MINIMAL RESOLUTIONS OF IDEALS ASSOCIATED TO TRIANGULATED HOMOLOGY MANIFOLDS

by

John A. Eagon

The Roberts-Weyman bicomplex is useful in studying minimal resolutions over A of A/I where $A = k[x_1, \ldots, x_n]$, x_i an indeterminate, k a field, is graded in the usual way and I is a proper homogeneous ideal (see [Rob]). We present a description of this bicomplex and also a closely related bicomplex with which we construct minimal resolutions of I, when I is minimally generated by polynomials of the same degree. We present examples of this latter bicomplex where I is a square free monomial ideal. We associate to I a certain abstract finite simplicial complex, (not the usual "Stanley-Reisner" association) and note what happens when it is a triangulated homology n-manifold without boundary. We look at the classical case of 2-manifolds, with special reference to the well-known six vertex triangulation of the real projective plane, torus with 7 and Klein bottle with 8 vertices.

The bicomplexes turn out to be "partially split" in the sense of [Eag] and we use the machinery described in that paper.

I. The Roberts-Weyman bicomplex and a related bicomplex

Let I_q denote the k-vector space of homogeneous polynomials in I of (total) degree q and $(A/I)_q$ the k-vector space of cosets of I that can be represented by homogeneous polynomials of degree q, so that as vector spaces we have the short exact sequences for all q:

$$0 \to I_q \to A_q \to (A/I)_q \to 0$$

and $\oplus_q I_q = I$, $\oplus_q A_q = A$, $\oplus (A/I)_q = A/I$ as direct sums over k. Let $\bar{x}_i = x_i + I$ be the coset of I containing x_i in A/I. From now on all tensor products are over k, unless otherwise stated.

We now consider the ring $A \otimes A/I$. In this ring $x_1 \otimes 1 + 1 \otimes \bar{x}_1, x_2 \otimes 1 + 1 \otimes \bar{x}_2, \ldots, x_n \otimes |t| \otimes \bar{x}_n$, is a regular sequence. This is because $A \otimes A/I$ is isomorphic to $A/I[x_1, \ldots, x_n]$ and in $R[x_1 \ldots x_n]$, $x_1 + a_1, \ldots, x_n + a_n$ is a regular sequence for any R and any elements a_i. Thus the Koszul complex $\mathcal{K}$, corresponding to the above sequence over $A \otimes A/I$ has all its homology concentrated in degree 0, and

$$H_0(\mathcal{K}) \cong \frac{A \otimes A/I}{(x_1 \otimes 1 + 1 \otimes \bar{x}_1, x_2 \otimes 1 + 1 \otimes \bar{x}_2, \ldots)}$$

which is isomorphic to A/I over $A \otimes A/I$ under the isomorphism which takes the coset of $x_i \otimes 1$ to $-\bar{x}_1$. Thus $\mathcal{K}$ is a resolution of A/I over $A \otimes A/I$. We want a minimal resolution over A. The general result in [Eag] permits us to construct one via spectral sequences if we have bicomplexes with certain properties. The bicomplexes we shall present have these properties.

The modules of the above mentioned Koszul complex $\mathcal{K}$ may be represented as follows: Let V be a k-vector space of dimension n with selected basis denoted by $x_1, \ldots, x_n$. Then $\mathcal{K}_m = A \otimes A/I \otimes \wedge^m V$, where $\wedge^m V$ is the mth exterior power of V over k. We will denote the basis of $\wedge^m V$ corresponding to the basis $x_1, \ldots, x_n$ by "square free monomials" $\tau = x_{i_1} x_{i_2} \ldots x_{i_m}$, $i_1 < i_2 < \ldots < i_m$. The usual wedge symbols are omitted. Exterior multiplication is denoted by juxtaposition. We have the usual "face operators"

$$\partial_i : \wedge^m V \to \wedge^{m-1} V$$

$$\partial_i(\tau) = \left\{ \begin{array}{cc} \pm\tau/x_i, & x_i | \tau \\ 0, & x_i \not| \tau \end{array} \right\}$$

where the signs are chosen in any manner so that

$$\partial_i \partial_i = 0, \;\; \partial_i \partial_j + \partial_j \partial_i = 0.$$

Composition of operators is also denoted by juxtaposition. The maps $\partial : \mathcal{K}_m \to \mathcal{K}_{m-1}$ of $\mathcal{K}$ may be described as

$$\partial = \sum_{i=1}^{n} (x_i \otimes 1 + 1 \otimes x_i) \otimes \partial_i$$

where $x_i \otimes 1 + 1 \otimes x_i$ denotes multiplciation by that element of $A \otimes A$ on $A \otimes A/I$ and ∂_i is the ith face operator on $\wedge V$.

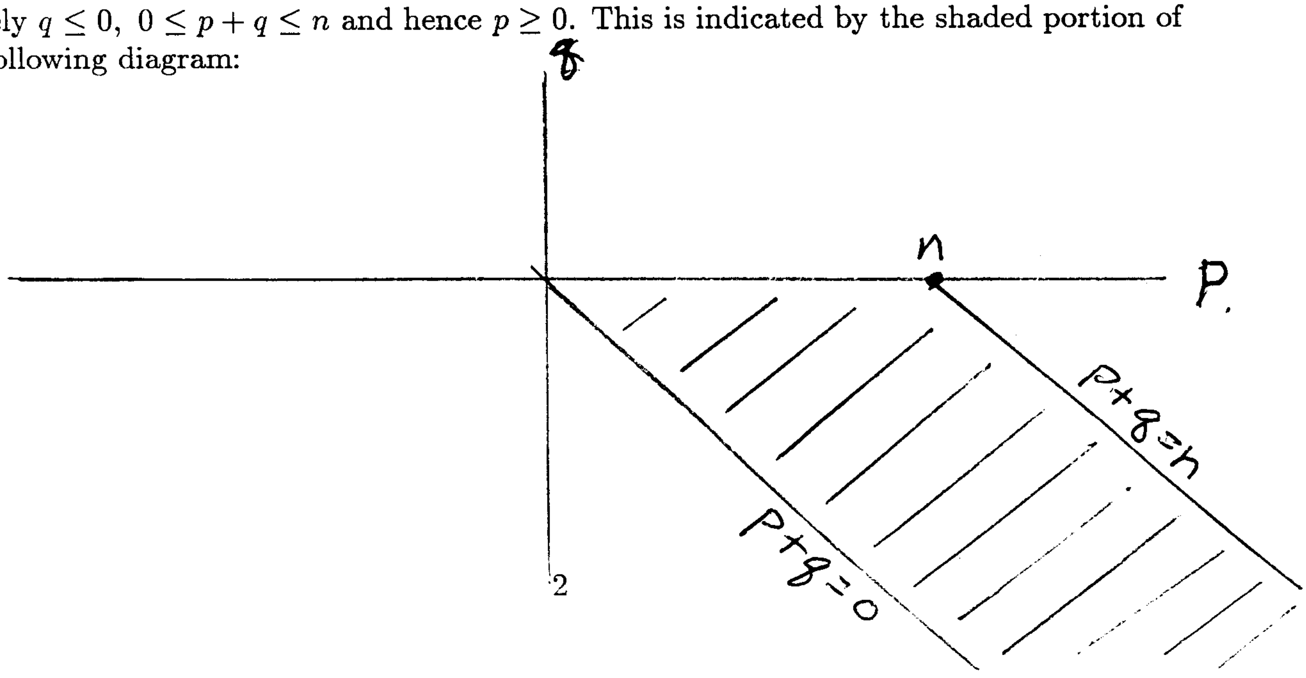
We now consider the following two doubly indexed arrays of vector spaces which are also A modules by left action for all integral values of p and q:

$$\mathcal{K}_{p,q} = A \otimes (A/I)_{-q} \otimes \wedge^{p+q} V$$

and

$$E_{p,q} = A \otimes I_{-q} \otimes \wedge^{p+q} V$$

Note that these arrays are non-zero only in a certain portion of the "fourth quadrant," namely $q \leq 0$, $0 \leq p + q \leq n$ and hence $p \geq 0$. This is indicated by the shaded portion of the following diagram:



Each of these arrays becomes a bicomplex (in the sense of [McL] page 340) by introduction of the A-module maps

$$d_0 = \sum_{i=1}^{n} 1 \otimes x_i \otimes \partial_i$$
$$d_1 = \sum_{i=1}^{n} x_i \otimes 1 \otimes \partial_i$$

e.g.

$$d_0 : A \otimes (A/I)_{-q} \otimes \wedge^{p+q} V \to A \otimes (A/I)_{-q+1} \otimes \wedge^{p+q-1} V$$

(We use the symbols d_0 and d_1 as in [Eag] instead of ∂' and ∂'' as in [McL].)

