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On f-Vectors and Relative Homology

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Abstract. We find strong necessary conditions on the f-vectors, Betti sequences, and relative Betti sequence of a pair of simplicial complexes. We also present an example showing that these conditions are not sufficient. If only the *difference* between two Betti sequences is specified, and not the individual Betti sequences, then the characterization is complete, and the characterization of all pairs of simplicial complexes matches the characterization of pairs of near-cones. Our necessary conditions rely upon a combinatorial decomposition of pairs of simplicial complexes that reflects the homology and relative homology of the complexes.

Keywords: f-vector, Betti sequence, relative homology, simplicial complex, decomposition

1. Introduction

Given a class of simplicial complexes, it is always interesting to ask which vectors can be the f-vector of some complex in that class (see, e.g., [1]). For instance, the Björner-Kalai theorem [2, Theorem 1.1] (restated here as Theorem 2.4) characterizes which vectors can be the f-vector of a simplicial complex with given Betti sequence. Put another way, the Björner-Kalai theorem characterizes which pairs of vectors can be the f-vector and Betti sequence of a single simplicial complex. Our main results, Theorems 1.1 and 1.2, describe strong necessary conditions on which 5-tuples of vectors can be the two f-vectors, two Betti sequences, and one relative Betti sequence (measuring relative homology) of a single *pair* of simplicial complexes. These theorems depend primarily upon the technique of combinatorial decompositions, used previously [3, 10] to sharpen the Björner-Kalai theorem.

For basic definitions of simplicial complexes and their homology and relative homology, see, e.g., [8, Chapter 1] or [11, Section 0.3]. We allow the empty simplicial complex $\emptyset$ consisting of no faces; all other complexes must contain the empty set as a (-1)-dimensional face. We also allow the complex $\{\emptyset\}$ consisting of only the empty face, but we do distinguish between the two complexes $\emptyset$ and $\{\emptyset\}$. Throughout this paper, a **sequence** θ will refer to the special case of a sequence of integers $\theta = (\theta_{-1}, \theta_0, \theta_1, \dots)$ starting with index -1, and having only a finite number of non-zero terms. The f**-vector** of a simplicial complex Δ is the sequence $f(\Delta) = (f_{-1}, f_0, f_1, \dots)$, where $f_i = \#\{F \in \Delta : \dim F = i\}$. The same notion of f-vector will apply in this paper to every finite collection of sets.

Let K be a field, fixed throughout the paper. The **Betti sequence** of a simplicial complex Δ is the sequence $\beta(\Delta) = (\beta_{-1}, \beta_0, \beta_1, \dots)$, where $\tilde{H}_i(\Delta) = \tilde{H}_i(\Delta; K)$ is the ith reduced homology group of Δ with respect to K, and $\beta_i = \dim_K \tilde{H}_i(\Delta)$. Similarly, the **relative Betti sequence** of a pair of simplicial complexes $\Gamma \subseteq \Delta$ is the sequence $\eta(\Delta, \Gamma) = (\eta_{-1}, \eta_0, \eta_1, \ldots)$, where $\tilde{H}_i(\Delta, \Gamma) = \tilde{H}_i(\Delta, \Gamma; K)$ is the ith reduced **relative homology group** of the pair (Δ, Γ) with respect to K, and $\eta_i = \dim_K \tilde{H}_i(\Delta, \Gamma)$. "Reduced" homology means precisely to treat the empty set as a face of any non-empty complex, so β_0 is one less than the number of connected components of Δ, and hence one less than the "unreduced" β_0. Furthermore, $\beta_{-1} = 0$, unless $\Delta = \{\emptyset\}$, in which case $\beta_{-1} = 1$. Reduced relative homology, which also treats the empty set as a face of any non-empty complex, is the same as unreduced relative homology, except that $\eta_{-1}(\{\emptyset\}, \emptyset) = 1$; for any other pair of complexes, $\eta_{-1} = 0$.

The necessary conditions in Theorems 1.1 and 1.2 use several sequence functions and relations which we now introduce. Define the usual componentwise partial order on sequences θ and σ by setting $\theta \leq \sigma$ when $\theta_i \leq \sigma_i$ for all $i \geq -1$, and the usual componentwise sum of sequences θ and σ by setting $(\theta + \sigma)_i = \theta_i + \sigma_i$ for all $i \geq -1$. Let θ_- be the sequence defined by $(\theta_-)_i = \theta_{i-1}$ for $i \geq 0$, and $(\theta_-)_{-1} = 0$, so $\theta_- = (0, \theta_{-1}, \theta_0, \ldots)$ (Stanley [9] uses the notation $E\theta = \theta_-$).

Given an integer $k \geq 1$, any integer $n \geq 1$ can be written uniquely in the form

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_i}{i}$$

such that $a_k > \cdots > a_i \geq i > 0$. Define

$$\partial_{k-1}(n) = \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_i}{i-1}$$

(see, e.g., [4]). Further, define

$$\partial_{-1}(n) = \begin{cases} 0 & \text{if } n = 1 \\ 1 & \text{if } n > 1 \end{cases}$$

and $\partial_{k-1}(0) = 0$ for $k \geq 0$. Finally, let $\partial\theta$ be the sequence defined by $(\partial\theta)_i = \partial_i\theta_i$ for $i \geq -1$. The significance of ∂ is given by the Kruskal-Katona theorem (Theorem 2.1).

Let θ^* denote the sequence whose elements are given by

$$\theta^*_{i-1} = \theta_i - \theta_{i+1} + \theta_{i+2} - \cdots = \sum_{n \geq i} (-1)^{n-i} \theta_n.$$

As a result,

$$\theta = \theta^* + \theta^*_-$$

(because $(\theta_-)^* = (\theta^*)_-$, we write θ^*_- for either). The $*$ operation is linear. See Eq. (11) for the significance of $*$.

Theorem 1.1 *If $\Gamma \subseteq \Delta$ are simplicial complexes, and $f = f(\Delta)$, $g = f(\Gamma)$, $\beta = \beta(\Delta)$, $\gamma = \beta(\Gamma)$, and $\eta = \eta(\Delta, \Gamma)$, then*

$$\partial(\chi + \beta) \leq \chi_{-}, \tag{1}$$
$$\partial(\psi + \gamma) \leq \psi_{-}, \tag{2}$$
$$\gamma' \leq \gamma, \tag{3}$$
$$\gamma' \leq \chi - \psi, \tag{4}$$
$$\gamma' \geq \gamma - \beta, \tag{5}$$
$$\chi, \psi, \eta, \gamma, \beta, \gamma' \geq 0, \tag{6}$$

where $\chi = (f - \beta)^*$, $\psi = (g - \gamma)^*$, and $\gamma' = (\eta - (\beta - \gamma))^*$.

These conditions are more easily stated in terms of χ and ψ, and it is not hard to see that knowing χ, ψ, β, and γ is equivalent to knowing f, g, β, and γ. We therefore call the set of vectors $(\chi, \psi, \beta, \gamma, \eta)$ the **parameters** of the pair $\Gamma \subseteq \Delta$.

Conditions (3)–(5) and the non-negativity of the new parameter γ' (and of η) are new, while the other conditions are immediate consequences of the Björner-Kalai theorem applied separately to Δ and Γ.

The necessary conditions of Theorem 1.1 are insufficient (Example 4.2), but if only $\beta - \gamma$ is specified, and not β and γ individually, then we can find necessary and sufficient conditions; this characterization is Theorem 1.2, below. In this case, the characterization for all pairs is the same as the characterization for pairs of "near-cones," a special combinatorially defined class of simplicial complexes that are each homotopic to a wedge of spheres (see Section 2).

