)$ minor obtained by removing the last column is degenerate, but there exists some $1\le m\le k-1$ such that the minor obtained by removing the mth column is non-degenerate.

For any $y, z \in \mathbb{A}^{n-1}$ let denote $d(y, z)$ the usual *Euclidean distance* of y and z.

Take a $y = (y_1, \ldots, y_{n-1}) \in X_{n-1,\mu}$. If y is sufficiently close to x, then the minor obtained by removing the mth column of $(y_i^{\mu_j})_{i=1,\ldots,k-1}^{j=1,\ldots,k}$ is non-degenerate. Some point λy on $\overline{0y}$ belongs to X if and only if $(1, \ldots, 1)$ is contained in the row space of the matrix $((\lambda y_i)^{\mu_j})_{i=1,\ldots,n-1}^{j=1,\ldots,k}$, that is, if λ is a root of the polynomial $\delta_\mu(ty_1, \ldots, ty_{k-1}, 1) \in \mathbb{C}[t]$. We have the equalities

$$\delta_\mu(ty_1, \ldots, ty_{k-1}, 1) = t^{\mu_1+\cdots+\mu_{k-1}}(\delta_{\hat{\mu}_k}(y_1, \ldots, y_{k-1}) - t^{\mu_k-\mu_{k-1}}\delta_{\hat{\mu}_{k-1}}(y_1, \ldots, y_{k-1})\pm \cdots + (-1)^{k+1}t^{\mu_k-\mu_1}\delta_{\hat{\mu}_1}(y_1, \ldots, y_{k-1})) = t^{\mu_1+\cdots+\mu_{k-1}}g(y_1, \ldots, y_{k-1})(t).$$

We may choose y on $X_{n-1,\mu}$ arbitrarily close to x satisfying also $\delta_{\mu'}(y_1, \ldots, y_{k-1}) \neq 0$. Indeed, let Z be an irreducible component of $X_{n-1,\mu}$ containing x. The polynomial $\delta_{\hat{\mu}_m}(u_1, \ldots, u_{k-1})$ is not identically zero on Z (since it does not vanish in x), hence the projection

$$\tau : \mathbb{A}^{n-1} \to \mathbb{A}^{k-1}$$
$$(z_1, \ldots, z_{n-1}) \mapsto (z_1, \ldots, z_{k-1})$$

has finite fibers over the Zariski dense open subset $\overline{\tau(Z)} \setminus V(\delta_{\hat{\mu}_m})$, hence by Proposition 2.1 $\dim(\overline{\tau(Z)}) = \dim(Z) = k - 1$, implying that $\mathrm{res}_Z\tau : Z \to \mathbb{A}^{k-1}$ is a dominating morphism, so $\delta_{\mu'}(u_1, \ldots, u_{k-1})$ is not identically zero on Z (we have used already this argument in the proof of Proposition 2.2). The proper algebraic subset $Z \cap V(\delta_{\mu'}(u_1, \ldots, u_{k-1}))$ of Z does not contain a neighborhood (in the Euclidean topology) of x, implying our claim.

Zero is a root of the non-constant polynomial $g(x_1, \ldots, x_{k-1})(t) \in \mathbb{C}[t]$. Since the coefficients of the polynomial $g(y_1, \ldots, y_{k-1})(t)$ depend continuously on y, if y is close enough to x, then the polynomial has a root close to zero. More precisely, for any natural n there exists $y^{(n)}$ on $X_{n-1,\mu}$ satisfying the following conditions:

(1) $d(x, y^{(n)}) < \frac{1}{n}$;
(2) $\delta_{\hat{\mu}_m}(y_1, \ldots, y_{k-1}) \neq 0$;
(3) $\delta_{\mu'}(y_1, \ldots, y_{k-1}) \neq 0$;
(4) $g(y_1^{(n)}, \ldots, y_{k-1}^{(n)})(t)$ has a non-zero root λ_n with $d(0, \lambda_n y^{(n)}) < \frac{1}{n}$.