One checks easily that $Tot(\mathcal{K}_{\cdot,\cdot})$ is the Koszul complex $\mathcal{K}$ above, since clearly $\partial = d_0 + d_1$. Since the two bicomplexes are in the fourth quadrant the spectral sequences obtained by filtering each by "columns" is convergent (see [McL], Chapter XI, Prop. 3.2). The spectral sequences obtained by filtering by "rows" will be ignored throughout.

Thus the E^∞ term of $\mathcal{K}_{\cdot,\cdot}$ (denoted $\mathcal{K}^\infty$) has the property that $\mathcal{K}^\infty_{p,q} = 0$ unless $p+q = 0$ and $\mathcal{K}^\infty_{p,-p}$ provides a filtration of $H_o(Tot(\mathcal{K}_{\cdot,\cdot})) \cong A/I$.

Note that the map d_0 is the same as $1 \otimes d$ where d is the k-vector space map $\sum_{i=1}^{n} x_i \otimes \partial_i$. As explained in [Eag] there exist (many) k-maps

$$d^+ : (A/I)_{-q+1} \otimes \wedge^{p+q-1} V \to (A/I)_{-q} \otimes \wedge^{p+q} V$$

with the three properties

1.) $dd^+d = d$
2.) $d^+dd^+ = d^+$
3.) $(d^+)^2 = 0$

Similarly for $E_{\cdot,\cdot}$. Such a map d^+ is called a differential quasi-inverse of d. The three properties

1.) $d_0(1 \otimes d^+)d_0 = d_0$
2.) $(1 \otimes d^+)d_0(1 \otimes d^+) = 1 \otimes d^+$
3.) $(1 \otimes d^+)^2 = 0$

follow immediately, and thus both bicomplexes are partially split. By the main theorem of [Eag, page 346] the derived wall complex of $\mathcal{K}_{\cdot,\cdot}$ provides a minimal resolution of A/I over A. This is proved in [Rob]. A key fact is that $\mathcal{K}^1_{p,-p} = 0$ unless $p = 0$, and $\mathcal{K}^1_{0,-0} \cong A$ where $\mathcal{K}^1_{\cdot,\cdot}$ is the E^1 term of the above mentioned spectral sequence associated to $\mathcal{K}_{\cdot,\cdot}$. Thus the E^∞ term is non-zero only at $\mathcal{K}^\infty_{0,0}$ which implies that $\mathcal{K}^\infty_{0,0} \cong A/I$.

From this point on we will concentrate on the bicomplex $E_{\cdot,\cdot}$. As noted above it is partially split once d^+ is chosen. The needed properties of $Tot(E_{\cdot,\cdot})$ may be derived from those of $Tot(\mathcal{K}_{\cdot,\cdot})$. It is clear that for ech p, q we have exact sequences

$$0 \to E_{p,q} \to \bar{\mathcal{K}}_{p,q} \to \mathcal{K}_{p,q} \to 0$$

where $\bar{\mathcal{K}}_{p,q} = A \otimes A_{-q} \otimes \wedge^{p+q} V$. These are bicomplex maps with d_0 and d_1 described as before and $\bar{\mathcal{K}}_{\cdot,\cdot}$ corresponds to the ideal $I = 0$. Thus the homology of $Tot(\bar{\mathcal{K}}_{\cdot,\cdot})$ is concentrated in degree 0 with $H^0(Tot\bar{\mathcal{K}}_{\cdot,\cdot}) \cong A/(0) \cong A$ taking the Tot of these exact sequences we have

$$0 \to Tot(E_{\cdot,\cdot}) \to Tot(\bar{\mathcal{K}}_{\cdot,\cdot}) \to Tot(\mathcal{K}_{\cdot,\cdot}) \to 0$$

The corresponding long exact sequence on homology is all 0 except for degree 0 where it is $0 \to I \to A \to A/I \to 0$. Thus all the homology of $Tot(E_{\cdot,\cdot})$ is concentrated in degree 0 and $H^0(Tot(E_{\cdot,\cdot})) \cong I$. Thus the E^∞ term of the spectral sequence associated to $E_{\cdot,\cdot}$ denoted $E^\infty_{\cdot,\cdot}$ is 0 except for $E^\infty_{p,-p}$ which provides a filtration of I. We need to calculate $E^1_{p,-p}$. The appropriate part of $E_{\cdot,\cdot}$ is

$$\begin{array}{c} E_{p,-p+1} = A \otimes I_{p-1} \otimes \wedge' V \\ \downarrow 1 \otimes \sum_{i=1}^{n} x_i \otimes \partial_i \\ E_{p,-p} = A \otimes I_p \otimes \wedge^0 V \\ \downarrow \\ 0 \end{array}$$

and we need to see the deviation from exactness at $E_{p,-p}$ which turns out to be

$$A \otimes \frac{I_p}{(x_1, \ldots x_n) I_{p-1}} = E^1_{p,-p}$$

As we stated at the beginning, we will assume that I is minimally generated by polynomials of fixed degree b. This implies that $I_p = 0$ for $p < b$ and $I_p = (x_1, \ldots, x_n) I_{p-1}$ for $p > b$. Thus $E^1_{p,-p} = 0$ for $p \neq b$ and $E^1_{b,-b} \cong A \otimes I_b$. Thus the E^∞ term is 0 except $E^\infty_{b,-b} \cong I$. It then follows from [Eag] that the derived wall complex provides a minimal resolution of I. Note that the map

$$D = (I - dd^+ - d^+ d) \sum_{j=0}^{\infty} d_1 (d^+ d_1)^j$$

is only formally infinite because $(d^+ d_1)^j = 0$ for $j > n$.

II. Monomial Ideals and the Notion of Content.

From now on we will assume I is minimally generated by monomials of fixed degree b. In that case $I_{-q} \otimes \wedge^{p+q} V$ has a k basis of the form $\{m \otimes \tau\}$ where m is a monomial in I of degree $-q$ and τ is a square free monomial in $x_1, \ldots, x_n$ of degree $p+q$ (remember $q \leq 0$).

Note: The product (in A) $m\tau$ of these two monomials has degree $-q + p + q = p$. The k-span in $I_{-q} \otimes \wedge^{p+q} V$ of all $m \otimes \tau$ such that $m\tau = C$ for fixed C will be said to have <u>content</u> C, as will every element in this k-span. The submodule of $A \otimes I_{-q} \otimes \wedge^{p+q} V$

generated by this k-span will also be said to have content C, as will every element in this submodule.

The point is that

$$(\sum_{i=1}^{n} x_i \times \partial_i)(m \otimes \tau) =$$

$$= \sum_{x_i 1 \tau} \pm x_i m \otimes \tau / x_i + 0$$

and $x_i m \ \tau/x_i = m\tau = C$ so that the map d preserves content. Thus the pth column $E_{p,\cdot} = A \otimes I_{-\cdot} \otimes \wedge^{p+\cdot} V$ of $E_{\cdot,\cdot}$ is the direct sum as complexes over all monomials C of degree p of its submodules of content C. Let $\Omega_C = \{\tau | \tau$ is square free, $\tau | C$ and $C/\tau \in I\}$.