Define the **reduced parameters** for a pair of simplicial complexes to be (χ, ψ, ϕ, η), where $\phi = \beta - \gamma$ and the other parameters are as defined in Theorem 1.1.

Theorem 1.2 *The following are equivalent*:
(a) *(χ, ψ, ϕ, η) are the reduced parameters for some pair of simplicial complexes $\Gamma \subseteq \Delta$;*
(b)

$$\partial(\chi + \beta') \leq \chi_{-}, \tag{7}$$
$$\partial(\psi + \gamma') \leq \psi_{-}, \tag{8}$$
$$\gamma' \leq \chi - \psi, \tag{9}$$
$$\chi, \psi, \eta, \gamma', \beta' \geq 0, \tag{10}$$

where $\gamma' = (\eta - \phi)^*$ and $\beta' = \phi + \gamma'$; and
(c) *(χ, ψ, ϕ, η) are the reduced parameters for some pair of near-cones $\Gamma \subseteq \Delta$.*

Knowing the reduced parameters (χ, ψ, ϕ, η) is *not* the same as knowing (f, g, ϕ, η), since we need β and γ individually to compute f and g from ψ and χ, but this still appears to be the best set of four parameters to pick in an incomplete characterization.

The proofs of Theorems 1.1 and 1.2 depend primarily upon Theorem 3.10, a decomposition of the larger complex Δ that captures most of the information about homology and relative homology of the pair (Δ, Γ). Section 3 is devoted to proving Theorem 3.10.

Some basic techniques of f-vectors and Betti sequences are discussed in Section 2. The proofs of Theorems 1.1 and 1.2 are in Sections 4 and 5, respectively.

2. Compression

A key technique in characterizing f-vectors, and one that we will use in all of our constructions, is compression, a canonical way to construct a simplicial complex with given f-vector.

Definition Let $S = \{x_{i_1} < \cdots < x_{i_k}\}$ and $T = \{x_{j_1} < \cdots < x_{j_k}\}$ be k-subsets of a totally ordered set $V = \{x_1 < \cdots < x_n\}$. Then $S <_{\mathrm{AL}} T$ under the **anti-lexicographic order** if there is a q such that $i_q < j_q$ and $i_p = j_p$ for $p > q$. A collection $\mathcal{C}$ of k-subsets of V is **compressed** if $S \leq_{\mathrm{AL}} T$ and $T \in \mathcal{C}$ together imply that $S \in \mathcal{C}$, and a simplicial complex Δ is **compressed** if Δ_k is compressed for every k.

Theorem 2.1 (Kruskal-Katona [7, 6]) *For a sequence f, the following are equivalent:*
(a) *there is a simplicial complex whose f-vector is f;*
(b) $\partial f \leq f_-$; *and*
(c) *there is a (unique) compressed simplicial complex whose f-vector is f.*

For a proof of the Kruskal-Katona theorem, and further discussion of the uses and generalizations of compression, see [4, Section 8]. We will use the following simple observation repeatedly in our constructions.

Lemma 2.2 *If Γ and Δ are compressed simplicial complexes, then $f(\Gamma) \leq f(\Delta)$ implies $\Gamma \subseteq \Delta$.*

Proof: The anti-lexicographic order $\leq_{\mathrm{AL}}$ use to build compressed complexes is a total order, so $f_k(\Gamma) \leq f_k(\Delta)$ implies $\Gamma_k \subseteq \Delta_k$, for every k, and the lemma follows. □

Corollary 2.3 *For sequences f and g, the following are equivalent:*
(a) *there is a pair of simplicial complexes $\Gamma \subseteq \Delta$ such that $f = f(\Delta)$ and $g = f(\Gamma)$;*
(b)

$$g \leq f,$$
$$\partial g \leq g_-,$$
$$\partial f \leq f_-.$$

Proof:

(a)$\Rightarrow$(b) That $g \leq f$ is immediate from $\Gamma \subseteq \Delta$; the other two conditions are just the Kruskal-Katona conditions (Theorem 2.1).
(b)$\Rightarrow$(a) Let Γ and Δ be the (unique) compressed simplicial complexes with f-vectors g and f respectively, guaranteed to exist because of the Kruskal-Katona theorem (Theorem 2.1). Then Lemma 2.2 implies $\Gamma \subseteq \Delta$. □

Remark The explicit statement of Corollary 2.3 appears to be new, though implicit in the literature. (I am grateful to Richard Stanley and Curtis Greene for pointing this out to me.)

Remark The statement and proof of Corollary 2.3 extend easily to chains of simplicial complexes.

Björner and Kalai improved upon the Kruskal-Katona theorem by characterizing the f-vector of a simplicial complex with prescribed Betti sequences. The characterization uses near-cones [2, Section 4].

Definition A **near-cone** with **apex** v_0 is a simplicial complex Δ satisfying the following property: For all $F \in \Delta$, if $v_0 \notin F$ and $w \in F$, then

$$(F - \{w\}) \cup \{v_0\} \in \Delta.$$

For Δ a near-cone with apex v_0, let $B(\Delta) = \{F \in \Delta : F \cup \{v_0\} \notin \Delta\}$ and $\Delta' = \{F \in \Delta : v_0 \notin F, F \cup \{v_0\} \in \Delta\}$; then

$$\Delta = (v_0 * \Delta') \dot{\cup} B(\Delta),$$

where $*$ denotes topological join (so $v_0 * \Delta' = \Delta' \dot{\cup} \{\{v_0\} \dot{\cup} F : F \in \Delta'\}$). Both Δ' and $\Delta' \dot{\cup} B(\Delta)$ are subcomplexes of Δ. If $B(\Delta) = \emptyset$, then Δ is simply a **cone**.

Note, in particular, that $\emptyset$ and $\{\emptyset\}$ are near-cones (the condition in the definition is vacuous in this case) and that $\emptyset = v_0 * \emptyset$ and $\{\emptyset\} = (v_0 * \emptyset) \dot{\cup} \{\emptyset\}$. If Δ is a near-cone with apex v_0, then v_0 is one of the vertices of Δ, unless $\Delta = \emptyset$ or $\{\emptyset\}$.

Every $F \in B(\Delta)$ is maximal in Δ, so the collection of faces in $B(\Delta)$ forms an antichain. Further, $f(B(\Delta)) = \beta(\Delta)$, which follows by contracting $v_0 * \Delta'$ to v_0, leaving a sphere for every face in $B(\Delta)$ [2, Theorem 4.3]. Also note [2, p. 292] that

$$f(\Delta') = (f(\Delta) - \beta(\Delta))^*. \tag{11}$$

Theorem 2.4 (Björner-Kalai [2]) *For sequences f and β, the following are equivalent*:
(a) *there is a simplicial complex Δ such that $f = f(\Delta)$ and $\beta = \beta(\Delta)$;*
(b) *there is a near-cone Δ such that $f = f(\Delta)$ and $\beta = \beta(\Delta)$;*
(c) $\partial(\chi + \beta) \leq \chi_-$, *where $\chi = (f - \beta)^*$.*

Proof: We will not reproduce the proof, but mention for our constructions that in the near-cone $\Delta = (v_0 * \Delta') \dot\cup B$ that Björner and Kalai construct in order to prove (c)⇒(b), Δ' and $\Delta' \dot\cup B$ are compressed subcomplexes, and $\chi = f(\Delta')$. □

Remark Björner and Kalai [2] state the numerical condition (c) of Theorem 2.4 slightly differently. In particular, they explicitly include the Euler-Poincaré relation: If Δ has at least one vertex, then

$$1 = (f_0 - f_1 + f_2 - \cdots) - (\beta_0 - \beta_1 + \beta_2 - \cdots).$$

To see why Theorem 2.4(c) implies this equation, rewrite the right-hand side as

$$(f_0 - \beta_0) - (f_1 - \beta_1) + (f_2 - \beta_2) - \cdots = (f - \beta)^*_{-1} = \chi_{-1}.$$

Thus the Euler-Poincaré relation is that $\chi_{-1} = 1$ (this is how it is stated in [2]); it is not hard to show that this condition is implicitly contained in condition (c).