These conditions imply that $z^{(n)} = \lambda_n y^{(n)}$ lie on $X \setminus \{0\}$, they tend to 0 as n goes to infinity, and the limit of the secant lines $\overline{0z^{(n)}}$ is the line $\overline{0x}$. □

3. Monomial ideals

A summary about Gröbner bases can be found for example in [E, Chapter 15]. To speak about Gröbner bases one has to fix an admissible order $\prec$ of the monomials in $F[t_1, \ldots, t_n]$. For any $f \in F[t_1, \ldots, t_n]$ denote by $\mathrm{In}_\prec(f)$ the *initial monomial* of f, that is, the monomial of f with non-zero coefficient which is maximal with respect to $\prec$. The *initial ideal* of an ideal I is

$$\mathrm{In}_\prec(I) = \langle \mathrm{In}_\prec(f) | f \in I \rangle,$$

the ideal generated by the initial monomials of the elements of I. The polynomials $f_1,\dots,f_m \in I$ form a *Gröbner basis* of I with respect to $\prec$ if $\mathrm{In}_\prec(I)$ is generated by $\mathrm{In}_\prec(f_1),\dots,\mathrm{In}_\prec(f_m)$. In this case I is generated by $f_1,\dots,f_m$. Any ideal has a finite Gröbner basis. We shall use the fact that the residue classes of monomials not contained in $\mathrm{In}_\prec(I)$ form an F-linear basis of the factor ring $F[t_1,\dots,t_n]/I$. In particular, the Hilbert polynomial of a homogeneous ideal I coincides with the Hilbert polynomial of its initial ideal. A subset of I is called a *universal Gröbner basis* if it is a Gröbner basis with respect to any admissible monomial order.

Critical monomial ideals have the following algebraic characterization.

Lemma 3.1. *A monomial ideal is critical if and only if all of its associated primes have the same dimension.*

Proof. Let $M = Q_1 \cap \dots \cap Q_p$ be an irredundant primary decomposition of a monomial ideal. Then $\sqrt{Q}_1,\dots,\sqrt{Q}_p$ are the associated primes of M. We may assume that the first m of them have maximal dimension. It is well known that $e(M) = e(Q_1) + \dots + e(Q_m)$.

If $m < p$, then $Q_1 \cap \dots \cap Q_m$ is a monomial ideal that strictly contains M and has the same degree as M, so M is not critical.

Conversely, assume that $m = p$. Let w be any monomial not contained in M. Then w is not contained in some primary component of M, say $w \notin Q_1$. The prime ideal $\sqrt{Q}_1$ is generated by $n-k$ variables from $\{t_1,\dots,t_n\}$, where $k = \dim(M)$. Thus there are k variables not contained in $\sqrt{Q}_1$, say $t_1,\dots,t_k \notin \sqrt{Q}_1$. Since Q_1 is primary, it follows that $wt_1^{\alpha_1}\cdots t_k^{\alpha_k} \notin Q_1 \supseteq M$ for any $\alpha_1,\dots,\alpha_k$. The number of degree d monomials of this form is given by a polynomial $P(d)$ for sufficiently large d, where the leading term of $P(d)$ is $\frac{d^{k-1}}{(k-1)!}$. This clearly implies that $\dim(\langle w, M\rangle) < \dim(M)$ or $\deg(\langle w, M\rangle) < \deg(M)$. □

Now we turn to the investigation of our concrete ideal.

Proposition 3.2. *(i) If $D = \{\delta_\mu(t_{i_1},\dots,t_{i_k}) | 1 \le i_1 < \dots < i_k \le n\}$ is a Gröbner basis of $\langle D\rangle$ for any admissible monomial order satisfying $t_1 \prec \dots \prec t_n$, then D is a universal Gröbner basis of $\langle D\rangle$.*

(ii) Let $\prec$ be an admissible monomial order with $t_1 \prec \dots \prec t_k$. Then the initial monomial of $\delta_\mu(t_1,\dots,t_k) = t_1^{\mu_1}\cdots t_k^{\mu_k}$.

