SIAM J. COMPUT.
Vol. 27, No. 2, pp. 356–400, April 1998


003

# ON THE COMPLEXITY OF COMPUTING MIXED VOLUMES*

MARTIN DYER†, PETER GRITZMANN‡, AND ALEXANDER HUFNAGEL§

**Abstract.** This paper gives various (positive and negative) results on the complexity of the problem of computing and approximating mixed volumes of polytopes and more general convex bodies in arbitrary dimension.

On the negative side, we present several #$\mathbb{P}$-hardness results that focus on the difference of computing mixed volumes versus computing the volume of polytopes. We show that computing the volume of zonotopes is #$\mathbb{P}$-hard (while each corresponding mixed volume can be computed easily) but also give examples showing that computing mixed volumes is hard even when computing the volume is easy.

On the positive side, we derive a randomized algorithm for computing the mixed volumes

$$V(\overbrace{K_1,\dots,K_1}^{m_1},\overbrace{K_2,\dots,K_2}^{m_2},\dots,\overbrace{K_s,\dots,K_s}^{m_s})$$

of well-presented convex bodies $K_1,\dots,K_s$, where $m_1,\dots,m_s \in \mathbb{N}_0$ and $m_1 \geq n-\psi(n)$ with $\psi(n)=o(\frac{\log n}{\log\log n})$. The algorithm is an interpolation method based on polynomial-time randomized algorithms for computing the volume of convex bodies.

This paper concludes with applications of our results to various problems in discrete mathematics, combinatorics, computational convexity, algebraic geometry, geometry of numbers, and operations research.

**Key words.** computational convexity, volume, mixed volumes, convex body, polytope, zonotope, parallelotope, computation, approximation, computational complexity, deterministic algorithm, randomized algorithm, polynomial-time algorithm, $\mathbb{NP}$-hardness, #$\mathbb{P}$-hardness, permanent, determinant problems, lattice point enumerator, partial order, Newton polytope, polynomial equations

**AMS subject classifications.** 52B55, 52A39, 68Q20, 68Q15, 68R05, 68U05, 52A20, 90C30, 90C25

**PII.** S0097539794278384

## Introduction.

The present paper deals with algorithmic questions related to the problem of computing or approximating volumes and mixed volumes of convex bodies by means of deterministic or randomized algorithms. The emphasis will be on the case of varying dimension (but we will also mention some results for fixed dimension).

As the terms are used here, a *convex body* in $\mathbb{R}^n$ is a nonempty compact convex set and a *polytope* is a convex body that has only finitely many extreme points. A convex body or a polytope in $\mathbb{R}^n$ is called *proper* if it is $n$-dimensional and hence has nonempty interior. A convenient way to deal algorithmically with general convex bodies is to assume that the convex body in question is "well presented" by an algorithm (called an *oracle*) that answers certain sorts of questions about the body and also gives some a priori information; see subsection 1.2 for precise definitions.

The problem of computing the volume $\mathrm{vol}_n(K)$ of an appropriately presented convex body $K$ of $\mathbb{R}^n$ is of fundamental importance from both a theoretical and computational point of view. If $K$ is of the form $K=\sum_{i=1}^{s}\lambda_i K_i$, where $K_1,\dots,K_s$

*Received by the editors December 9, 1994; accepted for publication (in revised form) January 23, 1996. Research of each author was supported in part by the Deutsche Forschungsgemeinschaft. Research of P. Gritzmann was supported in part by a Max-Planck Research Award.

http://www.siam.org/journals/sicomp/27-2/27838.html

†School of Computer Studies, University of Leeds, Leeds LS2 9JT, U.K. (dyer@dcs.leeds.ac.uk).

‡University of Technology Munich, Center for Mathematical Sciences, D-80290 Munich, Germany (gritzman@mathematik.tu-muenchen.de).

§Vogelherd 9, 90542 Eckental, Germany.

are convex bodies and $\lambda_1, \ldots, \lambda_s$ are positive reals, then $\mathrm{vol}_n(K)$ can be expressed in terms of the *mixed volumes* $V(K_{i_1}, K_{i_2}, \ldots, K_{i_n})$ of $K_1, \ldots, K_s$; in fact,

$$V\left(\sum_{i=1}^{s} \lambda_i K_i\right) = \sum_{i_1=1}^{s} \sum_{i_2=1}^{s} \cdots \sum_{i_n=1}^{s} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_n} V(K_{i_1}, K_{i_2}, \ldots, K_{i_n})$$

is a multivariate homogeneous polynomial of degree $n$ in the variables $\lambda_1, \ldots, \lambda_s$; see subsection 1.1.

The corresponding *Brunn–Minkowski* theory is the backbone of convexity theory (see [Sc93]), but it is also relevant for numerous applications in combinatorics, algebraic geometry and a number of other areas; see section 4 and [GK94].

As it is well known, $V(K, \ldots, K) = \mathrm{vol}_n(K)$, and hence, mixed volumes generalize the ordinary volume. From this observation it is already clear that, in general, any hardness result for volume computation carries over to mixed volumes. Specifically, the problem of computing the volume of polytopes (given in terms of their vertices—"$\mathcal{V}$-polytopes"—or in terms of their facet hyperplanes—"$\mathcal{H}$-polytopes"; see subsection 1.2) is known to be #P-hard (see [DF88]), whence computing mixed volumes of polytopes is also (at least) #P-hard.

The hardness issue of volume versus mixed volume computation is, however, more complicated than that. In the case where the number of bodies is not bounded beforehand but part of the input, the above multivariate polynomial typically has exponentially many coefficients and this implies that the task of computing *all* mixed volumes of a given set of bodies does require exponential time. But this fact also allows for the possibility that the volume of the Minkowski sum of convex bodies may be hard to compute even if each mixed volume can be computed easily. Indeed, when the bodies $K_1, \ldots, K_s$ are all line segments, each mixed volume computation is just the evaluation of a corresponding determinant; computing the volume of the zonotope $K = K_1 + \cdots + K_s$ is, in general however, hard.

THEOREM 1. *The following task is #P-hard: given $n, s \in \mathbb{N}$ and rational vectors $z_1, \ldots, z_s$ of $\mathbb{R}^n$, compute the volume of the zonotope $\sum_{i=1}^{s} [0,1] z_i$.*

A slight strengthening of this result is contained in Theorem 5. As a corollary to Theorem 1 we show in Theorem 2 that (approximately) computing the volume of the Minkowski sum of ellipsoids is also #P-hard, a result needed in subsection 2.4.

Conversely to Theorem 1, computing *a single* mixed volume may be hard even if the volume of the corresponding Minkowski sum is easy to compute.

THEOREM 3. *The following problem MIXED-VOLUME-OF-BOXES is #P-hard: given a positive integer $n$ and, for $i, j = 1, 2, \ldots, n$, positive rationals $\alpha_{i,j}$, determine the mixed volume $V(Z_1, \ldots, Z_n)$ of the axes-parallel parallelotopes $Z_i = \sum_{j=1}^{n} [0, \alpha_{i,j}] e_j$, $(i = 1, 2, , \ldots, n)$, where $e_j$ denotes the $j$th unit vector.*

*The #P-hardness persists even when the boxes are restricted to having just two different (and previously prescribed) edge lengths.*

Proofs of these theorems (and related results) are given in section 2. We further show that Theorem 3 can be strengthened to just two parallelotopes if one of them is permitted to deviate from being axes-parallel (Theorem 4). In view of these results it may be surprising that even though the computation of certain mixed volumes appears to be harder than volume computation, from the point of view of complexity theory it is not. Theorems 6 and 7 show that the problem of computing any specific mixed volume of polytopes (or zonotopes) is #P-easy.

Section 2 will also discuss the problem of how efficiently mixed volumes can be approximated by means of deterministic algorithms. [GLS88], [AK90], and [BH93]

give exponential upper bounds for the error of deterministic polynomial approximations of the volume, and [BF86] gives an almost matching lower bound in the oracular model. We discuss possible extensions to mixed volumes and derive a polynomial-time algorithm for estimating any mixed volume of two convex bodies to a relative error that depends only on the dimension but is independent of the "well-boundedness" parameters of the bodies (Theorem 9). As a necessary "by-product" we further show that it can be decided in polynomial time whether the mixed volume of convex bodies vanishes (Theorem 8). This is a nontrivial result since a mixed volume may be greater than zero even if each set is contained in a lower-dimensional affine subspace.

A natural approach to mixed volumes is to try to use values (or estimates thereof) of the polynomial $\mathrm{vol}_n(\sum_{i=1}^{s} \lambda_i K_i)$ for computing (or estimating) (some of) its coefficients, the mixed volumes of the convex bodies $K_i$. This approach works under reasonable assumptions provided the above polynomial can be evaluated (approximately) in polynomial time; see [GK94]. This is particularly true for polytopes in *fixed* dimension; see [AS86], [CH79]. For variable dimension there is not much hope in ever obtaining a polynomial-time deterministic algorithm for this task, but we may utilize the polynomial-time randomized volume algorithm of [DFK91].

PROPOSITION 1. *There is a polynomial-time randomized algorithm which solves the following problem:*

Instance: *A well-presented convex body $K$ in $\mathbb{R}^n$, positive rational numbers $\tau$ and $\beta$.*

Output: *A random variable $\hat{v} \in \mathbb{Q}$ such that*

$$prob\left\{\frac{|\hat{v} - \mathrm{vol}_n(K)|}{\mathrm{vol}_n(K)} \geq \tau\right\} \leq \beta.$$

Let us point out that after a preprocessing "rounding" step whose running time depends on the "a priori parameters" of the body, the running time of the main algorithm is bounded above by a polynomial in $n$, $\frac{1}{\tau}$, and $\log(\frac{1}{\beta})$.

[DFK91]'s algorithm was improved by [LS90], [AK90], [DF91], [LS93], [KLS97]; see [Kh93], [GK94], and [Lo95] for surveys. Let us point out that, when dealing with randomized algorithms of the above kind, it suffices to give the desired approximation to error probability, say $\frac{1}{4}$. Then after $O(\log(1/\beta))$ independent trials of the algorithm, the median of the results achieves the required probability $\beta$; see [JVV86], [SJ89], [KKLLL93], or [LS93].

Even for just two bodies there are two major difficulties in extending Proposition 1 to mixed volumes. First, in general there is *no* way of obtaining *relative* estimates of the coefficients from *relative* estimates of the values of a polynomial $p$. (This is easily seen by considering the one-parameter sequence of univariate polynomials $q_\beta(x) = 1 + \beta x + x^2$, where $\beta$ may be any arbitrary small positive rational number; cf. [GK94, section 6.2]). The special structure of the "mixed volume polynomial" $p(x) = \mathrm{vol}_n(K_1 + xK_2)$ will, however, allow us to handle this problem. Second, the absolute values of the entries of the "inversion" which is used for expressing the coefficients of the polynomial in terms of its approximated values are not bounded by a polynomial, while the randomized volume approximation algorithm is polynomial only in $\frac{1}{\tau}$ but not in $size(\tau)$. This difficulty is mirrored in the restrictions on $\psi$ in the following theorem.

THEOREM 10. *Suppose that $\psi : \mathbb{N} \to \mathbb{N}$ is nondecreasing with*

$$\psi(n) \leq n \qquad \text{and} \qquad \psi(n) \log \psi(n) = o(\log n).$$

*Then there is a polynomial-time algorithm for the following problem:*

Instance: *Well-presented convex bodies* $K_1$, $K_2$ *of* $\mathbb{R}^n$, *positive rational numbers* $\epsilon$ *and* $\beta$, *an integer* $m$ *with* $0 \le m \le \psi(n)$.

Output: *The information that the mixed volume*

$$a_m = V(\overbrace{K_1, \ldots, K_1}^{n-m}, \overbrace{K_2, \ldots, K_2}^{m})$$

*of* $K_1$ *and* $K_2$ *vanishes, iff* $a_m = 0$, *or, otherwise, a random variable* $\hat{a}_m \in \mathbb{Q}$, *satisfying*

$$prob\left\{\frac{|\hat{a}_m - a_m|}{a_m} \ge \epsilon\right\} \le \beta.$$

The complexity of the above algorithm is only marginally worse than the complexity of the volume oracle; see section 3 for a detailed analysis. Note that the function

$$\psi(n) = \left\lceil \frac{\log n}{\log^2 \log n} \right\rceil$$

satisfies the above condition (on $\mathbb{N} \setminus \{1, 2, 3\}$).

Theorem 10 can be extended to more than two bodies.

THEOREM 11. *Suppose that* $\psi : \mathbb{N} \to \mathbb{N}$ *is nondecreasing with*

$$\psi(n) \le n \qquad \text{and} \qquad \psi(n) \log \psi(n) = o(\log n).$$

*Then there is a polynomial-time algorithm for the following problem:*

Instance: $n, s \in \mathbb{N}$, $m_1, \ldots, m_s \in \mathbb{N}_0$ *with* $m_1 + m_2 + \cdots + m_s = n$ *and* $m_1 \ge n - \psi(n)$, *well-presented convex bodies* $K_1, \ldots, K_s$ *of* $\mathbb{R}^n$, *positive rational numbers* $\epsilon$ *and* $\beta$.

Output: *The information that the mixed volume*

$$V_{m_1,\ldots,m_s} = V(\overbrace{K_1, \ldots, K_1}^{m_1}, \ldots, \overbrace{K_s, \ldots, K_s}^{m_s})$$

*vanishes, iff* $V_{m_1,\ldots,m_s} = 0$, *or, otherwise, a random variable* $\hat{V}_{m_1,\ldots,m_s} \in \mathbb{Q}$ *such that*

$$prob\left\{\frac{|\hat{V}_{m_1,\ldots,m_s} - V_{m_1,\ldots,m_s}|}{V_{m_1,\ldots,m_s}} \ge \epsilon\right\} \le \beta.$$

Theorems 10 and 11 will be proved in section 3. But let us take a few words here to place their results into perspective. Both theorems are proved by using an interpolation (or numerical differentiation) method, which is based on Proposition 1. A special feature of such a method is that in order to compute *a specific* coefficient of the polynomial under consideration it computes essentially *all* (or at least "all previous") coefficients. Now, suppose that $\psi : \mathbb{N} \to \mathbb{N}$ is a functional with $\psi(n) \le n$ for all $n \in \mathbb{N}$; let

$$\mathcal{I}_\psi(n) = \{(m_1, \ldots, m_{\psi(n)}) : m_1, \ldots, m_{\psi(n)} \in \mathbb{N}_0,\ m_1 + \cdots + m_{\psi(n)} = n \text{ and } n - m_1 \le \psi(n)\},$$

and let $K_1, \dots, K_{\psi(n)}$ be convex bodies of $\mathbb{R}^n$. Then

$$|\mathcal{I}_\psi(n)| = \binom{2\psi(n)-1}{\psi(n)-1},$$

whence the number of different mixed volumes

$$V(\overbrace{K_1, \dots, K_1}^{m_1}, \overbrace{K_2, \dots, K_2}^{m_2}, \dots, \overbrace{K_{\psi(n)}, \dots, K_{\psi(n)}}^{m_{\psi(n)}})$$

is in general only bounded by a polynomial in $n$ if $\psi(n) \le \kappa \log n$ for some constant $\kappa$. This means that there can possibly be a polynomial-time algorithm for computing *all such* mixed volumes only if

$$\psi(n) \le \kappa \log n.$$

As we will see in section 3, the statements of Theorems 10 and 11 are much easier to prove for $\psi$ being constant. As the previous discussion shows, when the number of bodies is part of the input, no polynomial-time algorithm is capable of computing more than "very few" mixed volumes. This fact places severe limitations on interpolation methods that indicate that the restriction on $\psi$ in Theorem 11 is "essentially best-possible" for any such method.

Let us remark that, for general convex bodies, it is an open problem whether there exists any method that avoids these limitations and allows one to access *single specific* mixed volumes. Hence it is open, whether the above restrictions on $\psi$ can be lifted and whether there are polynomial-time randomized algorithms which, on arbitrarily *given* $n, s \in \mathbb{N}$, $m_1, \dots, m_s \in \mathbb{N}_0$ *with* $m_1 + m_2 + \cdots + m_s = n$, *well-presented convex bodies* $K_1, \dots, K_s$ *of* $\mathbb{R}^n$ *and positive rational numbers* $\epsilon$ *and* $\beta$, *compute a random variable* $\hat{V}_{m_1,\dots,m_s} \in \mathbb{Q}$ *such that* $\text{prob}\{|\hat{V}_{m_1,\dots,m_s} - V_{m_1,\dots,m_s}|/V_{m_1,\dots,m_s} \ge \epsilon\} \le \beta$.

Note specifically that even the case $s = n$, $m_1 = \cdots = m_s = 1$ is open.

Section 4 contains some problems related to mixed volumes and some applications of our results. In particular, we deal with the problem of counting the number of integer points in lattice polytopes and with some determinant problems involving minors of given matrices. Furthermore, we discuss possible applications of our results to problems in mixture management, combinatorics, and algebraic geometry.

**1. Basic geometric and computational aspects.** The following three subsections provide definitions, notation, background information, and some first results that are needed later in sections 2 and 3.

**1.1. Mixed volumes.** Let $\mathcal{K}^n$ denote the family of all convex bodies of $\mathbb{R}^n$.

A theorem of Minkowski [Mi11] (see also [BF34], [Sc93, section 5]) shows that for $K_1, K_2, \dots, K_s \in \mathcal{K}^n$ and nonnegative reals $\lambda_1, \lambda_2, \dots, \lambda_s$,

$$\text{vol}_n\left(\sum_{i=1}^{s} \lambda_i K_i\right)$$

is a homogeneous polynomial of degree $n$ in $\lambda_1, \dots, \lambda_s$, and can be written in the form

$$\text{vol}_n\left(\sum_{i=1}^{s} \lambda_i K_i\right) = \sum_{i_1=1}^{s} \sum_{i_2=1}^{s} \cdots \sum_{i_n=1}^{s} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n} V(K_{i_1}, K_{i_2}, \dots, K_{i_n}), \tag{1.1}$$

where the coefficients $V(K_{i_1}, K_{i_2}, \ldots, K_{i_n})$ are order-independent, i.e., invariant under permutations of their arguments. The coefficient $V(K_{i_1}, K_{i_2}, \ldots, K_{i_n})$ is called the *mixed volume* of $K_{i_1}, K_{i_2}, \ldots, K_{i_n}$. We will also use the term *mixed volume* for the functional

$$V : \overbrace{\mathcal{K}^n \times \cdots \times \mathcal{K}^n}^{n} \to \mathbb{R}, \qquad (K_1, \ldots, K_n) \mapsto V(K_1, \ldots, K_n),$$

as well as for restrictions of this functional to certain subsets of $\mathcal{K}^n \times \cdots \times \mathcal{K}^n$. Mixed volumes are nonnegative, monotone, multilinear, and continuous valuations; see [BZ88, Chapter 4], [Sa93], and [GK94] for the basic properties of mixed volumes, and see [Sc93] for an excellent detailed treatment of the Brunn-Minkowski theory.

The order-independence gives rise to the notation

$$V(\overbrace{K_1, \ldots, K_1}^{m_1}, \overbrace{K_2, \ldots, K_2}^{m_2}, \ldots, \overbrace{K_s, \ldots, K_s}^{m_s})$$

for the mixed volume $V(K_1, \ldots, K_s)$, where each $K_i$ occurs exactly $m_i$ times and $\sum_{i=1}^{s} m_i = n$. The following *Aleksandrov–Fenchel inequality*, [Al37], [Al38], [Fe36], plays a fundamental role in the Brunn-Minkowski theory and will be needed in the approximation algorithm of section 3.

$$V(K_1, K_2, K_3, \ldots, K_n)^2 \geq V(K_1, K_1, K_3, , \ldots, K_n)\, V(K_2, K_2, K_3, \ldots, K_n), \tag{1.2}$$

whenever $K_1, K_2, \ldots, K_n \in \mathcal{K}^n$; see [Sc93 ] for a proof and a discussion of this inequality.

The following "decomposition lemma" (see, e.g., [BZ88, section 19.4]) will also turn out to be useful in our analysis.

PROPOSITION 2. *Let $K_1, \ldots, K_n \in \mathcal{K}^n$ and suppose that $K_{n-m+1}, \ldots, K_n$ are contained in some $m$-dimensional affine subspace $U$ of $\mathbb{R}^n$. Let $V_U$ denote the mixed volume with respect to the $m$-dimensional volume measure on $U$, and let $V_{U^\perp}$ be defined similarly with respect to the orthogonal complement $U^\perp$ of $U$. Then*

$$\binom{n}{m} V(K_1, \ldots, K_{n-m}, K_{n-m+1}, \ldots, K_n) = V_{U^\perp}(K_1', \ldots, K_{n-m}')\, V_U(K_{n-m+1}, \ldots, K_m),$$

*where $K_1', \ldots, K_{n-m}'$ denote the orthogonal projections of $K_1, \ldots, K_{n-m}$ onto $U^\perp$, respectively.*

As a particular consequence, it follows that

$$V(K_1, \ldots, K_{n-m}, K_{n-m+1}, \ldots, K_n) = 0$$

if there is a proper subspace of $U$ that contains $K_{n-m+1}, \ldots, K_n$. (Note, however, that in general the mixed volume may be greater than zero even if each set lies in *some* lower-dimensional subspace of $\mathbb{R}^n$.) In the special case $m = 1$, $K_n = [0,1]v$, and $U = \operatorname{lin}\{v\}$, where $v \in \mathbb{R}^n \setminus \{0\}$, Proposition 2 reads

$$n \cdot V(K_1, \ldots, K_{n-1}, [0,1]v) = \|v\| \cdot V_{U^\perp}(K_1', \ldots, K_{n-1}').$$

If all bodies $K_1, \ldots, K_s$ are line segments, say

$$K_i = S_i = p_i + [0,1]z_i \quad (i = 1, \ldots, s),$$

with $p_i, z_i \in \mathbb{R}^n$, then $Z = \sum_{i=1}^{s} S_i$ is a *zonotope*. It follows that for any sequence $1 \le i_1, i_2, \dots, i_n \le s$ of mutually distinct indices,

$$V(S_{i_1}, S_{i_2}, \dots, S_{i_n}) = \frac{1}{n!}\left|\det(z_{i_1}, z_{i_2}, \dots, z_{i_n})\right|,$$

where $(z_{i_1}, z_{i_2}, \dots, z_{i_n})$ denotes the $n \times n$-matrix with columns $z_{i_1}, \dots, z_{i_n}$. With the aid of (1.1) this implies the well-known volume formula for zonotopes,

$$\text{(1.3)} \qquad \operatorname{vol}_n\left(\sum_{i=1}^{s} S_i\right) = \sum_{1 \le i_1 < i_2 < \cdots < i_n \le s} \left|\det(z_{i_1}, z_{i_2}, \dots, z_{i_n})\right|;$$

see [Sh74], [Mo89], or [St91].

