The index of $grad\, f(x, y)$

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Abstract

Let $f(x, y)$ be a real polynomial of degree d with isolated critical points, and let i be the index of $grad\, f$ around a large circle containing the critical points. An elementary argument shows that $|i| \leq d - 1$. In this paper we show that $i \leq max\{1, d-3\}$. We also show that if all the level sets of f are compact, then $i = 1$, and otherwise $|i| \leq d_R - 1$ where d_R is the sum of the multiplicities of all the real linear factors in the homogeneous term of highest degree in f. The technique of proof involves computing i from information at infinity. The index i is broken up into a sum of components $i_{p,c}$ corresponding to points p in the real line at infinity $\mathbb{L}$ and limiting values $c \in \mathbb{R} \cup \{\infty\}$ of the polynomial. We compute the numbers $i_{p,c}$ in three ways: geometrically, from a resolution of $f(x, y)$, and from a Morsification of $f(x, y)$. We also show that the $i_{p,c}$ provide a lower bound for the number of vanishing cycles of $f(x, y)$ at the point p and value c.

Contents

1 Introduction

Let $f(x, y)$ be a real polynomial with isolated critical points. Let i be the index of the gradient vector field of $f(x, y)$ around a large circle C containing the critical points, oriented in the counterclockwise direction. (Recall that the index is the topological degree of the map $C \to S^1$ defined by $t \mapsto grad\, f(\gamma(t))/|grad\, f(\gamma(t))|$ where $t \mapsto \gamma(t)$ is a parameterization of C.) If the critical points of f are nondegenerate, then the index i is the number of local extrema minus the number of saddles.

What bounds can be placed on the index i in terms of the degree d of the polynomial? It follows easily from Bezout's theorem that [DKM+93, Proposition 2.5]

$$|i| \leq d - 1$$

It is easy to find polynomials satisfying the lower bound of this inequality; for example if $f = l_1 \dots l_d$ where the l_i are equations of lines in general position, then $i = 1 - d$, as can be seen by looking at how the gradient vector field turns on the circle C or by counting critical points [DKM+93, Section 4].

The upper bound is more mysterious. In the first place, polynomials with $i > 1$ are hard to find. (The dubious reader should try to do so!) A simple example with two local extrema and no other critical points ($i = 2$) is $f(x, y) = y^5 + x^2y^3 - y$. A polynomial of degree five can have as many as sixteen critical points in the complex plane and a generic polynomial of degree five will have exactly this number. Note however that the above polynomial has only four critical points in the plane (two real and two complex), so it is not generic. In fact this behavior is typical for polynomials with $i > 1$.

There are polynomials of degree d with i arbitrarily large [DKM+93, Section 2], but they have $i \approx (1/3)d$. So evidently there is a large gap between the theoretical upper bound and examples. One of the goals of this paper is to give a modest improvement of this upper bound. We will show

Theorem 6.7. If $f(x, y)$ is a real polynomial of degree d with isolated critical points, and i is the index of $grad\, f$ around a large circle containing the critical points, then

$$i \leq max\{1, d-3\}$$

In particular this result implies that the minimum degree for a polynomial with $i > 1$ is five, as in the example above. In fact, the bound of the theorem can be improved in many cases, as we will see in the proof.

Let d_R be the sum of the multiplicities of all the real linear factors in the homogeneous term of highest degree in f. We will also show

Theorem 6.4. If all the level sets of the polynomial $f(x,y)$ are compact, then $i = 1$. Otherwise

$$|i| \leq d_R - 1$$

The basic idea is to compute the index i from "information at infinity". We write i as

$$i = 1 + \sum_{\substack{p \in \mathbb{L} \\ c \in \mathbb{R} \cup \{\infty\}}} i_{p,c}$$

The terms $i_{p,c}$ are defined as follows: The number $\pm 1/2$ is assigned to a point where the circle C is tangent to a level set of the polynomial according as whether the level set is locally inside or outside C. The circle is then made larger and larger. A point of tangency approaches a limiting point p on the line at infinity, and the value of the polynomial f approaches a limiting value c. The term $i_{p,c}$ is the sum of all the numbers $\pm 1/2$ associated to p and c in this manner. We show that the family of circles can be replaced by the level sets of any reasonable function, and the $i_{p,c}$ will remain the same.

The polynomial f extends to a function on projective space which is not well-defined at certain points on the line at infinity. Blowing up these points gives a well-defined function $\tilde{f}$. Using Morse theory, we show that the $i_{p,c}$ can be computed from the critical points of $\tilde{f}$ and information about the exceptional sets. The process of blowing up and computing the index is easy to carry out in specific examples.

The polynomial f can also be deformed into one which is more generic, and there is an simple formula relating the index of the original polynomial, the index of the new polynomial, and the newly created critical points. If the new polynomial is a "Morsification", i.e. it has the maximum number of critical points in the complex plane, then these formulas are particularly easy.

The computations of these two sections are used to establish bounds on the $i_{p,c}$. These local bounds are sharp. The global bounds on i follow from the local bounds and some delicate arguments. However, the global bounds are not sharp and there still is a big gap between the global bounds and the examples.

In the last section we show that $i^{abs}_{p,c}$ is a lower bound for the number of one-dimensional vanishing cycles of the (complex) function $f - c$ at p, where $i^{abs}_{p,c}$ is the sum of the absolute values of the numbers summed in $i_{p,c}$.

Throughout this paper the techniques are those of basic topology (Morse Theory) and algebraic geometry (Bezout's theorem, explicit computation of intersection multiplicities, etc.) Computer algebra programs were used to find countour plots and hence the $i_{p,c}$. They were also used to find critical points and their type, and to compute Milnor numbers. Although many of the results and techniques are valid in higher dimensions, the exposition is in dimension two for reasons of clarity.

The author's interest in these questions started in 1989 when he worked with a group of undergraduates in the Mount Holyoke Summer Research Institute in Mathematics [DKM+93]. Another group of students continued this work in 1992; one of their results was the construction of polynomials with an arbitrarily large number of local maxima and no other critical points [Rob92]. (These polynomials have $i \approx d/4$.) Shustin [Shu93] has studied polynomials all of whose critical points lie in the complex plane. (These polynomials have $i = 1 - d_R$.) He finds polynomials of this type with arbitrarily prescribed numbers of local maxima, minima and saddles.

This paper is part of a general study of the "critical points at infinity" of a polynomial; see [Dur95] for further references.

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Some notation which will be used throughout the paper: We let

$$\mathbb{L} = \{[x, y, z] \in \mathbb{P}^2 : z = 0\}$$

be the line at infinity in real projective space $\mathbb{P}^2$, and $\mathbb{L}_{\mathbb{C}}$ be the line at infinity in complex projective space.

We let d be the degree of the polynomial $f(x, y)$. We let f_d denote the homogeneous term of degree d in f. If $p = [a, b, 0] \in \mathbb{L}_{\mathbb{C}}$, we let d_p be the multiplicity of the factor $(bx - ay)$ in f_d. We define the *real degree* d_R or $d_R(f)$ of polynomial f by

$$d_R = \sum_{p \in \mathbb{L}} d_p$$

We let

$$f_d \cap \mathbb{L} = \{[x, y, 0] \in \mathbb{L} : f_d(x, y) = 0\}$$

This is the set of points where the zero locus of f_d intersects the line at infinity. We also let

$$f \cap \mathbb{L} = \{p \in \mathbb{L} : \text{ there is a } t \in \mathbb{R} \text{ such that } p \in \overline{\{f = t\}}\}$$

This is the set of points in $\mathbb{L}$ through which the closure of at least one real level curve passes. We let l be the number of points in $f \cap \mathbb{L}$. Note that

$$f \cap \mathbb{L} \subset f_d \cap \mathbb{L}$$

This inclusion is proper, as is shown by the example $f(x, y) = y^4 + x^2$.

We use $\mathbb{P}^1$ and $\mathbb{R} \cup \{\infty\}$ interchangebly, depending on the context.

Occasionally we split a real branch of an algebraic curve at a point p into two "half-branches" with common point p.

2 A zoo of polynomials

A number of polynomials with strange properties are used throughout the paper. These are described in this section.

The polynomial $y(xy - 1)$, which has no critical points in the plane, is the standard example of a polynomial with a "critical point at infinity" (at $[1, 0, 0]$).

The polynomial $x(y^2 - 1)$ has saddles at $(0, 1)$ and $(0, -1)$. The family of level curves at $[1, 0, 0]$ is equisingular; there is no "critical point" at $[1, 0, 0]$.

The "max-min polynomial" $y^5 + x^2y^3 - y$ from [DKM+93] has a local minimum at $(0, -1/\sqrt[4]{5})$ and a local maximum at $(0, 1/\sqrt[4]{5})$ and no other critical points. This polynomial can easily be generalized [DKM+93, Section 2].

The "two-min polynomial" $(xy^2 - y - 1)^2 + (y^2 - 1)^2$ from [Mat85] has local minima at $(2, 1)$ and $(0, -1)$, and no other critical points. Note the asymmetry of this polynomial compared with the previous one.

The polynomial $x^2(1 + y)^3 + y^2$ has its sole critical point at the origin, which is a local maximum. However this is not an absolute maximum [CV80].

The polynomial $y - (xy - 1)^2$ from [Kra] has a saddle at $(-1/2, 0)$ and no other critical points. At $[1, 0, 0]$ the level set $f = 0$ has one branch, but the general level set has two branches.