Proof. (i) See the proof of [Lo, Lemma 2.1]. The key fact is that D is stabilized by any permutation of the variables. On the other hand, any admissible monomial order can be obtained from an order with $t_1 \prec \dots \prec t_n$ using a permutation of the variables.

(ii) Any monomial of δ_μ is of the form $t_{\pi(1)}^{\mu_1}\cdots t_{\pi(k)}^{\mu_k}$ for some $\pi \in \mathrm{Sym}(k)$, so the statement is obvious. □

Thus we may restrict to monomial orders with $t_1 \prec \dots \prec t_n$, and then the ideal generated by the initial monomials of the elements in D is

$$M_{n,\mu} = \langle t_{i_1}^{\mu_1}\cdots t_{i_k}^{\mu_k} | 1 \le i_1 < \dots < i_k \le n\rangle.$$

For any $1 \le i_1 < \dots < i_{n-k+1} \le n$ we put

$$Q_{n,\mu}(i_1,\dots,i_{n-k+1}) = \langle t_{i_1}^{\mu_{i_1}},\dots,t_{i_s}^{\mu_{i_s-s+1}},\dots,t_{i_{n-k+1}}^{\mu_{i_{n-k+1}-n+k}}\rangle.$$

Proposition 3.3. *The irredundant primary decomposition of* $M_{n,\mu}$ *is*

$$M_{n,\mu} = \bigcap_{1\le i_1<\cdots<i_{n-k+1}\le n} Q_{n,\mu}(i_1,\dots,i_{n-k+1})$$

(if $\mu_1 = 0$*, then the intersection runs over* $2 \le i_1 < \cdots < i_{n-k+1} \le n$*).*

Proof. Clearly, $Q_{n,\mu}(i_1,\dots,i_{n-k+1})$ is a primary ideal with radical $\langle t_{i_1},\dots,t_{i_{n-k+1}}\rangle$.

$\subseteq$: Consider a monomial $w = t_{j_1}^{\mu_1}\cdots t_{j_k}^{\mu_k} \in M_{n,\mu}$. For any $1 \le i_1 < \cdots < i_{n-k+1} \le n$ the intersection $\{j_1,\dots,j_k\} \cap \{i_1,\dots,i_{n-k+1}\}$ is non-empty, let $j_r = i_s$ be the element of the intersection with r,s maximal. We have the inequality $i_s \le n-(k-r)-(n-k+1-s) = r+s-1$, implying that $w \in \langle t_{j_r}^{\mu_r}\rangle \subseteq \langle t_{i_s}^{\mu_{i_s-s+1}}\rangle \subseteq Q_{n,\mu}(i_1,\dots,i_{n-k+1})$.

$\supseteq$: Let $w = t_1^{\alpha_1}\cdots t_n^{\alpha_n}$ be a monomial contained in the intersection on the right-hand side. We define recursively $j_k > \cdots > j_1$ as follows. Since $w \in Q_{n,\mu}(k,k+1,\dots,n) = \langle t_k^{\mu_k}, t_{k+1}^{\mu_k},\dots,t_n^{\mu_k}\rangle$, there exists some j with $\alpha_j \ge \mu_k$ and $k \le j \le n$. We put j_k for the maximal such j. Assume that we have already defined $j_k > \cdots > j_{k-r+1} \ge k-r+1$ $(1 \le r < k)$. Then w is contained in $Q_{n,\mu}(i_1,\dots,i_{n-k+1})$, where $\{i_1,\dots,i_{n-k+1}\} \cup \{j_k,j_{k-1},\dots,j_{k-r+1}\} = \{k-r,k-r+1,\dots,n\}$. Therefore there exists some $k-r \le j < j_{k-r+1}$ with $\alpha_j \ge \mu_{k-r}$, and we put j_{k-r} for the maximal such j. Now w is divisible by $t_{j_1}^{\mu_1}\cdots t_{j_k}^{\mu_k}$ by construction, hence $w \in M_{n,\mu}$. □