The polynomial expression in (1.1) involves all $s$ variables $\lambda_1, \dots, \lambda_s$ but, of course, one of the variables, say $\lambda_1$, may be set to 1, whence the problem of computing all mixed volumes of $s$ sets in $\mathbb{R}^n$ can be reduced to the task of computing the coefficients of a (generally now inhomogenous) polynomial of degree $n$ in $s-1$ indeterminates. For $s = 2$ we obtain the univariate polynomial

$$p(x) = \operatorname{vol}_n(K_1 + xK_2) = \sum_{i=0}^{n} \binom{n}{i} a_i x^i,$$

where

$$a_i = V(\overbrace{K_1, \dots, K_1}^{n-i}, \overbrace{K_2, \dots, K_2}^{i}).$$

The *Aleksandrov–Fenchel inequality* implies that the sequence

$$q_m = \frac{a_{m-1}}{a_m} \qquad (m = 1, \dots, n)$$

is increasing, whence the sequence $a_0, \dots, a_n$ is *unimodal*.

Finally, note that when $s > 2$, $m_1, m_2, \dots, m_s \in \mathbb{N}_0$ with $\sum_{i=1}^{s} m_i = n$, and $L_x = K_0 + xK_1$ for nonnegative $x \in \mathbb{R}$,

$$q(x) = V(\overbrace{L_x, \dots, L_x}^{m_1}, \overbrace{K_2, \dots, K_2}^{m_2}, \dots, \overbrace{K_s, \dots, K_s}^{m_s})$$

is a polynomial in $x$ of degree $m_1$, and we have

$$\text{(1.4)} \qquad q(x) = \sum_{k=0}^{m_1} \binom{m_1}{k} V(\overbrace{K_0, \dots, K_0}^{m_1-k}, \overbrace{K_1, \dots, K_1}^{k}, \overbrace{K_2, \dots, K_2}^{m_2}, \dots, \overbrace{K_s, \dots, K_s}^{m_s}) x^k.$$

This fact will, in particular, be used in subsection 3.3.

**1.2. Algorithmic preliminaries.** The present subsection begins with some remarks on how to deal algorithmically with polytopes and more general convex bodies, and then collects a few results that are needed later.

The underlying model of computation is the binary Turing machine model, which—in case of convex bodies—will be augmented by certain oracles; see [GJ79], [GLS88].

From an algorithmic point of view, polytopes are dealt with much more easily than general convex bodies because polytopes can be presented in a finite manner, namely, in terms of their vertices or in terms of their facet halfspaces. Clearly, from an algorithmic point of view it is not the geometric object that is relevant but its presentation. Hence we use the following notation; see e.g., [GK94].

A string $(n, m; v_1, \ldots, v_m)$ with $n, m \in \mathbb{N}$, and $v_1, \ldots, v_m \in \mathbb{Q}^n$ is called a *$\mathcal{V}$-polytope* in $\mathbb{R}^n$; it represents the geometric object $P = \operatorname{conv}\{v_1, \ldots, v_m\}$; hence we will sometimes write $P = (n, m; v_1, \ldots, v_m)$. A string $(n, m; A, b)$, where $n, m \in \mathbb{N}$, $A$ is a rational $m \times n$ matrix and $b \in \mathbb{Q}^m$ such that $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is a polytope is called an *$\mathcal{H}$-polytope* in $\mathbb{R}^n$, and is again identified with the geometric object $P$. If we want to focus more on the geometric object $P$ we will call each corresponding $\mathcal{V}$- or $\mathcal{H}$-polytope a $\mathcal{V}$- or *$\mathcal{H}$-presentation* of $P$. The *binary size* (or short *size*) of a $\mathcal{V}$- or $\mathcal{H}$-polytope $P$ is the number of binary digits needed to encode the data of the presentation.

Let us point out that each rational polytope admits a presentation as a $\mathcal{V}$- or $\mathcal{H}$-polytope, and in fixed dimension one can be computed from the other in polynomial time. This is no longer true in general when the dimension is part of the input, since the number of vertices of a polytope may be exponential in its number of facets and vice versa; see [Mc70].

Zonotopes admit specifically "compact" presentations. A string $(n, s; c; z_1, \ldots, z_s)$ with $n, s \in \mathbb{N}$ and $c, z_1, \ldots, z_s \in \mathbb{Q}^n$ is called an *$\mathcal{S}$-zonotope* in $\mathbb{R}^n$; it represents the geometric object $Z = c + \sum_{i=1}^{s} [0, 1] z_i$. Sometimes we will also work with zonotopes whose relationship to the origin (and whose scaling) is different. Specifically, zontopes of the form $\sum_{i=1}^{s} [-1, 1] z_i$ will be used. To keep the notation simple, we refrain, however, from introducing an additional name for such a presentation. Note that, in general, neither the vertices nor the facets of a zonotope are readily accessible from an $\mathcal{S}$-presentation. In fact, for zonotopes generated by $s$ segments in general position, both the number of facets and the number of vertices grow exponentially as $m$ increases.

A zonotope $Z$ is called a *parallelotope* if the "generators" $z_1, \ldots, z_s$ are linearly independent; it is *rectangular* if they are pairwise othogonal, and *axes-parallel* if all generators are standard unit vectors.

A convenient way to deal algorithmically with general convex bodies $K$ is to assume that $K$ is given by an algorithm (called an *oracle*) that answers certain sorts of questions about the body. These oracles are designed in such a way that the standard polytope case is included, i.e., it is easy to construct the corresponding oracles for $\mathcal{V}$- or $\mathcal{H}$-polytopes. This oracular approach has been introduced and extensively studied for proper convex bodies in [GLS88]. In particular, [GLS88] shows that under suitable additional assumptions, "membership," "separation," and "optimization" are equivalent. Here we need a slight variant since we want to deal with mixed volumes of possibly improper convex bodies. Let $K \in \mathcal{K}^n$, and define for $\epsilon \geq 0$ the *outer parallel body* and the *inner parallel body* of $K$, respectively, by

$$K(\epsilon) = (K + \epsilon \mathbb{B}^n) \cap \operatorname{aff}(K) \quad \text{and} \quad K(-\epsilon) = K \setminus ((\operatorname{aff}(K) \setminus K) + \epsilon \mathbb{B}^n),$$

where $\mathbb{B}^n$ denotes the Euclidean unit ball in $\mathbb{R}^n$. The most natural algorithmic problem for convex bodies $K$ is the following.

WEAK MEMBERSHIP PROBLEM FOR $K \in \mathcal{K}^n$. *Given $y \in \mathbb{Q}^n$, and a rational number $\epsilon > 0$, assert that $y \in K(\epsilon)$ or that $y \notin K(-\epsilon)$.*

If a convex body $K$ is given by an algorithm that solves the weak membership problem, we say that $K$ is described by a *weak membership oracle*. It is quite evi-

dent that the information given by a weak membership oracle is insufficient for most algorithmic purposes. Hence we need some additional a priori information about the body in question; see [GLS88] for a discussion of these assumptions in case of a proper convex body. A convex body $K$ of $\mathbb{R}^n$ will be called *well presented* if it is given by a weak membership oracle and if the following additional information is provided: a nonnegative integer $d$ and vectors $a_0, \dots, a_d \in \mathbb{Q}^n$ such that $\operatorname{aff}(K) = \operatorname{aff}\{a_0, \dots, a_d\}$; a vector $b \in Kn\mathbb{Q}^n$, and positive rational numbers $\rho$ and $R$ such that $(b + \rho\mathbb{B}^n) \cap \operatorname{aff}(K) \subset K \subset R\mathbb{B}^n$.

Note that $d$ is the dimension of $K$ and that $\operatorname{aff}(K)$ is presented in terms of an affine basis. It is, however, easy to compute from $a_0, \dots, a_d$ a presentation of $\operatorname{aff}(K)$ as the solution space of a system of (rational) linear equations and vice versa. The *size* of a well-presented convex body $K \in \mathcal{K}^n$ is then defined as $n$ plus the sum of the binary sizes of the parameters $a_0, \dots, a_d$, $b$, $\rho$, and $R$, and the *input size* of the weak membership oracle for $K$ is the sum of $size(K)$ and $size(\epsilon)$.

It is not hard to see that well presentation carries over to Minkowski sums. In fact, let $K_1, \dots, K_s \in \mathcal{K}^n$ be well presented with parameters $n$, $d_i$, $a_{i,0}, \dots, a_{i,d_i}$, $b_i$, $\rho_i$, and $R_i$ for $i = 1, \dots, s$, and let $\lambda_1, \dots, \lambda_s$ be positive rationals whose sizes are bounded above by the sizes of $K_1, \dots, K_s$. Then it is easy to find an affine basis of the affine hull of $K = \lambda_1 K_1 + \cdots + \lambda_s K_s$, and

$$
\begin{aligned}
&d = \dim(\operatorname{lin}\{a_{i,j} - a_{i,0} : i = 1, \dots, s; j = 1, \dots, d_i\}),\\
&b = \lambda_1 b_1 + \cdots + \lambda_s b_s,\\
&R = \lambda_1 R_1 + \cdots + \lambda_s R_s
\end{aligned}
$$

are valid parameters for $K$. Further, one can compute in polynomial time a nontrivial lower bound $\rho$ such that $(b+\rho\mathbb{B}^n)\cap\operatorname{aff}(K) \subset K$. It is also true that a membership oracle for $K$ can be derived in polynomial time from membership oracles for $K_1, \dots, K_s$, but this result makes use of the nontrivial relation of the oracles studied in [GLS88].

PROPOSITION 3. *Let $K_1, \dots, K_s \in \mathcal{K}^n$ be well presented, and let $\lambda_1, \dots, \lambda_s$ be positive rationals whose sizes are bounded above by the sizes of $K_1, \dots, K_s$. Then, a well presentation for $K = \lambda_1 K_1 + \cdots + \lambda_s K_s$ can be computed in polynomial time.*

The following Löwner–John-type "rounding lemmas" in terms of $\mathbb{B}^n$ and $C_n = [-1, 1]^n$ are due to [GLS88] and [AK90], respectively; see the survey [GK94, section 6.2] for some additional results in this context.

PROPOSITION 4. *There are oracle polynomial-time algorithms which accept as input a well-presented convex body $K$ and construct affine transformations $\phi_1$ and $\phi_2$ such that $0 \in \operatorname{aff}(\phi_1(K))$, $0 \in \operatorname{aff}(\phi_2(K))$ and*

$$\operatorname{aff}(\phi_1(K))\cap\mathbb{B}^n \subset \phi_1(K) \subset n\sqrt{n+1}\mathbb{B}^n, \qquad \operatorname{aff}(\phi_2(K))\cap C_n \subset \phi_2(K) \subset 2(n+1)C_n.$$

**1.3. Some estimates for numerical differentiation.** As already mentioned above, computing (some/all) mixed volumes from the ordinary volume can be regarded as computing (some/all) of the coefficients of a polynomial from its values. This can in principle be done by numerical differentiation, and we will derive a few estimates now that will be used in section 3.

Let

$$q(x) = \sum_{i=0}^{n} c_i x^i$$

be a univariate polynomial of degree $n$, and let $\xi_0, \dots, \xi_n$ be pairwise different interpolation points. The *Lagrange-interpolation polynomials* $l_k(x) = \sum_{i=0}^{n} b_{ki} x^i$ on the

node set $X = \{\xi_0, \ldots, \xi_n\}$ satisfy

$$l_k(\xi_j) = \sum_{i=0}^{n} b_{ki}\xi_j^i = \delta_{jk},$$

where $\delta_{jk}$ is the usual Kronecker symbol. Then

$$q(x) = \sum_{k=0}^{n} q(\xi_k) l_k(x) = \sum_{i=0}^{n} \left( \sum_{k=0}^{n} q(\xi_k) b_{ki} \right) x^i;$$

hence

$$c_i = \sum_{k=0}^{n} b_{ki} q(\xi_k) \quad \text{for } i = 0, \ldots, n.$$

In general, only estimates $\hat{q}(\xi_j)$ of the function values $q(\xi_j)$ are available. In fact, for the purpose of section 3 we can only use estimates with bounded *relative* error. Here we suppose first that the *absolute* error is bounded beforehand, i.e., there is a positive $\delta$ such that

$$|\hat{q}(\xi_j) - q(\xi_j)| \leq \delta \quad \text{for } j = 0, \ldots, n.$$

Now, let $m \in \{0, \ldots, n\}$, and let us take

$$\hat{c}_m = \sum_{k=0}^{n} b_{km} \hat{q}(\xi_k)$$

as an estimate for $c_m$. Then we obtain

$$|c_m - \hat{c}_m| = \left| \sum_{k=0}^{n} b_{km} (q(\xi_k) - \hat{q}(\xi_k)) \right| \leq \delta \sum_{k=0}^{n} |b_{km}|. \tag{1.5}$$

Efficient methods for performing the computations in a systematical way (e.g., by using divided differences) can be found in any textbook on numerical analysis; see for example [BZ65]. The problem of how to choose the interpolation points to minimize the error terms $\sum_{k=0}^{n} |b_{km}|$ is discussed in (among others) [Ri75]; see also [MM85], [Sa74], [Ri90]; equidistant nodes are in general not optimal. We will use equidistant interpolation points anyway since, on the one hand, the subsequent analysis becomes more tractable and, on the other hand, the additional error introduced that way is dominated by other occurring error terms and hence is essentially irrelevant.

For the estimates in this subsection, we will normalize the nodes to the set $X = \{0, 1, \ldots, n\}$; in section 3 we will use the node set $hX$ for some suitable positive rational $h$.

Let $M$ denote the (infinite) Vandermonde matrix $M = (j^i)_{i,j \in \mathbb{N}_0}$, (with the setting $0^0 = 1$); for $r \in \mathbb{N}$ let $M^{(r)} = (j^i)_{i,j=0,\ldots,r-1}$ be the restriction of $M$ to its first $r$ rows and columns, and let $B^{(r)} = (b_{ij}^{(r)})_{i,j=0,\ldots,r-1}$ be the inverse of $M^{(r)}$. We will now derive an upper estimate for $\sum_{i=0}^{r-1} |b_{im}^{(r)}|$.

For $i, j \in \mathbb{N}$ with $i \geq j$, let $\sigma_{ij}$ denote the *Stirling numbers of the second kind*, i.e., the number of partitions of the set $\{1, \ldots, i\}$ into $j$ pairwise disjoint nonempty subsets (see, e.g., [St86]). In addition, let $\sigma_{00} = 1$, $\sigma_{i0} = 0$ for all $i > 0$ and $\sigma_{ij} = 0$

whenever $i < j$. Note that $j!\sigma_{ij}$ is the number of surjective mappings of $\{1,\dots,i\}$ into $\{1,\dots,j\}$; hence, it follows that

$$j^i = \sum_{k=0}^{\infty} \sigma_{ik} k! \left\{ \begin{matrix} j \\ k \end{matrix} \right\} = \sum_{k=0}^{\infty} \sigma_{ik} j(j-1)\cdot\ldots\cdot(j-k+1), \tag{1.6}$$

and, in particular,

$$\sigma_{ij} \le \frac{j^i}{j!}.$$

Thus, with the notation $(x)_k = x(x-1)(x-2)\cdot\ldots\cdot(x-k+1)$, the identity

$$x^i = \sum_{k=0}^{\infty} \sigma_{ik} (x)_k$$

holds for all integers $x = 0,\dots,i$ and hence holds for all $x \in \mathbb{R}$. Let

$$L = (\sigma_{ij})_{i,j\in\mathbb{N}_0}, \quad U = \left\{ \begin{matrix} j \\ i \end{matrix} \right\}_{i,j\in\mathbb{N}_0}, \quad \text{and} \quad D = \operatorname{diag}(0!, 1!, 2!, \dots).$$

Then $L$ and $U$ are an (infinite) lower and upper triangular matrix, respectively, and (1.6) can be written as

$$M = LDU.$$

Left multiplication by $L^{-1} = (s_{ij})_{i,j\in\mathbb{N}_0}$ yields

$$x(x-1)\cdot\ldots\cdot(x-i+1) = \sum_{k=0}^{\infty} s_{ik} x^k.$$

Hence, $s_{ij}$ has sign $(-1)^{i-j}$ (for $j \le i$) and, evaluating the above identity for $x = -1$ yields

$$\sum_{j=0}^{i} |s_{ij}| = i!.$$

The numbers $s_{ij}$ are called the *Stirling numbers of the first kind.* From

$$x^j = \sum_{i=0}^{\infty} \left\{ \begin{matrix} j \\ i \end{matrix} \right\} (x-1)^i \quad \text{and} \quad (x-1)^j = \sum_{i=0}^{\infty} \left\{ \begin{matrix} j \\ i \end{matrix} \right\} (-1)^{j-i} x^i,$$

we conclude for the inverse $U^{-1} = (w_{ij})_{i,j\in\mathbb{N}_0}$ of $U$ that

$$w_{ij} = (-1)^{j-i} \binom{j}{i}.$$

Now, note that

$$\left(L^{(r)}\right)^{-1} = \left(L^{-1}\right)^{(r)}, \quad \left(D^{(r)}\right)^{-1} = \left(D^{-1}\right)^{(r)}, \quad \left(U^{(r)}\right)^{-1} = \left(U^{-1}\right)^{(r)}$$
$$\text{and} \quad M^{(r)} = L^{(r)} D^{(r)} U^{(r)}.$$

Hence,

$$B^{(r)} = \left(U^{(r)}\right)^{-1}\left(D^{(r)}\right)^{-1}\left(L^{(r)}\right)^{-1} = \left(U^{-1}\right)^{(r)}\left(D^{-1}\right)^{(r)}\left(L^{-1}\right)^{(r)},$$

and this reads explicitly as

(1.7)
$$b_{ij}^{(r)} = \sum_{k=0}^{r-1}(-1)^{k-i}\binom{k}{i}\frac{1}{k!}s_{kj} = (-1)^{i+j}\sum_{k=0}^{r-1}\binom{k}{i}\frac{1}{k!}|s_{kj}| \qquad (i,j=0,\ldots,r-1).$$

This implies that for any $m \in \{0,\ldots,r-1\}$,

(1.8)
$$\sum_{i=0}^{r-1}|b_{im}^{(r)}| = \sum_{i=0}^{r-1}\sum_{k=0}^{r-1}\binom{k}{i}\frac{|s_{km}|}{k!} \le \sum_{k=0}^{r-1}2^k\frac{|s_{km}|}{k!} < 2^r.$$

We conclude the univariate case with an additional technical estimate that is needed in section 3. It gives an upper bound on the error induced by using only $B^{(r)}$ (for some $r \le n$) rather than the full matrix $B^{(n+1)}$ in the computation of the coefficients of a polynomial of degree $n$.

Let, for $i,j \in \mathbb{N}_0$ with $j < r$,

$$d_{ij} = \sum_{k=0}^{r-1} b_{kj}^{(r)}k^i.$$

Clearly $d_{ij} = \delta_{ij}$ for $i < r$. For $i \ge r$, combining (1.6) and (1.7) yields

(1.9)
$$\begin{aligned}\left|\sum_{k=0}^{r-1} b_{kj}^{(r)}k^i\right| &= \left|\sum_{k=0}^{r-1}\left(\sum_{p=0}^{r-1}(-1)^{p-k}\binom{p}{k}\frac{1}{p!}s_{pj}\right)\left(\sum_{q=0}^{\infty}\sigma_{iq}q!\binom{k}{q}\right)\right| \\ &= \left|\sum_{p=0}^{r-1}\sum_{q=0}^{r-1}s_{pj}\sigma_{iq}\frac{q!}{p!}(-1)^{q-p}\left(\sum_{k=0}^{r-1}(-1)^{k-q}\binom{k}{q}\binom{p}{k}\right)\right| \\ &\le \sum_{p=0}^{r-1}|s_{pj}|\sigma_{ip} \le \sum_{p=0}^{r-1}p^i \le r^i.\end{aligned}$$

Let us close this section with a few brief remarks about the general multivariate case. Let, for $n,s \in \mathbb{N}$,

$$Y_{n,s} = \left\{y = (m_1,\ldots,m_s) \in (\mathbb{N}_0)^s : \sum_{i=1}^{s} m_i = n\right\}.$$

Clearly,

$$N = |Y_{n,s}| = \binom{n+s-1}{n}.$$

Suppose that the elements of $Y_{n,s}$ are ordered (for instance lexicographically) so that $Y_{n,s} = \{y_1,\ldots,y_N\}$, where $y_j = (m_{j,1},\ldots,m_{j,s})$. Now we want to determine the coefficients of a homogeneous multivariate polynomial

$$q(x_1,\ldots,x_s) = \sum_{j=1}^{N} c_j x_1^{m_{j,1}}\cdot\ldots\cdot x_s^{m_{j,s}}$$

from its function values. So, we have to choose $N$ interpolation points in such a way that the $N \times N$ matrix

$$\left( (\xi_1^{(i)})^{m_{j,1}} \cdot (\xi_2^{(i)})^{m_{j,2}} \cdot \ldots \cdot (\xi_s^{(i)})^{m_{j,s}} \right)_{i,j=1,\ldots,N},$$

a higher-dimensional analogue of the classical Vandermonde matrix, is nonsingular. (Note that, as opposed to the univariate case, it does not suffice to choose the $N$ points mutually different.) Sufficient conditions for nonsingularity can be found in [CY77]; see also [Ol86]. In particular, one may take an $(s-1)$-dimensional simplex $S = \operatorname{conv}\{z_1, \ldots, z_s\}$ in $\mathbb{R}^s$ and choose $\xi^{(k_1,\ldots,k_s)} = \frac{1}{n} \sum_{j=1}^{s} k_j z_j$, where $(k_1, \ldots, k_s) \in Y_{n,s}$.