The "two-parabola polynomial" $f(x, y) = (x-y^2)(x(y^2+1)-y^2(y^2+1)-1)$ has its zero locus along the parabola $x = y^2$ and the curve $x = y^2+1/(y^2+1)$ which is asymptotic to this parabola. Its only critical point is a minimum at $(1/2, 0)$. The level curves intersect $\mathbb{L}$ only at $[1, 0, 0]$, and they are tangent to $\mathbb{L}$ at this point.

3 A formula for i from the geometry of $grad\, f$

Let $f(x, y)$ be a real polynomial with isolated critical points. Choose a large circle C centered at the origin which contains all the critical points. By moving the center of C we may assume that when the level curves of f are tangent to C that this tangency is nondegenerate, and that this is true for all larger circles with the same center. (See Lemma 3.4.) If u is a point on C where a level curve of f is tangent to C, we call u a "point of tangency". We assign the number $k_u = \pm 1/2$ to each point of tangency u according as whether the level curve is inside or outside C, as in Figure 1. If u is not a point of tangency, we let $k_u = 0$.

Since the index can be computed by counting the inverse images of a regular value, clearly

$$i = 1 + \sum_{u \in C} k_u$$

Note that this sum is over points on the circle where $grad\, f$ points both out of and into the circle; the process of decomposing the index described below does not work if the sum is just over those points where the gradient points out, as can be seen in the example $f(x, y) = y(x^2y - 1)$.

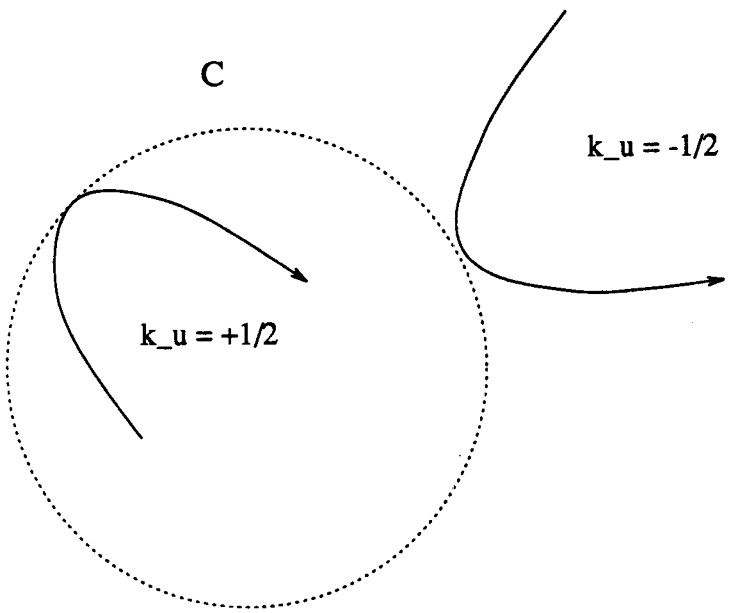

Figure 1: Assigning $k_u = \pm 1/2$ to a point of tangency u

Now make this circle larger and larger (keeping the same center). A point of tangency u will travel along a "curve of tangencies" and approach a point $p \in \mathbb{L}$, and $f(u)$ will approach a limiting value $c \in \mathbb{R} \cup \{\infty\}$. (These limits exist by Lemma 3.1.) Under these conditions we "associate (p, c) to u ". This process for the polynomial $y(xy-1)$ is pictured in Figure 2.

For $p \in \mathbb{L}$ and $c \in \mathbb{R} \cup \{\infty\}$, let

$$i_{p,c} = \sum k_u$$

where the sum is over all $u \in C$ with (p, c) associated to u. We let

$$i_p = \sum_{c \in \mathbb{R} \cup \{\infty\}} i_{p,c}$$

and

$$i_{\mathbb{L},\infty} = \sum_{p \in \mathbb{L}} i_{p,\infty}$$

The following lemma will be proved in the next section.

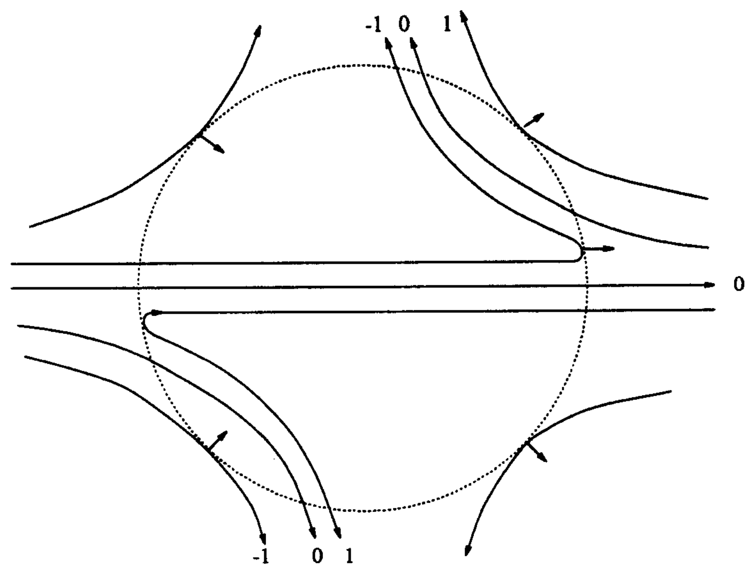

Figure 2: The index computation for the polynomial $y(xy-1)$

Lemma 3.1. *The numbers $i_{p,c}$ exist and are integers (not just half-integers).*

We thus have a formula for the index:

Formula 3.2. *If $f(x,y)$ is a real polynomial with isolated critical points, then*

$$i = 1 + \sum_{\substack{p \in \mathbb{L} \\ c \in \mathbb{R}}} i_{p,c} + i_{\mathbb{L},\infty}$$

Lemma 3.3. *If $c \in \mathbb{R}$ and $p \notin f \cap \mathbb{L}$, then $i_{p,c} = 0$.*

This lemma is a consequence of Lemma 4.2. However we can see now that if $c \in \mathbb{R}$ and $p \notin f_d \cap \mathbb{L}$, then $i_{p,c} = 0$: Since $f(u) \to c$ as $u \to p$, the curve $f = c$ passes through p, so p must be a zero of f_d.

The invariants of Formula 3.2 for some functions are given in Table 3; all the nonzero $i_{p,c}$ for $c \in \mathbb{R}$ are listed. Some of the reasons for combining the $i_{p,\infty}$ into $i_{\mathbb{L},\infty}$ are the formula of Proposition 4.4, Part (3), the comment about Lemma 3.4, part (3), and the fact that the estimates for i work better this way.

$f(x,y)$	$i_{\mathbb{L},\infty}$	$p \in f \cap \mathbb{L}$	$c \in \mathbb{R}$	$i_{p,c}$	i
$y(xy-1)$	-2	$[1,0,0]$	0	1	0
$x(y^2-1)$	-3				-2
y^2-x	-1				0
$y^5+x^2y^3-y$	-1	$[1,0,0]$	0	2	2
two-parabola	-1	$[1,0,0]$	0	1	1
$y-(xy-1)^2$	-2	$[1,0,0]$	0	0	-1
two-min	-1	$[1,0,0]$	1	1	2
			2	1	
$y(x^2y-1)$	-2	$[1,0,0]$	0	1	0

Table 1: Index invariants of selected polynomials

A more detailed analysis can be done by compactifying the plane by the circle $\{(a,b,0) \in (\mathbb{R}^3-0)/\mathbb{R}^+\}$ (the two-fold cover of $\mathbb{L}$) and seeing how the points of tangency approach it. For example, for $y-(xy-1)^2$ there are two u's associated to $([1,0,0],0)$, one with $k_u = +1/2$ associated to $((1,0,0),0)$ and another with $k_u = -1/2$ associated to $((-1,0,0),0)$. The two cancel out to give $i_{[1,0,0],0} = 0$.

Fix a real polynomial $f(x,y)$ with isolated critical points, and fix an even integer $e > 0$. We let $\mathcal{H}$ be the set of polynomials $h(x,y)$ of degree e whose homogeneous term of highest degree has no real linear factors, and we let $\mathcal{H}_f$ be the set of $h(x,y) \in \mathcal{H}$ such that the level curves of f and h do not have degenerate tangencies outside some compact set in the plane. (A degenerate tangency occurs at a point if the two curves have intersection number more than two at that point. For example, the functions $f = y^4 - x$ and $h = x^4 + y^4$ have degenerate tangencies along the x-axis.)

Instead of using a family of concentric circles (the level curves of the polynomial x^2+y^2) to define $i_{p,c}$, we could use instead the level curves of a polynomial $h \in \mathcal{H}_f$. We let $i^h_{p,c}$ be the decomposition of the index defined this way. Thus $i_{p,c} = i^h_{p,c}$ for $h(x,y) = x^2+y^2$.

Lemma 3.4. *Let f be a polynomial with isolated critical points.*

1. *$\mathcal{H}_f$ is a dense, connected subset of $\mathcal{H}$.*
2. *If $h \in \mathcal{H}_f$, then $i^h_{p,c} = i_{p,c}$ for all $p \in \mathbb{L}$ and $c \in \mathbb{R}$.*

Proof. A point p is a common tangent for f and h exactly when $g(p) = 0$, where

$$g = f_x h_y - f_y h_x$$

It is a degenerate tangency if in addition

$$grad\, g(p) = 0$$

Thus $h \notin \mathcal{H}_f$ exactly when $g_x = 0$ and $g_y = 0$ have a common unbounded component. If $h \notin \mathcal{H}_f$ and $f_y \neq 0$, then h may be replaced by $h + a$, where a is a function of x alone of degree $\leq e$ and $a'' \neq 0$, and similarly if $f_x \neq 0$. This proves the first assertion. The second will be proved in the next section. □

Note that the second part of the above lemma is not true for $c = \infty$ and $p = [a, b, 0]$ with $f_d(a, b) \neq 0$, but that $i_{\mathbb{L},\infty}$ remains the same by Formula 3.2.