3. Decomposition

In this section, we prove the decomposition theorem (Theorem 3.10) from which Theorems 1.1 and 1.2 follow. Our model is the following theorem that implies and sharpens the Björner-Kalai theorem (Theorem 2.4); see [3, Section 2] for details. The acyclic case is due to Stanley [10, Theorem 1.2].

Theorem 3.1 ([3, Theorem 1.1]) *Every (finite) simplicial complex Δ can be written as a disjoint union $\Delta = \Delta' \dot\cup B \dot\cup \Omega$, where:*
(a) *Δ' is a subcomplex of Δ;*
(b) *$f(B) = \beta(\Delta)$ and B is an antichain;*
(c) *$\Delta' \dot\cup B$ is a subcomplex of Δ; and*
(d) *there exists a bijection $\eta : \Delta' \to \Omega$ such that for all $F \in \Delta'$ we have $F \subset \eta(F)$ and $|\eta(F) - F| = 1$.*

Theorem 3.10 generalizes Theorem 3.1 to pairs of simplicial complexes and relative homology.

Corollary 3.2 *If $\Delta = \Delta' \dot\cup B \dot\cup \Omega$ is the decomposition of a simplicial complex Δ described in Theorem 3.1, then $f(\Delta) = f(\Delta') + \beta(\Delta) + f_-(\Delta')$*

Proof: From the given decomposition, condition (d) implies $f(\Omega) = f_-(\Delta')$, and condition (b) implies $f(B) = \beta(\Delta)$. Then the decomposition yields $f(\Delta) = f(\Delta') + f(B) + f(\Omega) = f(\Delta') + \beta(\Delta) + f_-(\Delta')$. □

The rest of this section is devoted to proving Theorem 3.10. We start with some algebraic preliminaries. Recall that K is a field. Let Δ be a simplicial complex on vertex set $V = \{x_1, \ldots, x_n\}$. Let $\Lambda(KV)$ denote the exterior algebra of the vector space KV; it has a K-vector space basis consisting of all the monomials $x^F := x_{i_1} \wedge \cdots \wedge x_{i_k}$ where $F = \{x_{i_1}, \ldots, x_{i_k}\} \subseteq V$. Let I_Δ be the ideal of $\Lambda(KV)$ generated by all $\{x^F : F \notin \Delta\}$. The quotient algebra $\Lambda[\Delta] := \Lambda(KV)/I_\Delta$ is called the **exterior face ring** of Δ (over K) (see [2] or [3] for more details).

Because K is a field, $\dim_K \tilde{H}^i(\Delta; K) = \dim_K \tilde{H}_i(\Delta; K)$ [8, Section 53], where $\tilde{H}^i(\Delta; K)$ denotes the ith reduced cohomology group with respect to K, and may be computed with a coboundary operator [8, Section 42]. It is not hard to see that, in the exterior face ring, the standard coboundary operator $\delta : \Lambda[\Delta] \to \Lambda[\Delta]$ is simply right multiplication by $v = x_1 + \cdots + x_n \in \Lambda[\Delta]$; i.e., $\delta y = y \wedge v$.

Let Γ be a subcomplex of Δ. Since $\Gamma \subseteq \Delta$ is a simplicial complex in its own right, we may define $\Lambda[\Gamma]$ and δ_Γ in exactly the same way.

Now let $\Sigma = \Delta - \Gamma$, the poset of faces in Δ but not Γ. Relative (co)homology of the pair (Δ, Γ) depends only Σ; in particular, we may compute relative (co)homology of (Δ, Γ) by a (co)boundary operator on Σ (see, e.g., [8, Section 43]). Define $\Lambda[\Sigma]$ to be the ideal in $\Lambda[\Delta]$ generated by the faces in Σ. (This is analogous to the commutative case [9, p. 205].) Because $\Lambda[\Sigma]$ is an ideal, $s \in \Lambda[\Sigma]$ implies $\delta s \in \Lambda[\Sigma]$. The coboundary operator $\delta_\Sigma : \Lambda[\Sigma] \to \Lambda[\Sigma]$ is just the restriction of δ to $\Lambda[\Sigma]$, i.e., $\delta_\Sigma x = \delta x$ (for all $x \in \Lambda[\Sigma]$).

Also because $\Lambda[\Sigma]$ is an ideal, we may interpret $\Lambda[\Gamma]$ as a quotient algebra $\Lambda[\Delta]/\Lambda[\Sigma]$; for $x \in \Lambda[\Delta]$, define $\tilde{x} = x + \Lambda[\Sigma] \in \Lambda[\Gamma]$. Then for $x \in \Lambda[\Delta]$, we have $\delta_\Gamma \tilde{x} = \delta x + \Lambda[\Sigma]$. From now on, we will interpret $\Lambda[\Gamma]$ and δ_Γ in this way.

Lemma 3.3 *For any* $\Gamma \subseteq \Delta$ *and* $\Sigma = \Delta - \Gamma$,
(a) $\ker \delta_\Gamma = \{x \in \Lambda[\Delta] : \delta x \in \Lambda[\Sigma]\}$; *and*
(b) $\operatorname{im} \delta_\Gamma = \operatorname{im} \delta + \Lambda[\Sigma]$.

Proof: By the definitions of $\Lambda[\Gamma]$ (as the quotient algebra $\Lambda[\Delta]/\Lambda[\Sigma]$) and δ_Γ,

$$\begin{aligned}\ker \delta_\Gamma &= \{\tilde{x} \in \Lambda[\Gamma] : \delta x + \Lambda[\Sigma] \in \Lambda[\Sigma]\} = \{x + \Lambda[\Sigma] : x \in \Lambda[\Delta],\ \ \delta x \in \Lambda[\Sigma]\} \\ &= \{x \in \Lambda[\Delta] : \delta x \in \Lambda[\Sigma]\} + \Lambda[\Sigma] \\ &= \{x \in \Lambda[\Delta] : \delta x \in \Lambda[\Sigma]\},\end{aligned}$$

proving (a).

The definitions of $\Lambda[\Gamma]$ and δ_Γ also imply $\operatorname{im} \delta_\Gamma = \{\delta x + \Lambda[\Sigma] : x \in \Lambda[\Delta]\} = \{\delta x : x \in \Lambda[\Delta]\} + \Lambda[\Sigma] = \operatorname{im} \delta + \Lambda[\Sigma]$, proving (b). □

Lemma 3.4 *If* $\Gamma \subseteq \Delta$, *then* $\Lambda[\Gamma]/(\ker \delta_\Gamma) \cong \Lambda[\Delta]/\{x \in \Lambda[\Delta] : \delta x \in \Lambda[\Sigma]\}$.