Corollary 3.4. *For any* $n \ge k \ge 2$ *the monomial ideal* $M_{n,\mu}$ *is* $(k-1)$*-dimensional, critical, and its degree equals to the* $(n+1-k)$*th complete symmetric function of* $\mu_1,\dots,\mu_k$*. In particular, we have the recursion* $e(M_{n,\mu}) = e(M_{n-1,\mu'}) + \mu_k e(M_{n-1,\mu})$*, where* $n > k \ge 3$ *and* $\mu' = (\mu_1,\dots,\mu_{k-1})$*.*

Proof. By Proposition 3.3 any associated prime of $M_{n,\mu}$ is of the form $\langle t_{i_1},\dots,t_{i_{n-k+1}}\rangle$, so $(k-1)$-dimensional, and hence by Lemma 3.1 $M_{n,\mu}$ is critical. The primary decomposition shows also that

$$e(M_{n,\mu}) = \sum \deg(Q_{n,\mu}(i_1,\dots,i_{n-k+1})) = \sum_{1\le i_1<\cdots<i_{n-k+1}\le n} \mu_{i_1}\mu_{i_2-1}\cdots\mu_{i_{n-k+1}-n+k}.$$

□

Proof of Theorem 1.2. Since the polynomial δ_μ has rational coefficients, if the theorem holds over $\mathbb{C}$ then it holds over any algebraically closed field of characteristic zero. Hence we may assume that $F = \mathbb{C}$, and we can use the results of Section 2.

Let $\prec$ be an admissible monomial order with $t_1 \prec \cdots \prec t_n$. Since the polynomials $\delta_\mu(t_{i_1},\dots,t_{i_k})$ vanish on $Y_{n,\mu}$, we have $\mathrm{In}_\prec(I(Y_{n,\mu})) \supseteq M_{n,\mu}$, and our aim is to show that equality holds here. By Corollary 3.4 $M_{n,\mu}$ is critical, therefore it is sufficient to show that $\dim(\mathrm{In}_\prec(I(Y_{n,\mu}))) = \dim(M_{n,\mu})$ and $e(\mathrm{In}_\prec(I(Y_{n,\mu}))) = e(M_{n,\mu})$. The Hilbert polynomials of $I(Y_{n,\mu})$ and $\mathrm{In}_\prec(I(Y_{n,\mu}))$ coincide, hence we have $\dim(\mathrm{In}_\prec(I(Y_{n,\mu}))) = \dim(Y_{n,\mu})+1 = k-1 = \dim(M_{n,\mu})$ and $e(\mathrm{In}_\prec(I(Y_{n,\mu}))) = \deg(Y_{n,\mu})$.

By double induction on k and n we prove that $e(M_{n,\mu}) = \deg(Y_{n,\mu})$.

Consider first the case $n = k \ge 2$. Then $e(M_{k,\mu}) = \mu_1 + \cdots + \mu_k$ by Proposition 3.4. The algebraic set $Y_{k,\mu}$ is a hypersurface defined by $\delta_\mu(t_1,\dots,t_k)$, and since this polynomial has no multiple factors by Lemma 1.1, we have $\deg(Y_{k,\mu}) = \deg(\delta_\mu) = \mu_1 + \cdots + \mu_k$.

Assume next that $n \geq k = 2$. If $\mu_1 = 0$, then $Y_{n,(0,\mu_2)}$ consists of the points $(x_1 : \cdots : x_n)$ whose any coordinate is a μ_2th root of unity, and there are μ_2^{n-1} such points. If $\mu_1 = 1$, then $Y_{n,(1,\mu_2)}$ consists of the points whose any coordinate is either zero or a $(\mu_2 - 1)$th root of unity, and there are $\frac{\mu_2^n - 1}{\mu_2 - 1}$ such points. In both cases we have $\deg(Y_{n,\mu}) = \frac{\mu_2^n - \mu_1^n}{\mu_2 - \mu_1} = e(M_{n,\mu})$.