This implies, in particular, that when the dimension $n$ is *fixed*, there is a polynomial-time algorithm which, given $s \in \mathbb{N}$ and ($\mathcal{V}$- or $\mathcal{H}$-) polytopes $P_1, \ldots, P_s$, computes all mixed volumes $V(P_{i_1}, \ldots, P_{i_n})$.

**2. Deterministic algorithms.** The present section discusses the problem of computing or approximating (mixed) volumes by means of deterministic algorithms. In particular we give results that focus on the difference of volume versus mixed volume computation.

**2.1. Computing the volume of zonotopes.** In this subsection we deal with the following problem.

VOLUME-OF-ZONOTOPES.

*Given an $\mathcal{S}$-zonotope $Z = (n, s; c; z_1, \ldots, z_s)$, compute its volume.*

Note that the problem asks for $\operatorname{vol}_n(Z)$, where $Z = c + \sum_{i=1}^{s} [0,1] z_i$. Since the volume is translation invariant, we can always assume that $c = 0$. Now, let $A$ denote the $n \times s$ matrix with columns $z_1, \ldots, z_s$ and let $\mathcal{J}$ denote the family of all subsets $I$ of $\{1, \ldots, s\}$ of cardinality $n$. Then (1.3) can be written in the form

$$\operatorname{vol}_n(Z) = \sum_{I \in \mathcal{J}} |\det B_I|,$$

where $B_I$ is the $n \times n$-minor of $A$ whose columns correspond to $I$. It is clear that for constant $n$ or constant $s-n$, the number $\binom{s}{n}$ of $n \times n$ subdeterminants is polynomially bounded, whence the volume of zonotopes can be computed in polynomial time. The general case is, however, #$\mathbb{P}$-hard.

THEOREM 1. *VOLUME-OF-ZONOTOPES is #$\mathbb{P}$-hard.*

*Proof.* The proof will use a reduction of the following #$\mathbb{P}$-complete problem.

#SUBSET-SUM (see [GJ79], [Jo90]). *Given positive integers $m$, $\alpha_1, \ldots, \alpha_m$, and $\alpha$, determine the number of different subsets $J$ of $\{1, \ldots, m\}$ such that $\sum_{j \in J} \alpha_j = \alpha$.*

So, suppose $(m; \alpha_1, \ldots, \alpha_m, \alpha)$ is an instance of #SUBSET-SUM, and let $n = m+2$ and $s = 2m+3$. Further, define

$$\begin{aligned}
z_{2k-1} &= e_k + \alpha_k e_{m+2}, && k = 1, \ldots, m; \\
z_{2k} &= e_k, && k = 1, \ldots, m+1; \\
z_{2m+1} &= e_{m+1} - \alpha e_{m+2}; \\
z_{2m+3}^{\delta} &= -\sum_{i=1}^{m+1} e_i + \delta e_{m+2}, && \delta \in \{-1, 0, 1\},
\end{aligned}$$

where $e_1, \ldots, e_n$ denote again the standard basis vectors of $\mathbb{R}^n$, and set

$$Z_\delta = \sum_{i=1}^{s-1} [0,1] z_i + [0,1] z_s^{\delta}.$$

Suppose now that there is a polynomial-time algorithm $\mathcal{A}$ for solving the problem VOLUME-OF-ZONOTOPES, and apply $\mathcal{A}$ to compute $\mathrm{vol}_n(Z_{-1}) - 2\mathrm{vol}_n(Z_0) + \mathrm{vol}_n(Z_1)$. In terms of the determinant formula, this means that we are only interested in those $n \times n$ submatrices $B_I$ of the $n \times s$ matrix

$$A_\delta = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & \dots & 0 & 0 & 0 & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 1 & -1 \\ \alpha_1 & 0 & \alpha_2 & 0 & \alpha_3 & 0 & \dots & \alpha_m & 0 & -\alpha & 0 & \delta \end{pmatrix}$$

which depend on $\delta$. Then, clearly, $B_I$ has to contain the last column $z^\delta_{2m+3}$ of $A_\delta$, and in choosing the remaining $m+1$ columns, we have to select exactly one vector from each pair $z_{2k-1}$, $z_{2k}$ $(k = 1, \dots, n-1)$. Therefore, the summands $|\det B_I|$ of the determinantal expansion of $\mathrm{vol}_n(Z_\delta)$ which are depending on $\delta$ are in one-to-one correspondence with the subsets $J$ of $\{1, \dots, m+1\}$ via

$$j \in J \Longleftrightarrow 2j - 1 \in I.$$

From this it follows easily that there is an integer $\kappa$ that depends only on $\alpha_1, \dots, \alpha_m$ and $\alpha$ but not on $\delta$ such that

$$\mathrm{vol}_n(Z_\delta) = \kappa + \sum_{J \subset \{1,\dots,m+1\}} |\delta + \sum_{i \in J} \alpha_j|,$$

where, for notational consistency, $\alpha_{m+1} = -\alpha$. Then,

$$\mathrm{vol}_n(Z_{-1}) - 2\mathrm{vol}_n(Z_0) + \mathrm{vol}_n(Z_1) = \sum_{J \subset \{1,\dots,m+1\}} \left( \left| -1 + \sum_{j \in J} \alpha_j \right| - 2 \left| \sum_{j \in J} \alpha_j \right| + \left| 1 + \sum_{j \in J} \alpha_i \right| \right).$$

Since for any nonzero integer $\gamma$,

$$|-1+\gamma| - 2|\gamma| + |1+\gamma| = 0,$$

it follows that

$$\frac{1}{2}\Big(\mathrm{vol}_n(Z_{-1}) - 2\mathrm{vol}_n(Z_0) + \mathrm{vol}_n(Z_1)\Big) = \Big|\{J \subset \{1, \dots, m+1\} : \sum_{j \in J} \alpha_i = 0\}\Big|.$$

But

$$\sum_{j \in J} \alpha_i = 0 \quad \text{if and only if} \quad m+1 \in J \quad \text{and} \quad \sum_{j \in J \cap \{1,\dots,m\}} \alpha_j = \alpha,$$

whence $\mathcal{A}$ gives rise to a polynomial-time algorithm for #SUBSET-SUM. □

Theorem 1 proves the #$\mathbb{P}$-hardness of evaluating $\sum_{I \in \mathcal{J}} |\det B_I|$. This result is in striking contrast to the fact that by the *Binet–Cauchy formula* (see, e.g., [BS83]),

$$\sum_{I \in \mathcal{J}} (\det B_I)^2 = \det(AA^T), \tag{2.1}$$

whence the sum of the *squares* of all $n \times n$ subdeterminants can be evaluated in polynomial time.

Note, further, that Proposition 1 can be applied to $\mathcal{S}$-zonotopes since it is standard fare to derive a well presentation for an $\mathcal{S}$-zonotope. So there is a polynomial-time randomized algorithm for VOLUME-OF-ZONOTOPES. Zonotopes come, however, with an additional structure (and in particular, with a natural dissection into parallelotopes) so it is conceivable that there are faster randomized algorithms for zonotopes than there are for general well-presented convex bodies. This question is, however, open.

For an easiness result complementing Theorem 1 see Theorem 6, and for an application of VOLUME-OF-ZONOTOPES to a problem in the oil industry see subsection 4.3.

We will now draw the first of a few consequences of Theorem 1 and prove a result that is relevant in subsection 2.4.

THEOREM 2. *The following problem* VOLUME-OF-SUM-OF-ELLIPSOIDS *is* $\#\mathbb{P}$*-hard: given* $s, n \in \mathbb{N}$, *nonsingular rational* $(n \times n)$*-matrices* $A_1, \ldots, A_s$, *an error bound* $\epsilon \in \mathbb{Q}$, $\epsilon > 0$, *compute a rational number* $\hat{V}$ *which satisfies*

$$\left|\hat{V} - \mathrm{vol}_n(E_1 + E_2 + \cdots + E_s)\right| < \epsilon,$$

*where* $E_i$ *is the ellipsoid* $E_i = \{x \in \mathbb{R}^n : x^T A_i^T A_i x \leq 1\}$.

*Proof.* Let $(n, s; c; z_1, \ldots, z_s)$ be an instance of VOLUME-OF-ZONOTOPES and set $Z = \sum_{i=1}^{s} [-1, 1] z_i$. Note that

$$\mathrm{vol}_n(Z) = 2^n \mathrm{vol}_n\left(c + \sum_{i=1}^{s} [0, 1] z_i\right),$$

whence it suffices to show how the computation of $\mathrm{vol}_n(Z)$ can be reduced to a suitable instances of VOLUME-OF-SUM-OF-ELLIPSOIDS.

For each $i = 1, \ldots, s$ we compute first an orthogonal basis $\{v_{i,1}, \ldots, v_{i,n}\}$ of $\mathbb{R}^n$ such that $v_{i,1} = z_i$. Let $B_i$ be the $n \times n$-matrix with rows $v_{i,1}^T, \ldots, v_{i,n}^T$, set for $\mu \in \mathbb{N}$,

$$D_i^\mu = \mathrm{diag}\left(\frac{1}{\langle z_i, z_i \rangle}, \frac{\mu}{\langle v_{i,2}, v_{i,2} \rangle}, \ldots, \frac{\mu}{\langle v_{i,n}, v_{i,n} \rangle}\right),$$

and define the ellipsoid

$$E_i^\mu = \{x \in \mathbb{R}^n : x^T (D_i^\mu B_i)^T (D_i^\mu B_i) x \leq 1\}.$$

Then we have

$$[-1, 1] z_i \subset E_i^\mu \subset [-1, 1] z_i + \frac{1}{\mu} \sum_{j=2}^{n} [-1, 1] v_{i,j}.$$

Now, let $Z' = \sum_{i=1}^{s} \sum_{j=2}^{n} [-1, 1] v_{i,j}$, and let $R \in \mathbb{N}$ such that $Z \cup Z' \subset R C_n$, where $C_n$ denotes again the standard unit cube. Then the above inclusions yield

$$Z \subset E_1^\mu + E_2^\mu + \cdots + E_s^\mu \subset Z + \frac{1}{\mu} Z' \subset Z + \frac{R}{\mu} C_n.$$

Now note that for any $\lambda > 0$,

$$\mathrm{vol}_n(Z + \lambda C_n) - \mathrm{vol}_n(Z) = \sum_{i=1}^{n} \binom{n}{i} V(\overbrace{Z, \ldots, Z}^{n-i}, \overbrace{C_n, \ldots, C_n}^{i}) \lambda^i,$$

and this implies that

$$\mathrm{vol}_n(E_1^\mu + \cdots + E_s^\mu) - \mathrm{vol}_n(Z) \le \sum_{i=1}^{n} \binom{n}{i} V(\overbrace{Z,\ldots,Z}^{n-i}, \overbrace{C_n,\ldots,C_n}^{i}) \left(\frac{R}{\mu}\right)^i \le \frac{(4R)^n}{\mu}.$$

Hence, if for $\mu_0 = \lceil \frac{2}{\epsilon}(4R)^n \rceil$ the volume of $E_1^{\mu_0} + \cdots + E_s^{\mu_0}$ is approximated to absolute error $\frac{\epsilon}{2}$, we obtain an estimate of $\mathrm{vol}_n(Z)$ to absolute error $\epsilon$. Further, it follows from (1.3) that $size(\mathrm{vol}_n(Z))$ is bounded by a polynomial in the input size. Therefore, it suffices to approximate $\mathrm{vol}_n(Z)$ to a sufficiently small absolute error $\epsilon$ whose size is polynomially bounded and then perform the usual rounding (with continued fractions) in order to obtain $\mathrm{vol}_n(Z)$ precisely.

Finally note that all constructions and computations can be done in polynomial time; this completes the transformation. □

**2.2. Mixed volumes of paralellotopes.** We give some hardness results for computing mixed volumes of parallelotopes. The first involves axes-parallel parallelotopes which (for brevity) will be called *boxes*.

Before we state the result we need two lemmas.

LEMMA 1. *Let the entries of* $A = (\alpha_{ij})_{i,j=1,2,\ldots,n}$ *be nonnegative rationals, and for* $i = 1,\ldots,n$ *set* $Z_i = \sum_{j=1}^{n} [0, \alpha_{ij}] e_j$. *Then*

$$n! V(Z_1,\ldots,Z_n) = \mathrm{per}(A),$$

*where* $\mathrm{per}(A)$ *denotes the permanent of* $A$.

*Proof.* Note that the $Z_i$ are all boxes, and so is $\sum_{i=1}^{n} \lambda_i Z_i$ for each $n$-tuple $(\lambda_1,\ldots,\lambda_n)$ of nonnegative reals. Hence,

$$\mathrm{vol}_n\left(\sum_{i=1}^{n} \lambda_i Z_i\right) = \mathrm{vol}_n\left(\sum_{j=1}^{n} \left[0, \sum_{i=1}^{n} \lambda_i \alpha_{ij}\right] e_j\right) = \prod_{j=1}^{n} \left(\sum_{i=1}^{n} \lambda_i \alpha_{ij}\right).$$

Comparing the coefficients of $\lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$ we see that

$$V(Z_1,\ldots,Z_n) = \frac{1}{n!} \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} \epsilon_{j_1,\ldots,j_n} \alpha_{1,j_1} \cdot \ldots \cdot \alpha_{n,j_n},$$

where

$$\epsilon_{j_1,\ldots,j_n} = \begin{cases} 1 & \text{if } j_1,\ldots,j_n \text{ is a permutation of } 1,2,\ldots,n, \\ 0 & \text{otherwise,} \end{cases}$$

and this proves the assertion. □

The problem of computing the permanent of 0-1-matrices is known to be #ℙ-complete [Va77]; see also [Va79], [Br86], [JS89], [LS90], [KKLLL93]. Hence, Lemma 1 implies that the problem of computing the mixed volume $V(Z_1,\ldots,Z_n)$ of $n$ faces $Z_i$ of the cube $[0,1]^n$ is also #ℙ-complete. (Note, on the positive side, that in view of Lemma 1, Theorem 11 yields a randomized polynomial-time algorithm for estimating the permanent of certain classes of matrices.) In order to extend this result to *proper* boxes, observe that if we replace the 0-entries of a given 0-1-matrix $B$ by a parameter $\alpha$ to obtain an $\alpha$-1-matrix $B_\alpha$, then $\mathrm{per}(B_\alpha)$ is a polynomial in $\alpha$; evaluation of this polynomial for $n+1$ different values of $\alpha$ (or for one sufficiently small value

of $\alpha$) allows us to compute its constant term $\mathrm{per}(B)$. In order to prove the sharper statement of Theorem 3, we use the following strengthening of [Va77]'s hardness result to the permanent of $\alpha$-$\beta$-matrices with prescribed $\alpha$ and $\beta$.

LEMMA 2. *The following problem is $\#\mathbb{P}$-hard for any pair $\alpha, \beta$ of (fixed) distinct rationals: given a positive integer $n$, and an $n \times n$ matrix $A$ with entries $\alpha, \beta$, compute* $\mathrm{per}(A)$.

*Proof.* We may assume that $\beta = \alpha + 1$ and $\alpha \neq 0$. Let $B = (b_{ik})_{i,k=1,\ldots,n}$ be an arbitrary 0-1-matrix. We will reduce the computation of $\mathrm{per}(B)$ to the computation of the permanent of several matrices with $\alpha, \alpha + 1$ entries.

Let $G$ denote the bipartite graph on $2n$ vertices whose adjacency matrix is $B$, and for $k = 1, \ldots, n$ let $M_k$ be the number of matchings of size $k$ in $G$. We want to compute $M_n = \mathrm{per}(B)$. For $j = 0, \ldots, n$ let $X^{(j)} = (x_{ik}^{(j)})$, denote the $(n+j) \times (n+j)$ matrix with entries

$$x_{ik}^{(j)} = \begin{cases} \alpha + b_{ik} & \text{for } i, k = 1, \ldots, n; \\ \alpha & \text{otherwise.} \end{cases}$$

Clearly, $X^{(j)}$ has only entries $\alpha, \alpha + 1$; whence using an oracle for evaluating the permanent of matrices with $\alpha, \alpha+1$ entries $n+1$ times, we can determine the permanents of all the $X^{(j)}$. On the other hand, we have

$$\sum_{k=0}^{n} M_k (n - k + j)! \alpha^{n-k+j} = \mathrm{per}(X^{(j)}) \qquad (j = 0, \ldots, n). \tag{2.2}$$

To see this, regard $\alpha$ as an indeterminate, and expand $\mathrm{per}(X^{(j)})$ as a polynomial in $\alpha$. Then the terms contributing to the coefficient of $\alpha^{n-k+j}$ arise as follows. For every $k$-matching in $G$ we obtain a product $(\alpha + 1)^k$, and this can be completed in $(n - k + j)!$ ways to give a lowest term $\alpha^{n-k+j}$. We do this by selecting the $\alpha$ term from the product of monomials (either $\alpha$ or $\alpha + 1$) represented by any matching on the complete bipartite graph induced by the remaining $n - k + j$ rows and columns.

Therefore, if the above system (2.2) of linear equations is nonsingular, we can solve it for $M_n$, and this establishes the $\#\mathbb{P}$-hardness result.

To see that (2.2) is indeed nonsingular, let us rewrite the system as follows:

$$\sum_{k=0}^{n} \binom{k+j}{j} (k! \alpha^k M_{n-k}) = \frac{j! \, \mathrm{per}(X^{(j)})}{\alpha^j} \qquad (j = 0, \ldots, n).$$

Introducing the new variables $y_k = k! \alpha^k M_{n-k}$, the question now reduces to deciding whether the $(n + 1) \times (n + 1)$ matrix $C$ with entries $c_{kj} = \binom{k+j}{j}$ is nonsingular. But this follows easily from the Vandermonde identity

$$\sum_{r=0}^{n} \binom{k}{r} \binom{j}{r} = \binom{k+j}{j} \quad \text{for } k, j \leq n,$$

since $C = UU^T$, where $U$ is the lower triangular matrix with entries $u_{ij} = \binom{i}{j}$ which has all its diagonal elements 1. □

Now we can prove Theorem 3.

THEOREM 3. *Let $\alpha, \beta$ be (fixed) distinct positive rationals. Then the following restriction of* MIXED-VOLUME-OF-BOXES *is $\#\mathbb{P}$-hard: given $n \in \mathbb{N}$, and for $i, j =$*

$1, 2, , \ldots, n$, *an element* $\alpha_{i,j}$ *of* $\{\alpha, \beta\}$, *compute the mixed volume* $V(Z_1, \ldots, Z_n)$ *for the proper boxes* $Z_i = \sum_{j=1}^{n} [0, \alpha_{i,j}] e_j$, $(i = 1, \ldots, n)$.

*Proof.* The result is a simple consequence of Lemmas 1 and 2. □

Theorem 3 implies directly the "instability" result that, while the case of $\epsilon = 0$ is trivial, it is #ℙ-hard for any $\epsilon > 0$ to compute the mixed volume of $n$ proper boxes, all containing the unit cube $C_n$ and all being contained in the cube $(1 + \epsilon)C_n$.

Note that MIXED-VOLUME-OF-BOXES can be solved in polynomial time if the number $s$ of different boxes is bounded beforehand. In this case there are only $O(n^{s-1})$ different mixed volumes, and they can all be computed by the approach of subsection 1.3 (see [GK94, subsection 4.1]), since their Minkowski sum is a box whose volume can be computed easily. We will show, however, that the corresponding problem for just two proper rectangular parallelotopes is #ℙ-hard if they are not both required to be axes-parallel.

THEOREM 4. *The following problem is* #ℙ*-hard: given* $n \in \mathbb{N}$, $m \in \{0, \ldots, n\}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$, *and integer vectors* $y_1, \ldots, y_n$ *which form an orthogonal basis of* $\mathbb{R}^n$, *compute* $V(\overbrace{Z_1, \ldots, Z_1}^{n-m}, \overbrace{Z_2, \ldots, Z_2}^{m})$, *where* $Z_1 = \sum_{i=1}^{n} [0, 1] y_i$ *and* $Z_2 = \sum_{i=1}^{n} [0, \alpha_i] e_i$.

*Proof.* We use the problem VOLUME-OF-ZONOTOPES of Theorem 1 for a reduction. Let $Z = (n, s; c, z_1, \ldots, z_s)$ be an $\mathcal{S}$-zonotope. We may assume without loss of generality that $z_1, \ldots, z_s \in \mathbb{Z}^n$, that $s > n$, and that $Z$ is proper. Now, let $A$ denote the $n \times s$ matrix with columns $z_1, \ldots, z_s$. Since, by (1.3), elementary row operations to $A$ do not change the volume of the zonotope generated by the columns of $A$, we may further assume that the rows $v_1, \ldots, v_n$ of $A$ are orthogonal.