Finally, is there an Eisenbud-Levine-type algebraic formula for i, i_p or $i_{p,c}$?

4 A formula for i in terms of a resolution

The polynomial

$$f : \mathbb{R}^2 \to \mathbb{R}$$

extends to a map of real projective spaces

$$\hat{f} : \mathbb{P}^2 \to \mathbb{P}$$

which is undefined at a finite number of points on the line at infinity $\mathbb{L}$. By blowing up these points one gets a manifold M and a map

$$\pi : M \to \mathbb{P}^2$$

such that the map

$$\tilde{f} : M \to \mathbb{P}$$

which is the lift of $\hat{f}$ is everywhere defined. We call the map $\tilde{f}$ a *resolution of f*.

For example, a resolution (the minimal resolution) of $y(xy - 1)$ is given in Figure 3. The symbol $c^{(m)}$ next to a divisor means that at each smooth

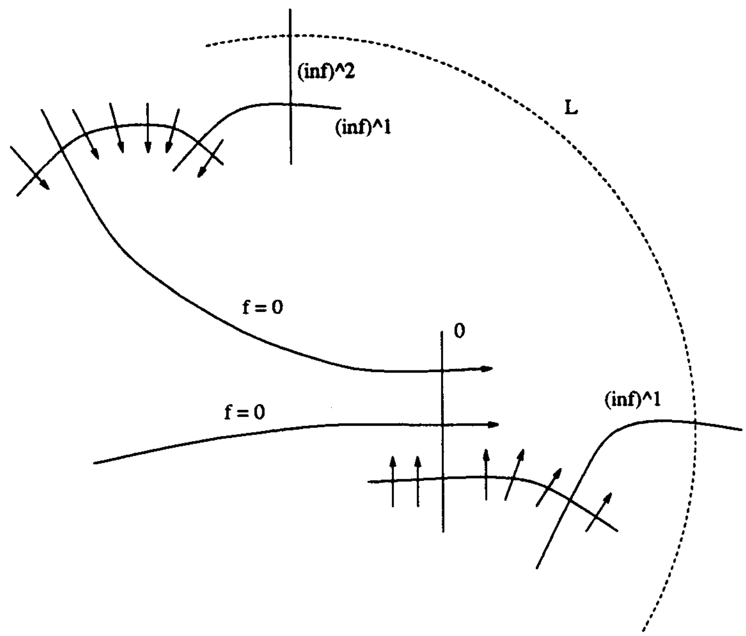

Figure 3: A resolution of $y(xy-1)$

point of the divisor there are local coordinates (u,v) in a neighborhood of the point such that the divisor is $u=0$ and $\tilde{f}(u,v)=(u-c)^m$. The symbol (inf) means ∞. The proper transform of level curves of f have arrowheads on them; the exceptional sets do not.

Proof of Lemma 3.1: The limiting point p of a curve of tangency is an intersection of a branch of the curve $yf_x - xf_y = 0$ with $\mathbb{L}$. Hence p exists. Since each half-branch at p is a curve of tangencies and there are an even number of half-branches, the $i_{p,c}$ are integers, not just half-integers.

Let $\tilde{f} : M \to \mathbb{P}$ be a resolution of f. The real branch can be lifted to a curve on M; this curve intersects $\pi^{-1}(p)$ at a single point, say q. Then $c = \tilde{f}(q)$; taking $\tilde{f}$ to be the minimal resolution shows that c is well-defined. □

For $p \in \mathbb{L}$ and $c \in \mathbb{R} \cup \{\infty\}$, let

- $i_{p,c}(\tilde{f})$ be the sum of the indices of $\tilde{f}$ at points $q \in M$ such that $\tilde{f}(q) = c$ and $\pi(q) = p$.

- $i_{\mathbb{L}}(\tilde{f}) = \sum_{\substack{p \in \mathbb{L} \\ c \in \mathbb{R}}} i_{p.c}(\tilde{f})$

- $\xi_{p,c}(\tilde{f})$ be the number of exceptional sets E with $\pi(E) = p$ and $\tilde{f}|E = c$.
- $\xi_{p,nc}(\tilde{f})$ be the number of exceptional sets E with $\pi(E) = p$ and $\tilde{f}|E$ nonconstant. (In the pictures, these exceptional sets are cross hatched by curves with arrowheads.)
- $\xi_{\mathbb{L},c}(\tilde{f}) = \sum_{p\in\mathbb{L}} \xi_{p,c}$
- $\xi_{\mathbb{L},nc}(\tilde{f}) = \sum_{p\in\mathbb{L}} \xi_{p,nc}$
- $\xi_{\mathbb{L}}(\tilde{f}) = \sum_{\substack{p\in\mathbb{L}\\ c\in\mathbb{R}}} \xi_{p,c}$

Lemma 4.1. *Let $f(x,y)$ be a polynomial with isolated critical points, and let $\tilde{f}$ be a resolution of f.*

1. *For all exceptional sets E and $A \gg 0$, $\{\tilde{f} = +A\} \cup \{\tilde{f} = -A\}$ intersects E if and only if $\tilde{f}|E$ is nonconstant.*
2. *There is a two-to-one correspondence between connected components of $\{f = +A\} \cup \{f = -A\}$ in $\mathbb{R}^2$ and non-constant exceptional sets.*

Proof. If E is an exceptional set and $\tilde{f}|E$ is not constant, there is an $x \in E$ such that $\tilde{f}|\{E - \{x\}\}$ is a polynomial. For all exceptional sets E and $A \gg 0$, the following is true: If $\tilde{f}|E$ is constant then $A > |(\tilde{f}|E)|$, and if $\tilde{f}|E$ is nonconstant then $\tilde{f}|E$ takes the value $+A$ or $-A$. This proves the first part.

In fact, in the latter case $\tilde{f}|E$ either takes the value $+A$ exactly twice, the value $-A$ exactly twice, or the values $+A$ and $-A$ each once. The level sets $\tilde{f} = \pm A$ for $A \gg 0$ are transverse to E and hence must be closures of level sets of f. This proves the second part. □

Lemma 4.2. *For $p \in \mathbb{L}$, the following statements are equivalent:*

1. *$\xi_{p,nc}(\tilde{f}) > 0$ for some resolution $\tilde{f}$ of f.*
2. *For $A \gg 0$, $p \in \overline{\{f = +A\} \cup \{f = -A\}}$.*
3. *$p \in f \cap \mathbb{L}$.*

4. *There is a sequence of points u_k in $\mathbb{R}^2$ such that $u_k \to p$ and $f(u_k)$ is bounded.*

Corollary 4.3. $l \leq \xi_{\mathbb{L},nc}$.

Proof. (1) $\Longrightarrow$ (2): This follows from Lemma 4.1, and the fact that the level sets $\tilde{f} = t$ are transverse to E, and hence must be closures of level sets of f.
(2) $\Longrightarrow$ (3): By definition.
(3) $\Longrightarrow$ (1): The exceptional sets over p form a connected tree. If the closure of some level set $f = t$ intersects p, then it intersects this tree. Since the value of $\tilde{f}$ is t at this point on the tree, and is ∞ at the points where the tree intersects the proper transform of $\mathbb{L}$, there is a component E of the tree such that $\tilde{f}|E$ takes a continuum of values.
(3) $\Longrightarrow$ (4): Suppose $t \in \mathbb{R}$ is such that $p \in \overline{\{f = t\}}$. Choose the sequence of points on the curve $f = t$.
(4) $\Longrightarrow$ (1) or (3): By the curve selection lemma for points at infinity (see for instance [NZ92] or [Ha91], Lemma 3.1), the sequence of points can be replaced by an analytic curve. Suppose that the curve lifted to M approaches $q \in E$ for some exceptional set E in M. Then $\tilde{f}(q) = t$ for some $t \in \mathbb{R}$. Then either: (i) $q \in \overline{\{f = t\}}$, so (3) is true; or (ii) $\tilde{f}|E = t$, in which case the same argument as (3) implies (1) gives (1); or (iii) q is an isolated real point of the curve $\{\tilde{f} = t\}$, but then $\tilde{f}|E$ is nonconstant and (1) is true. □

An example of the inequality $l \leq \xi_{\mathbb{L},nc}$ is provided by the polynomial $(y(x^2+1)-1)(y(x^2+2)-1)\ldots(y(x^2+k)-1)$, which has $l = 2$ and $\xi_{[1,0,0],nc} = k$, hence $\xi_{\mathbb{L},nc} = k+1$.