Finally, assume that $n > k \geq 3$. By Corollary 3.4 and the induction hypothesis we have

$$e(M_{n,\mu}) = e(M_{n-1,\mu'}) + \mu_k e(M_{n-1,\mu}) = \deg(Y_{n-1,\mu'}) + \mu_k \deg(Y_{n-1,\mu}), \tag{3.1}$$

and by Proposition 2.3 we have

$$\deg(Y_{n,\mu}) = \mathrm{mult}_p(Y_{n,\mu}) + \mu_k \deg(Y_{n-1,\mu}). \tag{3.2}$$

We use the notation of Proposition 2.4. The multiplicity $\mathrm{mult}_p(Y_{n,\mu}) = \mathrm{mult}_0(X)$ equals $(k-3)!$ times the leading coefficient of the Hilbert polynomial of the ideal $\langle \mathrm{lead}_0(f) | f \in I(X)\rangle$ (see for example [H, p.258]). The common zero set of $\langle \mathrm{lead}_0(f) | f \in I(X)\rangle$ is the tangent cone TC_0X, which by Proposition 2.4 coincides with $X_{n-1,\mu'}$. On the other hand, we saw in the proof of Proposition 2.4 that

$$\langle \mathrm{lead}_0(f) | f \in I(X)\rangle \supseteqq \langle \delta_{\mu'}(t_{i_1}, \ldots, t_{i_{k-1}}) | 1 \leq i_1 < \cdots < i_{k-1} \leq n-1\rangle,$$

and by the induction hypothesis this latter ideal is exactly the ideal of $X_{n-1,\mu'}$. It follows that $F[u_1, \ldots, u_{n-1}]/\langle \mathrm{lead}_0(f) | f \in I(X)\rangle$ is the coordinate ring of $X_{n-1,\mu'}$, and so

$$\mathrm{mult}_p(Y_{n,\mu}) = \deg(Y_{n-1,\mu'}). \tag{3.3}$$

Comparing (3.1), (3.2), and (3.3) we obtain the desired equality $e(M_{n,\mu}) = \deg(Y_{n,\mu})$.

So we proved that $\mathrm{In}_\prec(I(Y_{n,\mu})) = M_{n,\mu}$, and by Proposition 3.2 this implies the universal Gröbner basis property. □

Corollary 3.4 and Theorem 1.2 have the following corollary:

Corollary 3.3. *The degree of $X_{n,\mu}$ equals the $(n+1-k)$th complete symmetric function of $\mu_1, \ldots, \mu_k$, that is,*

$$\deg(X_{n,\mu}) = \sum_{j_1+\cdots+j_k = n-k+1} \mu_1^{j_1} \cdots \mu_k^{j_k}.$$

□

Remark. Since $M_{n,\mu}$ is critical, to verify the equality $\mathrm{In}_\prec(I(X_{n,\mu})) = M_{n,\mu}$ it was sufficient to show that the dimension and the degree of $M_{n,\mu}$ computed by combinatorial arguments is what was predicted by the geometry of $X_{n,\mu}$. So it is a natural question for which affine cones it is true that the initial ideal of their vanishing ideal is critical. Let X be an affine cone over the origin. Obviously, if $\mathrm{In}_\prec(I(X))$ is critical then X must be unmixed (otherwise $\mathrm{In}_\prec(I(X')) \supsetneqq \mathrm{In}_\prec(I(X))$ and $e(\mathrm{In}_\prec(I(X'))) = e(\mathrm{In}_\prec(I(X)))$, where X' is the union of the irreducible components of X having maximal dimension). The converse is not true, that is, the unmixedness of X does not imply that $\mathrm{In}_{\prec}(I(X))$ is critical, as the following example shows. We put

$$X = \{(x,x,y,y),(x,y,x,y),(x,y,y,x) | x,y \in \mathbb{C}\} \subset \mathbb{A}^4,$$