Proof: Let $\Sigma = \Delta - \Gamma$ as before. Then by Lemma 3.3(a),

$$\begin{aligned}\Lambda[\Gamma]/\ker \delta_\Gamma &\cong (\Lambda[\Delta]/\Lambda[\Sigma])/\ker \delta_\Gamma \cong \Lambda[\Delta]/(\ker \delta_\Gamma + \Lambda[\Sigma]) \\ &\cong \Lambda[\Delta]/(\{x \in \Lambda[\Delta] : \delta x \in \Lambda[\Sigma]\} + \Lambda[\Sigma]) \\ &\cong \Lambda[\Delta]/\{x \in \Lambda[\Delta] : \delta x \in \Lambda[\Sigma]\},\end{aligned}$$

as desired. □

Lemma 3.5 *For any* $\Gamma \subseteq \Delta$ *and* $\Sigma = \Delta - \Gamma$,
(a) $\ker \delta_\Sigma = \ker \delta \cap \Lambda[\Sigma]$; *and*
(b) $\operatorname{im} \delta_\Sigma = \{\delta x : x \in \Lambda[\Sigma]\}$.

Proof: Part (b) is immediate from the definition of δ_Σ. The definition of δ_Σ also implies $\ker \delta_\Sigma = \{x \in \Lambda[\Sigma] : \delta x = 0\} = \ker \delta \cap \Lambda[\Sigma]$. □

Extend the f_- notation from sequences to graded vector spaces in the obvious way. Namely, for a graded vector space V whose ith graded component is V_i, let V_- denote the graded vector space with ith graded component $(V_-)_i = V_{i-1}$.

Lemma 3.6 *As graded vector spaces,*

$$\{x \in \Lambda[\Delta] : \delta x \in \Lambda[\Sigma]\}_- / (\ker \delta + \Lambda[\Sigma])_- \cong (\operatorname{im} \delta \cap \Lambda[\Sigma]) / \{\delta s : s \in \Lambda[\Sigma]\}.$$

Proof: The isomorphism is induced by δ. The lemma then follows from the following four simple claims, the first two of which establish that the map δ is well-defined on the whole space and on the quotient space, respectively, and the last two of which establish injectivity and surjectivity, respectively.

Claim 1. $\delta\{x \in \Lambda[\Delta] : \delta x \in \Lambda[\Sigma]\} \subseteq \operatorname{im} \delta \cap \Lambda[\Sigma]$.
This is obvious.

Claim 2. $\delta(\ker \delta + \Lambda[\Sigma]) \subseteq \{\delta s : s \in \Lambda[\Sigma]\}$.
Let $x \in \ker \delta + \Lambda[\Sigma]$, so $x = z + s$, where $z \in \ker \delta$ and $s \in \Lambda[\Sigma]$. Then $\delta x = \delta z + \delta s = 0 + \delta s \in \{\delta s : s \in \Lambda[\Sigma]\}$.

Claim 3. If $\delta x \in \{\delta s : s \in \Lambda[\Sigma]\}$, then $x \in \ker \delta + \Lambda[\Sigma]$.
If $\delta x \in \{\delta s : s \in \Lambda[\Sigma]\}$, then $\delta x = \delta s$ for some $s \in \Lambda[\Sigma]$. Therefore $\delta(x - s) = 0$, so $x - s \in \ker \delta$, and thus $x = (x - s) + s \in \ker \delta + \Lambda[\Sigma]$.

Claim 4. If $y \in \operatorname{im} \delta \cap \Lambda[\Sigma]$, then there is an $x \in \{x \in \Lambda[\Delta] : \delta x \in \Lambda[\Sigma]\}$ such that $\delta x = y$. If $y \in \operatorname{im} \delta \cap \Lambda[\Sigma]$, then $y \in \operatorname{im} \delta$, so there is an x such that $\delta x = y$. But then $\delta x = y \in \Lambda[\Sigma]$, so $x \in \{x \in \Lambda[\Delta] : \delta x \in \Lambda[\Sigma]\}$. □

We need an improved version of the following lemma.

Lemma 3.7 ([3, Lemma 3.2]) *Let* Δ *be any simplicial complex with simplicial coboundary operator* $\delta : \Lambda[\Delta] \to \Lambda[\Delta]$. *If* $k \in \ker \delta$ *and* x *is a vertex of* Δ, *then* $k \wedge x \in \operatorname{im} \delta$.

Corollary 3.8 *If* $\delta k \in \Lambda[\Sigma]$ *and* x *is a vertex of* Δ, *then* $k \wedge x \in \operatorname{im} \delta + \Lambda[\Sigma]$.

Proof: If $x \in \Sigma$, then $k \wedge x \in \Lambda[\Sigma]$. Otherwise, $x \in \Gamma$, and we may apply Lemma 3.7 to Γ and coboundary operator $\delta_\Gamma : \Lambda[\Gamma] \to \Lambda[\Gamma]$. By Lemma 3.3, $\ker \delta_\Gamma = \{k \in \Lambda[\Delta] : \delta k \in \Lambda[\Sigma]\}$ and $\operatorname{im} \delta_\Gamma = \operatorname{im} \delta + \Lambda[\Sigma]$, and then the corollary follows. □

The acyclic version ($\operatorname{im}\mu = \ker\mu$) of the following result is due to Stanley [10, Lemma 1.1].

Lemma 3.9 ([3, Lemma 3.1]) *Let D be a directed graph on the vertex set X, and let KX be the K-vector space with basis X. Suppose there is a linear transformation $\mu : KX \to KX$ satisfying*

(a) *if $x \in X$, then $\mu(x) \in \operatorname{span}_K\{y \in X : (x, y)$ is an edge of $D\}$; and*

(b) $\operatorname{im}\mu \subseteq \ker\mu$ *(i.e., $\mu^2 = 0$).*

Also assume that Y is a subset of X whose image in $KX/(\operatorname{im}\mu)$ is a basis for $KX/(\operatorname{im}\mu)$ and that Z is a subset of Y whose image in $KX/(\ker\mu)$ is a basis for $KX/(\ker\mu)$. Then there is a matching of Z and $X - Y$ in D.

The following theorem is at the heart of our main results (Theorems 1.1 and 1.2). The statement and proof reduce to those of Theorem 3.1 in the case of $\Gamma = \emptyset$.

Theorem 3.10 *Let $\Gamma \subseteq \Delta$ be a pair of simplicial complexes. Then Δ can be written as a disjoint union $\Delta = \Gamma' \dot\cup G' \dot\cup \Sigma' \dot\cup \bar{G} \dot\cup B' \dot\cup \Omega$ such that (see figure 1)*

(a) *Γ', $\Gamma' \dot\cup G'$, $\Gamma' \dot\cup G' \dot\cup \Sigma'$, $\Gamma' \dot\cup G' \dot\cup \Sigma' \dot\cup \bar{G}$, and $\Gamma' \dot\cup G' \dot\cup \Sigma' \dot\cup \bar{G} \dot\cup B'$ are subcomplexes;*

(b) *G' and $\bar{G} \dot\cup B'$ are antichains;*

(c)

$$f(\Gamma) = f(\Gamma') + f_-(\Gamma') + f(G') + f(\bar{G}), \tag{12}$$

$$\beta(\Gamma) = f(G') + f(\bar{G}), \tag{13}$$

$$\beta(\Delta) = f(\bar{G}) + f(B'), \tag{14}$$

$$\eta(\Delta, \Gamma) = f_-(G') + f(B'); \tag{15}$$

and

(d) *there exists a bijection $\mu : (\Gamma' \dot\cup G' \dot\cup \Sigma') \to \Omega$ such that $F \subseteq \mu(F)$ and $|\mu(F) - F| = 1$ for all $F \in \Gamma' \dot\cup G' \dot\cup \Sigma'$.*

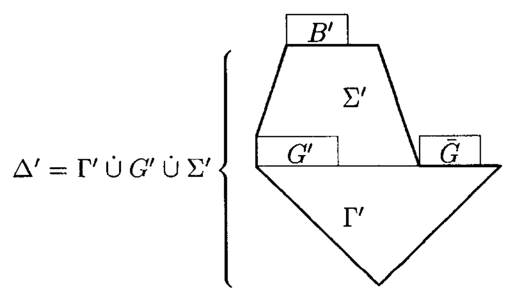

Figure 1. Theorem 3.10.