Then we have the following:

Remark 1. $C/\tau \otimes \tau$ has content C, $C/\tau \otimes \tau$ is a basis element of $I_{-q} \otimes \wedge^{p+q} V$ where $q = -deg(C/\tau)$ and $p = deg\tau - q = degC$, and C/τ is in I.

Remark 2. If $C/\tau \in I$ and $\mu | \tau$ then $C/\mu = C/\tau \cdot \tau/\mu \in I$ so Ω_C is an abstract finite simplicial complex.

Remark 3. $(\sum_{i=1}^{n} x_i \otimes \partial_i)(C/\tau \otimes \tau) = \sum_{i=1}^{n} (C/\tau) x_i \otimes \partial_i(\tau)$ and thus the subcomplex of $E_{p,\cdot}$ of content C is isomorphic to $Ch(\Omega_c)$, the augmented A-chains on the simplicial complex Ω_c, since $C/\tau \in I$ implies $(C/\tau)x_i \in I$. Since this is a direct sum of complexes we have the following:

Fact 1. We may construct d^+ on each subcomplex. For if we take the direct sum of such d^+'s the three equations of a differential quasi-inverse will still hold.

Fact 2. The homology of each column, from which the wall complex is constructed, will be the direct sum of the reduced homology $\tilde{H}(\Omega_C)$ of the simplicial complexes, over A. I.e.

$$\begin{aligned} W_m &\cong \bigoplus_{p+q=m} E^1_{p,q} \\ &\cong \bigoplus_{p+q=m} H_q(E_{p,\cdot}) \\ &\cong \bigoplus_{p+q=m} \ \bigoplus_{degC=p} \tilde{H}_{p+q-1}(\Omega_C) \\ &\cong \bigoplus_{C} \tilde{H}_{m-1}(\Omega_C) \text{ over } A \end{aligned}$$

where W_m is the mth module of the wall complex (see [Eag]).

Fact 3. For all but finitely many C, Ω_C is a cone and so $\tilde{H}(\Omega_C)$ vanishes identically. Indeed we have the following:

Proposition: Let $G(I)$ be the minimal set of monomial generators of I. If $C \neq \underset{G|C,\ G \in G(I)}{l.c.m.} G$ then Ω_C is a cone.

Proof: The hypothesis means that there is an indeterminate x_i whose exponent in C is larger than in any $G \in G(I)$ such that $G|C$. Let τ be a maximal face of Ω_C. Suppose $x_i \not| \tau$. $\exists G \in G(I)$ with $G|C/\tau \Rightarrow \tau G|C$. Then $x_i \tau G|C$ since x_i's exponent is larger in C than in τG. Thus $G|C/x_i\tau$ so $x_i\tau \in \Omega_C$ contradicting the maximality of τ. So x_i is in every maximal face so Ω_C is a cone.

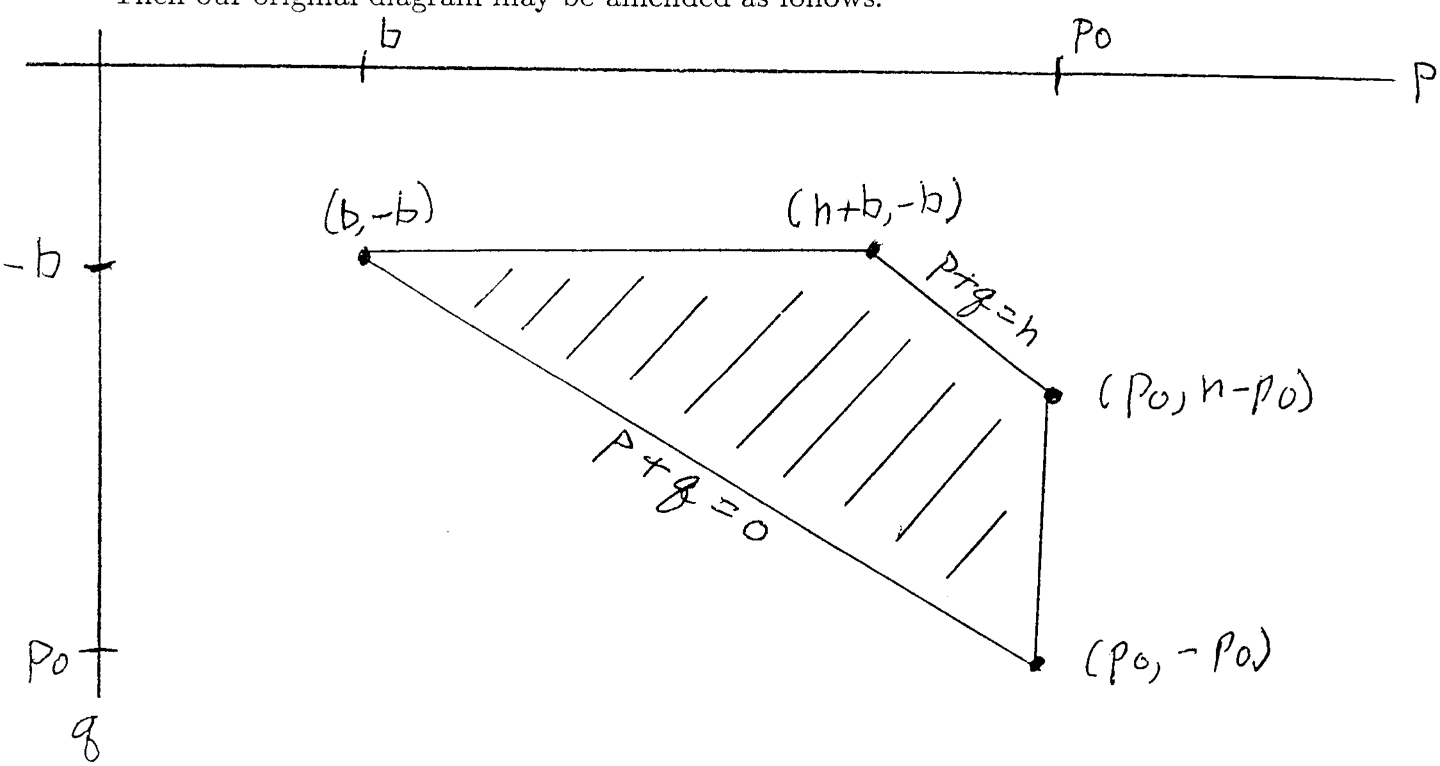
This fact means that $\tilde{H}(\Omega_C)$, the reduced homology is identically zero most of the time. But the modules of the Wall complex which is the minimal resolution of I as in [Eag] are formed from the E^1 term which is a direct sum of the above reduced homology.

When appropriate indices are kept track of, it turns out that all Ω_C are sub-simplicial complexes of Ω_{C_0} where C_0 is the least common multiple of all G in $G(I)$.