Proposition 4.4. *If $f(x,y)$ is a real polynomial with isolated critical points and $\tilde{f}$ is a resolution of f, then*

1. $i = 1 - i_{\mathbb{L}}(\tilde{f}) - \xi_{\mathbb{L}}(\tilde{f})$
2. $i_{p,c} = -i_{p,c}(\tilde{f}) - \xi_{p,c}(\tilde{f})$ *for* $p \in \mathbb{L}$ *and* $c \in \mathbb{R}$.
3. $i_{\mathbb{L},\infty} = -\xi_{\mathbb{L},nc}(\tilde{f})$

For example, the resolution $\tilde{f}$ of the polynomial $y(xy-1)$ shown above has two saddle points with critical value 0 over $[1,0,0]$, so $i_{[1,0,0],0} = -2$, and one exceptional set E over $[1,0,0]$ with $\tilde{f}|E = 0$, so $\xi_{[1,0,0],0} = 1$. Thus $i_{[1,0,0],0} = -(-2) - 1 = 1$.

Proof. Part (1) follows from a straight-forward application of Morse theory, part (2) follows from Morse theory on a manifold with boundary, and part (3) follows from the geometry of the large level curves. Any two parts of this proposition imply the third, but proving each part separately is more instructive.

Proof of (1): We will do Morse theory on the function $\tilde{f} : M \to \mathbb{R} \cup \{\infty\}$. Suppose $c_1 < c_2 \ldots < c_r$ in $\mathbb{R}$ are the critical values of $\tilde{f}$ restricted to the inverse image of $\mathbb{R}$. We may assume (by a slight perturbation of the function $\tilde{f}$) that each critical value corresponds to either a critical point in $\mathbb{R}^2$ or a critical point on the preimage of the line at infinity. Choose $\epsilon > 0$ so that $c_i + \epsilon < c_{i+1} - \epsilon$ for $1 \leq i < r$. Choose $A > 0$ so that $-A < c_1$ and $c_r < A$. Since a level set of $\tilde{f}$ corresponding to a regular value is a union of circles,

$$\chi(M) = \chi(\{\tilde{f} \leq -A\} \cup \{\tilde{f} \geq A\}) + \sum_i \chi(\{c_i - \epsilon \leq \tilde{f} \leq c_i + \epsilon\}) \quad (1)$$

where $\chi(X)$ is the Euler characteristic of the set X. The set $\{\tilde{f} \leq -A\} \cup \{\tilde{f} \geq A\}$ is homotopy equivalent to the set $\tilde{f}^{-1}(\infty)$. This is a connected set, and is homotopy equivalent to a join of circles. These circles are the exceptional sets where $\tilde{f} = \infty$ together with the proper transform of $\mathbb{L}$. Thus

$$\chi(\{\tilde{f} \leq -A\} \cup \{\tilde{f} \geq A\}) = -\xi_{\mathbb{L},\infty}(\tilde{f})$$

Next, M is a connected sum of copies of $\mathbb{P}^2$, so

$$\chi(M) = 1 - (\xi_{\mathbb{L}}(\tilde{f}) + \xi_{\mathbb{L},\infty}(\tilde{f}))$$

At a critical value c_i, $\chi(\{c_i - \epsilon \leq \tilde{f} \leq c_i + \epsilon\})$ is the index of the critical point by Morse theory. These indices can be summed, and the sum split into the parts coming from critical points in the finite plane and the line at infinity. Thus Equation (1) becomes

$$1 - \xi_{\mathbb{L}}(\tilde{f}) - \xi_{\mathbb{L},\infty}(\tilde{f}) = -\xi_{\mathbb{L},\infty}(\tilde{f}) + i + i_{\mathbb{L}}(\tilde{f})$$

which proves (1).

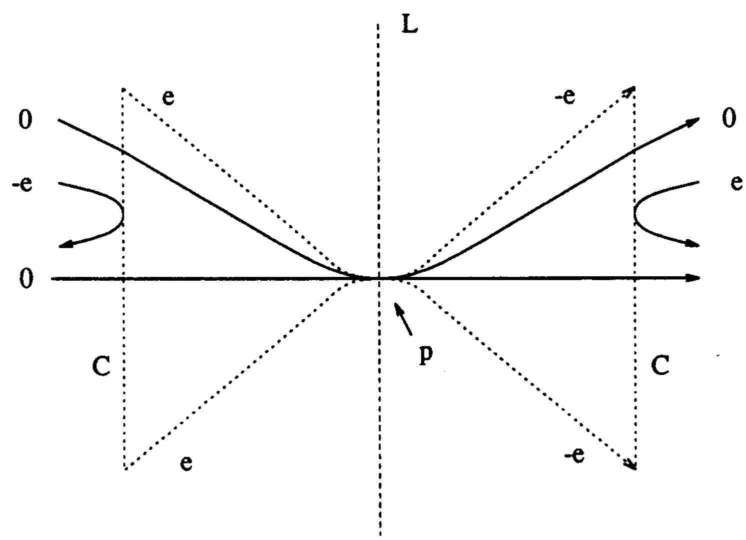

Figure 4: The region N' (bounded by dotted lines) for the polynomial $y(xy-1)$ at $p=[1,0,0]$ and $c=0$.

Proof of (2): (See Figures 4 and 5.) Choose $\epsilon > 0$ so that c is the only critical value in $(c-\epsilon, c+\epsilon)$. Let C' be the (closed) exterior of the circle C in the plane. Let N' be the connected component of $\{(x,y) \in \mathbb{R}^2 : c-\epsilon \leq f(x,y) \leq c+\epsilon\} \cap C'$ containing p in its closure. Choose the circle C large enough so that each boundary component of N' consists of an arc of $f = \pm\epsilon$ followed by an arc of C followed by an arc of $f = \pm\epsilon$.

Let

$$N = \overline{\pi^{-1}(N')} \subset M$$

We assume that N is connected; if it is disconnected the proof is similar.

We need a variant of the Poincaré-Hopf Theorem for vector fields on a manifold, or more properly, a variant of Morse theory on manifolds with boundary. (See, for instance, [Mil65], p. 35). For an oriented manifold X with boundary, the Euler characteristic $\chi(X)$ is given by

$$\chi(X) = \sum\{\text{indices of internal critical points}\} + \{\text{index on boundary}\}$$

where the index of the vector field on the boundary is measured with respect to the outward pointing normal vector. This result is true for a gradient vector field on a nonorientable two-manifold X without boundary, provided

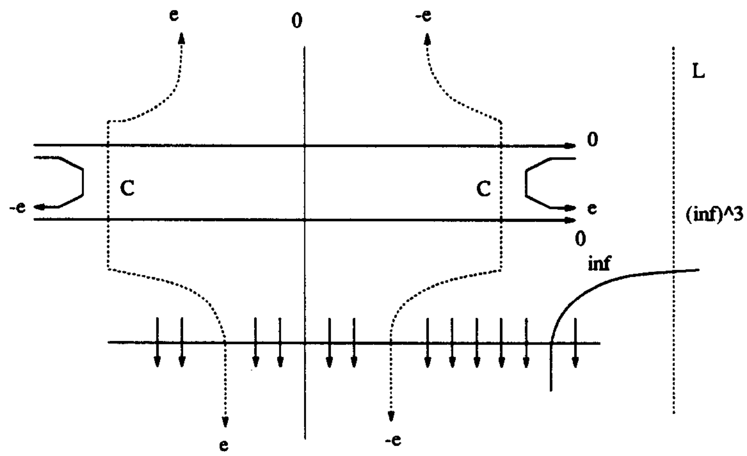

Figure 5: The region N

that the index is defined to be $+1$ at a local extremum and -1 at a saddle, or more generally defined at an arbitrary critical point using the result of Arnold [Arn78] the index of a polynomial $f(x, y)$ at a point p is $1-r$, where r is the number of real branches at p of the curve $f(x, y) = f(p)$. If X has a boundary with an orientable collar neighborhood, then the result is still true, provided that the index on the boundary is measured according as Figure 1. Finally, the form we will use for N is

$$\chi(N) = 1 + \sum\{\text{indices of internal critical points}\} + \{\text{index on boundary}\}$$

The term $+1$ comes from the fact that N has four corners (see Figure 5).

Choose a Riemannian metric on N so that it agrees on the boundary components of N consisting of arcs of C with the standard one on the plane. We apply the above result to the gradient vector field of $\tilde{f}$. In the interior of N there are the exceptional sets with $\tilde{f} = c$ and those critical points of $\tilde{f}$ which have critical value c. Since N retracts to the exceptional sets contained in it,

$$\chi(N) = 1 - \xi_{p,c}(\tilde{f})$$

The index of the internal critical points of $\tilde{f}$ is $i_{p,c}(\tilde{f})$. Finally, the index of the gradient vector field on the boundary of N is $i_{p,c}$. Combining these facts proves the result.