so X is the union of three 2 dimensional linear subspaces, and $h_X(t) = 3t+$ some constant . The polynomials $(t_{i_1} - t_{i_2})(t_{i_1} - t_{i_3})(t_{i_2} - t_{i_3})$ and $(t_{i_4} - t_{i_1})(t_{i_4} - t_{i_2})(t_{i_4} - t_{i_3})$ with $\{i_1, i_2, i_3, i_4\} = \{1,2,3,4\}$ vanish on X. Let $\prec$ be the lexicographic order induced by $t_1 \prec t_2 \prec t_3 \prec t_4$. Then

$$\mathrm{In}_{\prec}(I(X)) \supseteq \langle t_2t_3^2,\ t_2t_4^2,\ t_3t_4^2,\ t_2^2t_4,\ t_3^2t_4,\ t_2t_3t_4,\ t_4^3\rangle$$

($t_2^2t_4$ is the initial monomial of $(t_1-t_2)(t_1-t_3)(t_1-t_4)+(t_2-t_1)(t_2-t_3)(t_2-t_4)$). Those degree d monomials outside $\mathrm{In}_{\prec}(I(X))$ which are divisible by t_4 are all contained in the set $\{t_1^{d-2}t_4^2,\ t_1^{d-2}t_2t_4,\ t_1^{d-2}t_3t_4\}$, hence

$$\#\{\text{degree } d \text{ monomials } \notin \mathrm{In}_{\prec}(I(X))\} - \#\{\text{degree } d \text{ monomials } \notin (\mathrm{In}_{\prec}(I(X)), t_4)\} \leq 3.$$

It follows that $e(\mathrm{In}_{\prec}(I(X)), t_4) = e(\mathrm{In}_{\prec}(I(X)))$, though t_4 is clearly not contained in the ideal $\mathrm{In}_{\prec}(I(X))$.

4. Subspace arrangements related to reflection groups

A finite collection $\mathcal{H}$ of $n-1$ dimensional linear subspaces of $\mathbb{C}^n$ is called a *complex hyperplane arrangement.* For any $1 \leq k \leq n-1$ denote by $\mathcal{H}(k)$ the collection of k dimensional linear subspaces which are intersections of elements of $\mathcal{H}$. It is called the $(n-k)$-*truncation of* $\mathcal{H}$, following the terminology of [B, 5.1.(iv)]. An interesting example is the so called *braid arrangement,* consisting of the hyperplanes $V(t_i - t_j)$ with $1 \leq i < j \leq n$. The union of the elements of the $(n-k)$-truncation of the braid arrangement is

$$\{x \in \mathbb{C}^n | x \text{ has at most } k \text{ different coordinates}\},$$

and the special case $(\mu_1, \ldots, \mu_{k+1}) = (0,1,2,\ldots,k)$ of Theorem 1.2 gives a Gröbner basis of the vanishing ideal of this subspace arrangement. The reflections with respect to the hyperplanes in the braid arrangement generate the standard representation of the symmetric group $\mathrm{Sym}(n)$. There are other interesting hyperplane arrangements belonging to the other Weyl groups, or more generally, to the complex pseudo-reflection groups. Recall that a linear transformation of $\mathbb{C}^n$ is called a *pseudo-reflection,* if it is of finite order and it fixes a hyperplane. A subgroup of $Gl_n(\mathbb{C})$ generated by pseudo-reflections is called a *pseudo-reflection group.*

Now let G be a finite pseudo-reflection group, and denote by $\mathcal{H}_G$ the set of the reflecting hyperplanes. The problem to describe a Gröbner basis of the vanishing ideal of the G-orbit of a k dimensional subspace which is the intersection of reflecting hyperplanes (orbit arrangement) was raised in [B, 13.5]. Note that the union of these orbit arrangements is the $(n-k)$-truncation of $\mathcal{H}_G$. We would like to point out that the corresponding special cases of Theorem 1.2 give Gröbner bases of the vanishing ideal of the $(n-k)$-truncation of $\mathcal{H}_G$ for all but finitely many irreducible pseudo-reflection groups G.