Thus if we let $C_0 = \underset{G \in G(I)}{l.c.m.G}$ we have $\tilde{H}(\Omega_C) \neq 0 \Rightarrow C = \underset{G \in S \subset G(I)}{l.c.m.G}$ so $C|C_0$. Thus $W_m \cong \bigoplus_C \tilde{H}_{m-1}(\Omega_C)$. The direct sum is over such C that $C = \underset{G \in S \subset G(I)}{l.c.m.G}$ for some subset $S \subset G(I)$. Furthermore, $d_1(a \otimes m \otimes \tau) = \sum_{i=1}^n x_i \otimes 1 \otimes \delta_i (a \otimes m \otimes \tau) = \sum_{i=1}^n \pm x_i a \otimes m \otimes \tau / x_i$. So the map d_1 takes elements of content C to elements of content dividing C. If we choose d^+ to preserve content as indicated above, then even for the map $D = (I - dd^+ - d^+d) \sum_{j=1}^\infty d_1(d^+d_1)^j$ we need only consider contents dividing C_0. Let $p_0 = degC_0$.

$$I_{-q} \neq 0 \Rightarrow -q \geq b \Rightarrow q \leq -b \Rightarrow p+q \leq p-b$$

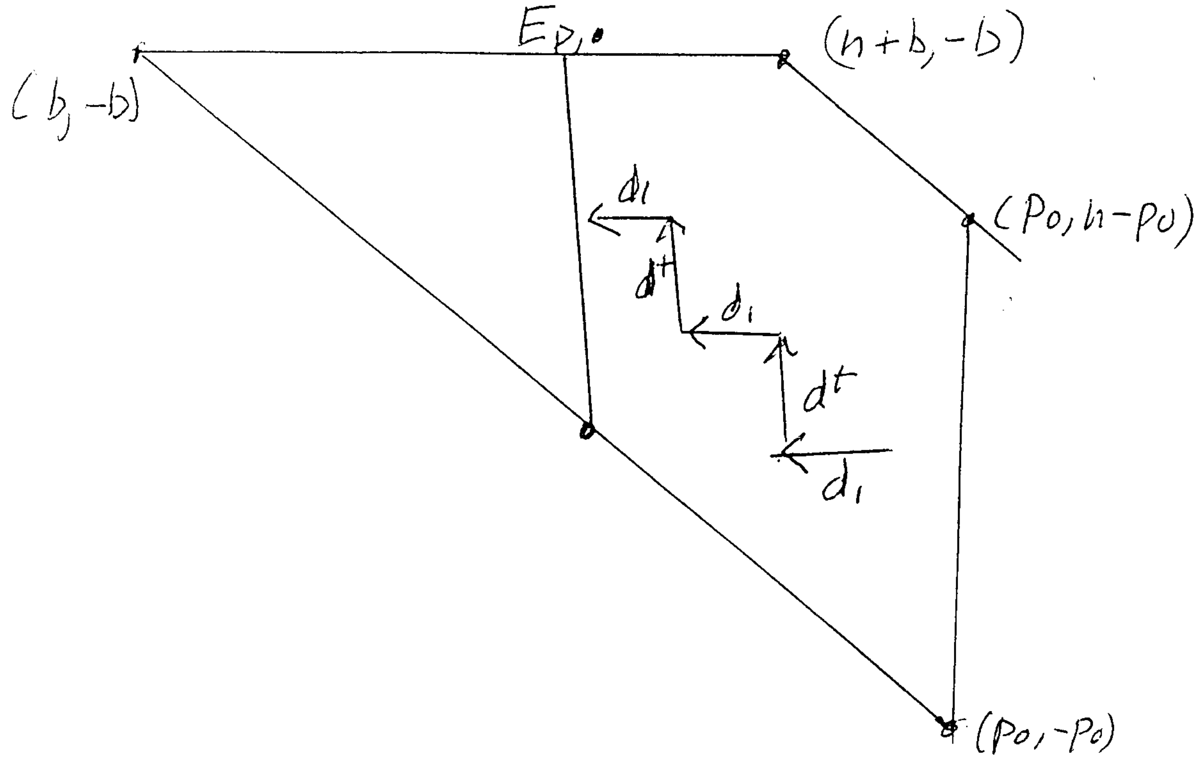
Then our original diagram may be amended as follows:



To construct the modules of the minimal resolution we need only consider $E^1_{p,q}$ for (p,q) in the shaded trapezoidal region.

If the $\deg C_0 = p_o$ is less than or equal to $n + b$ where n is the number of variables, the region becomes triangular.

To construct the map we need only consider $E_{p,q}(=E^o_{p,q})$ in the same region.



As remarked before the pth column $E_{p,\cdot} = \bigoplus_{deg\ C=p} Ch(\Omega_C)$ with $E_{p,q} = \bigoplus_{deg\ C=p} Ch_{p+q-1}(\Omega_c)$

Proposition: If $C'|C$ are two monomials then $\Omega_{C'} \subset \Omega_C$ and in fact if $T = C/C'$ then $\Omega_{C'} = \{\tau \in \Omega_C \text{ such that } T\tau|C,\ C/T\tau \in I\}$

Proof: $\tau|C' = C/T \Rightarrow T\tau|C \Rightarrow \tau|C$ and if $C'/\tau = C/T\tau$ is in I then $C/\tau = (C/T\tau)T$ is in I.

Corollary 1. $C|C_0$ implies $\Omega_C \subset \Omega_{C_0}$ so all simplicial complexes we need consider are subcomplexes of Ω_{C_0}.

Finally, if we assume that I is generated by square free monomials, the subcomplexes Ω_C of Ω_{C_0} turn out to be links of faces of Ω_{C_0} in the usual sense.

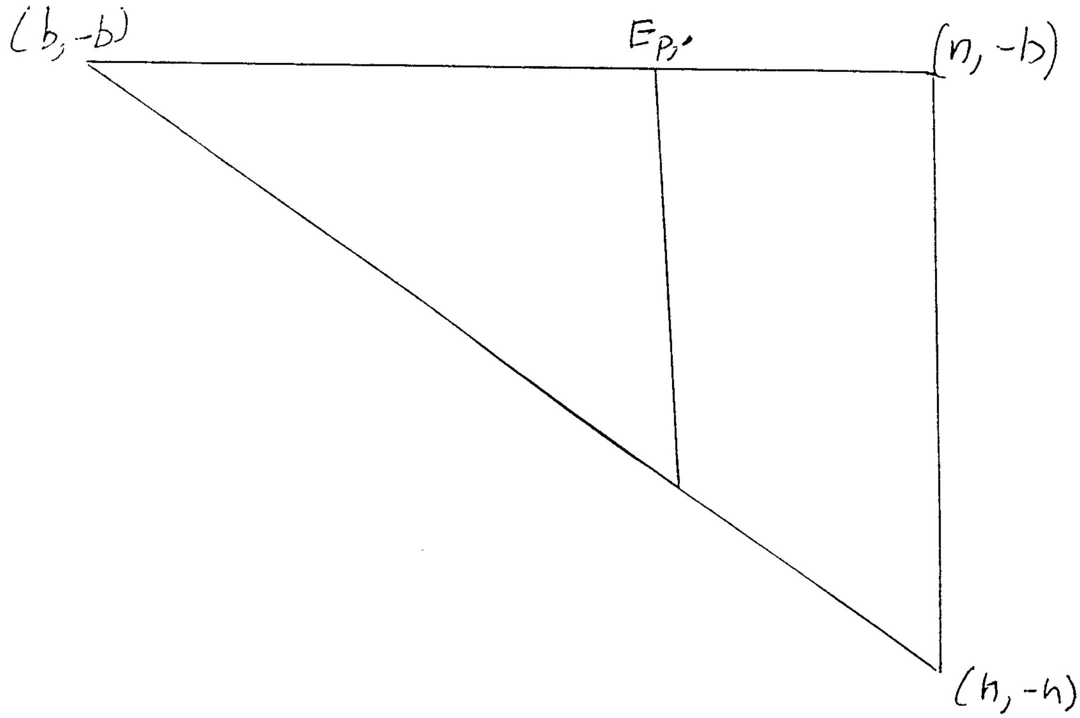
Corollary 2. If $T = C/C'$, then $\Omega_{C'}$ is something like the "link" of T in Ω_C and in fact if C is square free then T is a face of Ω_C and $\Omega_{C'} = \ell k(T, \Omega_C)$ where $\ell k(\sigma, \Omega)$ = link of σ in Ω is defined as usual, i.e. $\ell k(\sigma,\Omega) = \{\tau \in \Omega : \sigma \cap \tau = \phi,\ \sigma \cup \tau \in \Omega\}$ for any face σ of any abstract finite simplicial complex Ω

Proof: $T|C \Rightarrow T$ square free and $\tau T|C \Rightarrow T$ and τ have no variable in common. $C/T\tau \in I \Rightarrow C/T \in I \Rightarrow T$ and $T\tau \in \Omega_C$.