Proof of (3): (See Figure 6.) If $\gamma(t)$ is a point of tangency and $f(\gamma(t)) \to \infty$, then $\gamma(t)$ intersects either $f = A$ or $f = -A$ for $A >> 0$. Let I be a

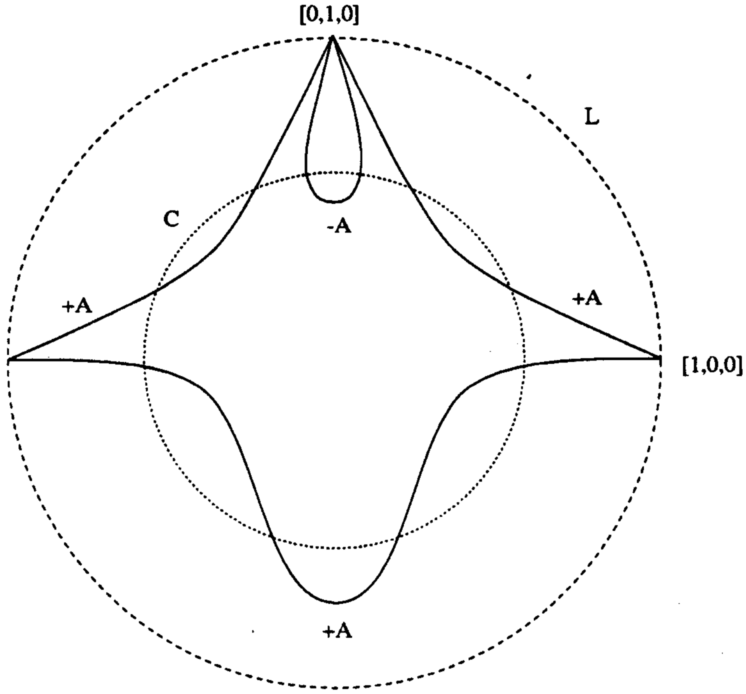

Figure 6: The level curves $y(x^2y-1) = \pm A$ in $\mathbb{P}^2$

connected component of $f = \pm A$ in $\mathbb{R}^2$. Since I begins and ends outside C, clearly $\sum_{u\in I} k_u = -1/2$. By Lemma 4.1, there is a two-to-one correspondence between connected components of $f = \pm A$ and nonconstant exceptional sets. $\square$

Corollary 4.5. *If the real linear factors of f_d are irreducible, then $i = 1 - d_R$.*

Proof. This is "geometrically obvious" but also easily follows from Part (1) of Proposition 4.4, since the resolution at each $p \in f \cap \mathbb{L}$ has exactly one exceptional set where $\tilde{f}$ is nonconstant. $\square$

Proof of the second part of Proposition 3.4: Let $\tilde{f} : M \to \mathbb{P}^1$ be a resolution of f. Since the homogeneous term of highest degree in h has no real linear factors, h lifts to a well-defined function $\tilde{h}$ on M. Let $\gamma(t)$ be a parameterization of a curve of tangencies of f and h, and assume that

$\gamma(t) \to q \in M$ as $t \to \infty$. By assumption, $\pi(q) = p$ and $\tilde{f}(q) = c$. If q is in an exceptional set locally defined by $u = 0$, but not an intersection point, then the tangent to the level curve of $\tilde{h}$ at $\gamma(t)$ approachs the tangent to the exceptional set at q, since locally $\tilde{h}$ has the form u^a for some a. Thus the level curve of $\tilde{f}$ at $\gamma(t)$ also approaches the tangent to the exceptional set at q. Thus either $\tilde{f}$ is constant ($= c$) on the exceptional set, or $\tilde{f}$ has a critical point at q.

We have that $i_{p,c} = i^{h'}_{p,c}$ for $h' = (x^2 + y^2)^{e/2}$. Choose a path from h to h' in $\mathcal{H}_f$. The end point q of the arc $\gamma(t)$ varies as h changes to h'. If q is a critical point of $\tilde{f}$ the end point stays fixed. Otherwise it varies in the component of the exceptional set over p where $\tilde{f} = c$. □

Is there a polynomial f with a resolution $\tilde{f}$ and a point q in the exceptional set such that $\tilde{f}$ has a local extremum at q?

5 A formula for i in terms of a deformation

Definition 5.1. A *deformation* of a real polynomial $f(x,y)$ of degree d with isolated critical points is a real polynomial $h(x,y,s)$ of degree d in x and y with $h(x,y,0) = f(x,y)$. We let $f^s(x,y) = h(x,y,s)$.

For $p \in \mathbb{L}$ and $c \in \mathbb{R} \cup \{\infty\}$ we let

- $i^{\infty}_{p,c}(f^s)$ be the index of the critical points of f^s which go to p and whose critical value goes to c as $s \to 0$
- $i^{\infty}_{p}(f^s) = \sum_{c \in \mathbb{R} \cup \{\infty\}} i^{\infty}_{p,c}(f^s)$
- $i^{\infty}(f^s) = \sum_{p \in \mathbb{L}} i^{\infty}_{p}(f^s)$

Proposition 5.2. *If $f^s(x,y)$ is a deformation of a real polynomial $f(x,y)$ with isolated critical points, then*

1. $i = i(f^s) - i^{\infty}(f^s)$
2. $i_p = i_p(f^s) - i^{\infty}_p(f^s)$ *for* $p \in \mathbb{L}$.

There is no obvious formula for $i_{p,c}$ (see below).

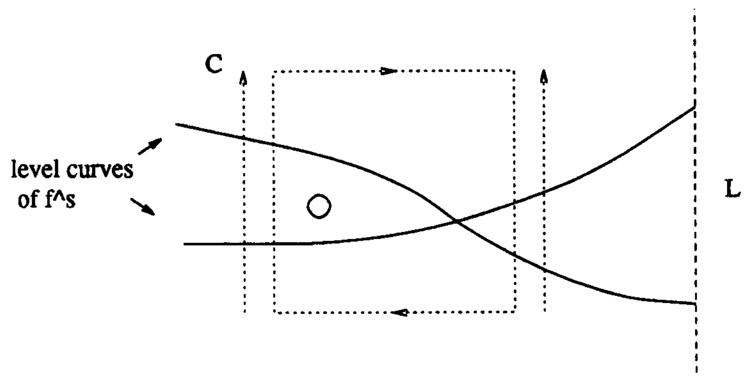

Figure 7: Computing i using a deformation

Proof. (1): Summing the indices of all the critical points of f^s gives $i(f^s) = i^\infty(f^s) + i$. (2): We may assume that $p = [1,0,0]$. There are two segments of the circle C containing the u's whose tangencies approach p, one on the right and the other on the left. The right segment of the circle can be replaced by a clockwise-oriented region containing the critical points of the deformation which are near p and a segment on its opposite side as in Figure 7. Similarly the left segment can be replaced. The index of f^s about the two regions is $i_p^\infty(f^s)$, and along the new segments is $i_p(f^s)$. Thus $i_p(f^s) = i_p^\infty(f^s) + i_p$. □

Definition 5.3. A deformation f^s of a polynomial f of degree d is a *Morsification* if f^s has $(d-1)^2$ nondegenerate critical points in $\mathbb{C}^2$, for all $s \neq 0$.

This definition, appropriate for complex f, is actually stronger than we need. If f^s is a Morsification, then $(f^s)_d$ has distinct linear factors.

Proposition 5.4. *A polynomial of degree d has a Morsification. The set of Morsifications is a dense open subset of the set of polynomials of degree* $\leq d$.

Proof. The partial derivatives of the homogenization of a deformation f^s of f are a deformation of the partial derivatives of the homogenization of the original function f. If f^s is chosen so that its term of highest degree for $s \neq 0$ has no repeated complex factors, then both partials are of degree $d-1$

and their zero loci have all their intersections in the affine plane. Finally, f^s can be chosen so that its critical points are nondegenerate for $s \neq 0$. □

If f^s is a Morsification of f and $p \in \mathbb{L}$, we let $\tilde{d}_p(f^s)$ be the number of real linear factors in the homogeneous term of highest degree of f^s which are deformations of the factor corresponding to p in the homogeneous term of highest degree of f. The number $\tilde{d}_p(f^s)$ is also the number of points on $\mathbb{L}$ near p through which the level sets of the Morsification pass.

Corollary 4.5 implies that

$$i(f^s) = 1 - d_R(f^s)$$

and similar argument shows that

$$i_p(f^s) = 1 - \tilde{d}_p(f^s) \tag{2}$$

These equations thus imply the following.

Corollary 5.5. *Let f^s be a Morsification of f.*

1. $i = 1 - d_R(f^s) - i^\infty(f^s)$
2. $i_p = 1 - \tilde{d}_p(f^s) - i_p^\infty(f^s)$ *for* $p \in \mathbb{L}$.

For example the polynomial $f(x,y) = y(xy-1)$ has a Morsification $f^s(x,y) = (y-sx)(xy-1)$. The level sets of the original function are as in Figure 2, and the level sets of the deformation are as in Figure 8. The index computation is given on the first line of Table 5. The table shows some other deformations of various polynomials.