Proof: Define ideals of $\Lambda[\Delta]$

$$I_1 = \{x \in \Lambda[\Delta] : \delta x \in \Lambda[\Sigma]\},$$
$$I_2 = \ker\delta + \Lambda[\Sigma],$$
$$I_3 = \ker\delta,$$
$$I_4 = \ker\delta \cap (\operatorname{im}\delta + \Lambda[\Sigma]), \text{ and}$$
$$I_5 = \operatorname{im}\delta$$

(these definitions are not so odd, in light of Lemmas 3.3–3.6). Also let

$$Q_i = \Lambda[\Delta]/I_i$$

for each i. It is not hard to see that, for each i, $I_i \supseteq I_{i+1}$, and thus $Q_i \subseteq Q_{i+1}$.

We now inductively define a set of face monomials, using the lexicographic total order from Section 2. Let L_i be the lexicographically least set of face monomials such that $L_1 \dot\cup \cdots \dot\cup L_i$ is a basis for Q_i ($i = 1, \ldots, 5$). (So L_1 is the lexicographically least basis for Q_1; L_2 is lexicographically least such that $L_1 \dot\cup L_2$ is a basis for Q_2; etc.) Thus, if $F \in \Delta$, then $x^F \notin L_1 \dot\cup \cdots \dot\cup L_i$ if and only if

$$x^F = a_1 x^{F_1} + \cdots + a_t x^{F_t} + k, \tag{16}$$

where $k \in I_i$ and, for each j, we have $a_j \in K$, $F_j \in \Delta$, and $F_j <_L F$. It also follows that L_i is a basis for I_{i-1}/I_i (let $I_0 = \Lambda[\Delta]$). Finally, let

$$\Gamma' = \{F \in \Delta : x^F \in L_1\},$$
$$G' = \{F \in \Delta : x^F \in L_2\},$$
$$\Sigma' = \{F \in \Delta : x^F \in L_3\},$$
$$\bar{G} = \{F \in \Delta : x^F \in L_4\}, \quad \text{and}$$
$$B' = \{F \in \Delta : x^F \in L_5\}$$

(see figure 1).

Proof of (a). Suppose $x^F \notin L_1 \dot\cup \cdots \dot\cup L_i$ and $F \subset H$. We need to show that $x^H \notin L_1 \dot\cup \cdots \dot\cup L_i$. Multiply Eq. (16) on the right by x^{H-F}:

$$\begin{aligned} x^H &= \pm(x^F \wedge x^{H-F}) = \pm\left(\sum_{i=1}^{t} a_i x^{F_i} \wedge x^{H-F}\right) \pm (k \wedge x^{H-F}) \\ &= \left(\sum_{\substack{F_i \dot\cup (H-F) \in \Delta \\ F_i \cap (H-F) = \emptyset}} \pm a_i x^{F_i \dot\cup (H-F)}\right) \pm (k \wedge x^{H-F}). \end{aligned} \tag{17}$$

Now, $F_i \dot\cup (H - F) <_L F \dot\cup (H - F) = H$, and $k \wedge x^{H-F} \in I_i$, so Eq. (17) implies $x^H \notin L_1 \dot\cup \cdots \dot\cup L_i$.

Proof of (b). The proofs that G' and $\bar{G} \dot\cup B'$ are antichains are similar, so we prove them simultaneously, with the argument for G' in brackets. The technique is similar to that used in (a). To show that $\bar{G} \dot\cup B'$ is an antichain [G' is an antichain], suppose that $x^F \in L_4 \dot\cup L_5$ [$x^F \in L_2$] and $F \subset H$. We need to show that $x^H \notin L_4 \dot\cup L_5$ [$x^H \notin L_2$]. Let $x_j \in H - F$, and let $F' = H - \{x_j\}$. Since $\Gamma' \dot\cup G' \dot\cup \Sigma'$ is a subcomplex [Γ' is a subcomplex] and $F \notin \Gamma' \dot\cup G' \dot\cup \Sigma'$ [$F \notin \Gamma'$], it follows that $F' \notin \Gamma' \dot\cup G' \dot\cup \Sigma'$ [$F' \notin \Gamma'$] also, so $x^{F'} \notin L_1 \dot\cup L_2 \dot\cup L_3$ [$x^{F'} \notin L_1$] and

$$x^{F'} = \sum_{i=1}^{t} a_i x^{F_i} + k,$$

where $k \in \ker\delta$ [$\delta k \in \Lambda[\Sigma]$], $a_i \in K$, and $F_i <_L F'$.

Thus,

$$\begin{aligned} x^H &= \pm(x^{F'} \wedge x_j) = \pm\left(\sum_{i=1}^{t} a_i x^{F_i} \wedge x_j\right) \pm (k \wedge x_j) \\ &= \left(\sum_{\substack{F_i \cup \{x_j\} \in \Delta \\ x_j \notin F_i}} \pm a_i x^{F_i \cup \{x_j\}}\right) \pm (k \wedge x_j). \end{aligned} \tag{18}$$

Now, $F_i \cup \{x_j\} <_L F' \cup \{x_j\} = H$, and $k \wedge x_j \in \operatorname{im}\delta$ [$k \wedge x_j \in \operatorname{im}\delta + \Lambda[\Sigma] \subseteq \ker\delta + \Lambda[\Sigma]$] by Lemma 3.7 [Corollary 3.8], so Eq. (18) implies $x^H \notin L_4 \dot\cup L_5$ [$x^H \notin L_2$].

Proof of (d). Let D be the directed graph whose vertex set is Δ, and whose edges are the pairs (F, H) with $F \subset H \in \Delta$ and $|H - F| = 1$. Because $\delta : \Lambda[\Delta] \to \Lambda[\Delta]$ is a coboundary operator, $\mu = \delta$ satisfies all the conditions of Lemma 3.9. Taking $Z = \Gamma' \dot\cup G' \dot\cup \Sigma'$ and $Y = \Gamma' \dot\cup G' \dot\cup \Sigma' \dot\cup \bar{G} \dot\cup B'$, Lemma 3.9 gives a matching

$$\mu : \Gamma' \dot\cup G' \dot\cup \Sigma' \to \Delta - (\Gamma' \dot\cup G' \dot\cup \Sigma' \dot\cup \bar{G} \dot\cup B') = \Omega$$

satisfying the conditions in (d).

Proof of (c). The union $L_4 \dot\cup L_5$ is a basis of $\ker\delta/\operatorname{im}\delta$, and thus $\beta(\Delta) = f(\bar{G}) + f(B')$, proving Eq. (14).