Corollary 2 suggests that we further restrict our attention to ideals I generated by square free monomials such that every monomial $G \in G(I)$, the minimal generating set of I, has a fixed degree b. We will assume further that every variable "occurs" in some $G \in G(I)$ in the sense that $C_0 = \underset{G \in G(I)}{l.c.m.\,G} = x_1 x_2 \dots x_n$, the product of all the variables.

The relation between $G(I)$ and Ω_{C_0} is that τ is a maximal face of Ω_{C_0} if and only if $\tau = \frac{x_1 \cdots x_n}{G}$ for some $G \in G(I)$. Notice that A/I is not the Stanley-Reisner ring of Ω_{C_0} but of a kind of "canonical simplicial Alexander dual" of Ω_{C_0}, i.e. embed Ω_{C_0} in the canonical minimal triangulation of a sphere (of dim $n-2$) consisting of all proper subsets of $\{x_1, \ldots, x_n\}$.

Take the simplicial Alexander dual of this embedding i.e. the set of all σ in the sphere such that $\frac{x_1 \cdots x_n}{\sigma} \notin \Omega_{C_0}$. Then A/I is the Stanley-Reisner or face ring of this dual.
In this case $p_0 = n$, the diagram becomes the triangle



From now on let $\ell k(\sigma, \Omega_{C_0}) = \ell k(\sigma)$ and $\Omega_{C_0} = \Omega$. Then

$$E^1_{p,\cdot} = \bigoplus_{dim\sigma = n-p-1} \tilde{H}(\ell k(\sigma))$$

and $W_m = \bigoplus_{\sigma} \tilde{H}_{m-1}(\ell k(\sigma)$.

Homology Manifolds without Boundary

A particularly simple case is that of a triangulated homology manifold without boundary, for the links of non-empty faces are then homology spheres.

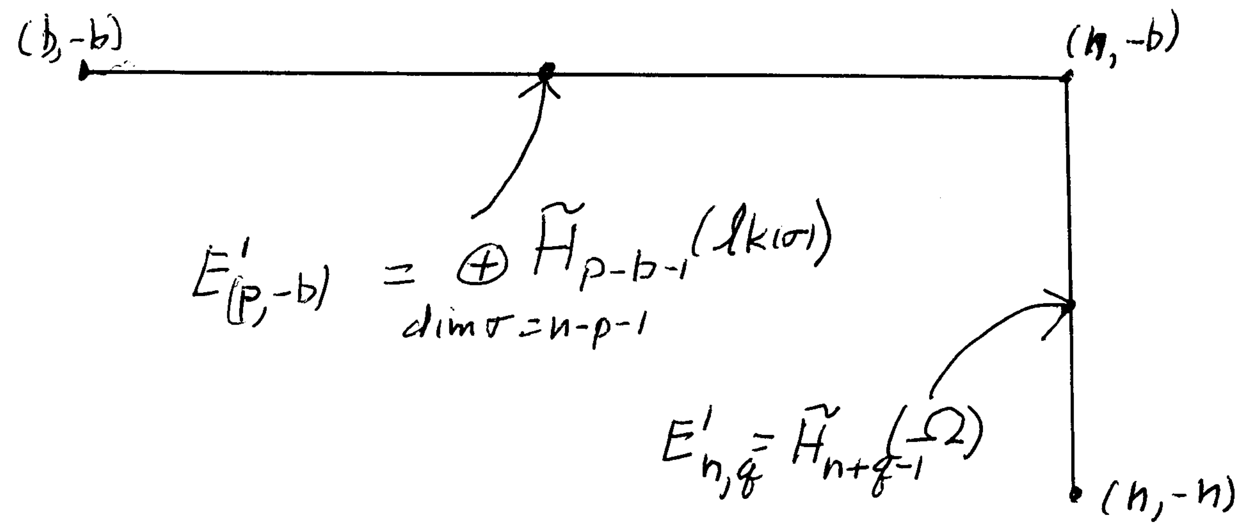
We follow the notation and conventions of [Mun] beginning on page 374. From now on assume $\Omega_{C_0} = \Omega$ is a triangulation of a homology manifold (without boundary). It then follows that its dimension is $n-b-1$. By Theorem 63.2 and Exercise 2 page 377 of [Mun] it follows that if $\sigma \neq 1$ (i.e. not the empty set in the augmented chains), $\ell k(\sigma)$ is a homology sphere of dim$(p-b-1)$ when dim$\sigma = n-p-1$. If $\sigma = 1$ $\ell k(\sigma) = \Omega$ and its reduced homology is that of the manifold.

Thus if $\sigma \neq 1$ and dim$\sigma = n-p-1$ then $\tilde{H}_{p-b-1}(\ell k \sigma) \cong A$ and all other reduced homology is 0. This implies that $E^1_{p,-b} \cong A^{f(n-p-1)}$ where $f(x)$ is the number of faces of Ω of dimension x. If $\sigma = 1$ so dim$\sigma = -1$ so $n = p$ and $E^1_{n,q} \cong \tilde{H}_{n+q-1}(\Omega)$ (this is only possibly non-zero for $-n \leq q \leq -b$). Thus $W_m = A^{f(n-m-b-1)} \oplus \tilde{H}_{m-1}(\Omega)$. The maximal faces of Ω are the $\{\tau : \tau = \frac{x_1 \cdots x_n}{G},\ G \in G(I)\}$ so deg$(\tau) = n-b$ and dim$(\tau) = n-b-1$

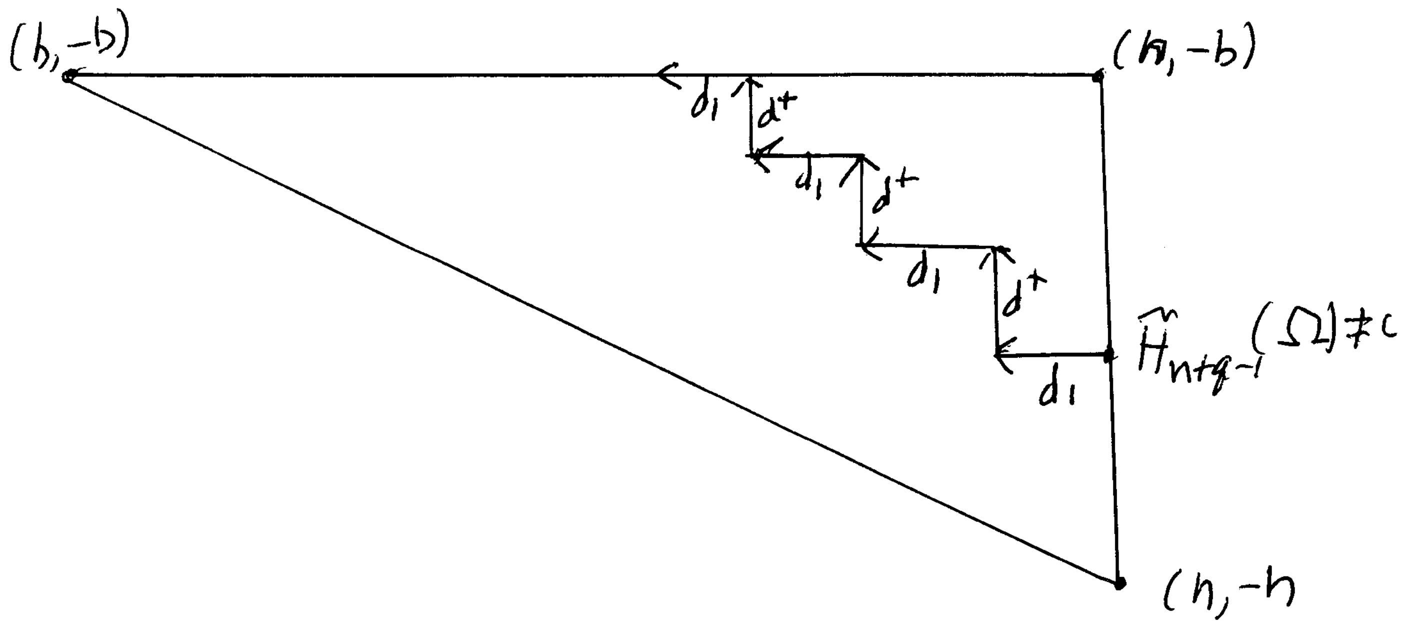
$$W_0 = A^{f(n-b-1)} \oplus \tilde{H}_{-1}(\Omega)$$

$\tilde{H}_{-1}(\Omega) = 0$ except in the trivial case where $\Omega = \{1\}$, which we ignore.