There is no obvious formula for $i_{p,c}$, as can be seen from the deformation $xy(y-1)+sx^3$ of $x(y^2-1)$. However in all the examples there are natural numbers $\tilde{d}_{p,c}(f^s)$ with $\tilde{d}_{p,\infty}(f^s) > 0$, for $p \in \mathbb{L}$ and $c \in \mathbb{R} \cup \{\infty\}$, such that

$$\tilde{d}_p(f^s) = \sum_c \tilde{d}_{p,c}(f^s)$$

and

$$i_{p,c} = -i_{p,c}^\infty(f^s) - \tilde{d}_{p,c}(f^s)$$

for $c \in \mathbb{R}$, and

$$i_{p,\infty} = 1 - i_{p,\infty}^\infty(f^s) - \tilde{d}_{p,\infty}(f^s)$$

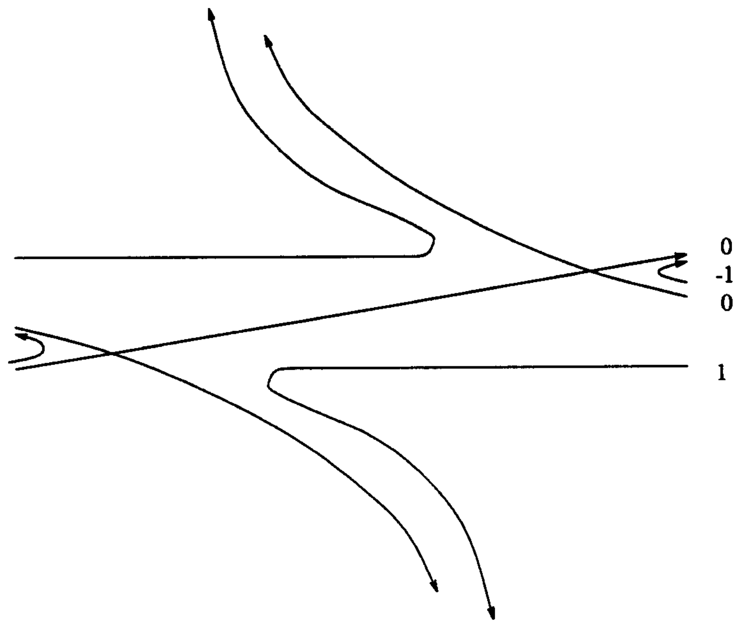

Figure 8: Level sets of the deformation $(y - sx)(xy - 1)$

f	f^s	p	$-\tilde{d}_p(f^s)$	c	$-i^{\infty}_{p,c}(f^s)$	$i_{p,c}$	i_p
$y(xy-1)$	$(y-sx)(xy-1)$	$[1,0,0]$	-2	∞	0	0	1
				0	2	1	
$x(y^2-1)$	$x(y+sx)(y-sx)$	$[1,0,0]$	-2	∞	-1	-1	-1
				0	1	0	
	$xy(y-1)+sx^3$ $(s>0)$	$[1,0,0]$	0	∞	-2	-1	
	$xy(y-1)+sx^3$ $(s<0)$	$[1,0,0]$	-2	∞	0	-1	
$y^5+y^3x^2-y$	$(y+s)(y^4+y^2x^2$ $-y(1+sx^2)^2)$	$[1,0,0]$	-3	∞	0	0	2
				0	4	2	
$y-(xy-1)^2$	$y-x(x+s)y(y+s)$ $+2xy-1$	$[1,0,0]$	-2	∞	0	0	0
				-1	1	0	
				0	0	0	
		$[0,1,0]$	-2	∞	1	0	0
y^2-x	y^2+sx^2-x $(s>0)$	$[1,0,0]$	0	∞	-1	0	0
	y^2+sx^2-x $(s<0)$	$[1,0,0]$	-2	∞	1	0	

Table 2: The computation of i_p using deformations

However, it seems unlikely that this is true in general.

Finally, an odd property of deformations which will be used in the next section.

Lemma 5.6. *If* $\tilde{d}_p(f^s) = d_p$, *then* $i_p^\infty(f^s) \leq 0$. *If* $d_R(f^s) = d_R$, *then* $i^\infty(f^s) \leq 0$.

Proof. The first equation follows from Equation (2) and Lemma 6.1. The second follows from the first. □

For a deformation f^s of f it is easy to find bounds on the number of local maxima, minima and saddles near a point $p \in \mathbb{L}$. It would be interesting to see what possible combinations of these can occur, similar to the investigation in [DKM+93] or [Shu93].

6 Bounds on i

Let $f(x, y)$ be a real polynomial of degree d with isolated critical points. First we give the local analogue of the estimate $|i| \leq d - 1$ from the Introduction. For $p \in f_d \cap \mathbb{L}$ and $c \in \mathbb{R} \cup \{\infty\}$, recall that

$$i_{p,c} = \sum k_u$$

where the sum is over u associated with p and c, and that $k_u = \pm 1/2$. We let

$$i_{p,c}^+ = \sum k_u$$

where the sum is over those u with $k_u = +1/2$, and

$$i_{p,c}^- = \sum k_u$$

where the sum is over those u with $k_u = -1/2$, so that

$$i_{p,c} = i_{p,c}^+ + i_{p,c}^-$$

We let

$$i_{p,c}^{abs} = i_{p,c}^+ - i_{p,c}^- = \sum |k_u|$$

These invariants can be computed from a resolution of f, and in particular are integers (although this is not evident from the proof of 4.4): If $\tilde{f}$ has only nondegenerate critical points, and these are $e_{p,c}(\tilde{f})$ local extrema and $s_{p,c}(\tilde{f})$ saddles at points q with $\tilde{f}(q) = c$ and $\pi(q) = p$, then

$$i^{+}_{p,c} = s_{p,c}(\tilde{f})$$

and

$$i^{-}_{p,c} = -e_{p,c}(\tilde{f}) - \xi_{p,c}(\tilde{f})$$

The invariants $i^{N,+}_{p,c}$ and so forth later in this section can also be computed from a resolution, but they apparently cannot be computed from a deformation of f. Note also that the $i^{abs}_{p,c}$ are not related to the total number of critical points of f in the plane; if f has nondegenerate critical points, and these are e local extrema and s saddles, then in general

$$e + s \neq 1 + \sum_{\substack{p \in \mathbb{L} \\ c \in \mathbb{R}}} i^{abs}_{p,c} - i_{\mathbb{L},\infty}$$

For example, the "monkey saddle" has a deformation with $s = 3$ and $e = 1$, and another deformation with $s = 2$ and $e = 0$.

Lemma 6.1. *For $p \in f_d \cap \mathbb{L}$,*

$$\sum_{c \in \mathbb{R} \cup \{\infty\}} i^{abs}_{p,c} \leq d_p - 1$$

Proof. Without loss of generality we may assume that $p = [1, 0, 0]$. We have that $f = y^{d_p} h(x, y) +$ {terms of lower order} where y does not divide $h(x, y)$ and $d_p \geq 1$. Since the circle C is large and the points are approaching p, we may replace the circle by the lines $x = +c$ and $x = -c$, for $c \gg 0$. The number $\sum i^{abs}_{p,c}$ is one-half the number of points near p on these lines where the gradient vector field is horizontal.

If X and Y are algebraic curves, we use the notation $X \,.\, Y_p$ for their intersection number at a point p. In the following, the subscript " $\to p$" means the intersections which approach p as $A \to \infty$, and $(X)_{\mathbb{R}}$ means the real component of X.

We have that

$$\sum i^{abs}_{p,c} = (1/2)\Big(\{(f_y)_{\mathbb{R}} = 0\} \cdot \{x = +A\}_{\to p} + \{(f_y)_{\mathbb{R}} = 0\} \cdot \{x = -A\}_{\to p}\Big)$$

$$\leq (1/2)\Big(\{(f_y) = 0\} \cdot \{x = +A\}_{\to p} + \{(f_y) = 0\} \cdot \{x = -A\}_{\to p}\Big)$$
$$= \{f_y = 0\} \cdot \{z = 0\}_p$$

This intersection number can be computed as in [Ful69, III.3]; since f_y in local coordinates (y, z) at p is given by $y^{d_p-1}h(1, y) +$ {terms with z}, the intersection number is $d_p - 1$. (Alternatively, we can Morsify f.) □

This Proposition is sharp: The upper bound is realized by the polynomial

$$(y(x^2 + 1) - 1)(y(x^2 + 2) - 1) \dots (y(x^2 + k) - 1)$$

at $p = [1, 0, 0]$, which has $d_p = k$ and $i_p = i_{p,0} = i^+_{p,0} = k-1$. (This polynomial has $k - 1$ local extrema and no other real critical points; for $k = 2$ there is a local minimum; for $k = 3$ there is a local minimum and a local maximum; for $k = 4$ there are two local minima and a local maximum, and so forth [DKM+93, Section 2].) The lower bound is realized by the polynomial

$$x(y + 1)(y + 2) \dots (y + k) \tag{3}$$

at $p = [1, 0, 0]$, which has $d_p = k$ and $i_p = i_{p,0} = i^-_{p,0} = 1 - k$.

Recall that $l = \#\{f \cap \mathbb{L}\}$, and that $l \leq \xi_{\mathbb{L},nc}$ by Corollary 4.3.

Lemma 6.2. *If all the level sets of f are compact ($l = 0$), then $i = 1$.*

Proof. No level sets of f intersect $\mathbb{L}$. Let A be a large positive constant. On the complement of a large compact set, either $f \geq A$ or $f \leq -A$; let us assume the former. If A is larger than the critical values of f, the level set $f = A$ is an embedded circle. The vector field *grad* f restricted to C is homotopic to the vector field *grad* f restricted to $f = A$, which has index 1. □

Lemma 6.3. *If $f(x, y)$ is a real polynomial with isolated critical points, then*

$$i \leq 1 + d_R - 2l$$

Proof. We have that

$$i = 1 + \sum_{\substack{p \in \mathbb{L} \\ c \in \mathbb{R}}} i_{p,c} + i_{\mathbb{L},\infty}$$

$$\leq 1 + \sum_{\substack{p\in f\cap \mathbb{L} \\ c\in \mathbb{R}}} i^{abs}_{p,c} - \xi_{\mathbb{L},nc}$$

$$\leq 1 + \sum_{\substack{p\in f\cap \mathbb{L} \\ c\in \mathbb{R}}} i^{abs}_{p,c} - l$$

$$\leq 1 + \sum_{p\in f\cap \mathbb{L}} (d_p - 1) - l$$

$$\leq 1 + \sum_{p\in f\cap \mathbb{L}} d_p - 2l$$

where the first line follows from Formula 3.2, the second from Lemma 3.3 and Part (3) of Proposition 4.4, the third from Corollary 4.3, and the fourth from Lemma 6.1. □