The finite irreducible unitary pseudo-reflection groups were classified in [ST]. We discussed already the infinite series of the symmetric groups $\mathrm{Sym}(n)$. (To get an irreducible representation one has to restrict the standard representation onto the hyperplane $V(t_1+\cdots+t_n)$.) For the dihedral groups (the symmetry groups of regular m-gons) our question is not interesting, because then $n=2$ and the only truncations of $\mathcal{H}_G$ are itself and $\{0\}$. Besides finitely many exceptional groups the remaining groups in the list are the groups $G(m,q,n)$, where $m\geq 2$ and q divides m, defined as follows:

Consider $\mathrm{Sym}(n)$ as a subgroup of $Gl_n(\mathbb{C})$ consisting of permutation matrices, and denote by $D(m,q,n)$ the group of diagonal matrices whose diagonal entries are mth roots of unity, and whose determinant is an m/qth root of unity. Obviously, $\mathrm{Sym}(n)$ normalizes $D(m,q,n)$, so they generate in $Gl_n(\mathbb{C})$ their semi-direct product $G(m,q,n)=D(m,q,n)\rtimes \mathrm{Sym}(n)$.

Case I: $G=G(m,q,n)$, $m\geq 2$, $q\neq m$. In this case the reflecting hyperplanes are

$$\mathcal{H}_G=\{V(t_i-\zeta t_j),\ V(t_l)|1\leq i<j\leq n,\ l=1,\dots,n,\ \zeta^m=1\}.$$

We claim that for any $1\leq k\leq n-1$ the union of the linear subspaces in the $(n-k)$-truncation of $\mathcal{H}_G$ is $X_{n,\mu}$ with $\mu=(1,m+1,2m+1,\dots,km+1)$. By definition we have

$$X_{n,\mu}=\{x\in\mathbb{C}^n|x_{i_1}\cdots x_{i_{k+1}}\prod_{1\leq r<s\leq k+1}(x_{i_s}^m-x_{i_r}^m)=0 \text{ for any } 1\leq i_1<\cdots<i_{k+1}\leq n\}.$$

On the other hand, take an element L of $\mathcal{H}_G(k)$. There are k "free coordinates" on L, say $t_1,\dots,t_k$, and L is defined by $n-k$ linear equations $l_{k+1}=\cdots=l_n=0$, where $l_j=t_j$ or $l_j=t_j-\zeta_j t_{i_j}$ with $\zeta_j^m=1$ and $i_j\in\{1,\dots,k\}$. It is easy to see that the union of all such subspaces is

$$\{x\in\mathbb{C}^n|\text{ among any } k+1 \text{ coordinates of } x \text{ there is a } 0$$
$$\text{or there are two whose } m\text{th powers are equal}\}=X_{n,\mu}$$

Note that in the special case $m=2$ we get the Weyl group $G(2,1,n)$ associated with the Dynkin diagram B_n. (In this case the subspace arrangement is defined over the real numbers, and the conclusion of Theorem 1.2 clearly holds with $F=\mathbb{R}$.)

Case II: $G=G(m,m,n)$, $m\geq 2$. In this case we have

$$\mathcal{H}_G=\{V(t_i-\zeta t_j)|1\leq i<j\leq n,\ \zeta^m=1\},$$

and similarly to the above case for any $1\leq k\leq n-1$ we have $\cup\mathcal{H}_G(k)=X_{n,\mu}$ with $\mu=(0,m,2m,\dots,km)$. In the special case $m=2$ the group $G(2,2,n)$ is the Weyl group belonging to the Dynkin diagram D_n.

So the special cases $\mu=(1,m+1,2m+1,\dots,km+1)$ and $\mu=(0,m,2m,\dots,km)$ of Theorem 1.2 give Gröbner bases of the vanishing ideals of $(n-k)$-truncations of the reflection arrangements belonging to the pseudo-reflection groups $G(n,m,q)$.

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