Let $\{v_{n+1}, \ldots, v_s\} \subset \mathbb{Q}^s$ be an orthogonal basis of the orthogonal complement of the linear hull of $\{v_1, \ldots, v_n\}$ such that the sizes of $v_{n+1}, \ldots, v_s$ are bounded by a polynomial in $size(Z)$. Note that such a basis can be computed essentially by solving a system of linear equations. Let $B$ denote the $s \times s$ matrix that is obtained from $A$ by augmenting the rows $v_{n+1}, \ldots, v_s$, and let $y_1, \ldots, y_s$ be the column vectors of $B$. Since the rows of $A$ are orthogonal, so are the columns. Hence,

$$Z_1 = \sum_{i=1}^{s} [0, 1] y_i$$

is a proper rectangular parallelotope in $\mathbb{R}^s$. Set, further,

$$C = \sum_{i=n+1}^{s} [0, 1] e_i, \qquad \text{and for } 0 < \mu < 1, \quad Z_2^{\mu} = C + \mu \sum_{i=1}^{n} [0, 1] e_i.$$

By Proposition 2, applied with $U = \{0\}^n \times \mathbb{R}^{s-n}$ (and hence $U^{\perp} = \mathbb{R}^n \times \{0\}^{s-n}$), we obtain

$$\binom{s}{n} V(\overbrace{Z_1, \ldots, Z_1}^{n}, \overbrace{C, \ldots, C}^{s-n}) = \mathrm{vol}_n(Z),$$

and this gives already an #ℙ-hardness result for the case that one of the parallelotopes is permitted to be lower dimensional. To complete the proof of Theorem 4, observe that (by (1.4))

$$V(\overbrace{Z_1, \ldots, Z_1}^{n}, \overbrace{Z_2^{\mu}, \ldots, Z_2^{\mu}}^{s-n}) = \sum_{i=0}^{s-n} \binom{s-n}{i} V(\overbrace{Z_1, \ldots, Z_1}^{n}, \overbrace{C, \ldots, C}^{s-n-i}, \overbrace{\hat{C}, \ldots, \hat{C}}^{i}) \mu^i,$$

where $\hat{C} = [0,1]^n \times \{0\}^{s-n}$. Since there is a positive integer $R$ of size bounded by a polynomial in $size(Z)$ such that $Z_1 \subset R[0,1]^s$, it follows that

$$V(\overbrace{Z_1,\dots,Z_1}^{n},\overbrace{C,\dots,C}^{s-n}) \le V(\overbrace{Z_1,\dots,Z_1}^{n},\overbrace{Z_2^{\mu},\dots,Z_2^{\mu}}^{s-n}) \le V(\overbrace{Z_1,\dots,Z_1}^{n},\overbrace{C,\dots,C}^{s-n}) + 2^{s-n}R^n\mu.$$

Now, let $\mu_0$ be a positive rational of size bounded by a polynomial in the input size such that $1/\mu_0 > 2 \cdot 2^{s-n}R^n\binom{s}{n}$, and set $Z_2 = Z_2^{\mu_0}$. Then $Z_2$ is a proper rectangular parallelotope, and

$$\left|\mathrm{vol}_n(Z) - \binom{s}{n} V(\overbrace{Z_1,\dots,Z_1}^{n},\overbrace{Z_2,\dots,Z_2}^{s-n})\right| < \frac{1}{2}.$$

Since $\mathrm{vol}_n(Z)$ is an integer, this shows that it suffices to compute the mixed volume $V(\overbrace{Z_1,\dots,Z_1}^{n},\overbrace{Z_2,\dots,Z_2}^{n-s})$ in order to obtain $\mathrm{vol}_n(Z)$. To conclude the transformation just apply a suitable scaling to make $\mu_0$ integer. □

As a simple consequence of Theorem 4 we can derive a sharpening of Theorem 1.

THEOREM 5. *The following problem is #ℙ-hard: given $n \in \mathbb{N}$, and two $n$-tuples $v_1,\dots,v_n$ and $w_1,\dots,w_n$ of integer vectors that form orthogonal bases of $\mathbb{R}^n$, compute the volume of the Minkowski sum*

$$\mathrm{vol}_n\left(\left(\sum_{i=1}^{n}[0,1]v_i\right) + \left(\sum_{j=1}^{n}[0,1]w_j\right)\right).$$

*Proof.* Let $Z_1 = \sum_{i=1}^{n}[0,1]v_i$ and $Z_2 = \sum_{j=1}^{n}[0,1]w_j$. For the proof of the theorem, just note that all mixed volumes of $Z_1$ and $Z_2$ can be computed by the method indicated in subsection 1.3 by evaluating $\mathrm{vol}_n(Z_1 + \xi Z_2)$ for $n+1$ mutually disjoint interpolation points $\xi_0,\dots,\xi_n$, and apply Theorem 4. □

**2.3. Easiness of mixed volume computation.** The results of the previous subsection show that mixed volume computation is in general at least as hard as any problem in #ℙ. The present subsection addresses the question of whether computing mixed volumes is possibly even harder.

As shown in [DF88], using any oracle which solves some #ℙ-complete problem in constant time, the volume of a $\mathcal{V}$-polytope can be computed in polynomial time; this is stated by saying that volume computation for $\mathcal{V}$-polytopes is #ℙ-*easy*.

$\mathcal{H}$-presented polytopes come with the additional difficulty that the size of their volume is not bounded by a polynomial in the input size. An example was given by [La91], showing that there is no polynomial-space algorithm for *exact* computation of the volume of $\mathcal{H}$-polytopes. However, approximation to any positive rational absolute error $\epsilon$ is again #ℙ-easy for $\mathcal{H}$-polytopes, [DF88].

It is clear from section 1.3 (see also [GK94]) that the easiness results for computing or approximating the volume can be extended to mixed volumes if the number $s$ of sets under consideration is bounded beforehand. If, however, $s$ is part of the input the number of volume computations needed for the numerical differentiation approach to compute a single mixed volume cannot be bounded by a polynomial in $n$ and $s$. The

reason is that this method "essentially" computes all mixed volumes at once and their number is exponential.

We will show in the following, however, that even in this general case computation (for $\mathcal{V}$-polytopes or $\mathcal{S}$-zonotopes) or approximation (for $\mathcal{H}$-polytopes) of any single mixed volume is $\#\mathbb{P}$-easy. We begin with the easier case of $\mathcal{S}$-zonotopes.

THEOREM 6. *Let $\Pi$ be any $\#\mathbb{P}$-complete problem. Then any oracle $\mathcal{O}_\Pi$ for solving $\Pi$ can be used to produce an algorithm that runs in time that is oracle-polynomial in the input size for solving the following problem:*

Instance: $n, s \in \mathbb{N}$, *and* $m_1, \ldots, m_s \in \mathbb{N}$ *such that* $\sum_{i=1}^{s} m_i = n$, $\mathcal{S}$-*zonotopes* $Z_i = (n, s_i; c_i; z_{i,1}, \ldots, z_{i,s_i})$, *for* $i = 1, \ldots, s$.

Task: *Compute the mixed volume*

$$V(\overbrace{Z_1, \ldots, Z_1}^{m_1}, \overbrace{Z_2, \ldots, Z_2}^{m_2}, \ldots, \overbrace{Z_s, \ldots, Z_s}^{m_s}).$$

*Proof.* The proof reduces the problem to the task of approximating the volume of a (typically nonconvex) finite union of parallelotopes.

For $i \in S = \{1, \ldots, s\}$, let $J_i = \{(i,1), \ldots, (i, s_i)\}$, set $J = J_1 \cup \cdots \cup J_s$, and

$$\mathcal{J}_{m_1,\ldots,m_s} = \{I \subset J : |I \cap J_i| = m_i, \text{ for } i \in S\}.$$

Further, let $r = \sum_{i=1}^{s} s_i$, and let $A$ denote the $n \times r$ matrix

$$A = (z_{1,1}, \ldots, z_{1,s_1}, \ldots, z_{s,1}, \ldots, z_{s,s_s}).$$

Then it is easy to see, by expanding $\mathrm{vol}_n(\sum_{i=1}^{s} \lambda_i Z_i)$, that

$$\binom{n}{m_1, \ldots, m_s} V(\overbrace{Z_1, \ldots, Z_1}^{m_1}, \ldots, \overbrace{Z_s, \ldots, Z_s}^{m_s}) = \sum_{I \in \mathcal{J}_{m_1,\ldots,m_s}} |\det B_I|, \tag{2.3}$$

where $B_I$ denotes the $n \times n$ submatrix of $A$ with column indices in $I$, and $\binom{n}{m_1,\ldots,m_s}$ is the usual multinomial coefficient, i.e.,

$$\binom{n}{m_1, \ldots, m_s} = \frac{n!}{m_1! \cdot \ldots \cdot m_s!}.$$

To prove the theorem, we will now interpret (2.3) geometrically. In fact, let

$$Z = \sum_{(i,j) \in J} [0,1] z_{i,j},$$

and let again $\mathcal{J}$ denote the family of all subsets $I$ of $J$ of cardinality $n$. Using a simple inductive argument (with respect to $r$), we see that there is a subset $\mathcal{I}$ of $\mathcal{J}$ and that there are vectors $p_I$ $(I \in \mathcal{I})$ such that the parallelotopes

$$P_I = p_I + \sum_{i \in I} [0,1] z_i \qquad (I \in \mathcal{I})$$

form a dissection of $Z$ into proper parallelotopes; see [Sh74]. Further, for each $x \in Z \cap \mathbb{Q}^n$, a subset $I \in \mathcal{I}$ with $x \in P_I$ can be found in time bounded by a polynomial in $size(Z)$ and $size(x)$. Note that these parallelotopes are in one-to-one correspondence with the nonsingular matrices $B_I$ with $I \in \mathcal{J}$. Hence, with $Z_{m_1,\ldots,m_s} = \bigcup_{I \in \mathcal{J}_{m_1,\ldots,m_s}} P_I$, we have

$$\binom{n}{m_1, \ldots, m_s} V(\overbrace{Z_1, \ldots, Z_1}^{m_1}, \ldots, \overbrace{Z_s, \ldots, Z_s}^{m_s}) = \mathrm{vol}_n(Z_{m_1,\ldots,m_s}),$$

and membership in $Z_{m_1,\ldots,m_s}$ of a point $x \in \mathbb{Q}^n$ can be checked in polynomial time.

Now, let $R = \sum_{(i,j)\in J} \|z_{i,j}\|_\infty$, whence $Z \subset R[-1,1]^n$. Further, let $\epsilon$ be a positive rational, let

$$\alpha = \left\lceil \frac{2r^n n^2 (2R)^n}{\epsilon} \right\rceil, \quad \text{and} \quad \delta = \frac{R}{\alpha}.$$

For each integer vector $t = (\tau_1, \ldots, \tau_n)$, let

$$x_t = \delta \left(\tau_1 + \frac{1}{2}, \ldots, \tau_n + \frac{1}{2}\right)^T, \quad \text{and} \quad C_t = x_t + \frac{\delta}{2}[-1,1]^n.$$

For each $x_t$, membership in $Z_{m_1,\ldots,m_s}$ can be decided in polynomial time, so $\mathcal{O}_\Pi$ can be used to construct a counting machine that outputs the number $N$ of integer vectors $t$ for which $x_t \in Z_{m_1,\ldots,m_s}$. So, if $\nu$ is the number of cubes $C_t$ that intersect the boundary of $Z_{m_1,\ldots,m_s}$, we have

$$|N\delta^n - \mathrm{vol}_n(Z_{m_1,\ldots,m_s})| \le \nu\delta^n.$$

It is readily seen that each facet of any $Z_I$ ($I \in \mathcal{J}_{m_1,\ldots,m_s}$) is intersected by at most $2n(2\alpha)^{n-1}$ such cubes, whence (after some standard calculations)

$$|N\delta^n - \mathrm{vol}_n(Z_{(m_1,\ldots,m_s)})| \le 4r^n n^2 (2\alpha)^{n-1}\delta^n \le \epsilon \le \binom{n}{m_1,\ldots,m_s}\epsilon.$$

Therefore,

$$\left| V(\overbrace{Z_1,\ldots,Z_1}^{m_1},\ldots,\overbrace{Z_s,\ldots,Z_s}^{m_s}) - N\delta^n \binom{n}{m_1,\ldots,m_s}^{-1} \right| \le \epsilon.$$

Now, $size(\mathrm{vol}_n(Z_{m_1,\ldots,m_s}))$ is bounded above by a polynomial in the size of the input. So a suitable choice of $\epsilon$ and subsequent rounding yields $V(\overbrace{Z_1,\ldots,Z_1}^{m_1},\ldots,\overbrace{Z_s,\ldots,Z_s}^{m_s})$ exactly. □

Note that as a corollary we see that VOLUME-OF-ZONOTOPES is #ℙ-easy.

THEOREM 7. *Let $\Pi$ be any #ℙ-complete problem. Then any oracle $\mathcal{O}_\Pi$ for solving $\Pi$ can be used to produce an algorithm that runs in time that is oracle-polynomial in the input size (including size($\epsilon$) in the second case) for solving the following problems:*

Instance 1: *$n, s \in \mathbb{N}$ and $m_1, \ldots, m_s \in \mathbb{N}$ such that $\sum_{i=1}^s m_i = n$, $\mathcal{V}$-polytopes $P_i = (n, n_i; v_{i,1}, \ldots, v_{i,n_i})$, for $i = 1, \ldots, s$.*

Task 1: *Compute the mixed volume*

$$V(\overbrace{P_1,\ldots,P_1}^{m_1},\overbrace{P_2,\ldots,P_2}^{m_2},\ldots,\overbrace{P_s,\ldots,P_s}^{m_s}).$$

Instance 2: *$n, s \in \mathbb{N}$ and $m_1, \ldots, m_s \in \mathbb{N}$ such that $\sum_{i=1}^s m_i = n$, $\mathcal{H}$-polytopes $P_i = (n, n_i; A_i, b_i)$, for $i = 1, \ldots, s$, a positive rational number $\epsilon$.*

Task 2: *Compute a rational number $\hat{V}_{m_1,m_2,\ldots,m_s}$ such that*

$$\left| \hat{V}_{m_1,m_2,\ldots,m_s} - V(\overbrace{P_1,\ldots,P_1}^{m_1},\overbrace{P_2,\ldots,P_2}^{m_2},\ldots,\overbrace{P_s,\ldots,P_s}^{m_s}) \right| \le \epsilon.$$

*Proof.* For $\mathcal{V}$- or $\mathcal{H}$-polytopes it is not so clear (as it was for $\mathcal{S}$-zonotopes) that mixed volumes can be reduced to a volume computation, yet it is possible. The proof makes substantial use of a formula of [Sc94] (a generalization of [Be92]), and we will begin by restating [Sc94]'s approach.

For a polytope $P$ in some $\mathbb{R}^m$ and an integer $k$ with $0 \le k \le m$, let, as usual, $\mathcal{F}_k(P)$ denote the set of $k$-faces of $P$, and let $\mathcal{F}(P) = \bigcup_{k=0}^{m} \mathcal{F}_k(P)$. Further, for a face $F$ of $P$, let $N(P,F)$ denote the cone of outer normals of $P$ at $F$.

Now, let $P_1, \dots, P_s$ be polytopes in $\mathbb{R}^n$. Set $r = s \cdot n$ and

$$\tilde{P} = P_1 \times P_2 \times \cdots \times P_s \subset \overbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}^{s} = \mathbb{R}^r.$$

It is easy to see that

$$\mathcal{F}(\tilde{P}) = \{F_1 \times F_2 \times \cdots \times F_s \ : F_1 \in \mathcal{F}(P_1), \dots, F_s \in \mathcal{F}(P_s)\}$$

and that

$$\mathcal{F}_k(\tilde{P}) = \bigcup_{\substack{k_1,\dots,k_s \in \mathbb{N}_0 \\ k_1+\cdots+k_s=k}} \{F_1 \times F_2 \times \cdots \times F_s : F_1 \in \mathcal{F}_{k_1}(P_1), \dots, F_s \in \mathcal{F}_{k_s}(P_s)\}.$$

Let

$$\Delta = \{(x^T, x^T, \dots, x^T)^T \in \mathbb{R}^r : x \in \mathbb{R}^n\};$$

$\Delta$ is a linear subspace of $\mathbb{R}^r$ of dimension $n$. For $\tilde{v} = (v_1, \dots, v_r)^T \in \Delta^\perp \setminus \{0\}$, let $\Delta_{\tilde{v}} = \mathrm{lin}(\Delta \cup \{\tilde{v}\})$, and let

$$\Delta_{\tilde{v}}^+ = \{\tilde{w} \in \Delta_{\tilde{v}} : \langle \tilde{w}, \tilde{v} \rangle > 0\},$$

the corresponding "positive" open halfspace. Further, let $\pi_\Delta$ and $\pi_{\Delta_{\tilde{v}}}$ denote the orthogonal projections onto $\Delta$ and $\Delta_{\tilde{v}}$, respectively, let $\pi'_\Delta$ be the restriction of $\pi_\Delta$ to the set $\Delta_{\tilde{v}}$, and set $P_{\tilde{v}} = \pi_{\Delta_{\tilde{v}}}(\tilde{P})$. Note that

$$\pi_\Delta(x_1, \dots, x_s) = \frac{1}{s}\left(\sum_{i=1}^{s} x_i^T, \dots, \sum_{i=1}^{s} x_i^T\right)^T.$$

Then $\mathrm{vol}_n(\pi_\Delta(\tilde{P}))$ is just the sum of the volumes of the projections of those facets of $P_{\tilde{v}}$ with outer normal vector $\tilde{w}$ in $\Delta_{\tilde{v}}^+$.

Suppose that none of the $\tilde{w} \in \Delta_{\tilde{v}}^+$ is orthogonal to a hyperplane in $\mathbb{R}^r$ that supports $\tilde{P}$ in a face of dimension greater than $n$. Then each facet of $P_{\tilde{v}}$ with outer normal $\tilde{w}$ in $\Delta_{\tilde{v}}^+$ is the projection of exactly one $n$-dimensional face $\tilde{F} \in \mathcal{F}_n(\tilde{P})$ such that $\tilde{w} \in N(\tilde{P}, \tilde{F})$. Let $\tilde{\mathcal{F}}_n^+$ be the set of all faces $\tilde{F} \in \mathcal{F}_n(\tilde{P})$ for which

$$N(\tilde{P}, \tilde{F}) \cap \Delta_{\tilde{v}}^+ \neq \emptyset.$$

It follows that

$$\pi_\Delta(\tilde{P}) = \bigcup_{\tilde{F} \in \tilde{\mathcal{F}}_n^+} \pi_\Delta(\tilde{F})$$

and

$$\mathrm{vol}_n(\pi_\Delta(\tilde{F}) \cap \pi_\Delta(\tilde{G})) = 0 \quad \text{for all } \tilde{F}, \tilde{G} \in \tilde{\mathcal{F}}_n^+,\ \tilde{F} \neq \tilde{G}.$$

Now, let

$$F_{m_1,\dots,m_s} = \bigcup_{\substack{\tilde{F} = F_1 \times F_2 \times \cdots \times F_s \in \tilde{\mathcal{F}}_n^+ \\ \dim(F_1)=m_1,\dots,\dim(F_s)=m_s}} \mathrm{int}(F_1 + \cdots + F_s),$$

where int is taken with respect to $\mathbb{R}^n$. (Clearly, in terms of volume computations, taking the interior does not matter, and we do it only for technical reasons that become clear when we develop a method for checking membership in $F_{m_1,\dots,m_s}$ later.) By replacing $P_i$ by $\lambda_i P_i$ for $\lambda_i > 0$, and comparing coefficients we obtain [Sc94]'s formula

$$\binom{n}{m_1,\dots,m_s} V(\overbrace{P_1,\dots,P_1}^{m_1},\dots,\overbrace{P_s,\dots,P_s}^{m_s}) = \mathrm{vol}_n(F_{m_1,\dots,m_s}) = \sum_{\substack{\tilde{F}=F_1\times F_2\times\cdots\times F_s\in\tilde{\mathcal{F}}_n^+ \\ \dim(F_1)=m_1,\dots,\dim(F_s)=m_s}} \mathrm{vol}_n(F_1+\cdots+F_s).$$

Suppose that rational vectors $v_1, v_2, \dots, v_s \in \mathbb{R}^n$ can be computed in polynomial time with $\tilde{v} = (v_1^T, v_2^T, \dots, v_s^T)^T \in \Delta^\perp$ and such that no $\tilde{w} \in \Delta_{\tilde{v}}^+$ supports $\tilde{P}$ in a face of dimension greater than $n$. We can then apply the same proof technique as in the proof of Theorem 6, if we can check membership of a point $z$ in $F_{m_1,\dots,m_s}$ in polynomial time. But this can be done as follows (in both cases where $P_1, \dots, P_s$ are $\mathcal{V}$- or $\mathcal{H}$-polytopes).

Given $z \in \mathbb{R}^n$, we first check whether $z \in P_1 + P_2 + \cdots + P_s$. Clearly, this can be done by linear programming. If the answer is affirmative, we compute the vector $\tilde{z}_0$ that is given by

$$\{\tilde{z}_0\} = \{\tilde{z} + \lambda\tilde{v} : \lambda \geq 0\} \cap \mathrm{relbd}(P_{\tilde{v}}), \quad \text{where } \tilde{z} = (z^T, \dots, z^T)^T.$$

To see that this can be done in polynomial time, observe that the corresponding parameter $\lambda_0$ is the solution of the linear program

$$\max\langle \tilde{v}, \tilde{x}\rangle \quad \text{s.t.} \quad \tilde{x} \in \tilde{P} \cap (\tilde{z} + \Delta^\perp).$$

Since $\tilde{P} = \{\tilde{x} = (x_1^T, \dots, x_s^T)^T : x_1 \in P_1, \dots, x_s \in P_s\}$ and $\tilde{z} + \Delta^\perp$ can be easily expressed in the form $A\tilde{x} = b$, where $A$ is an $n \times r$ matrix with 0-1 coefficients, $b \in \mathbb{Q}^n$, and the size is bounded by a polynomial in $r$ and $size(z)$, the given linear program can be solved in polynomial time.