Theorem 6.4. *Let $f(x, y)$ be a real polynomial of real degree d_R with isolated critical points, and let i be the index of grad f around a large circle containing the critical points. If all the level sets of f are compact, then $i = 1$. Otherwise*

$$|i| \leq d_R - 1$$

Proof. If $l = 0$ the result follows from Lemma 6.2. The upper bound for $l > 0$ follows from Lemma 6.3. For the lower bound, choose a Morsification f^s of f with $d_R(f^s) = d_R$. Then Corollary 5.5 and Lemma 5.6 give the result. (Also see Corollary 4.5.) □

We now further decompose $i_{p,c}$ and its refinements defined above. For $p \in f_d \cap \mathbb{L}$ and $c \in \mathbb{R} \cup \{\infty\}$, recall from Section 3 that $i_{p,c} = \sum k_u$, summed over all points of tangency u. Each of the points is on a curve of tangencies $\gamma(t)$. We let $i^T_{p,c}$ (respectively, $i^N_{p,c}$) be the sum of the k_u's such that the corresponding curve $\gamma(t)$ is tangent (respectively, not tangent) to $\mathbb{L}$ at p. Thus

$$i_{p,c} = i^N_{p,c} + i^T_{p,c}$$

We similarly decompose $i^+_{p,c}$, $i^-_{p,c}$, and $i^{abs}_{p,c}$. As before, these numbers are all integers. For example, the polynomial $y(xy - 1)$ has $i_{[1,0,0],0} = i^{N,+}_{[1,0,0],0} = 1$.

The following lemma is a refinement of Lemma 6.1.

Lemma 6.5. *For $p \in f_d \cap \mathbb{L}$,*

$$\sum_{c \in \mathbb{R} \cup \{\infty\}} i_{p,c}^{N,abs} + 2\Big(\sum_{c \in \mathbb{R} \cup \{\infty\}} i_{p,c}^{T,abs} \Big) \leq d_p - 1$$

Proof. We let f_y^T (respectively, f_y^N) be the branches of f_y tangent (respectively, not tangent) to $\mathbb{L}$ at p, so that $f_y = f_y^T f_y^N$ near p. As in the proof of Lemma 6.1, we have

$$\sum i_{p,c}^{N,abs} = (1/2)\Big(\{(f_y^N)_{\mathbb{R}} = 0\} \cdot \{x = +A\}_{\to p} + \{(f_y^N)_{\mathbb{R}} = 0\} \cdot \{x = -A\}_{\to p} \Big)$$

$$\leq (1/2)\Big(\{(f_y^N) = 0\} \cdot \{x = +A\}_{\to p} + \{(f_y^N) = 0\} \cdot \{x = -A\}_{\to p} \Big)$$

$$= \{f_y^N = 0\} \cdot \{z = 0\}_p$$

We also have

$$\sum i_{p,c}^{T,abs} = (1/2)\Big(\{(f_y^T)_{\mathbb{R}} = 0\} \cdot \{x = +A\}_{\to p} + \{(f_y^T)_{\mathbb{R}} = 0\} \cdot \{x = -A\}_{\to p} \Big)$$

For each real branch B of f_y^T at p we have

$$B \cdot \{x = +A\}_{\to p} + B \cdot \{x = -A\}_{\to p} = 2$$

For the complexification C of B we have

$$C \cdot \{x = +A\}_{\to p} \geq 2$$

and

$$C \cdot \{x = -A\}_{\to p} \geq 2$$

since C is tangent to $\mathbb{L}$. Thus the right side of the equation seven lines above is

$$\leq (1/4)\Big(\{(f_y^T) = 0\} \cdot \{x = +A\}_{\to p} + \{(f_y^T) = 0\} \cdot \{x = -A\}_{\to p} \Big)$$

$$= (1/2)\{f_y^T = 0\} \cdot \{z = 0\}_p$$

Thus

$$\sum_{c\in\mathbb{R}\cup\{\infty\}} i_{p,c}^{N,abs} + 2\Big(\sum_{c\in\mathbb{R}\cup\{\infty\}} i_{p,c}^{T,abs}\Big) \le \{f_y^N = 0\}\cdot\{z=0\}_p + \{f_y^T = 0\}\cdot\{z=0\}_p$$

$$= \{f_y = 0\}\,.\,\{z=0\}_p$$
$$= d_p - 1$$

□

The following lemma will not be needed in the proof of the main theorem, but is included to show another technique of making estimates. It seems reasonable that $i_{p,c}^N$ could somehow be included in this estimate, too.

Lemma 6.6. *If $i_{p,c}^T > 0$ for some $p \in \mathbb{L}$ and $c \in \mathbb{R}$, then*

$$i_{p,c}^{T,+} \le (1/2)d_p - 1$$

Proof. The curve of tangencies has at least $2i_{p,c}^{T,+}$ real half-branches tangent to $\mathbb{L}$ at p. Suppose u is a point on one of these curves of tangencies. Since $k_u = +1/2$ and $c \in \mathbb{R}$, the curves $f(x,y) = f(u)$ converge to two half-branches of $f(x,y) = c$ with common point p as $u \to p$. Thus $f = c$ has at least $2i_{p,c}^{T,+} + 1$ half-branches tangent to $\mathbb{L}$ at p, and hence at least $i_{p,c}^{T,+} + 1$ branches. As in the proof of Lemma 6.5,

$$i_{p,c}^{T,+} - 1 \le (1/2)\{f=0\}\cdot\{z=0\}_p = (1/2)d_p$$

□

For example, the two-parabola polynomial has $i_{[1,0,0],0} = i_{[1,0,0],0}^{T,+} = 1$ and $d_p = 6$.

Theorem 6.7. *If $f(x,y)$ is a real polynomial of degree d with isolated critical points, and i is the index of grad f around a large circle containing the critical points, then*

$$i \le max\{1, d-3\}$$

Proof. If $l = 0$ then $i = 1$ by Lemma 6.2. If $l \geq 2$ then $i \leq d_R - 3$ by Lemma 6.3. Thus we must treat the case $l = 1$. Suppose $f \cap \mathbb{L} = \{p\}$, and that $p = [1,0,0]$. Note that $d_R = d_p$ and $\xi_{\mathbf{L},nc} = \xi_{p,nc}$. From Formula 3.2 and Part (3) of Proposition 4.4 we have that

$$\begin{aligned} i &= 1 + \sum_{c \in \mathbb{R}} i_{p,c} + i_{\mathbf{L},\infty} \\ &= 1 + \sum_{c \in \mathbb{R}} i_{p,c} - \xi_{p,nc} \end{aligned} \tag{4}$$

Since $\xi_{\mathbf{L},nc} \geq l = 1$ by Corollary 4.3,

$$i \leq \sum_{c \in \mathbb{R}} i_{p,c}$$

If $d_p < d$, then $d_p \leq d - 2$ since the roots of f_d other than p are complex and hence conjugate. Thus by Lemma 6.1

$$\sum_{c \in \mathbb{R}} i_{p,c} \leq \sum_{c \in \mathbb{R}} i_{p,c}^{abs} \leq d_p - 1 \leq d - 3$$

Thus we may assume that $d_p = d$, so that

$$f(x,y) = y^d + h(x,y)$$

where h has degree $e < d$. If h is a function of x alone, then $i = 1 - e$: If h_x has distinct zeros then f has $e - 1$ saddles; the general f can be perturbed to this case. Thus we may assume that h is a nonconstant function of both x and y.

If $i_{p,c}^N = 0$ for all $c \in \mathbb{R}$, then

$$\sum_{c \in \mathbb{R}} i_{p,c} = \sum_{c \in \mathbb{R}} i_{p,c}^T \leq \sum_{c \in \mathbb{R}} i_{p,c}^{T,abs} \leq (1/2)(d_p - 1) \leq max\,\{1, d_R - 3\}$$

where the second inequality follows from Lemma 6.5.

If $i_{p,c}^T = 0$ for all $c \in \mathbb{R}$, we proceed as follows: As before let f_y^T (respectively, f_y^N) be the branches of f_y tangent (respectively, not tangent) to $\mathbb{L}$ at p, so that $f_y = f_y^T f_y^N$ at p. Since h is a nonconstant function of both x and y, f_y^T is not empty: We have that $f_y(x,y) = dy^{d-1} + h_y(x,y)$ where $h_y \neq 0$, and changing to coordinates (y,z) at $[1,0,0]$ shows that z divides the term of lowest degree. Thus the intersection number $\{f_y^T = 0\} \cdot \{z = 0\}_p \geq 2$. Thus

$$\begin{aligned}\sum_{c\in\mathbb{R}} i_{p,c} &\leq \sum_{c\in\mathbb{R}} i_{p,c}^{N,abs} \leq \{f_y^N = 0\} \cdot \{z = 0\}_p \\ &= \{f_y = 0\} \cdot \{z = 0\}_p - \{f_y^T = 0\} \cdot \{z = 0\}_p \\ &\leq (d_p - 1) - 2 \\ &= d_p - 3\end{aligned}$$

where the second inequality follows from the proof of Lemma 6.5, and the third from the proof of Lemma 6.1.