Similarly, L_2 is a basis for

$$\{x \in \Lambda[\Delta] : \delta x \in \Lambda[\Sigma]\}/(\ker\delta + \Lambda[\Sigma]);$$

meanwhile, L_4 is a basis for

$$\begin{aligned} \ker\delta/(\ker\delta \cap (\operatorname{im}\delta + \Lambda[\Sigma])) &\cong (\ker\delta + (\operatorname{im}\delta + \Lambda[\Sigma]))/(\operatorname{im}\delta + \Lambda[\Sigma]) \\ &\cong (\ker\delta + \Lambda[\Sigma])/(\operatorname{im}\delta + \Lambda[\Sigma]), \end{aligned}$$

where the first isomorphism is via the standard isomorphism $M/(N \cap M) \cong (M+N)/N$ (e.g., [5, p. 176]) and the second follows from $\operatorname{im}\delta \subseteq \ker\delta$. Therefore

$$\begin{aligned} f(G') + f(\bar{G}) &= \dim\{x \in \Lambda[\Delta] : \delta x \in \Lambda[\Sigma]\}/(\ker\delta + \Lambda[\Sigma]) \\ &\quad + \dim(\ker\delta + \Lambda[\Sigma])/(\operatorname{im}\delta + \Lambda[\Sigma]) \\ &= \dim\{x \in \Lambda[\Delta] : \delta x \in \Lambda[\Sigma]\}/(\operatorname{im}\delta + \Lambda[\Sigma]) \\ &= \beta(\Gamma), \end{aligned} \tag{19}$$

by Lemma 3.3, proving Eq. (13).

Now, L_2 is a basis for $\{x \in \Lambda[\Delta] : \delta x \in \Lambda[\Sigma]\}/(\ker\delta + \Lambda[\Sigma])$, which is isomorphic to $(\operatorname{im}\delta \cap \Lambda[\Sigma])/\{\delta x : x \in \Lambda[\Sigma]\}$ with a dimension shift, by Lemma 3.6, so

$$f_-(G') = \dim(\operatorname{im}\delta \cap \Lambda[\Sigma])/\{\delta x : x \in \Lambda[\Sigma]\}.$$

And L_5 is a basis for

$$\begin{aligned} (\ker\delta \cap (\operatorname{im}\delta + \Lambda[\Sigma]))/\operatorname{im}\delta &\cong (\operatorname{im}\delta + (\ker\delta \cap \Lambda[\Sigma]))/\operatorname{im}\delta \\ &\cong (\ker\delta \cap \Lambda[\Sigma])/(\operatorname{im}\delta \cap (\ker\delta \cap \Lambda[\Sigma])) \\ &\cong (\ker\delta \cap \Lambda[\Sigma])/(\operatorname{im}\delta \cap \Lambda[\Sigma]), \end{aligned}$$

where the first and third isomorphisms are because $\operatorname{im}\delta \subseteq \ker\delta$, and the second isomorphism is $M/(N \cap M) \cong (M+N)/N$ again, so

$$f(B') = \dim((\ker\delta \cap \Lambda[\Sigma])/(\operatorname{im}\delta \cap \Lambda[\Sigma])).$$

Therefore,

$$\begin{aligned} f(B') + f_-(G') &= \dim((\ker\delta \cap \Lambda[\Sigma])/(\operatorname{im}\delta \cap \Lambda[\Sigma])) \\ &\quad + \dim((\operatorname{im}\delta \cap \Lambda[\Sigma])/\{\delta x : x \in \Lambda[\Sigma]\}) \\ &= \dim((\ker\delta \cap \Lambda[\Sigma])/\{\delta x : x \in \Lambda[\Sigma]\}) \\ &= \eta(\Delta, \Gamma), \end{aligned}$$

by Lemma 3.5, proving Eq. (15).

Finally, to establish Eq. (12), note that L_1 is the lexicographically least basis of $\Lambda[\Delta]/\{x \in \Lambda[\Delta] : \delta x \in \Lambda[\Sigma]\}$, which is isomorphic to $\Lambda[\Gamma]/\ker\delta_\Gamma$, by Lemma 3.4. This is precisely the definition of L_1, and hence of Γ', needed to ensure that $\Gamma = \Gamma' \dot\cup G^* \dot\cup \Omega_\Gamma$ (for some G^* and Ω_Γ) is the decomposition of Γ described by Theorem 3.1. Corollary 3.2 and Eq. (19) then imply $f(\Gamma) = f(\Gamma') + \beta(\Gamma) + f_-(\Gamma') = f(\Gamma') + f(G') + f(\bar{G}) + f_-(\Gamma')$. □

4. Numerical conditions

In this section, we use the decomposition of Theorem 3.10 to prove the necessary numerical conditions of Theorem 1.1, and a corollary (Corollary 4.1) stating necessary and sufficient conditions for $(\chi, \psi, \beta, \gamma)$ only. We also demonstrate that the necessary conditions of Theorem 1.1 are not sufficient (Example 4.2).

Proof of Theorem 1.1: Conditions (1) and (2) are immediate from the Björner-Kalai theorem (Theorem 2.4), since both Δ and Γ are simplicial complexes.

We next establish combinatorial interpretations of ψ, χ, and γ'. First, note that the matching of Theorem 3.10 implies

$$f(\Omega) = f_-(\Gamma') + f_-(G') + f_-(\Sigma'). \tag{20}$$

The first line in each of the following equations is a direct application of Theorem 3.10, and the remaining lines use simple sequence manipulations and Eq. (20):

$$\begin{aligned}
\psi = (g-\gamma)^* &= ((f(\Gamma') + f_-(\Gamma') + f(G') + f(\bar{G})) - (f(G') + f(\bar{G})))^* \\
&= (f(\Gamma') + f_-(\Gamma'))^* \\
&= f(\Gamma'); \\
\chi = (f-\beta)^* &= ((f(\Gamma') + f(G') + f(\Sigma') + f(\bar{G}) + f(B') + f(\Omega)) \\
&\quad - (f(\bar{G}) + f(B')))^* \\
&= (f(\Gamma') + f(G') + f(\Sigma') + f(\Omega))^* \\
&= (f(\Gamma') + f(G') + f(\Sigma') + (f_-(\Gamma') + f_-(G') + f_-(\Sigma'))^* \\
&= f(\Gamma') + f(G') + f(\Sigma'); \\
\gamma' = (\eta - (\beta - \gamma))^* &= (f_-(G') + f(B') - ((f(\bar{G}) + f(B')) - (f(G') + f(\bar{G})))^* \\
&= (f_-(G') + f(G'))^* \\
&= f(G').
\end{aligned}$$

Inequalities (3), (4), and (5) now follow easily from the non-negativity of $f(\bar{G})$, $f(\Sigma')$, and $f(B')$, respectively

$$\begin{aligned}
\gamma' &= f(G') \\
&\le f(G') + f(\bar{G}) = \gamma; \\
\chi - \psi &= (f(\Gamma') + f(G') + f(\Sigma')) - f(\Gamma') = f(G') + f(\Sigma') \\
&\ge f(G') = \gamma'; \\
\gamma - \beta &= (f(G') + f(\bar{G})) - (f(\bar{G}) + f(B')) = f(G') - f(B') \\
&\le f(G') = \gamma'.
\end{aligned}$$

Finally, we establish condition (6). That η, γ, and β are non-negative is trivial. And the non-negativity of χ, ψ, and γ' follows from their combinatorial interpretation as f-vectors. (Of course, the non-negativity of χ and ψ is also a result of the Björner-Kalai theorem, Theorem 2.4, since Δ and Γ are simplicial complexes.) □

Corollary 4.1 *For sequences f, β, g, and γ, the following are equivalent:*
(a) *there is a pair of simplicial complexes $\Gamma \subseteq \Delta$ such that $f = f(\Delta)$, $\beta = \beta(\Delta)$, $g = f(\Gamma)$, and $\gamma = \beta(\Gamma)$;*
(b)

$$\psi \leq \chi, \tag{21}$$
$$\psi + \gamma \leq \chi + \beta, \tag{22}$$
$$\partial(\chi + \beta) \leq \chi_-, \tag{23}$$
$$\partial(\psi + \gamma) \leq \psi_-, \tag{24}$$

where $\chi = (f - \beta)^*$ and $\psi = (g - \gamma)^*$; and
(c) *there is a pair of near-cones $\Gamma \subseteq \Delta$ with common apex, such that $f = f(\Delta)$, $\beta = \beta(\Delta)$, $g = f(\Gamma)$, and $\gamma = \beta(\Gamma)$.*