Now, if $\lambda_0 = 0$ we know that $z \in \mathrm{bd}(P_1 + P_2 + \cdots + P_s)$, and we report that $z \notin F_{m_1,\dots,m_s}$.

Otherwise we compute an outer normal $\tilde{w} \in \Delta_{\tilde{v}}^+$ of $P_{\tilde{v}}$ at $\tilde{z}_0$. This can be done in polynomial time.

Let $F_{\tilde{w}}$ denote the face of $P_{\tilde{v}}$ that corresponds to the supporting hyperplane determined by $\tilde{w}$. It may or may not be the case that $F_{\tilde{w}}$ is a facet of $P_{\tilde{v}}$ (we will find out in the final step); and we know that $z \notin F_{m_1,\dots,m_s}$ if it is not. (This situation is the reason for considering only the interiors of the sets $F_1 + \cdots + F_s$ in the definition of $F_{m_1,\dots,m_s}$.)

Next we determine the face $\tilde{F}$ of $\tilde{P}$ which is induced by the supporting hyperplane orthogonal to $\tilde{w}$. This is done by solving for $i = 1, \dots, s$ the linear program

$$\max\langle w_i, x\rangle \quad \text{s.t.} \quad x \in P_i,$$

where $w_1, \dots, w_s \in \mathbb{R}^n$ such that $\tilde{w} = (w_1^T, \dots, w_s^T)^T$. Note that is is not enough to find *a* solution; we need to find a $\mathcal{V}$- or $\mathcal{H}$-presentation of the set of *all* solutions. But this can be done in polynomial time. So, let $F_1, \dots, F_s$ be the respective solution sets. Then $\tilde{F} = F_1 \times F_2 \times \cdots \times F_s$ is the face of $\tilde{P}$ in question. Now we need to check whether $\dim F_i = m_i$ for all $i = 1, \dots, s$, a task involving just linear algebra, and, if

this is the case, finally, whether $\pi_{\Delta_{\tilde{v}}}$ does not reduce $\dim \tilde{F}$, again a simple task from linear algebra.

Hence we have derived a polynomial-time algorithm for checking membership in $F_{m_1,\dots,m_s}$ which we can now apply to the points $x_t$ used in the proof of the easiness results for zonotopes, and we may proceed as before.

In order to finish the proof of Theorem 7, all that is left to be done is to show that an appropriate choice of the vector $\tilde{v}$ can be made in polynomial time.

The condition on $\tilde{v}$ is satisfied if

$$\bigcap_{i=1}^{s} (\operatorname{relint}(N(P_i, F_i)) - v_i) = \emptyset$$

for all $s$-tuples $(F_1, F_2, \dots, F_s)$ of faces $F_i$ of $P_i$ such that

$$\dim F_1 + \dim F_2 + \cdots + \dim F_s > n.$$

We will actually produce (in polynomial time) vectors $v_1, \dots, v_s$ such that

$$\bigcap_{i=1}^{s} (\operatorname{lin}(N(P_i, F_i)) - v_i) = \emptyset \tag{2.4}$$

for all $s$-tuples $(F_1, F_2, \dots, F_s)$ of faces $F_i$ of $P_i$ such that

$$\dim F_1 + \dim F_2 + \cdots + \dim F_s > n.$$

Let $(F_1, F_2, \dots, F_s)$ be such a choice of faces, i.e.,

$$k_1 + \cdots + k_s \geq n + 1, \quad \text{where } k_i = \dim F_i \text{ for } i = 1, \dots, s.$$

Suppose that for $i = 1, \dots, s$ the vectors $a_{i,1}, \dots, a_{i,n-k_i}$ are facet normals of $P_i$ that span $\operatorname{lin}(N(P_i, F_i))$. Then (2.4) is violated for some choice of $\tilde{v}$, if and only if the following inhomogenous system of linear equations (in the variables $x$ and $\lambda_{i,j}$) is feasible.

$$x + \sum_{j=1}^{n-k_i} \lambda_{i,j} a_{i,j} = v_i \qquad (i = 1, \dots, s). \tag{2.5}$$

Note that this system is overdetermined; it consists of $r$ equations in $n + \sum_{i=1}^{s}(n-k_i) = r + (n - \sum_{i=1}^{s} k_i) \leq r - 1$ variables and is, hence, generically infeasible. In order to find a specific vector $\tilde{v}$ of size that is bounded by a polynomial in the input size, which renders *all* such systems infeasible, we have to analyze the condition a bit more carefully, since in general there are doubly exponentially many such systems. Note, however, that the coefficient matrices have the property that all entries are of size that is bounded by a polynomial in the input size. Now suppose $\tilde{v}$ is of the form

$$\tilde{v}_\xi = (v_1^T, v_2^T, \dots, v_s^T)^T = (\xi, \xi^2, \dots, \xi^r)^T \quad \text{for some } \xi > 1.$$

Note that, in general, $\tilde{v}_\xi \notin \Delta^\perp$, but since $\tilde{v}_\xi \notin \Delta$, it is of the form $(y_\xi^T, \dots, y_\xi^T)^T + \tilde{v}'_\xi$ with $\tilde{v}'_\xi \in \Delta^\perp$, whence it suffices to show that for a suitable choice of $\xi$ the system (2.5) is infeasible for $\tilde{v}_\xi$. Now, since (2.5) is overdetermined, it is only feasible if the components of $\tilde{v}_\xi$ satisfy a linear relation with coefficients that come as subdeterminants of (2.5)'s coefficient matrices, whence are bounded in size by an integer polynomial $\pi(\Lambda)$ in the input size $\Lambda$, i.e., we have a relation

$$\xi^r + \sum_{i=1}^{r-1} \alpha_i \xi^i = 0 \quad \text{with } |\alpha_1|, \dots, |\alpha_{r-1}| \leq 2^{\pi(\Lambda)}.$$

Hence, with $\xi_0 = 2r2^{\pi(\Lambda)}$ the vector $\tilde{v}_{\xi_0}$ makes all systems (2.5) infeasible.

This completes the proof of the two asserted easiness results. □

**2.4. Deterministic methods for approximating mixed volumes.** The problem of how well the volume of a well-presented convex body can be approximated in polynomial time was investigated by various authors; see [GK94] for a survey.

For a positive functional $\phi$ on $\mathcal{K}^n$ (or on appropriate subsets of $\mathcal{K}^n$) and a functional $\lambda : \mathbb{N} \to \mathbb{R}$, a *(relative) $\lambda$-approximation* of $\phi$ is a functional $\hat{\phi}$ defined on the domain of $\phi$ such that

$$\frac{\phi(K)}{\hat{\phi}(K)} \leq \lambda \quad \text{and} \quad \frac{\hat{\phi}(K)}{\phi(K)} \leq \lambda.$$

(Note that the *relative error* $|(\hat{\phi}(K) - \phi(K))/\phi(K)|$ is only appropriate if one is confronted with small errors since taking $\hat{\phi}(K) = 0$ always gives an estimate with relative error 1.)

When looking for relative estimates for mixed volumes, the first question is if one can efficiently check whether the mixed volume under consideration is greater than zero.

THEOREM 8. *There is a polynomial time algorithm which solves the following problem: given $n, s \in \mathbb{N}$, $m_1, \ldots, m_s \in \mathbb{N}_0$ with $\sum_{i=1}^{s} m_i = n$, and well-presented convex bodies $K_1, \ldots, K_s$, decide whether*

$$V(\overbrace{K_1, \ldots, K_1}^{m_1}, \overbrace{K_2, \ldots, K_2}^{m_2}, \ldots, \overbrace{K_s, \ldots, K_s}^{m_s}) = 0.$$

*Proof.* For $i = 1, \ldots, s$, let $d_i = \dim(K_i)$, and let $a_{i,0}, \ldots, a_{i,d_i} \in K_i$ such that $\operatorname{aff}(K_i) = \operatorname{aff}\{a_{i,0}, \ldots, a_{i,d_i}\}$. Note that these vectors $a_{i,j}$ are part of the input of the problem. Since our task is clearly translation invariant, we may assume that $a_{1,0} = \cdots = a_{s,0} = 0$, and also that $b_1 = \cdots = b_s = 0$, where $b_i$ is the given "center" of $K_i$.

Now, for $i = 1, \ldots, s$, let $Z_i = \sum_{j=1}^{d_i} [-1, 1] a_{i,j}$. Then clearly, for some $\rho, R > 0$ (which we do not have to know explicitly),

$$\rho Z_i \subset K_i \subset R Z_i.$$

Hence, by the monotonicity of mixed volumes,

$$V(\overbrace{K_1, \ldots, K_1}^{m_1}, \overbrace{K_2, \ldots, K_2}^{m_2}, \ldots, \overbrace{K_s, \ldots, K_s}^{m_s}) = 0,$$

if and only if

$$V(\overbrace{Z_1, \ldots, Z_1}^{m_1}, \overbrace{Z_2, \ldots, Z_2}^{m_2}, \ldots, \overbrace{Z_s, \ldots, Z_s}^{m_s}) = 0.$$

Using the notation introduced in the previous subsection, let $J_i = \{(i,1), \ldots, (i,d_i)\}$ for all $i = 1, \ldots, s$, set $J = J_1 \cup \cdots \cup J_s$, set

$$\mathcal{J}_{m_1,\ldots,m_s} = \{I \subset J : |I \cap J_i| = m_i, \text{ for } i = 1, \ldots, s\}$$

and let $A_I = \{a_{i,j} : (i,j) \in I\}$ for $I \subset J$.

It follows from Proposition 2 that $V(\overbrace{K_1, \ldots, K_1}^{m_1}, \overbrace{K_2, \ldots, K_2}^{m_2}, \ldots, \overbrace{K_s, \ldots, K_s}^{m_s}) \neq 0$ if and only if there is a linear independent subset $A_I$ of $A_J$ which, for $i = 1, \ldots, s$, contains exactly $m_i$ elements from $A_{J_i}$. This is equivalent to the existence of a common basis for two matroids, the linear matroid and the partition matroid on $A_J$. The existence of such a common basis can be determined in polynomial time by the matroid intersection algorithm of [Ed70]; see also [GLS88, Theorem 7.5.16]. □

Now, assume that $K_1, \ldots, K_s$ are well-presented convex bodies and we are longing for relative approximations to

$$V(\overbrace{K_1, \ldots, K_1}^{m_1}, \ldots, \overbrace{K_s, \ldots, K_s}^{m_s}),$$

where $m_1 + \cdots + m_s = n$. Using Proposition 4, we easily obtain a $(\min\{\rho_n^{n/2}, \nu_n^{n/2}\})$-approximation of $\mathrm{vol}_n(K)$, where

$$\rho_n = n\sqrt{n+1} \quad \text{and} \quad \nu_n = 2(n+1).$$

(In the rest of the paper we will use these abbreviations to emphasize that improvements in Proposition 4 (in general or for subclasses of $\mathcal{K}^n$) carry over to our approximation results. For such an improvement for $\mathcal{H}$-polytopes see [KT93], and see [GK94] for a survey.) Note that $\rho_n^{n/2}$ and $\nu_n^{n/2}$ depend only on the dimension $n$ and are independent of the bounds of the in- and circumradii given in the input. On the negative side, it has been shown by [BF86] that for each polynomial-time algorithm which produces a $\lambda$-approximation of the volume of well-presented convex bodies there exists a constant $c$ such that $\lambda(n) \geq (\frac{cn}{\log n})^{n/2}$ for all $n \in \mathbb{N}$.

It is clear that we cannot expect anything better for mixed volumes, but can we at least get polynomial-time approximations whose error depends only on the dimension $n$? Note that the "obvious" approach of approximating the bodies $K_1, \ldots, K_s$ by parallelotopes each and then using the mixed volume of the parallelotopes as estimates fails in view of Theorem 4, and Theorem 2 indicates some limits for a similar approach using ellipsoids. The following result, however, gives a positive answer for $s = 2$; the general case is open and posed here as a problem.

THEOREM 9. *Let $m : \mathbb{N}_0 \to \mathbb{N}_0$ with $m(n) \leq n$ for all $n \in \mathbb{N}_0$, and let $\lambda : \mathbb{N} \to \mathbb{R}$ be defined by*

$$\lambda(n) = \min\left\{ \rho_n^{\frac{m(n)}{2}} \nu_n^{\frac{n-m(n)}{2}}, \rho_n^{\frac{n-m(n)}{2}} \nu_n^{\frac{m(n)}{2}} \right\}.$$

*Then there is a polynomial-time algorithm which produces a $\lambda$-approximation of*

$$V(\overbrace{K_1, \ldots, K_1}^{n-m(n)}, \overbrace{K_2, \ldots, K_2}^{m(n)})$$

*for well-presented proper convex bodies $K_1, K_2$.*

*Proof.* Given $K_1$ and $K_2$, let $\phi_1$ and $\phi_2$ be affine transformations such that

$$\mathbb{B}^n \subset \phi_1(K_1) \subset \rho_n \mathbb{B}^n \quad \text{and} \quad C_n \subset \phi_2(K_2) \subset \nu_n C_n.$$

This implies, with $Z = \phi_1(\phi_2^{-1}(C_n))$ and $m = m(n)$, that

$$\begin{aligned} V(\overbrace{\mathbb{B}^n, \ldots, \mathbb{B}^n}^{n-m}, \overbrace{Z, \ldots, Z}^{m}) &\leq V(\overbrace{\phi_1(K_1), \ldots, \phi_1(K_1)}^{n-m}, \overbrace{\phi_1(K_2), \ldots, \phi_1(K_2)}^{m}) \\ &\leq V(\overbrace{\rho_n \mathbb{B}^n, \ldots, \rho_n \mathbb{B}^n}^{n-m}, \overbrace{\nu_n Z, \ldots, \nu_n Z}^{m}). \end{aligned}$$

Since the common affine transformation $\phi_1$ changes the mixed volume only by the absolute value of the corresponding determinant as a factor, we obtain the desired bound by taking the geometric mean of the lower and upper estimates and noticing that the roles of $K_1$ and $K_2$ can be interchanged. The polynomiality of the algorithm

follows from Proposition 4 and from the fact that the quermassintegrals of a parallelotope can be approximated to absolute positive rational error $\epsilon$ in polynomial time in $n$ and $size(\epsilon)$; see, e.g., [GK94, Theorem 4.4.4]. □

Let us point out that Theorem 9 can be extended to improper sets $K_1$ and $K_2$ by first using Theorem 8 to check whether the mixed volume under consideration is 0, and if this is not the case, by applying Theorem 9 to the bodies $K_1 + \epsilon\mathbb{B}^n$ and $K_1 + \epsilon\mathbb{B}^n$ for suitably small positive rational $\epsilon$.

The final result of this subsection is needed as preprocessing for the inductive step in the main algorithm of section 3. It is included here because it is approximative in the sense that it gives an algorithmic solution to the (properly phrased variant of the) question of how well a specific mixed volume $V(\overbrace{K_1,\dots,K_1}^{n-k+1},\overbrace{K_2,\dots,K_2}^{k-1})$ of two bodies approximates the "next" one, $V(\overbrace{K_1,\dots,K_1}^{n-k},\overbrace{K_2,\dots,K_2}^{k})$. First we state a theoretical bound which holds after some preliminary normalizations, then we will show how these assumptions can be satisfied in polynomial time.

LEMMA 3. *Let $K_1, K_2 \in \mathcal{K}^n$, let $E$ be an ellipsoid centered at 0 such that $E \subset K_1 \subset \rho_n E$, and let $v_1,\dots,v_n$ be the semi-axis vectors of $E$, such that $\|v_1\| \le \cdots \le \|v_n\|$. Further, suppose that $\mathbb{B}^n \subset K_2 \subset \rho_n\mathbb{B}^n$ and that $\|v_m\| = 1$. Then*

$$(n+1)^{-4m+5/2} \le \frac{a_{m-1}}{a_m} \le (n+1)^{4m-3/2},$$

*where, for $k = m-1, m$,*

$$a_k = V(\overbrace{K_1,\dots,K_1}^{n-k},\overbrace{K_2,\dots,K_2}^{k}).$$

*Proof.* For $i = 1,\dots,n$, set $w_i = v_i/\|v_i\|$. Further, for $j = 1,\dots,m$, let $U_j = \mathrm{lin}\{v_1,\dots,v_j\}$, let $\pi_j : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection on $U_j^\perp$, and let $V_{U_j}$ and $V_{U_j^\perp}$ denote the (mixed) volume taken in $U_j$, $U_j^\perp$ (with respect to the standard $j$- or $(n-j)$-measure in $U_j$ or $U_j^\perp$), respectively.

Let us begin by giving a simple lower estimate for $V(\overbrace{K_1,\dots,K_1}^{n-k},\overbrace{K_2,\dots,K_2}^{k})$, when $k = m-1, m$. Let

$$Q_k = \mathrm{conv}\{\pm w_1,\dots,\pm w_k\}.$$

Then we obtain, with the aid of Proposition 2,

$$\begin{aligned} V(\overbrace{K_1,\dots,K_1}^{n-k},\overbrace{K_2,\dots,K_2}^{k}) &\ge V(\overbrace{K_1,\dots,K_1}^{n-k},\overbrace{Q_k,\dots,Q_k}^{k}) \\ &= \binom{n}{k}^{-1} V_{U_k^\perp}(\overbrace{\pi_k(K_1),\dots,\pi_k(K_1)}^{n-k}) V_{U_k}(\overbrace{Q_k,\dots,Q_k}^{k}) \\ &= \binom{n}{k}^{-1} \mathrm{vol}_{n-k}(\pi_k(K_1))\mathrm{vol}_k(Q_k) = \frac{2^k}{(n-k+1)\cdot\ldots\cdot n}\mathrm{vol}_{n-k}(\pi_k(K_1)) \\ &\ge \left(\frac{2}{n}\right)^k \mathrm{vol}_{n-k}(\pi_k(K_1)). \end{aligned} \tag{2.6}$$

Next we derive upper bounds. Note, first, that

$$\begin{aligned}\|v_1\|K_2 &\subset \rho_n\|v_1\|\mathbb{B}^n \subset \rho_n E \subset \rho_n K_1;\\ K_1 &\subset \pi_1(K_1) + \rho_n[-1,1]v_1;\\ K_2 &\subset \pi_1(K_2) + \rho_n[-1,1]w_1.\end{aligned}$$

Now, again let $k = m-1, m$. Using the monotonicity of mixed volumes we obtain

$$\begin{aligned}&V(\overbrace{K_1,\dots,K_1}^{n-k},\overbrace{K_2,\dots,K_2}^{k})\\ &\quad\le V(\overbrace{\pi_1(K_1)+\rho_n[-1,1]v_1,\dots,\pi_1(K_1)+\rho_n[-1,1]v_1}^{n-k},\\ &\qquad\overbrace{\pi_1(K_2)+\rho_n[-1,1]w_1,\dots,\pi_1(K_2)+\rho_n[-1,1]w_1}^{k})\\ &\quad=\sum_{i=0}^{n-k}\sum_{j=0}^{k}\binom{n-k}{i}\binom{k}{j}V(\overbrace{\pi_1(K_1),\dots,\pi_1(K_1)}^{n-k-i},\overbrace{\rho_n[-1,1]v_1,\dots,\rho_n[-1,1]v_1}^{i},\\ &\qquad\overbrace{\pi_1(K_2),\dots,\pi_1(K_2)}^{k-j},\overbrace{\rho_n[-1,1]w_1,\dots,\rho_n[-1,1]w_1}^{j}).\end{aligned}$$

Proposition 2 then yields the following estimate.

$$\begin{aligned}&V(\overbrace{K_1,\dots,K_1}^{n-k},\overbrace{K_2,\dots,K_2}^{k})\\ &\quad\le (n-k)V(\overbrace{\pi_1(K_1),\dots,\pi_1(K_1)}^{n-k-1},\rho_n[-1,1]v_1,\overbrace{\pi_1(K_2),\dots,\pi_1(K_2)}^{k})\\ &\qquad+kV(\overbrace{\pi_1(K_1),\dots,\pi_1(K_1)}^{n-k},\overbrace{\pi_1(K_2),\dots,\pi_1(K_2)}^{k-1},\rho_n[-1,1]w_1)\\ &\quad=\frac{2(n-k)\rho_n\|v_1\|}{n}V_{U_1^\perp}(\overbrace{\pi_1(K_1),\dots,\pi_1(K_1)}^{n-k-1},\overbrace{\pi_1(K_2),\dots,\pi_1(K_2)}^{k})\\ &\qquad+\frac{2k\rho_n}{n}V_{U_1^\perp}(\overbrace{\pi_1(K_1),\dots,\pi_1(K_1)}^{n-k},\overbrace{\pi_1(K_2),\dots,\pi_1(K_2)}^{k-1})\\ &\quad\le\frac{2(n-k)\rho_n^2+2k\rho_n}{n}V_{U_1^\perp}(\overbrace{\pi_1(K_1),\dots,\pi_1(K_1)}^{n-k},\overbrace{\pi_1(K_2),\dots,\pi_1(K_2)}^{k-1})\\ &\quad\le 2\rho_n^2V_{U_1^\perp}(\overbrace{\pi_1(K_1),\dots,\pi_1(K_1)}^{n-k},\overbrace{\pi_1(K_2),\dots,\pi_1(K_2)}^{k-1}).\end{aligned}$$

The same estimate can now be applied inductively; if we do this $k-1$ times for $k=m$ and $k$ times for $k=m-1$ we obtain