If $i_{p,a}^N > 0$ for some $a \in \mathbb{R}$, and $i_{p,b}^T > 0$ for some $b \in \mathbb{R}$ (or more generally, $i_{p,a}^{N,abs} > 0$ and $i_{p,b}^{T,abs} > 0$), we claim that $\xi_{p,nc} \geq 2$. Assuming this, by Equation (4)

$$\begin{aligned}i &\leq \sum_{c\in\mathbb{R}} i_{p,c} - 1 \\ &\leq \sum_{c\in\mathbb{R}} i_{p,c}^{abs} - 1 \\ &= \sum_{c\in\mathbb{R}} i_{p,c}^{N,abs} + 2\Big(\sum_{c\in\mathbb{R}} i_{p,c}^{T,abs}\Big) - \sum_{c\in\mathbb{R}} i_{p,c}^{T,abs} - 1 \\ &\leq (d-1) - 1 - 1 = d - 3\end{aligned}$$

where the last inequality follows from Lemma 6.5.

It remains to show that $\xi_{p,nc} \geq 2$ under these conditions. We suppose for notational simplicity that the tangent cone to $(f_y^N)_{\mathbb{R}}$ is $y = 0$, that the branches of $(f_y^T)_{\mathbb{R}}$ at p lie on the positive-x side of $\mathbb{L}$, and that $i_{p,c}^{N,abs} = 0$ for $c \neq a$ and that $i_{p,c}^{T,abs} = 0$ for $c \neq b$. The graph (see Figure 9) of $f(C, y) = z$ in the yz-plane for $C \gg 0$ has critical points near $(0, a)$ which converge to $(0, a)$ as $C \to \infty$, and other critical points near $(\pm B, b)$ with B large which converge to $(\pm\infty, b)$ as $C \to \infty$. The values of $f(C, y)$ for y between 0 and $\pm B$ converge to $\pm\infty$ by [DKM+93, 1.3]. Thus (see Figure 10) for $A \gg 0$, the set $\{f = +A\} \cup \{f = -A\}$ has at least six ends coming into p on the $x > 0$ side of $\mathbb{L}$, hence at least eight ends on both sides, and hence at least four connected components in $\mathbb{R}^2$. Since there is a two-to-one correspondence between connected components of $\{f = \pm A\}$ in $\mathbb{R}^2$ and nonconstant exceptional sets by Lemma 4.1, this shows that $\xi_{p,nc} \geq 2$. □

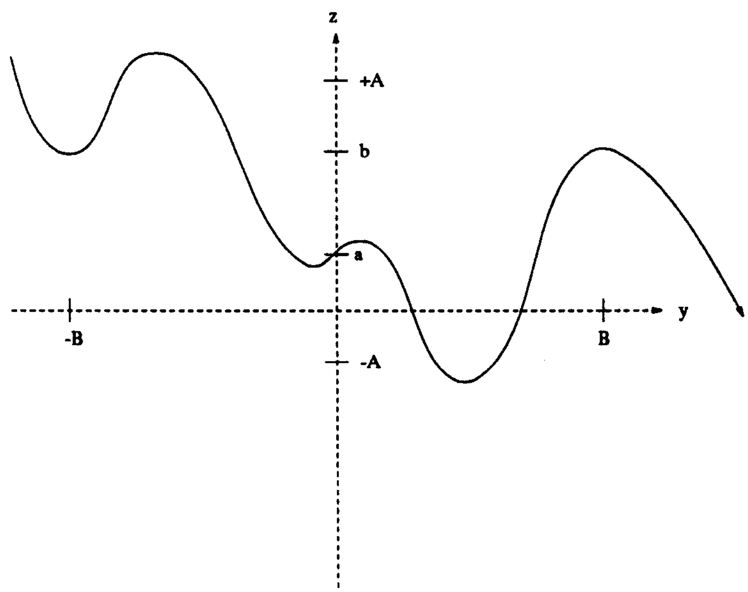

Figure 9: The graph of $z = f(C, y)$, for C large

We finally remark on an inequality for i_p and m_p, where m_p is the intersection multiplicity at p of the completions of the sets $\{f_x = 0\}$ and $\{f_y = 0\}$.

The proof of Proposition 6.2 of [DKM+93] (due to Jeff Roy) gives:

$$2i_p \leq m_p$$

This estimate is not optimal; for example the function $y^5 + y^3x^2 - y$ at $p = [1, 0, 0]$ has $i_p = 2$ and $m_p = 12$. Proposition 6.2 from [DKM+93] easily follows from the above and 3.2.

7 Vanishing cycles

Let $f(x, y)$ be a complex polynomial with isolated critical points. Given p in the complex line at infinity and $c \in \mathbb{C} \cup \{\infty\}$, we define the number of *vanishing cycles* of f at (p, c) by

$$\nu_{p,c} = rank\, H_1(\overline{\{(x, y) \in \mathbb{C}^2 \mid f(x, y) = t\}} \cap B)$$

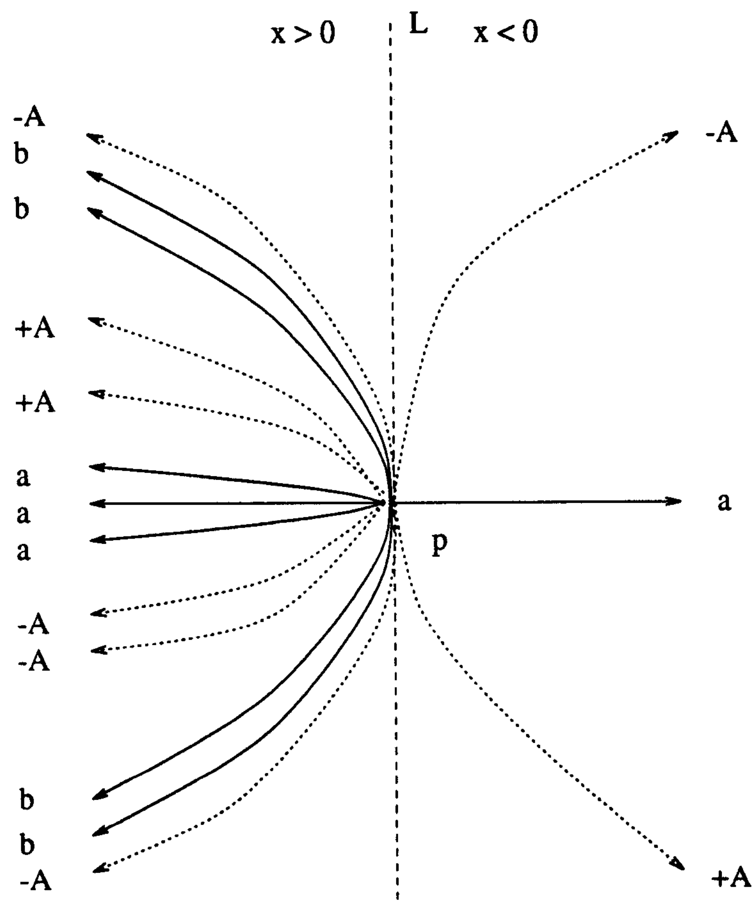

Figure 10: Level sets of f near p

where B is a small ball about p in complex projective space and t is near to, but not equal to, c. (If $c = \infty$, we take $|t|$ large.) If $p = [1, 0, 0]$ and

$$g_t(y, z) = z^d f(1/z, y/z) - tz^d$$

we have that

$$\nu_{p,c} = \mu_{p,c} - \mu_{p,gen}$$

where $\mu_{p,c}$ is equal to the Milnor number of $g_c(y, z)$ at $(0, 0)$ and $\mu_{p,gen}$ is equal to the 'generic Milnor number', which is the Milnor number of $g_t(y, z)$ at $(0, 0)$ for t near c.

Proposition 7.1. *Let $f(x, y)$ be a real polynomial with isolated critical points. For $p \in f_d \cap \mathbb{L}$ and $c \in \mathbb{R} \cup \{\infty\}$,*

$$i^{abs}_{p,c} \leq \nu_{p,c}$$

Proof. If u is a point of tangency associated to (p, c), then for large $|u|$ we have either $f(u) \geq c$ or $f(u) < c$. For large u, let

$$\begin{aligned} i^{abs,\uparrow}_{p,c} &= \sum |k_u| \text{ for } f(u) < c \\ i^{abs,\downarrow}_{p,c} &= \sum |k_u| \text{ for } f(u) \geq c \end{aligned}$$

Thus

$$i^{abs}_{p,c} = i^{abs,\uparrow}_{p,c} + i^{abs,\downarrow}_{p,c} \leq 2i^{abs,\downarrow}_{p,c}$$

where we assume without loss of generality that $i^{abs,\uparrow}_{p,c} \leq i^{abs,\downarrow}_{p,c}$ (replace f by $-f$). We may assume that $p = [1, 0, 0]$. By deforming the circle C, the number $2i^{abs,\downarrow}_{p,c}$ is equal to the number of real intersections of the curves $\{f = t\}$ and $f_y = 0$ near p, where t is near (but not equal to) c. This number is less than or equal to the number b of complex intersections of these curves near p.

The set $\overline{\{(x, y) \in \mathbb{C}^2 | f(x, y) = t\}} \cap B$ is a connected branched cover of B. Two sheets come together at each branch point, and all the sheets come together over p. Hurwitz's formula then implies that $b = \nu_{p,c}$.

A similar argument gives the lower bound. (Of course, f_y is a polar curve.) □

The inequality of the proposition is not an equality; for example the polynomial $y(x^a y - 1)$ at $p = [1, 0, 0]$ and $c = 0$ has $i_{p,c} = 1$ and $\nu_{p,c} = a + 1$.

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