Proof:

(c)$\Rightarrow$(a) is obvious.
(a)$\Rightarrow$(b) Conditions (23) and (24) are simply conditions (1) and (2). Equation (4) and the non-negativity of γ' together imply $0 \leq \gamma' \leq \chi - \psi$, and hence $\psi \leq \chi$. Equations (4) and (5) together imply $\gamma - \beta \leq \gamma' \leq \chi - \psi$, and hence $\psi + \gamma \leq \chi + \beta$.
(b)$\Rightarrow$(c) By the Björner-Kalai theorem (Theorem 2.4) and conditions (23) and (24), we can construct near-cones $\Delta = (v_0 * \Delta') \dot\cup B$ and $\Gamma = (v_0 * \Gamma') \dot\cup G$, each with the appropriate f-vector and Betti sequence, such that Δ', $\Delta' \dot\cup B$, Γ', and $\Gamma' \dot\cup G$ are compressed simplicial complexes. By Lemma 2.2, inequalities (21) and (22) imply $\Gamma' \subseteq \Delta'$ and $\Gamma' \dot\cup G \subseteq \Delta' \dot\cup B$, respectively. Then $\Gamma' \subseteq \Delta'$ implies $v_0 * \Gamma' \subseteq v_0 * \Delta'$, and thus $\Gamma = (v_0 * \Gamma') \dot\cup G \subseteq (v_0 * \Delta') \dot\cup B = \Delta$. □

Remark The implication (a)$\Rightarrow$(b) can also be easily proved by the techniques of algebraic shifting (e.g., [2]) (the proof is omitted). This allows the generalization of Corollary 4.1 to chains of simplicial complexes to follow readily.

The necessary conditions of Theorem 1.1 are not sufficient:

Example 4.2 The following parameters satisfy the necessary conditions in Theorem 1.1, but there is no pair of simplicial complexes with these parameters: $\psi = (1, 3)$, $\beta = \gamma = (0, 0, 3)$, $\chi = (1, 4, 3, 1)$, and $\eta = 0$.

Proof: First, we verify that the parameters satisfy Theorem 1.1. From $\beta = \gamma$ and $\eta = 0$, it follows that $\gamma' = (0-0)^* = 0$. Conditions (3)–(6) are then easily verified. By the Kruskal-Katona theorem (Theorem 2.1), conditions (1) and (2) are equivalent to the existence of simplicial complexes $\Gamma' \dot\cup G$ and $\Delta' \dot\cup B$, where G and B are each maximal antichains, with f-vectors $f(\Gamma') = \psi$, $f(G) = \gamma$, $f(\Delta') = \chi$, $f(B) = \beta$. The simplicial complexes in figure 2 have the appropriate f-vectors, and thus show that conditions (1) and (2) are satisfied.

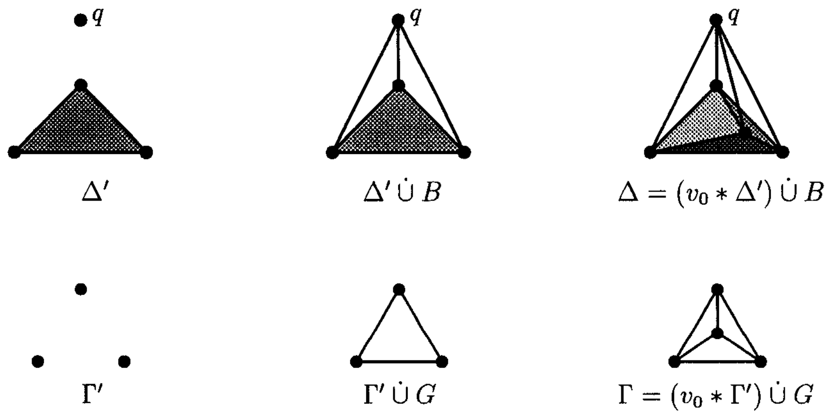

Figure 2. Complexes for Example 4.2.

Yet there is no pair of simplicial complexes with these parameters. Any such pair $\Gamma \subseteq \Delta$ would have f-vectors $f(\Gamma) = \psi + \psi_- + \gamma = (1, 3) + (0, 1, 3) + (0, 0, 3) = (1, 4, 6)$ and $f(\Delta) = \chi + \chi_- + \beta = (1, 4, 3, 1) + (0, 1, 4, 3, 1) + (0, 0, 3) = (1, 5, 10, 4, 1)$. The only simplicial complexes with these f-vectors are $\Gamma = (v_0 * \Gamma') \dot\cup G$ and $\Delta = (v_0 * \Delta') \dot\cup B$ as in figure 2. And, up to symmetry, there are only two ways to choose Γ as a subcomplex of Δ; the relative Betti sequence is either $(0, 0, 1, 1)$ or $(0, 0, 3, 3)$, depending on whether or not Γ is chosen to contain vertex q, contradicting $\eta = 0$. □

5. Reduction

Although Example 4.2 shows that the necessary conditions on parameters of pairs of simplicial complexes in Theorem 1.1 are not sufficient, we can establish "partial sufficiency" by constructing a pair of near-cones with almost the right parameters (Lemma 5.2). This construction and Theorem 1.1 readily lead to an almost complete characterization of the parameters of pairs of simplicial complexes (Theorem 1.2).

First, we generalize Björner and Kalai's argument (Section 2) that the Betti sequence of a near-cone $\Delta = (v_0 * \Delta') \dot\cup B$ is $\beta(\Delta) = f(B)$.

Lemma 5.1 *If $\Gamma \subseteq \Delta$ are a pair of near-cones with a common apex, $\Delta = (v_0 * \Delta') \dot\cup B$ and $\Gamma = (v_0 * \Gamma') \dot\cup G$, then $\eta(\Delta, \Gamma) = f(B) - f(G \cap B) + f_-(G \cap \Delta')$.*

Proof: We will slightly alter Δ and Γ, in ways whose effects on relative homology are easy to measure, replacing them by a pair of near-cones whose relative homology is easy to determine. (See [8, Sections 9 and 43] for details of the basic relative homology techniques we use.)