(2.7)

$$\begin{aligned}&V(\overbrace{K_1,\dots,K_1}^{n-m},\overbrace{K_2,\dots,K_2}^{m})\le(2\rho_n^2)^{m-1}V_{U_{m-1}^\perp}(\overbrace{\pi_{m-1}(K_1),\dots,\pi_{m-1}(K_1)}^{n-m},\pi_{m-1}(K_2))\\ &V(\overbrace{K_1,\dots,K_1}^{n-m+1},\overbrace{K_2,\dots,K_2}^{m-1})\le(2\rho_n^2)^{m-1}\mathrm{vol}_{n-m+1}(\pi_{m-1}(K_1)).\end{aligned}$$

Now we combine the estimates (2.6) and (2.7) with the fact that

$$\pi_{m-1}(K_2) \subset \rho_n \mathbb{B}^{n-m+1} \subset \rho_n \pi_{m-1}(K_1),$$
$$\pi_{m-1}(K_1) \subset \pi_m(K_1) + \rho_n[-1,1]v_m,$$

and obtain

(2.8)
$$V(\overbrace{K_1,\dots,K_1}^{n-m},\overbrace{K_2,\dots,K_2}^{m}) \le (2\rho_n^2)^{m-1} V_{U_{m-1}^{\perp}}(\overbrace{\pi_{m-1}(K_1),\dots,\pi_{m-1}(K_1)}^{n-m},\pi_{m-1}(K_2))$$
$$\le (2\rho_n^2)^{m-1}\rho_n \mathrm{vol}_{n-m+1}(\pi_{m-1}(K_1)) \le n^{m-1}\rho_n^{2m-1} V(\overbrace{K_1,\dots,K_1}^{n-m+1},\overbrace{K_2,\dots,K_2}^{m-1})$$

and

(2.9)
$$V(\overbrace{K_1,\dots,K_1}^{n-m},\overbrace{K_2,\dots,K_2}^{m}) \ge \left(\frac{2}{n}\right)^m \mathrm{vol}_{n-m}(\pi_m(K_1))$$
$$\ge \frac{2^{m-1}}{n^m\rho_n}\mathrm{vol}_{n-m+1}(\pi_{m-1}(K_1)) \ge \frac{1}{n^m\rho_n^{2m-1}} V(\overbrace{K_1,\dots,K_1}^{n-m+1},\overbrace{K_2,\dots,K_2}^{m-1}).$$

When $\rho_n$ is replaced by its upper bound $(n+1)^{3/2}$, the estimates (2.8) and (2.9) yield the assertion. □

LEMMA 4. *There is a polynomial-time algorithm which constructs, for given well-presented proper convex bodies $K_1, K_2$ of $\mathbb{R}^n$ and a given $m \in \{1,\dots,n\}$, an affine transformation $\phi$ and a rational number $\kappa > 0$ such that*

$$1 \le \frac{a'_{m-1}}{a'_m} \le (n+1)^{8m},$$

*where, for $k = m-1, m$,*

$$a'_k = V(\overbrace{K'_1,\dots,K'_1}^{n-k},\overbrace{K'_2,\dots,K'_2}^{k})$$

*is the corresponding mixed volume of the transformed bodies $K'_1 = \kappa\phi(K_1)$ and $K'_2 = \phi(K_2)$.*

*Proof.* Proposition 4 allows us to construct affine transformations $\phi$ and $\hat{\phi}$ such that

$$\mathbb{B}^n \subset \phi(K_2) \subset \rho_n\mathbb{B}^n \quad \text{and} \quad \mathbb{B}^n \subset \hat{\phi}(\phi(K_1)) \subset \rho_n\mathbb{B}^n.$$

So, let us assume for simplicity of notation that, already,

$$E \subset K_1 \subset \rho_n E \quad \text{and} \quad \mathbb{B}^n \subset K_2 \subset \rho_n\mathbb{B}^n,$$

where $E = A^{-1}\mathbb{B}^n$ for a nonsingular matrix $A$ whose entries are bounded in size by a polynomial in the input size. Now, using the multilinearity of the mixed volumes, Lemma 3 implies

$$(n+1)^{-4m+5/2}\|v_m\| \le \frac{a_{m-1}}{a_m} \le (n+1)^{4m-3/2}\|v_m\|.$$

The problem of computing $\|v_m\|$ is essentially the task of computing the eigenvalues of $A^TA$, and this can be done in time that is polynomial in the input data and in the binary size of the required precision $\epsilon$. (A conceptually simple way to find the largest eigenvalue of a positive definite matrix $A$ is to perform a binary search on $A - \lambda I$ (with respect to a parameter $\lambda$) using the criterion for positive definiteness that the determinants of the $k \times k$ submatrices of the first $k$ rows and columns are positive. The rest is then standard fare in linear algebra.) However, all these quantities are only available up to a polynomially bounded precision. So suppose that $\nu$ is a positive rational such that $|\nu - \|v_m\|| \leq \epsilon$. Then we obtain

$$(n+1)^{-4m+5/2}(\nu - \epsilon) \leq \frac{a_{m-1}}{a_m} \leq (n+1)^{4m-3/2}(\nu + \epsilon),$$

whence, with a sufficiently small (but polynomially bounded) positive $\epsilon$,

$$(n+1)^{-4m}\nu \leq \frac{a_{m-1}}{a_m} \leq (n+1)^{4m}\nu.$$

So, if we rescale $K_2$ by a factor $(n+1)^{4m}/\nu$, we obtain the asserted inequality. □

**3. Randomized algorithms.** In this section, we give a randomized algorithm for computing relative approximations of certain mixed volumes of well-presented convex bodies to relative error $\epsilon$ whose running time is polynomial in $1/\epsilon$ and the size of the input. We begin with the case of two bodies $K_1$ and $K_2$. Our algorithm uses the polynomial-time randomized volume algorithm of Proposition 1 to obtain relative estimates of the values of the polynomial

$$p(x) = \mathrm{vol}_n(K_1 + xK_2) = \sum_{j=0}^{n} c_j x^j = \sum_{j=0}^{n} \binom{n}{j} a_j x^j$$
$$= \sum_{j=0}^{n} \binom{n}{j} V(\overbrace{K_1, \ldots, K_1}^{n-j}, \overbrace{K_2, \ldots, K_2}^{j}) x^j$$

at certain interpolation points. After deriving a basic estimate in subsection 3.1 and showing that the general case of possibly improper convex bodies can be reduced to the case of all bodies in question being proper, we describe a randomized algorithm in subsection 3.2 that computes approximations $\hat{a}_m$ of the mixed volumes $a_m$ of two proper convex bodies recursively. The scaling of Lemma 4 is used as a preprocessing step; it gives a first rough estimate for $a_m$. The first part of the algorithm uses a search procedure to produce an approximation of the ratio $a_{m-1}/a_m$ to constant error; the second step gives the desired relative approximation of $a_m$ to error $\epsilon$. Subsection 3.2 concludes with the analysis of the complexity of the algorithm, thus establishing Theorem 10 (as stated in the introduction). Subsection 3.3 generalizes the randomized algorithm to more than two convex bodies and proves Theorem 11 (as stated in the introduction).

**3.1. A basic estimate and a reduction lemma.** The first part of this subsection gives an estimate that is fundamental for the algorithm presented in subsection 3.2.

Let $\xi_0, \ldots, \xi_n$ be (equidistant) interpolation nodes and for $i = 0, \ldots, n$, let $\hat{y}_i$ denote the relative estimate of $y_i = p(\xi_i)$ to error $\tau$. Setting

$$\hat{c}_m = \sum_{k=0}^{n} b_{km} \hat{y}_k,$$

where the $b_{km}$ are again the coefficients of the Lagrange polynomials, (1.5) yields

$$|\hat{c}_m - c_m| \le \tau \max_{i=0,\dots,n}\{q(\xi_i)\}\sum_{k=0}^{n}|b_{km}|.$$

We are, however, interested in a relative approximation, i.e., an estimate of the form

$$|\hat{c}_m - c_m| \le \tau'|c_m|.$$

Using the results of subsection 1.3, it is not hard to see that, in general,

$$\frac{1}{|c_m|}\max_{i=0,\dots,n}\{q(\xi_i)\}\sum_{k=0}^{n}|b_{km}|$$

grows exponentially in $n$. Unfortunately, the running time of the approximation algorithm of Proposition 1 is polynomial only in the approximation error and not in its *size*. Hence the relative approximations of $y_i$ to error $\tau$ that are produced via Proposition 1 cannot be used in this way to give estimates for *all* coefficients in polynomial time. This is the reason for using a small (left upper corner) $r \times r$ submatrix $B^{(r)}$ of the full matrix $B^{(n+1)}$; to allow polynomiality, $(rm)^m$ must be bounded by a polynomial in $n$.

The following lemma gives a bound for the error $|\hat{c}_m - c_m|$, where the estimate $\hat{c}_m$ is now computed from $B^{(r)}$. The parameters used are all generated later by the algorithm.

LEMMA 5. *Let $m \in \{1,\dots,n\}$, let $r \in \mathbb{N}$ with $r \ge 4m+7$, let $\alpha$, $\gamma$, and $\sigma$ be positive reals with $\alpha \ge 1$ such that*

$$\gamma^k a_k \le \begin{cases} \alpha^r\gamma^m\sigma & \text{for } k \le m-1; \\ \gamma^m\sigma & \text{for } k \ge m, \end{cases} \tag{3.1}$$

*let $0 < \eta \le 1$, $h = \eta\frac{\gamma}{rn}$, and for $j = 0,\dots,r-1$ let $\xi_j = j \cdot h$. Further, let $\tau > 0$, and for $j = 0,\dots,r-1$ let $\hat{y}_j \in \mathbb{Q}$ such that $|\hat{y}_j - y_j| \le \tau y_j$. Then, taking the estimate*

$$\hat{c}_m = h^{-m}\sum_{i=0}^{r-1} b_{im}^{(r)}\hat{y}_i,$$

*we have*

$$|\hat{c}_m - c_m| \le \frac{\sigma}{\eta^m}\binom{n}{m}\Big((2\alpha)^r e\tau(rm)^m + \frac{2\eta^r}{7!}\Big). \tag{3.2}$$

*Proof.* It follows from the choice of interpolation nodes and from (3.1) that

$$\begin{aligned} y_j = |p(\xi_j)| \le |p(\xi_r)| &= \sum_{i=0}^{n} c_i(rh)^i = \sum_{i=0}^{n} c_i\eta^i\gamma^i n^{-i} \\ &\le \sigma\gamma^m\alpha^r\sum_{i=0}^{n} n^{-i}\binom{n}{i} \le \sigma\gamma^m\alpha^r\sum_{i=0}^{n}\frac{1}{i!} \le \sigma\gamma^m\alpha^r e. \end{aligned} \tag{3.3}$$

Now let

$$\tilde{c}_m = h^{-m} \sum_{i=0}^{r-1} b_{im}^{(r)} y_i.$$

Since $\binom{n}{m} \geq (\frac{n}{m})^m$, it follows from (1.8) and (3.3) that

$$\begin{aligned} |\tilde{c}_m - \hat{c}_m| &\leq h^{-m}\tau \max\{y_0, \ldots, y_{r-1}\} \cdot \sum_{j=0}^{r-1} |b_{jm}^{(r)}| \leq h^{-m}\tau\sigma\gamma^m\alpha^r e \cdot 2^r \\ &\leq (2\alpha)^r \tau\eta^{-m}\sigma e(rm)^m \binom{n}{m}. \end{aligned} \tag{3.4}$$

Now,

$$h^m \tilde{c}_m = \sum_{j=0}^{r-1} b_{jm}^{(r)} y_j = \sum_{i=0}^{n} c_i h^i \sum_{j=0}^{r-1} b_{jm}^{(r)} j^i = c_m h^m + \sum_{i=r}^{n} c_i h^i \sum_{j=0}^{r-1} b_{jm}^{(r)} j^i.$$

Since $r \geq 4m + 7$, whence $7!\, r^{2m} \leq r!$, we obtain, with the aid of (1.9) and (3.1),

$$\begin{aligned} |\tilde{c}_m - c_m| &= h^{-m} \left| \sum_{i=r}^{n} c_i h^i \sum_{j=0}^{r-1} b_{jm}^{(r)} j^i \right| \leq h^{-m} \left| \sum_{i=r}^{n} c_i h^i r^i \right| \\ &= \eta^{-m}\gamma^{-m}(rn)^m \sum_{i=r}^{n} c_i \gamma^i \eta^i n^{-i} \leq \eta^{r-m}\sigma(rn)^m \sum_{i=r}^{n} \binom{n}{i} n^{-i} \\ &\leq \eta^{r-m}\sigma(rm)^m \binom{n}{m} \sum_{i=r}^{n} \frac{1}{i!} \leq \eta^{r-m}\sigma \binom{n}{m} \frac{2}{7!}. \end{aligned} \tag{3.5}$$

Clearly, (3.4) and (3.5) yield the asserted inequality (3.2). □

The next lemma will allow us to reduce the general case of mixed volume computation to the case of proper convex bodies. We use the notation of Theorem 11.

LEMMA 6. *Let $k \in \mathbb{N}$, $k \leq s$, and suppose $\mathcal{A}_k$ is a polynomial-time randomized algorithm that performs the task stated in Theorem* 11 *under the additional assumption that $K_1, \ldots, K_k$ are proper (while $K_{k+1}, \ldots, K_s$ may be improper). Then there exists a polynomial-time randomized algorithm $\mathcal{A}_{k-1}$ that performs the same task under the assumption that $K_1, \ldots, K_{k-1}$ are proper.*

*Proof.* Let $K_1, \ldots, K_s \in \mathcal{K}^n$ be well presented, let $K_1, \ldots, K_{k-1}$ be proper, and suppose we want to compute the mixed volume

$$v = V(\overbrace{K_1, \ldots, K_1}^{m_1}, \ldots, \overbrace{K_s, \ldots, K_s}^{m_s}).$$

Let us first use Theorem 8 to determine whether $v = 0$. If this is the case, we are done. So suppose that $v \neq 0$. Then, of course, there is a fixed integer polynomial $\pi$ in the size $\Lambda$ of the input such that

$$v \geq 2^{-\pi(\Lambda)}.$$

Now, consider for $0 \leq \delta \leq 1$ the mixed volume

$$p(\delta) = V(\overbrace{K_1, \ldots, K_1}^{m_1}, \ldots, \overbrace{K_{k-1}, \ldots, K_{k-1}}^{m_{k-1}}, \overbrace{K_k + \delta\mathbb{B}^n, \ldots, K_k + \delta\mathbb{B}^n}^{m_k}, \overbrace{K_{k+1}, \ldots, K_{k+1}}^{m_{k+1}}, \ldots, \overbrace{K_s, \ldots, K_s}^{m_s}).$$

Clearly,

$$p(\delta) = \sum_{i=0}^{m_k} \binom{m_k}{i} p_i \delta^i,$$

where

$$p_i = V(\overbrace{K_1, \ldots, K_1}^{m_1}, \ldots, \overbrace{K_k, \ldots, K_k}^{m_k - i}, \ldots, \overbrace{K_s, \ldots, K_s}^{m_s}, \overbrace{\mathbb{B}^n, \ldots, \mathbb{B}^n}^{i}),$$

and $p_0 = p(0) = v$. Let $R \in \mathbb{N}$ such that $K_1, \ldots, K_s \subset R[-1,1]^n$. Note that such a bound is part of the input. Then we have

$$p(\delta) - v = \delta \sum_{i=1}^{m_k} \binom{m_k}{i} p_i \delta^{i-1} \leq \delta (2R)^n \sum_{i=1}^{m_k} \binom{m_k}{i} \leq \delta (4R)^n.$$

Let $\epsilon \in \mathbb{Q}$ with $0 < \epsilon \leq 1$ be given; set

$$\delta_0 = \frac{\epsilon}{3(4R)^n 2^{\pi(\Lambda)}} \quad \text{and} \quad \tau = \frac{\epsilon}{3}.$$

From the given well-presentation of $K_k$ we can easily derive in polynomial time a well-presentation of $K_k' = K_k + \delta \mathbb{B}^n$; hence we can apply $\mathcal{A}_k$ to the bodies

$$K_1, \ldots, K_{k-1}, K_k', K_{k+1}, \ldots, K_s.$$

We call $\mathcal{A}_k$ with error parameter $\tau$ to compute an approximation $\hat{p}$ of $p = p(\delta_0)$ to relative error $\tau$. We take $\hat{v} = \hat{p}$ as an approximation of $v$, and obtain

$$\left| \frac{\hat{v} - v}{v} \right| = \left| \frac{\hat{p} - v}{p} \right| \left| \frac{p}{v} \right| \leq \left( \frac{|\hat{p} - p|}{p} + \frac{|p - v|}{p} \right) \frac{p}{v} \leq \tau \frac{p}{v} + \frac{p - v}{v} = \tau + (\tau + 1) \frac{p - v}{v}$$
$$\leq \tau + 2\delta_0 (4R)^n 2^{\pi(\Lambda)} \leq \frac{\epsilon}{3} + 2\frac{\epsilon}{3} = \epsilon.$$

Hence $\hat{v}$ is the desired approximation of $v$ to relative error $\epsilon$. □

**3.2. Mixed volumes of two proper bodies.** We will now describe a randomized algorithm for computing the mixed volumes $a_0, \ldots, a_k$ of two proper convex bodies $K_1$ and $K_2$ of $\mathbb{R}^n$ recursively, where $k \leq \psi(n)$, with

$$\psi(n) \leq n \quad \text{and} \quad \psi(n) \log \psi(n) = o(\log n).$$

We use Proposition 1 to compute relative estimates of $\mathrm{vol}_n(K_1 + xK_2)$ for suitable choices of nonnegative rational $x$. In particular, $a_0 = \mathrm{vol}_n(K_1)$ is already (approximately) available. For the inductive step suppose that, for some $m \in \{1, \ldots, \psi(n)\}$, estimates $\hat{a}_0, \ldots, \hat{a}_{m-1}$ of the mixed volumes $a_0, \ldots, a_{m-1}$, respectively, have already been obtained to relative error, say $\frac{1}{10}$.

By Lemma 4 we may assume that

$$1 \leq q_m = \frac{a_{m-1}}{a_m} \leq (n+1)^{8m},$$

since the transformation underlying Lemma 4 changes $a_m$ by a constant factor and does not affect relative approximation. Clearly,

$$\frac{10}{11}(n+1)^{-8m} \hat{a}_{m-1} \leq a_m \leq \frac{10}{9} \hat{a}_{m-1},$$

and this gives a first $\sqrt{(11/9)}(n+1)^{4m}$ approximation of $a_m$.

The main routine is divided into two parts. First, we apply a search technique to improve the above approximation of $q_m$ to constant error. Then we run a similar procedure to obtain the required approximation of $\alpha_m$ to relative error $\epsilon$.

1. Search procedure: Set $q_0 = 1$, and let

$$\gamma_0 = \begin{cases} 1 & \text{if } m = 1; \\ \max\left\{\frac{9\hat{a}_{m-2}}{11\hat{a}_{m-1}}, 1\right\} & \text{otherwise.} \end{cases}$$

Now, note that

$$\frac{11\hat{a}_{m-2}}{9\hat{a}_{m-1}} \geq q_{m-1} \geq \frac{9\hat{a}_{m-2}}{11\hat{a}_{m-1}} \quad \text{for } m \geq 2$$

and that the Aleksandrov–Fenchel inequality (1.2) implies that $q_m \geq \max\{q_{m-1}, 1\}$. This yields, for $\gamma = \gamma_0$,

$$q_m \geq \gamma \geq \left(\frac{9}{11}\right)^2 q_{m-1} \geq \frac{2}{3} q_{m-1}. \tag{3.6}$$

In the $k$th iteration of our search procedure we have $\gamma = \gamma_k$, also satisfying (3.6), hence,

$$\sigma_k := \frac{a_{m-1}}{\gamma_k} \geq \frac{a_{m-1}}{q_m} = a_m.$$

Now the Aleksandrov–Fenchel inequality implies that $\gamma_k \leq q_j$ for all $j \geq m$, hence, inductively,

$$\gamma_k^j a_j \leq \gamma_k^{m-1} a_{m-1} = \gamma_k^m \sigma_k \quad \text{for all } j \geq m.$$

Similarly for $j \leq m-2$, we deduce from $q_{j+1}, \ldots, q_{m-1} \leq \frac{3}{2}\gamma$ that

$$\gamma_k^j a_j \leq \left(\frac{3}{2}\right)^{m-j-1} \gamma_k^{m-1} a_{m-1} \leq \left(\frac{3}{2}\right)^r \gamma_k^m \sigma_k \quad \text{for all } j \leq m-2,$$

whenever $r \geq m$. Let us choose $r \geq 4m + 7$ such that $r = O(\psi)$. Now we apply Lemma 5 with the parameters

$$\gamma = \gamma_k, \quad \sigma = \sigma_k, \quad \alpha = \frac{3}{2}, \quad \eta = 1, \quad \text{and} \quad \tau = \frac{1}{20 \cdot 3^{r+1}(rm)^m}.$$

So, using the volume algorithm of Proposition 1 with interpolation nodes $\xi_j = j \cdot h$ for $j = 0, \ldots, r-1$, where $h = h_k = \frac{\gamma_k}{rn}$, and error bound $\tau$ we obtain relative estimates $\hat{y}_j \in \mathbb{Q}$ of $y_j = p(\xi_j)$ that can be used to produce in polynomial time an estimate $\hat{c}_m$ of $c_m$ that satisfies (3.2), whence

$$|\hat{c}_m - c_m| \leq \sigma_k \binom{n}{m} \left(\frac{1}{20} + \frac{2}{7!}\right) \leq \sigma_k \binom{n}{m} \frac{1}{19}.$$

But then we have for

$$s_k = \hat{c}_m / \binom{n}{m},$$

$$|s_k - a_m| \leq \frac{1}{19}\sigma_k.$$

Now, if

$$s_k \leq \frac{4 \cdot 10}{9 \cdot 11} \frac{\hat{a}_{m-1}}{\gamma} \leq \frac{4}{9}\sigma_k,$$

then

$$a_m \le \left(\frac{4}{9}+\frac{1}{19}\right)\sigma_k < \frac{\sigma_k}{2},$$

whence $\gamma_k < \frac{q_m}{2}$. So $\gamma_{k+1} = 2\gamma_k$ still satisfies (3.6), and we can repeat the above procedure with $\gamma_{k+1}$.