Thus $W_0 = A^N$ where N is the number of maximal faces of Ω, the same as the minimal number of generators of I, or the Betti number $b_1(I)$ ($b_0 = 1$ always). The Betti number $b_i = dim W_{i-1} = f(n-b-i) + dim \tilde{H}_{i-2}(\Omega)$ for $i > 1$ so, for example, if the manifold is connected $b_2 = f(n-b-2)$, the number of submaximal faces of Ω. The diagram for the relevant terms $E^1_{p,q}$ is now



For the map D in the minimal resolution we need $E_{p,q}$ in the region



whenever there is non-zero homology, $\tilde{H}_{n+q-1}(\Omega) \neq 0$.

Low-Dimensional Cases

It is easy to work out the cases of 1-dim manifolds Ω_{C_0} i.e. n-gons. These have Betti numbers $1, n, n, 1$.

For 2-manifolds we are looking at triangulations of spheres with various "handles" and "cross-caps" attached. The well known six-vertex triangulation of a single cross-cap is quite easy. (It is self-dual in the above canonical duality). Its Betti numbers are 1, 10, 15, 6 except in characteristic two and 1, 10, 15, 7, 1 in characteristic two. The characteristic two case is the first case where construction of d^+ is necessary, but construction of suitable d^+'s is easy for triangulations of 2-manifolds since the links of vertices are just k-gons for various k.

For $\dim(\Omega) = 1$ we have n-gons. $1 = n - b - 1$ so $n = b + 2$ and the diagram for $E^1_{p,q}$ is

$(b,-b)$ $(b+1,-b)$ $(b+2,-b)$

$(b+2,-b-1)$

$(b+2,-b-2)$

$$\begin{aligned}
E^1_{b,-b} &= \bigoplus_{dim\ \sigma=1} \tilde{H}_{-1}(1) \\
E^1_{b+1,-b} &= \bigoplus_{dim\ \sigma=0} \tilde{H}_0(twopoints) \\
E^1_{b+2,-b} &= \tilde{H}_1(ngon) \\
E^1_{b+2,-b-1} &= \tilde{H}_0(ngon) = 0 \quad (connected) \\
E^1_{(b+2,-b-2)} &= \tilde{H}_{-1}(ngon) = 0 \quad (non-empty)
\end{aligned}$$

Of $E_{p,q}$ all we need is

$$A \otimes I_b \otimes \wedge^0 V \underset{d_1}{\leftarrow} A \otimes I_b \otimes \wedge^1 V \underset{d_1}{\leftarrow} A \otimes I_b \otimes \wedge^2 V$$

The minimal resolution will be the maps induced on homology by d_1.

$$C_0 = x_1 \dots x_n \quad \Omega = \Omega_{C_0} = \{x_1x_2, x_2x_3, x_3x_4, \dots, x_{n-1}x_n, x_nx_1\}$$

so

$$I = (\frac{x_1 \dots x_n}{x_1x_2}, \frac{x_1 \dots x_n}{x_2x_3}, \dots \frac{x_1 \dots x_n}{x_{n-1}x_n}, \frac{x_1 \dots x_n}{x_nx_1}).$$

Let $G_{i,i+1} = \frac{x_1 \dots x_n}{x_ix_{i+1}}$. (Read subscripts mod n). The basis of the A-module of content C_0 is

$$\{1 \otimes G_{12} \otimes x_1x_2,\ 1 \otimes G_{23} \otimes x_2x_3, \dots, 1 \otimes G_{n-1,n} \otimes x_{n-1}x_n,\ 1 \otimes G_{n,1} \otimes x_nx_1\}$$

A basis for the homology in Ω_{C_0} may be taken to be represented by the cycle $1 \otimes G_{12} \otimes x_1x_2 + 1 \otimes G_{23} \otimes x_2x_3 + \dots + 1 \otimes G_{n-1,n} \otimes x_{n-1}x_n + 1 \otimes G_{n,1} \otimes x_nx_1 = c$

$$\begin{aligned}
d_1(c) = x_1 \otimes G_{12} \otimes x_2 - x_2 \otimes G_{12} \otimes x_1 + x_2 \otimes G_{23} \otimes x_3 \\
- x_3 \otimes G_{23} \otimes x_2 + \dots + x_n \otimes G_{n,1} \otimes x_1 - x_1 \otimes G_{n1} \otimes x_n
\end{aligned}$$

Rearranging by content get

$$d_1(c) = x_1 \otimes (G_{12} \otimes x_2 - G_{n1} \otimes x_n) + x_2 \otimes (G_{23} \otimes x_3 - G_{12} \otimes x_1)$$
$$+ \dots + x_n \otimes (G_{n,1} \otimes x_1 - G_{n-1,n} \otimes x_{n-1})$$

$\Omega \dfrac{x_1 \dots x_n}{x_i} = \{x_{i-1}, x_{i+1}\}$, two points. So a basis for homology of $\Omega \dfrac{x_1 \dots x_n}{x_1}$ in $A \otimes I_b \otimes \wedge^1 V$ is $1 \otimes (G_{12} \otimes x_2 - G_{n_1 \otimes x_n})$ etc. Thus the last matrix in the resolution is

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Computing $d_1(1 \otimes G_{12} \otimes x_2 - 1 \otimes G_{n,1} \otimes x_n) = x_2 \otimes G_{12} \otimes 1 - x_n \otimes G_{n,1} \otimes 1$. $d_1(1 \otimes G_{23} \otimes x_3 - 1 \otimes G_{12} \otimes x_1) = x_3 \otimes G_{23} \otimes 1 - x_1 \otimes G_{12} \otimes 1$ etc. Taking the ordered basis $1 \otimes G_{12} \otimes 1, 1 \otimes G_{23} \otimes 1, \dots$ of $A \otimes I_b \otimes \wedge^o V$ we get the n by n matrix below. The ideal is generated by the 1 by n matrix given below.

$$\left(\frac{x_1 \dots x_n}{x_1 x_2}, \frac{x_1 \dots x_n}{x_2 x_3}, \dots, \frac{x_1 \dots x_n}{x_{n-1} x_n}, \frac{x_1 \dots x_n}{x_n x_1}\right)$$

$$\begin{pmatrix} x_2, & -x_1 & & & & 0's \\ & x_3, & -x_2 & & & \\ & & x_4, & -x_3 & & \\ & & & x_5, & -x_4 & \\ & 0's & & & \ddots & \\ & & & & x_n, & -x_{n-1} \\ -x_n & & & & & x_1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}$$

```
% pres S
```

Macaulay print-out for hexagon.

```
-----------------------------------

uvwx vwxy wxyz uxyz uvyz uvwz

-----------------------------------

-z 0  0  0  0  -y
0  0  0  0  -z u
0  0  0  u  v  0
0  0  v  -w 0  0
0  w  -x 0  0  0
x  -y 0  0  0  0

-----------------------------------

-y
-x
-w
-v
u
z

-----------------------------------

% exit
```