The first step is to remove some common faces. Faces in G cannot contain the common apex v_0, so $G \subseteq \Delta' \dot\cup B$, and thus $G = (G \cap \Delta') \dot\cup (G \cap B)$. Since B and G are sets of maximal faces in Δ and Γ respectively, $G \cap B = G \backslash \Delta'$ is a set of faces that are maximal in both Δ and Γ; therefore $\Delta^{(1)} = \Delta - (G \cap B)$ and $\Gamma^{(1)} = \Gamma - (G \cap B)$ are simplicial complexes. Removing the same set of faces from a pair of simplicial complexes does not change their relative homology, so

$$\eta(\Delta, \Gamma) = \eta(\Delta^{(1)}, \Gamma^{(1)}). \tag{25}$$

Let $G' = G - (G \cap B) = G \cap \Delta'$, so $G' \subseteq \Delta'$ and $\Gamma^{(1)} = (v_0 * \Gamma') \dot{\cup} G'$. Also let $B' = B - (G \cap B)$, so $\Delta^{(1)} = (v_0 * \Delta') \dot{\cup} B'$, and

$$\beta(\Delta^{(1)}) = f(B') = f(B) - f(G \cap B). \tag{26}$$

The second step is to turn $\Gamma^{(1)}$ into a cone by adding faces, keeping track of which faces are added. Let $\Gamma^{(2)} = v_0 * (\Gamma' \dot{\cup} G')$, a cone. The only difference between $(\Delta^{(1)}, \Gamma^{(2)})$ and $(\Delta^{(1)}, \Gamma^{(1)})$ is that for every i-dimensional face $F \in G'$, there is an $(i+1)$-dimensional face $v_0 * F \in \Delta^{(1)}$ whose entire boundary is in both $\Gamma^{(1)}$ and $\Gamma^{(2)}$, but is itself in $\Gamma^{(2)}$ and not in $\Gamma^{(1)}$. Thus, the difference between $\eta(\Delta^{(1)}, \Gamma^{(1)})$ and $\eta(\Delta^{(1)}, \Gamma^{(2)})$ is counted, with a dimension shift of 1, by $f(G') = f(G \cap \Delta')$, so

$$\eta(\Delta^{(1)}, \Gamma^{(1)}) = \eta(\Delta^{(1)}, \Gamma^{(2)}) + f_{-}(G \cap \Delta'). \tag{27}$$

Finally, since $\Gamma^{(2)}$ is a cone and hence acyclic, [8, Theorem 43.1] implies

$$\eta(\Delta^{(1)}, \Gamma^{(2)}) = \beta(\Delta^{(1)}). \tag{28}$$

The lemma follows by stringing together Eqs. (25)–(28). □

The proof of Lemma 5.1 shows that, at least for near-cones, faces in $G \cap B$ are somewhat extraneous, in that they affect neither η nor $f - g$. For a pair of complexes that are not near-cones, we will remove the "equivalent" of $G \cap B$, in order to construct a pair of near-cones with almost the right parameters. Equations (13) and (14) suggest that the "equivalent" of $G \cap B$ in Theorem 3.10 is $\bar{G}$; the proof of Theorem 1.1 shows that $f(G') = \gamma'$, and then Eq. (13) implies that $f(\bar{G}) = \gamma - f(G') = \gamma - \gamma'$. We therefore subtract $\gamma - \gamma'$ from both γ and β, and use γ' and $\beta' = \beta - (\gamma - \gamma')$ in place of γ and β, respectively.

Lemma 5.2 *If $(\chi, \psi, \beta, \gamma, \eta)$ satisfy conditions* (1)–(6), *then $(\chi, \psi, \beta', \gamma', \eta)$ are the parameters for a pair of near-cones, where $\gamma' = (\eta - (\beta - \gamma))^*$ and $\beta' = (\beta - \gamma) + \gamma'$.*

Proof: By (5) and (6), γ' and β' are non-negative; by (3), $\gamma' \leq \gamma$ and $\beta' \leq \beta$. It follows from (1) and (2), respectively, then, that

$$\partial(\chi + \beta') \leq \partial(\chi + \beta) \leq \chi_{-},$$

and

$$\partial(\psi + \gamma') \leq \partial(\psi + \gamma) \leq \psi_{-}.$$

Also, condition (4) may be restated as

$$\psi + \gamma' \le \chi,$$

and hence $\psi + \gamma' \le \chi \le \chi + \beta'$ and $\psi \le \psi + \gamma' \le \chi$, since γ' and β' are non-negative. Corollary 4.1 then implies that we may construct a pair of near-cones, $\Gamma = (v_0 * \Gamma') \dot\cup G'$ and $\Delta = (v_0 * \Delta') \dot\cup B'$, each with the proper f-vector and Betti sequence, such that $\Gamma \subseteq \Delta$.

It only remains to show that $\eta(\Delta, \Gamma) = \eta$. Examining the proof of Corollary 4.1, we see that $f(B') = \beta(\Delta) = \beta'$ and $f(G') = \beta(\Gamma) = \gamma'$, and so, by Eq. (11), $f(\Delta') = (f - \beta')^* = \chi$ and $f(\Gamma') = (g - \gamma')^* = \psi$. We also see that $\Gamma' \dot\cup G'$ and Δ' are each compressed. Now, $f(\Gamma' \dot\cup G') = \psi + \gamma' \le \chi = f(\Delta')$, so by Lemma 2.2, $\Gamma' \dot\cup G' \subseteq \Delta'$. Then $G' \cap \Delta' = G'$, and, since Δ' and B' are disjoint, $G' \cap B' = \emptyset$. Lemma 5.1 then implies that

$$\begin{aligned}\eta(\Delta, \Gamma) &= f(B') + f_-(G') = \beta' + \gamma'_- = \beta - \gamma + \gamma' + \gamma'_- \\ &= \beta - \gamma + (\eta - (\beta - \gamma))^* + (\eta - (\beta - \gamma))^*_- = \beta - \gamma + (\eta - (\beta - \gamma)) \\ &= \eta.\end{aligned}$$

□

Subtracting the same quantity from both γ and β does not change $\beta - \gamma$, or χ, ψ, η, or even γ', since the definition of γ' is in terms of $\beta - \gamma$ where it involves β and γ at all. Thus, if we only care about $\beta - \gamma$, and not β and γ separately, we do have a complete characterization of parameters (Theorem 1.2); furthermore, the characterization for all pairs of simplicial complexes reduces to the characterization for pairs of near-cones, as in the single simplicial complex case.

Proof of Theorem 1.2:

(c)⇒(a) is trivial.

(a)⇒(b) Let Δ and Γ be the pair of complexes achieving the reduced parameters (χ, ψ, ϕ, η), and let Δ and Γ have (non-reduced) parameters $(\chi, \psi, \beta, \gamma, \eta)$, so $\phi = \beta - \gamma$. We show that the necessary conditions (1)–(6) of Theorem 1.1 imply the conditions in (b). Conditions (2) and (3) imply condition (8). Condition (3) and the definition of β' imply $\beta' \le \beta$; along with condition (1), this implies condition (7). The non-negativity of β' follows from condition (5), and the non-negativity of the other parameters in (10) follows from condition (6). Finally, condition (9) is simply condition (4).

(b)⇒(c) Since β' is defined to be $\phi + \gamma'$, it suffices to show that $(\chi, \psi, \beta', \gamma', \eta)$ are the parameters for a pair of near-cones. But by Lemma 5.2, then, we only need find β and γ so that: $\gamma' = (\eta - (\beta - \gamma))^*$; $\beta' = (\beta - \gamma) + \gamma'$; and $(\chi, \psi, \beta, \gamma, \eta)$ satisfy conditions (1)–(6). Let $\beta = \beta'$ and $\gamma = \gamma'$; then $(\eta - (\beta - \gamma))^* = (\eta - (\beta' - \gamma'))^* = (\eta - \phi)^* = \gamma'$ and $(\beta - \gamma) + \gamma' = (\beta' - \gamma') + \gamma' = \beta'$, as desired, so it only remains to show that $(\chi, \psi, \beta', \gamma', \eta)$ satisfy conditions (1)–(6). Conditions (1), (2), (4), and (6) are simply restatements of conditions (7), (8), (9), and (10), respectively. The non-negativity of β' implies condition (5). And condition (3) is satisfied with equality. □

Remark The implication (a)$\Rightarrow$(c) of Theorem 1.2 also follows directly from the decomposition of Theorem 3.10: Simply construct a pair of near-cones using G' and B' as in Lemma 5.2, ignoring $\bar{G}$.

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