Note that $\sigma_0 \le a_{m-1}$ and $\sigma_{k+1} = \frac{1}{2}\sigma_k$; this implies that after at most $8m\log(n+1)$ iterations the process stops with a $\hat{\gamma} \in \mathbb{Q}$ such that

$$s_k \ge \frac{4\hat{a}_{m-1}}{10\hat{\gamma}} \ge \frac{4}{11}\sigma_k.$$

This implies that

$$\sigma_k \ge a_m \ge \left(\frac{4}{11}-\frac{1}{19}\right)\sigma_k \ge \frac{1}{4}\sigma_k, \tag{3.7}$$

hence $q_m/4 \le \hat{\gamma} \le q_m$. Note that $10\hat{a}_{m-1}/(11\hat{\gamma})$ is already a 4-approximation of $a_m$.

2. Approximation: Now that we know $q_m$ approximately, we are able to compute $\hat{a}_m$, the desired approximation of $a_m$ to the relative error $\epsilon$. We assume that $0 < \epsilon \le 1$ and choose a positive rational $\eta_0$ of size that is bounded by the size of the input such that

$$\eta_0 \le \epsilon^{\frac{1}{r-m}}.$$

As before, we apply Lemma 5, this time with the parameters

$$\gamma = \hat{\gamma}, \quad \sigma = \sigma_k, \quad \alpha = \frac{3}{2}, \quad \eta = \eta_0, \quad \text{and} \quad \tau = \frac{\eta_0^r}{15\cdot 3^r (rm)^m}.$$

Then we use again the volume algorithm of Proposition 1, now with interpolation nodes $\xi_j = j\cdot h$ for $j = 0,\dots,r-1$, where $h = \frac{\hat{\gamma}}{rn}\eta_0$, and error bound $\tau$, and we obtain relative estimates $\hat{y}_j \in \mathbb{Q}$ of $y_j = p(\xi_j)$ that lead in polynomial time to an estimate $\hat{c}_m$ of $c_m$ that satisfies (3.2), whence

$$|\hat{c}_m - c_m| \le \sigma_k \eta_0^{r-m}\binom{n}{m}\left(\frac{e}{15}+\frac{2}{7!}\right) \le \binom{n}{m}\frac{\epsilon}{4}\sigma_k.$$

In conjunction with (3.7) this yields

$$|\hat{a}_m - a_m| \le \frac{\epsilon}{4}\sigma_k \le \epsilon a_m$$

for $\hat{a}_m = \hat{c}_m/\binom{n}{m}$ as required.

As for its running time, under the stated assumptions on $\psi$ the algorithm uses

$$O(rm\log(n+1)) = o(\log^3 n) \quad \text{calls to the volume estimator}$$

with error, where

$$\frac{1}{\tau} = O(3^r r^{2m}) = n^{o(1)} \quad \text{in the first part,}$$
$$\frac{1}{\tau} = \left(\frac{1}{\epsilon}\right)^{1+o(1)} n^{o(1)} \quad \text{in the final step.}$$

It follows that the algorithm is polynomial. Note that the running time is only marginally worse than the running time of the volume estimator. In fact, suppose that the volume algorithm (after rounding) has complexity

$$O\left(\frac{1}{\epsilon^k} n^l \log\left(\frac{1}{\beta}\right)\right).$$

Then the running time of our mixed volume algorithm is bounded by

$$O\left(\frac{1}{\epsilon^{k+o(1)}} n^{l+o(1)} \log\left(\frac{1}{\beta}\right)\right).$$

Since by Lemma 6 the initial assumption that $K_1$ and $K_2$ are proper is irrelevant, we have completed the proof of Theorem 10.

**3.3. Extension to more than two bodies.** Now we extend the algorithm to the case of more than two bodies, thus proving Theorem 11. So, let us consider approximating

$$V(\overbrace{K_1,\ldots,K_1}^{m_1},\overbrace{K_2,\ldots,K_2}^{m_2},\ldots,\overbrace{K_{s-1},\ldots,K_{s-1}}^{m_{s-1}},\overbrace{K_s,\ldots,K_s}^{m_s}),$$

where $\sum_{i=1}^{s} m_i = n$. We may assume again that $K_1,\ldots,K_s$ are proper.

Suppose, recursively, we have an approximation procedure whenever only $s-1$ different bodies occur with $3 \le s \le n$. We want to extend it then to all $s$ bodies by considering the polynomial

$$q(x) = \sum_{k=0}^{m} \binom{m}{k} a_k x^k,$$

for $m = m_{s-1} + m_s$, where

$$a_k = V(\overbrace{K_1,\ldots,K_1}^{m_1},\ldots,\overbrace{K_{s-2},\ldots,K_{s-2}}^{m_2},\overbrace{K_{s-1},\ldots,K_{s-1}}^{m-k},\overbrace{K_s,\ldots,K_s}^{k}),$$

and using estimates of

$$q(\xi_j) = V(\overbrace{K_1,\ldots,K_1}^{m_1},\ldots,\overbrace{K_{s-2},\ldots,K_{s-2}}^{m_2},\overbrace{K_{s-1}+\xi_j K_s,\ldots,K_{s-1}+\xi_j K_s}^{m})$$

for suitably chosen interpolation points $\xi_0,\ldots,\xi_m$. Note that for the coefficients $a_k$ we still have the Aleksandrov–Fenchel inequality (1.2).

Suppose, now, that $\psi$ satisfies the condition given in Theorem 11, and let $m_1 \ge n - \psi(n)$. Then the degrees of the corresponding mixed volume polynomials are bounded above by $\psi(n)$. This allows us to simplify the procedure by using the whole coefficient matrix. There is, however, one additional difficulty now. We do not have a polynomial-time procedure for obtaining a "good" initial scaling of the sets anymore (as Lemma 4 for $s = 2$) such that the ratio $a_{m-1}/a_m$ of successive coefficients in the polynomial is suitably bounded. We leave it as an open question whether there is an analogue of Lemma 4 for $s \ge 3$ (with a bound that is independent of the well-presentedness parameters $\rho_i, R_i$). However, the input yields vectors $b_i$ and numbers $\rho, R$ such that for each $K_i$ we have

$$b_i + \rho\mathbb{B}^n \subset K_i \subset b_i + R\mathbb{B}^n,$$

and we may assume without loss of generality that all $b_i$ are 0. Using the monotonicity of mixed volumes we obtain

$$\frac{\rho}{R} \le \frac{a_{m-1}}{a_m} \le \frac{R}{\rho},$$

and this implies that the number of iterations in the binary search part of the procedure is bounded by $O\big(\log(R/\rho)\big)$. Moreover it follows that for each $\xi \ge 0$,

$$\rho(1+\xi)\mathbb{B}^n \subset K_{s-1} + \xi K_s \subset R(1+\xi)\mathbb{B}^n.$$

Hence, we have an additional factor $\log(R/\rho)$ as part of the input to the volume approximator. With these modifications, Theorem 11 is just a corollary to Theorem 10.

**4. Related problems and applications.** The present section contains various applications of our results to problems in discrete mathematics, combinatorics, computational convexity, algebraic geometry, geometry of numbers, and operations research.

**4.1. Counting integer points in integer polytopes.** A polytope is called *integer* if all vertices are integer vectors. We denote by $\mathcal{P}^n(\mathbb{Z})$ the set of all integer polyopes of $\mathbb{R}^n$. The lattice point enumerator $G : \mathcal{P}^n(\mathbb{Z}) \to \mathbb{N}$ is counting the number of lattice points of lattice polytopes $P$, i.e., $G(P) = |P \cap \mathbb{Z}^n|$. The following polynomial expansion of $G(kP)$ is due to [Eh67], [Eh77].

PROPOSITION 5. *There are functionals $G_i : \mathcal{P}^n(\mathbb{Z}) \to \mathbb{N}_0$ such that for every $P \in \mathcal{P}^n(\mathbb{Z})$ and $k \in \mathbb{N}$,*

$$G(kP) = \sum_{i=0}^{n} k^i G_i(P).$$

The polynomial on the right-hand side is often referred to as *Ehrhart-polynomial*; see [St86] for basic facts on this polynomial. In case of lattice zonotopes one can give an explicit formula for the Ehrhart-polynomial; this was used in [St91] to find a generating function for the number of *degree sequences* of simple $m$-vertex graphs. (In fact, there is a one-to-one correspondence between these degree sequences and the integer points of a suitable zonotope.) The functionals $G_i$ have some interesting properties; see e.g., the survey [GW93]. They may be viewed as "discrete" analogues of the *quermassintegrals*

$$W_i(P) = V(\overbrace{P, \ldots, P}^{n-i}, \overbrace{\mathbb{B}^n, \ldots, \mathbb{B}^n}^{i}).$$

What is particularly important for our purpose is the fact that $G_n(P)$ is just the volume of $P$. Hence, it follows from Proposition 5 that determining the number of integer points of an integer polytope (that is presented in any of the standard ways) is (at least) as hard as computing its volume. In fact, it is easy to obtain from any standard presentation of an integer polytope $P$ the same kind of presentation for $kP$ of size that is bounded by a polynomial in $size(P)$ and $size(k)$. Hence, if we had a polynomial-time procedure for determining the number of lattice points of an integer polytope, we could run the algorithm for each polytope $0 \cdot P, 1 \cdot P, \ldots, n \cdot P$, and we would then obtain $\mathrm{vol}_n(P) = G_n(P)$ by computing the leading coefficient of the Ehrhard-polynomial, just by solving the corresponding system of linear equations. Hence, we obtain the following $\#\mathbb{P}$-hardness result as a consequence of the hardness results for volume computation of [DF88] (for $\mathcal{V}$- and $\mathcal{H}$-polytopes) and of Theorem 1 (for $\mathcal{S}$-zonotopes).

THEOREM 12. *The problem of evaluating $G(P)$ is $\#\mathbb{P}$-complete for integer $\mathcal{V}$-, integer $\mathcal{H}$-polytopes, and for integer $\mathcal{S}$-zonotopes.*

Let us remark that in fixed dimension $G(P)$ can be computed in a polynomial time, [Ba94]; see also [DK97]. Note, further, that while this task is easy for $\mathcal{V}$-polytopes, deciding whether a given $\mathcal{H}$-polytope $P$ is an integer polytope is co$\mathbb{NP}$-complete; see [PY90]. For a survey of various other results on lattice point enumeration see [GW93].

**4.2. Some determinant problems and their relatives.** We will proceed by determining the complexity of the following determinant problems (using similar

methods as in the proof of Theorem 1), and then draw some consequences for a problem in computational convexity.

Let $\kappa$ be an integer constant. Then $\kappa$-DETERMINANT is the following decision problem: *given positive integers $n, s$ with $s \geq n$, and an integer $n \times s$-matrix $A$, is there an $n \times n$-submatrix $B$ of $A$ such that* $\det B = \kappa$?

$\#(\kappa$-DETERMINANT) asks for the number of different such matrices.

THEOREM 13. *The problem $\#(\kappa$-DETERMINANT) is $\#\mathbb{P}$-complete for any $\kappa \in \mathbb{Z}$; $\kappa$-DETERMINANT is $\mathbb{NP}$-complete for $\kappa \neq 0$.*

*Proof.* Clearly, the first problem is in $\#\mathbb{P}$ while the second is in $\mathbb{NP}$. To prove the hardness results, we use reductions from # SUBSET-SUM and SUBSET-SUM, respectively; see the proof of Theorem 1.

Let $(m; \alpha_1, \ldots, \alpha_m, \alpha)$ be an instance of SUBSET-SUM (or, equivalently, of its counting version), and define the matrix $A'_\delta$ as in the proof of Theorem 1, but with each $\alpha_j$ replaced by $\beta_j = (|\kappa| + 2)\alpha_j$ for $j = 1, \ldots, m$, $\alpha_{m+1} = -\alpha$ replaced by $\beta_{m+1} = -(|\kappa| + 2)\alpha + 1$, and $\delta = \kappa - 1$. It follows readily that for each maximal square submatrix $B_I$ of $A'_\delta$ whose determinant does not depend on $\delta$ we have

$$\det B_I \in \{0, \pm\beta_1, \ldots, \pm\beta_m, \pm\beta_{m+1}\}.$$

In particular, $\det B_I \neq \kappa$, unless $\kappa = 0$.

Now, suppose $\kappa \neq 0$. Recall from the proof of Theorem 1 that in the remaining cases the index sets $I$ of the matrices $B_I$ in the determinantal expansion of $\mathrm{vol}_n(Z_\delta)$ are in one-to-one correspondence with the subsets $J$ of $\{1, \ldots, m+1\}$ via

$$j \in J \Longleftrightarrow 2j - 1 \in I.$$

Further, it is easy to see that $\det B_I = \kappa$ implies $m+1 \in J$, and, hence, $\det B_I = \kappa$ if and only $J \setminus \{m+1\}$ is a solution of the given instance of SUBSET-SUM. This settles the problem for $\kappa \neq 0$.

Now, let $\kappa = 0$, and let us use the original matrix $A_\delta$ of the proof of Theorem 1. Among the $\binom{2m+3}{m+2} - 2^{m+1}$ subdeterminants for which $\det B_I$ is independent of $\delta$, we have for each $i = 1, \ldots, m+1$ exactly $2^m + m2^{m-1}$ subsets $I$ of $\{1, \ldots, 2m+3\}$ of cardinality $m+2$ such that $|\det B_I| = |\alpha_i|$, and all other cases give $\det B_I = 0$. Hence, with the choice of $\delta = 0$, the number of singular $(m+2) \times (m+2)$ submatrices would allow us to compute the number of subsets $J \subset \{1, \ldots, m+1\}$ for which $\sum_{i \in I} \alpha_i = 0$, and this is the number of solutions of #SUBSET-SUM. □

Let us point out that the (seemingly) similar problem of finding an $n \times n$ submatrix $B$ of maximal determinant of a given $n \times m$ matrix $A$ with entries in $\{0, \pm 1\}$ is also $\mathbb{NP}$-hard even for quite small classes of such matrices $A$; [see GKL95, Theorem 5.2]. Clearly, this problem is closely related to the problem of finding a largest (with respect to the volume) $n$-simplex in a $\mathcal{V}$-polytope, while the $\mathbb{NP}$-hardness result of Theorem 13 implies that, given a $\mathcal{V}$-polytope $P$ and a positive integer $\kappa$, finding an $n$-simplex $S$ with vertices at vertices of $P$ and $\mathrm{vol}_n(S) = \kappa$ is $\mathbb{NP}$-complete.

We remark that Theorem 13 stops short of proving that SINGULAR-SUBMATRIX, the case $\kappa = 0$ of $\kappa$-DETERMINANT, is also $\mathbb{NP}$-hard. To extend Theorem 13 to SINGULAR-SUBMATRIX would be interesting since, in a geometric context, the existence of singular submatrices corresponds to configurations which are not in general position. However, general position assumptions are made frequently and it would be useful to know whether these assumptions can be checked efficiently; see [CKM82] for some related results.

**4.3. Volume of zonotopes and mixture management.** The following mixture management problem from the oil industry is studied in [GV89]. A seller has

a stock of $m$ bins which contain a mixture of chemical substances. Suppose that for $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$, the $i$th bin contains a nonnegative rational $z_{ij}$ of volume units of chemical $j$. To satisfy the customer's demand of a special mixture of $b_j$ volume units of each chemical $j$, the seller takes a proportion $\lambda_i$ with $0 \le \lambda_i \le 1$ of volume from each container such that

$$b_j = \sum_{i=1}^{m} \lambda_i z_{ij} \quad \text{for all } j \in \{1,\dots,n\}.$$

Typically, the mixtures in each bin come with associated costs, and a linear programming approach is used to satisfy the customer's demand at minimum total cost. It is pointed out in [GV89], however, that this is not a reasonable optimality criterium if all bins have (approximately) the same costs, and this is the case for particular applications in the oil industry.

Therefore [GV89] suggest the following approach. Clearly, the zonotope

$$Z = \sum_{i=1}^{m} [0,1]z_i, \quad \text{where } z_i = (z_{i1},\dots,z_{in})^T \text{ for } i = 1,\dots,m,$$

describes all possible demands the seller can satisfy. Now, typically, there are many possibilities to satisfy a demand $b = (b_1,\dots,b_m)^T$, and the question is how to do it in such a way that "the widest possible variety of possible future demands" can still be satisfied. More precisely, [GV89] suggests choosing for each $b \in \mathbb{Q}^m$ a vector

$$l = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} \in L(b) := \left\{ \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} \in [0,1]^m : b = \sum_{i=1}^{m} \lambda_i z_i \right\}$$

such that after taking $\lambda_i$ volume proportions from bin $i$, respectively, the set

$$Z(l) := \sum_{i=1}^{m} [0,1](1-\lambda_i)z_i$$

of the remaining possible mixtures has maximal volume. The volume as objective functions is justified by the fact that if the seller has no information about the future demands, it is reasonable to assume that the future demand is uniformly distributed.

Note that the function $f(l) := \mathrm{vol}_n(Z(l))$ is a homogeneous polynomial in the $(1-\lambda_i)$'s and (due to the *Brunn–Minkowski theorem*) has nice analytic properties. But as we have seen in Theorem 1, evaluating $f(\lambda)$ is $\#\mathbb{P}$-hard. This means that, while intriguing, this approach is not practical for large numbers of bins. However, Proposition 1 suggests that a randomized variant of the algorithm presented in [GV89] may be worth considering.

**4.4. Two applications of mixed volumes in combinatorics.** Two interesting applications in combinatorics can be found in [St81].

For the first, suppose that $\mathcal{M}$ is a unimodular matroid of rank $n$ with representation $v_1, v_2, \dots, v_m \in \mathbb{R}^n$ over the reals; see [We76], [Wh87]. Let $S_1,\dots,S_s$ be a partition of $\{1,2,\dots,m\}$ into proper subsets, and let $t_1,\dots,t_s$ be nonnegative integers such that $\sum_{i=1}^{s} t_i = n$. Then the number of bases of $\mathcal{M}$ with $t_i$ elements in $S_i$ equals

$$\binom{n}{t_1,\dots,t_s} V(\overbrace{Z_1,\dots,Z_1}^{t_1},\dots,\overbrace{Z_s,\dots,Z_s}^{t_s}),$$

where $Z_i$ is the zonotope

$$Z_i = \sum_{j \in S_i} [0,1]v_j.$$

Note that the total number $b$ of bases of $\mathcal{M}$ can be computed easily because the corresponding matrix $A$ with columns $v_1, \ldots, v_m$ is unimodular and hence, by (2.1),

$$b = \sum_{I \in \mathcal{J}} |\det B_I| = \sum_{I \in \mathcal{J}} (\det B_I)^2 = \det(AA^T).$$

As we will see now, this polynomial-time computability is destroyed for mixed volumes of unimodularly generated zonotopes (unless $\#\mathbb{P} = \mathbb{P}$).

THEOREM 14. *The following problem is $\#\mathbb{P}$-hard:*

Instance: *$n, s \in \mathbb{N}$ and $m_1, \ldots, m_s \in \mathbb{N}$ such that $\sum_{i=1}^{s} m_i = n$, $\mathcal{S}$-zonotopes $Z_i = (n, s_i; c_i; z_{i,1}, \ldots, z_{i,s_i})$, for $i = 1, \ldots, s$ such that the $(n \times r)$-matrix $A$ with columns $z_{i,j}$ is unimodular, where $r = \sum_{i=1}^{s} s_i$.*

Task: *Compute the mixed volume*

$$V(\overbrace{Z_1, \ldots, Z_1}^{m_1}, \overbrace{Z_2, \ldots, Z_2}^{m_2}, \ldots, \overbrace{Z_s, \ldots, Z_s}^{m_s}).$$

*Proof.* We reduce the problem of computing the number of perfect matchings in bipartite graphs to the given problem. For $i = 1, 2$, let $V_i = \{v_{i1}, \ldots, v_{in}\}$, let $V_1 \cap V_2 = \emptyset$, let $E \subset \{\{v_{1,j}, v_{2,k}\} : j, k = 1, \ldots, n\}$, and set $V = V_1 \cup V_2$. Let us now consider the bipartite graph $G = (V, E)$. Since it can be checked in polynomial time whether $G$ admits a perfect matching, we may assume that the number of perfect matchings of $G$ is not 0. We add an additional vertex $v_{2,n+1}$ to $V_2$ and the edges $E_{n+1} = \{\{v_{1,j}, v_{2,n+1}\} : j = 1, \ldots, n\}$ to $E$ and obtain a new bipartite graph $G' = (V', E')$. The node-edge incidence-matrix $A'$ of $G'$ is totally unimodular. It has $2n + 1$ rows but is only of rank $2n$. So we delete the row that corresponds to the new vertex $v_{2,n+1}$, and we obtain a totally unimodular matrix $A''$ of rank $2n$ with $2n$ rows and $|E'| = |E| + n$ columns. The nonsingular $(2n) \times (2n)$-submatrices $B$ of $A''$ are in one-to-one correspondence with the spanning trees of $G'$. (Note that in the totally unimodular case $GF(2)$-singularity is equivalent to $\mathbb{R}$-singularity; see, e.g., [Sc86, section 21.1].) Now, we partition $E'$ into $E_{n+1}$ and the $n$ subsets $E_1, \ldots, E_n$ where $E_j$ is the set of those edges of $E$ which contain $v_{2,j}$ $(j = 1, \ldots, n)$. Further, for $j = 1, \ldots, n + 1$, let $Z_j$ be the zonotope that is generated by the column vectors of $A''$ that correspond to $E_j$. Then by (2.3), the mixed volume

$$\frac{(2n)!}{n!} V(Z_1, Z_2, \ldots, Z_n, \overbrace{Z_{n+1}, \ldots, Z_{n+1},}^{n})$$

is just the number of those spanning trees of $G'$ that contain all edges of $E_{n+1}$ and for $j = 1, \ldots, n$ exactly one edge of $E_j$.