For $\dim(\Omega) = 2$ we have $2 = n - b - 1$ so $n = b + 3$. The diagram for $E^1_{p,q}$ is

$$\begin{array}{cccc}
(b, -b) & (b+1, -b) & (b+2, -b) & (b+3, -b) \\
\bullet & \bullet & \bullet & \bullet \\
 & & & \bullet (b+3, -b-1) \\
 & & & \bullet (b+3, -b-2) \\
 & & & \bullet (b+3, -b-3)
\end{array}$$

$$E^1_{b,-b} = \bigoplus_{dim\sigma = 2} \tilde{H}_{-1}(1)$$

$$E^1_{b+1,-b} = \bigoplus_{dim\sigma = 1} \tilde{H}_0(\text{two points})$$

$$E^1_{b+2,-b} = \bigoplus_{dim\sigma = 0} \tilde{H}_1 \ (\text{various q} - \text{gons})$$

$$E^1_{b+3,-b} = \tilde{H}_2(\Omega)$$

$$E^1_{b+3,-b-1} = \tilde{H}_1(\Omega)$$

$$E^1_{b+3,-b-2} = \tilde{H}_0(\Omega) = 0 \ (\text{connected})$$

$$E^1_{b+3,-b-3} = \tilde{H}_{-1}(\Omega) = 0 \ (\text{non} - \text{empty})$$

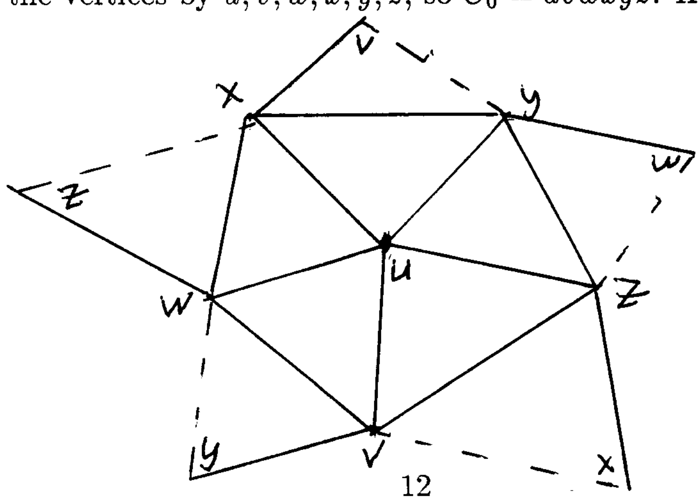
Of $E_{p,q}$ all we need is

$$\begin{array}{ccccccc}
A \otimes I_b \otimes \wedge^0 V & \xleftarrow[d_1]{} & A \otimes I_b \otimes \wedge^1 V & \xleftarrow[d_1]{} & A \otimes I_b \otimes \wedge^2 V & \xleftarrow[d_1]{} & A \otimes I_b \otimes \wedge^3 V \\
 & & & & \uparrow d^+ & & \\
 & & & & A \otimes I_{b+1} \otimes \wedge^1 V & \xleftarrow[d_1]{} & A \otimes I_{b+1} \otimes \wedge^2 V
\end{array}$$

If we now took Ω to be a triangulation of a 2-sphere, the situation would be quite analogous to the n-gons. We would only need the top line above, since $\tilde{H}_1(2 \text{ sphere}) = 0$

More interestingly, we will take Ω to be the well-known six point triangulation of the real projective plane.

We denote the vertices by u, v, w, x, y, z, so $C_0 = uvwxyz$. Here is the picture



If one computes $\frac{uvwxyz}{face}$ for each of the ten faces one gets

$$I = (uvx, uwy, uxz, uvy, uwz, vwx, wxy, xyz, vyz, vwz)$$

If we take k to be any field of characteristic $\neq 2$, the real projective plane has no reduced homology whatever. In this case all we need of $E_{p,q}$ is

$$A \otimes I_b \otimes \wedge^0 V \underset{d_1}{\leftarrow} A \otimes I_b \otimes \wedge^1 V \underset{d_1}{\leftarrow} A \otimes I_b \otimes \wedge^2 V$$

Here $n = 6$, so $b = n - 3 = 3$

For d_1 on the right using bases of the form $1 \otimes \varsigma_\tau \otimes \tau$ and taking map induced on homology with appropriate selection of cycles representing a basis for homology we get the 15 by 6 matrix

lks of edges	lk u	lk v	lk w	lk x	lk y	lk z	Chain 2
uv	$-v$	u					uv
uw	w		$-u$				uw
ux	x			$-u$			
uy	y				$-u$		
uz	$-z$					u	
vw		$-w$	v				
wx			$-x$	w			wx
xy				$-y$	x		
yz					$-z$	y	
vz						$-v$	vz
vy		$-y$			$-v$		
wz			$-z$			$-w$	
vx		$-x$		$-v$			
wy			y		w		
xz				$-z$		$-x$	xz

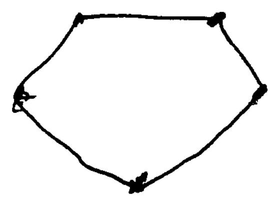
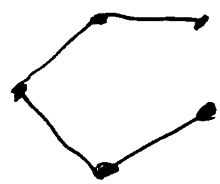
links of edges															links of faces ↓
u v	u w	u x	u y	u z	v w	w x	x y	y z	v z	v y	w z	v x	w y	x z	
−w	−v				−u										u v w
	x	−w				−u									u w x
		y	−x				−u								u x y
			z	y				−u							u y z
z				−v					−u						u v z
					y					−w			−v		v w y
								w			y		z		w y z
						z					−x			w	w x z
									x			z		−v	v x z
							v			x		−y			v x y

Similar calculations for d_1 on the left lead to the above 10 by 15 matrix

Compare with the Macaulay print out on the next page.

Macaulay print-out for IP, Char ≠2

```
% pres S

-----------------------------------

uvx uwy uxz uvy uwz vwx wxy xyz vyz vwz

-----------------------------------

0  0  0  0  0  0  0  0  0  -w 0  0  0  -y -z
0  0  0  0  0  0  0  -x 0  0  0  -z v  0  0
0  0  0  0  0  -y 0  0  0  0  w  0  0  0  v
0  0  0  -z 0  0  0  0  0  0  0  0  -w x  0
-v 0  0  0  0  0  0  0  0  0  -x y  0  0  0
0  -z 0  0  0  0  0  0  -y u  0  0  0  0  0
0  0  0  0  0  0  -z u  v  0  0  0  0  0  0
0  0  0  0  v  u  w  0  0  0  0  0  0  0  0
0  0  w  u  -x 0  0  0  0  0  0  0  0  0  0
u  x  -y 0  0  0  0  0  0  0  0  0  0  0  0

-----------------------------------

0  0  0  0  y  -x
0  0  -y 0  0  u
0  0  -x 0  u  0
0  0  0  x  -w 0
0  0  -w u  0  0
0  -w 0  -v 0  0
0  u  v  0  0  0
-v z  0  0  0  0
u  0  z  0  0  0
y  0  0  0  0  z
0  -y 0  0  0  v
0  -x 0  0  v  0
-x 0  0  0  z  0
-w 0  0  z  0  0
0  0  0  -y 0  -w

-----------------------------------


% lpr -Pcicle -h Z
; command not found: lpr

% exit
```

In the characteristic two case both $\tilde{H}_1(\Omega)$ and $\hat{H}_2(\Omega)$ are 1-dimensional and we need the full diagram on the middle of page 12. For the first time we need to choose a map d^+. It is possible to construct certain maps d^+ based on the choice of a "maximal tree" from a given basis for the modules $E_{p,q}$. A subset T' of the basis is called a "maximal tree" if $d(T)$ is a linearly independent set of chains and T is maximal with respect to that property. In the case of a graph (and here we have only some pentagons) this concept is the same as "rooted spanning tree" Such a tree in a pentagon  looks like  ← choose this as root.