It is easy to see that the spanning trees with this property are in one-to-one correspondence with the perfect matchings of $G$. □

For the second application let $P = \{p_1, p_2, \ldots, p_s, q_1, q_2, \ldots, q_{n-s}\}$ be a poset, and suppose that $p_1 < p_2 < \cdots < p_s$. For $j = 1, 2, \ldots, s$ let $N(i_1, i_2, \ldots, i_s)$ denote the number of linear extensions $\sigma$ of $P$ such that $\sigma(p_j) = i_j$; see [St86]. Then, with $i_0 = 0$ and $i_{s+1} = n + 1$,

$$N(i_1, i_2, \ldots, i_s) = (n - s)! V(\overbrace{K_0, \ldots, K_0}^{i_1 - i_0 - 1}, \ldots, \overbrace{K_s, \ldots, K_s}^{i_{s+1} - i_s - 1}),$$

where $K_j \subset \mathbb{R}^{n-s}$ $(j = 0, 1, \ldots, s)$ are the *order polytopes*, i.e., $x \in K_j$ if and only if for all $i = 1, 2, \ldots, n-s$,

$0 \le x_i \le 1$,
$x_i \le x_k$ if $q_i < q_k$ $(k = 1, 2, \ldots, n-s)$,
$x_i = 0$ if $j > 0$ and $q_i < p_j$,
$x_i = 1$ if $j < s$ and $q_i > p_{j+1}$.

These polytopes reflect the poset "between" $p_j$ and $p_{j+1}$ on the subset $\{q_1, \ldots, q_{n-s}\}$. By the Aleksandrov–Fenchel inequality (applied to in the case $s = 1$) it follows that $N(i)^2 \ge N(i-1)N(i+1)$ for $i = 1, \ldots, n-1$ and, hence, the sequence $N(1), \ldots, N(n)$ is unimodal. Observe that the evaluation of $N(i_1, i_2 \ldots, i_r)$ is $\#\mathbb{P}$-complete even when $s = 0$, [BW92]; in this case, $N$ is the number of linear extensions of the poset. It follows that computing the volume of $\mathcal{H}$-polytopes is $\#\mathbb{P}$-hard in the strong sense.

**4.5. An application of mixed volumes in algebraic geometry.** Let $S_1, S_2, \ldots, S_n$ be subsets of $\mathbb{Z}^n$, and consider a system $F = (f_1, \ldots, f_n)$ of Laurent polynomials in $n$ variables such that the exponents of the monomials in $f_i$ are in $S_i$ for all $i = 1, \ldots, n$. Suppose, further, that $F$ is *sparse* in that the number of monomials having nonzero coefficients is "small" as compared to the degree of the $f_i$. To fix the notation, let, for $i = 1, \ldots, n$,

$$f_i(x) = \sum_{q \in S_i} c_q^{(i)} x^q,$$

where $f_i \in \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$, and $x^q$ is an abbreviation for $x_1^{q_1} \cdots x_n^{q_n}$; $x = (x_1, \ldots, x_n)$ are the indeterminates and $q = (q_1, \ldots, q_n)$ the exponents. Further, let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Now, if the coefficients $c_q^{(i)}$ $(q \in S_i)$ are chosen "generically," the number $L(F)$ of distinct common roots of the system $F$ in $(\mathbb{C}^*)^n$ depends only on the *Newton polytopes* $P_i = \operatorname{conv} S_i$ of the polynomials (see [GKZ90]); more precisely,

$$L(F) = n! \cdot V(P_1, P_2, \ldots, P_n). \tag{4.1}$$

Moreover, if $F$ has less then $n!V(P_1, \ldots, P_n)$ distinct roots, there must exist a nonzero integer vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that the "homogenized" system

$$F_\alpha = (f_1^\alpha, \ldots, f_n^\alpha),$$

where

$$f_i^\alpha(x) = \sum_{q \in S_i^\alpha} c_q^{(i)} x^q, \qquad S_i^\alpha = \{q \in S_i : \alpha^T q = \min\{\alpha^T q : q \in S_i\}\}$$

has a root in $(\mathbb{C}^*)^n$. These results become more intuitive by noting that both sides of (4.1) are invariant under unimodular transformations of the exponent vectors and under translations by integer vectors. (Each translation of a set $S_i$ by a vector $p^{(i)}$ corresponds to a multiplication of $f_i$ with the monomial $x^{p^{(i)}}$.) Observe, further, that the Minkowki sum of the Newton polytopes $P_1, \ldots, P_n$ is the Newton polytope of the product of the corresponding polynomials whence both sides of the equation are also additive in each component.

The above theorem was first proved in [Be75]; see also [BZ88, Chapter 27].

A convex geometric approach (utilizing the above connections) was recently developed for computing the isolated solutions of sparse polynomial systems; see [HS95],

[VG95], and [Ro94]. The mixed volumes are determined by computing a "mixed subdivision" of the $P_i$ using lifting methods similar to those of [Sc94] stated in subsection 2.3. See also [GKZ90] and [GS93] for further results on Newton polytopes and [VC92], [PS93], [CE93], [CR91], [VVC94], [ER94], [EC95], [LRW96], [Ro94], [Ro97], and the papers quoted therein for further results on counting the roots of polynomial systems.

**Acknowledgment.** We are grateful to Mark Jerrum for providing the proof of Lemma 2.

## REFERENCES

[Al37] A.D. ALEKSANDROV, *On the theory of mixed volumes of convex bodies,* II. *New inequalities between mixed volumes and their applications*, Math. Sb. N.S., 2 (1937), pp. 1205–1238 (in Russian).

[Al38] A.D. ALEKSANDROV, *On the theory of mixed volumes of convex bodies,* IV. *Mixed discriminants and mixed volumes*, Math. Sb. N.S., 3 (1938), pp. 227–251 (in Russian).

[AS86] E.L. ALLGOWER AND P.M. SCHMIDT, *Computing volumes of polyhedra*, Math. of Comp., 46 (1986), pp. 171–174.

[AK90] D. APPLEGATE AND R. KANNAN, *Sampling and integration of near log-concave functions*, in Proc. 23rd ACM Symp. on Theory of Computing, ACM, New York, 1990, pp. 156–163.

[BF86] I. BÁRÁNY AND Z. FÜREDI, *Computing the volume is difficult*, in Proc. 18th ACM Symp. on Theory of Computing, ACM, New York, 1986, pp. 442–447. Reprinted in Discrete Comput. Geom., 2 (1987), pp. 319–326.

[Ba94] A.I. BARVINOK, *A polynomial time algorithm for counting integral points in polyhedra when the dimension is fixed*, Math. Oper. Res., 19 (1994), pp. 769–779.

[BZ65] I.S. BEREZIN AND N.P. ZHIDKOV, *Computing Methods*, Vol. 1, Pergamon Press, Oxford, 1965.

[Be75] D.N BERNSHTEIN, *The number of roots of a system of equations*, Funct. Anal. Appl., 9 (1975), pp. 183–185.

[Be92] U. BETKE, *Mixed volumes of polytopes*, Arch. Math., 58 (1992), pp. 388–391.

[BH93] U. BETKE AND M. HENK, *Approximating the volume of convex bodies*, Discrete Comput. Geom., 10 (1993), pp. 15–21.

[BF34] T. BONNESEN AND W. FENCHEL, *Theorie der konvexen Körper*, Springer-Verlag, Berlin, 1934 (in German); (reprinted: Chelsea, New York, 1948); *Theory of Convex Bodies*, BCS Associates, Moscow, Idaho, 1987, (in English).

[BW92] G. BRIGHTWELL AND P. WINKLER, *Counting linear extensions is #P-complete*, Order, 8 (1992), pp. 225–242.

[Br86] A. Z. BRODER, *How hard is it to marry at random? (On the approximation of the permanent)*, in Proc. 18th ACM Symp. on Theory of Computing, ACM, New York, 1986, pp. 50–58.

[BS83] R. A. BRUALDI AND H. SCHNEIDER, *Determinantal identities: Gauss, Schur, Cauchy, Sylvester, Kronecker, Jacobi, Binet, Laplace, Muir, and Cayley*, Linear Algebra Appl., 52–53 (1983), pp. 769–791.

[BZ88] YU. D. BURAGO AND V. A. ZALGALLER, *Geometric Inequalities*, Springer-Verlag, Berlin, 1988.

[CE93] J. CANNY AND I.Z. EMIRIS, *An efficient algorithm for the sparse mixed resultant*, in Proc. 10th Intl. Symp. Appl. Algebra, Algebraic Alg., Error-Corr. Codes, Lecture Notes in Comput. Sci. 263, Springer-Verlag, New York, 1993, pp. 89–104.

[CR91] J. CANNY AND J.M. ROJAS, *An optimality condition for determining the exact number of roots of a polynomial system*, in Proc. ACM Intl. Symp. Algebraic Symbolic Comput., Bonn, Germany, ACM, New York, 1991, pp. 96-102.

[CH79] J. COHEN AND T. HICKEY, *Two algorithms for determining volumes of convex polyhedra*, J. Assoc. Comput. Mach., 26 (1979), pp. 401–414.

[CY77] K.C. CHUNG AND T.H. YAO, *On lattices admitting unique Lagrange interpolation*, SIAM J. Numer. Anal., 14 (1977), pp. 735–743.

[CKM82] R. CHANDRASEKARAN, S.N. KABADI, AND K.G. MURTY, *Some NP-complete problems in linear programming*, Oper. Res. Lett., 1 (1982), pp. 101–104.

[Va79] L.G. VALIANT, *The complexity of enumeration and reliability problems*, SIAM J. Comput., 8 (1979), pp. 410–421.

[VC92] J. VERSCHELDE AND R. COOLS, *Nonlinear reduction for solving deficient polynomial systems by continuation methods*, Numer. Math., 63 (1992), pp. 263–282.

[VG95] J. VERSCHELDE AND K. GATERMANN, *Symmetric Newton polytopes for solving sparse polynomial systems*, Adv. Appl. Math, 16 (1995), pp. 95–127.

[VVC94] J. VERSCHELDE, P. VERLINDEN, AND R. COOLS, *Homotopies exploiting Newton polytopes for solving sparse polynomial systems*, SIAM J. Numer. Anal., 31 (1994), pp. 915–930.

[We76] D.J.A. WELSH, *Matroid Theory*, Academic Press, London, 1976.

[Wh87] N. WHITE, *Unimodular matroids*, in Combinatorial Geometries, N. White, eds., Cambridge University Press, Cambridge, MA, 1987, pp. 40–52.

[KT93] L.G. KHACHIYAN AND M.J. TODD, *On the complexity of approximating the maximal inscribed ellipsoid for a polytope*, Math. Programming, 61 (1993), pp. 137–159.

[La91] J. LAWRENCE, *Polytope volume computation*, Math. Comput., 57 (1991), pp. 259–271.

[LRW96] T.Y. LI, J.M. ROJAS, AND X. WANG, *Counting affine roots of polynomial systems via pointed Newton polytopes*, J. Complexity, 12 (1996), pp. 116–133.

[Lo95] L. LOVÁSZ, *Random walks on graphs: A survey*, in Combinatorics: Paul Erdös is 80, Vol. 2, D. Miklós, V. T. Sós and T. Szönyi, eds., Bolyai Soc. Math. Stud. 2, János Bolyai Math. Soc., Budapest, 1995, pp. 353–397.

[LS90] L. LOVÁSZ AND M. SIMONIVITS, *The mixing rate of Markov chains, an isoperimetric inequality and computing the volume*, in Proc. IEEE 31st Annual Symp. Found. Comput. Sci., IEEE Computer Society Press, Los Alamitos, CA, 1990, pp. 364–355.

[LS93] L. LOVÁSZ AND M. SIMONIVITS, *Random walks in a convex body and an improved volume algorithm*, Random Structures Algorithms, 4 (1993), pp. 359–412.

[Mc70] P. MCMULLEN, *The maximum number of faces of a convex polytope*, Mathematika, 17 (1970), pp. 179–184.

[MM85] G. MIEL AND R. MOONEY, *On the condition number of Lagrangian numerical differentiation*, Appl. Math. Comput., 16 (1985), pp. 241–252.

[Mi11] H. MINKOWSKI, *Theorie der konvexen Körper, insbesondere Begründung ihres Oberflächenbegriffs*, in Collected Works Vol. II, Leipzig, Berlin, 1911, pp. 131–229.

[Mo89] H. L. MONTGOMERY, *Computing the volume of a zonotope*, Amer. Math. Monthly, 96 (1989), p. 431.

[Ol86] C. OLMSTED, *Two formulas for the general multivariate polynomial which interpolates a regular grid on a simplex*, Math. Comput., 47 (1986), pp. 275–284.

[PY90] C.H. PAPADIMITRIOU AND M. YANNAKAKIS, *On recognizing integer polyhedra*, Combinatorica, 10 (1990), pp. 107–109.

[PS93] P. PEDERSEN AND B. STURMFELS, *Product formulas for sparse resultants*, Math. Z., 214 (1993), pp. 377–396.

[Ri75] T.J. RIVLIN, *Optimally stable Lagrangian numerical differentiation*, SIAM J. Numer. Anal., 12 (1975), pp. 712–725.

[Ri90] T.J. RIVLIN, *Chebyshev Polynomials. From Approximation Theory to Algebra and Number Theory*, 2nd ed., John Wiley, New York, 1990.

[Ro94] J.M. ROJAS, *A convex geometric approach to counting the roots of a polynomial system*, Theoret. Comput. Sci., 133 (1994), pp. 105–140.

[Ro97] J.M. ROJAS, *Toric intersection for affine root counting*, J. Pure Appl. Math., 133 (1997), to appear.

[Sa74] H.E. SALZER, *Some problems in optimally stable Lagrangian differentiation*, Math. Comput., 28 (1974), pp. 1105–1115.

[Sa93] J.R. SANGWINE-YAGER, *Mixed volumes*, in Handbook of Convex Geometry, P.M. Gruber and J.M. Wills, eds., Elsevier, Amsterdam, 1993, pp. 43–72.

[Sc93] R. SCHNEIDER, *Convex Bodies: The Brunn-Minkowski Theory*, Encyclopedia of Mathematics and its Applications, Vol. 44, Cambridge University Press, Cambridge, MA, 1993.

[Sc94] R. SCHNEIDER, *Polytopes and Brunn-Minkowski theory*, in Polytopes: Abstract, Convex and Computational, T. Bisztriczky, P. McMullen, R. Schneider, and A. Ivic Weiss, eds., Kluwer, Boston, 1994, pp. 273–300.

[Sc86] A. SCHRIJVER, *Linear and Integer Programming*, Wiley-Interscience, New York, 1986.

[Sh74] G.C. SHEPHARD, *Combinatorial properties of associated zonotopes*, Canad. J. Math., 26 (1974), pp. 302–321.

[SJ89] A. SINCLAIR AND M. JERRUM, *Approximate counting, uniform generation and rapidly mixing Markov chains*, Inform. Comput., 82 (1989), pp. 93–133.

[St81] R.M. STANLEY, *Two combinatorial applications of the Aleksandrov-Fenchel inequalities*, J. Combin. Theory Ser. A, 17 (1981), pp. 56–65.

[St86] R.M. STANLEY, *Enumerative Combinatorics*, Vol. 1, Wadsworth & Brooks/Cole, Pacific Grove, CA, 1986.

[St91] R. STANLEY, *A zonotope associated with graphical degree sequences*, in Applied Geometry and Discrete Mathematics: The Victor Klee Festschrift, P. Gritzmann and B. Sturmfels, eds., Amer. Math. Soc. and Assoc. Comput. Mach., Providence, RI, 1991, pp. 555–570.

[Va77] L.G. VALIANT, *The complexity of computing the permanent*, Theoret. Comput. Sci., 8 (1977), pp. 189–201.

[DF88] M.E. DYER AND A.M. FRIEZE, *The complexity of computing the volume of a polyhedron*, SIAM J. Comput., 17 (1988), pp. 967–974.

[DF91] M.E. DYER AND A.M. FRIEZE, *Computing the volume of convex bodies: a case where randomness provably helps*, in Probabilistic Combinatorics and its Applications, Proceedings of Symposia in Applied Mathematics, Vol. 44, Béla Bollobás, ed., American Mathematical Society, Providence, RI, 1991, pp. 123–169.

[DFK91] M.E. DYER, A.M. FRIEZE, AND R. KANNAN, *A random polynomial time algorithm for approximating the volume of a convex body*, J. Assoc. Comput. Mach., 38 (1991), pp. 1–17.

[DK97] M.E. DYER AND R. KANNAN, *On Barvinok's algorithm for counting lattice points in fixed dimension*, Math. Oper. Res., 22 (1997), to appear.

[Ed70] J. EDMONDS, *Submodular functions, matroids, and certain polyhedra*, in Combinatorial Structures and their Applications, R. Guy, H. Hanani, N. Sauer, and J. Schönheim, eds., Gordon and Breach, New York, 1970, pp. 69–87.

[Eh67] E. EHRHART, *Sur un probléme de géometrie diophantienne linéaire*, J. Reine Angew. Math., 226 (1967), pp. 1–29; 227 (1967), pp. 25–49.

[Eh77] E. EHRHART, *Polynômes arithmétiques et méthode des polyédres en combinatoire*, Birkhäuser, Basel, 1977.

[EC95] I.Z. EMIRIS AND J.F. CANNY, *Efficient incremental algorithms for the sparse resultant and the mixed volume*, J. Symbolic Comput., 20 (1995), pp. 117–149.

[ER94] I.Z. EMIRIS AND A. REGE, *Monomial bases and polynomial system solving*, in Proc. 8th ACM Intl. Symp. Algebraic Symbolic Comput. '94, Oxford, UK, ACM, New York, 1994, pp. 114–122.

[Fe36] W. FENCHEL, *Inégalités quadratique entre les volumes mixtes des corps convexes*, C.R. Acad. Sci. Paris, 203 (1936), pp. 647–650.

[GJ79] M.R. GAREY AND D.S. JOHNSON, *Computers and Intractability*, W.H. Freeman, San Francisco, CA, 1979.

[GKZ90] I.M. GELFAND, M.M. KAPRANOV, AND A.V. ZELEVINSKY, *Newton polytopes and the classical resultant and discriminant*, Adv. Math., 84 (1990), pp. 237–254.

[GV89] D. GIRAD AND P. VALENTIN, *Zonotopes and mixture management*, in New Methods in Optimization and their Industrial Uses, J.P. Penot, ed., ISNM87, Birkhäuser, Basel, 1989, pp. 57–71.

[GK94] P. GRITZMANN AND V. KLEE, *On the complexity of some basic problems in computational convexity:* II. *Volume and mixed volumes*, in Polytopes: Abstract, Convex and Computational, T. Bisztriczky, P. McMullen, R. Schneider, and A. Ivic Weiss, eds., Kluwer, Boston, 1994, pp. 373–466.

[GKL95] P. GRITZMANN, V. KLEE, AND D. LARMAN, *Largest k-simplices in d-polytopes*, Discrete Comput. Geom., 13 (1995), pp. 477–515.

[GS93] P. GRITZMANN AND B. STURMFELS, *Minkowski addition of polytopes: computational complexity and applications to Gröbner bases*, SIAM J. Discrete Math., 6 (1993), pp. 246–269.

[GW93] P. GRITZMANN AND J.M. WILLS, *Lattice points*, in Handbook of Convex Geometry, P.M. Gruber and J.M. Wills, eds., Elsevier, Amsterdam, 1993, pp. 765–798.

[GLS88] M. GRÖTSCHEL, L. LOVÁSZ, AND A. SCHRIJVER, *Geometric Algorithms and Combinatorial Optimization*, Springer-Verlag, Berlin, 1988.

[HS95] B. HUBER AND B. STURMFELS, *A polyhedral method for solving sparse polynomial systems*, Math. Comput., 64 (1995), pp. 1541–1555.

[JS89] M. R. JERRUM AND A. J. SINCLAIR, *Approximating the permanent*, SIAM J. Comput., 18 (1989), pp. 1149–1178.

[JVV86] M. R. JERRUM, L. G. VALIANT, AND V.V. VAZIRANI, *Random generation of combinatorical structures from a uniform distribution*, Theoret. Comput. Sci., 43 (1986), pp. 169–188.

[Jo90] D.S. JOHNSON, *A catalog of complexity classes*, in Handbook of Theoretical Computer Science. vol. A: Algorithms and Complexity, J. van Leeuwen, ed., Elsevier and M.I.T. Press, Amsterdam and Cambridge, MA, 1990 , pp. 67–161.

[KLS97] R. KANNAN, L. LOVÁSZ, AND M. SIMONIVITS, *Random walks and an $O^*(n^5)$ volume algorithm for convex bodies*, Random Structures Algorithms, 11 (1997), pp. 1–96.

[KKLLL93] N. KARMARKAR, R. KARP, R. LIPTON, L. LOVÁSZ, AND M. LUBY, *A Monte-Carlo algorithm for estimationg the permanent*, SIAM J. Comput., 22 (1993), pp. 284–293.

[Kh93] L.G. KHACHIYAN, *Complexity of polytope volume computation*, in New Trends in Discrete and Computational Geometry, J. Pach, ed., Springer-Verlag, Berlin, 1993, pp. 91–101.