Choosing these in a particular way we get the column labelled Char 2 on page 13, when we calculate $(I - dd^t - d^t d) d_1 d^t d_1$

Finally calculating d_1 on the far right for given choice of homology representative we get the column vector:
$$\begin{pmatrix} u \\ v \\ w \\ x \\ y \\ z \\ 0 \end{pmatrix}$$

Macaulay print-out for $\mathbb{P}^2$, char=2

```
% pres S

-----------------------------------

uvx uwy uxz uvy uwz vwx wxy xyz vyz vwz

-----------------------------------

0 0 0 0 0 0 0 0 0 w 0 0 0 y z
0 0 0 0 0 0 0 x 0 0 0 z v 0 0
0 0 0 0 0 y 0 0 0 0 w 0 0 0 v
0 0 0 z 0 0 0 0 0 0 0 0 w x 0
v 0 0 0 0 0 0 0 0 0 x y 0 0 0
0 z 0 0 0 0 0 0 y u 0 0 0 0 0
0 0 0 0 0 0 z u v 0 0 0 0 0 0
0 0 0 0 v u w 0 0 0 0 0 0 0 0
0 0 w u x 0 0 0 0 0 0 0 0 0 0
u x y 0 0 0 0 0 0 0 0 0 0 0 0

-----------------------------------

0 0 0 0 y x 0
0 0 y 0 0 u 0
0 0 x 0 u 0 0
0 0 0 x w 0 0
0 0 w u 0 0 0
0 w 0 v 0 0 0
0 u v 0 0 0 0
v z 0 0 0 0 0
u 0 z 0 0 0 0
y 0 0 0 0 z 0
0 y 0 0 0 v vy
0 x 0 0 v 0 vx
x 0 0 0 z 0 xz
w 0 0 z 0 0 wz
0 0 0 y 0 w wy

-----------------------------------

z
v
u
w
x
y
0

-----------------------------------
```

References

[Eag] Eagon, John A., "Partially Split Double Complexes ... etc", Jour. of Alg., Vol. 135 No. 2, Dec. 1990.

[McL] MacLane, Saunders, "Homology" Academic Press, Springer, 1963.

[Mun] Munkres, James R., "Elements of Algebraic Topology", Benjamin/Cummings 1984.

[Rob] Roberts, Joel, "Minimal Resolutions Derived from Bicomplexes and Other Wall Complexes", in preparation.

7 vertex triangulation of torus

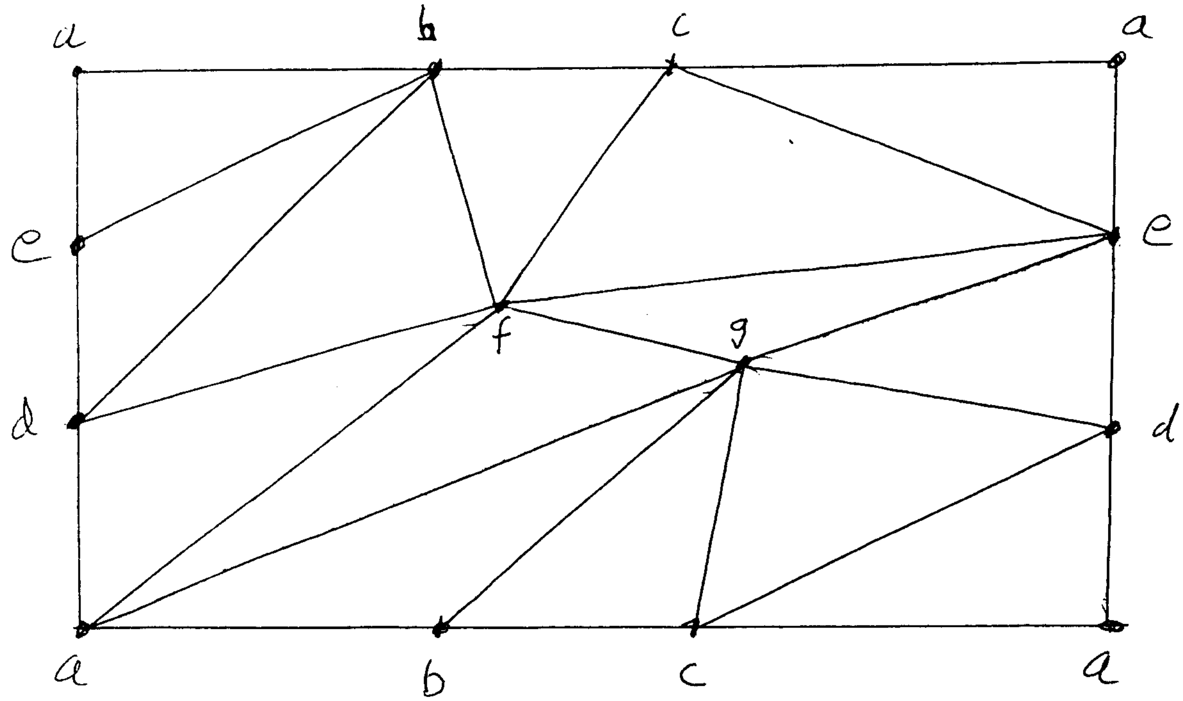


Diagram:

1 14 21 7 1

2

Faces: abe, bde, bdf, bcf, fce, ace, feg
ged, acd, gdc, bcg, agb, agf, afd

Generators of ideal: $\left(\frac{abcdefg}{xyz} \right)$

cdfg, acfg, aceg, adeg, abdg, bdfg, abcd
abcf, befg, abef, adef, cdef, bcde, bceg

pres S *Macaulay output, Torus, any characteristic*

```
----------------------------------

cdfg acfg aceg adeg abdg bdfg abcd abcf befg abef adef cdef bcde bceg

----------------------------------

0  0  0  0  0  0  e  0  0  0  0  0  0  0  0  0  b  0  0  0  a
0  0  0  0  0  0  0  0  0  0  0  0  0  b  0  0  0  0  0  e  -d
-b 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  -d -f 0
0  0  0  0  0  0  0  0  f  0  0  0  0  0  0  0  0  b  c  0  0
0  0  0  0  0  0  0  0  0  0  0  0  0  0  c  -f 0  -e 0  0  0
0  0  0  0  0  0  0  0  0  0  0  -e 0  0  0  a  -c 0  0  0  0
0  0  0  -e 0  0  0  0  0  0  0  0  -f 0  -g 0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  -e 0  0  d  -g 0  0  0  0  0  0  0
0  0  c  0  0  0  0  0  0  0  a  d  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  -d 0  c  -g 0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  -c 0  b  -g 0  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  b  a  -g 0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  -g 0  a  -f 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
a  d  -f 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0

----------------------------------

0  0  0  0  f  0  -d 0   0
0  0  0  -f 0  0  a  0   0
0  0  0  -d a  0  0  0   0
0  0  0  0  0  f  g  0   0
0  0  0  g  0  a  0  0   0
0  g  0  0  0  -b 0  0   0
0  a  0  b  0  0  0  0   0
0  0  g  0  0  -c 0  0   cg
0  -c b  0  0  0  0  0   bc
0  0  0  0  -g -d 0  0   dg
0  0  -d 0  -c 0  0  0   0
0  0  a  c  0  0  0  0   0
-g 0  0  0  0  -e 0  -eg eg
-d 0  0  0  e  0  0  -de 0
f  0  0  0  0  0  -e ef  -ef
c  0  e  0  0  0  0  0   0
a  0  0  -e 0  0  0  0   0
0  0  -f 0  0  0  -c cf  -cf
0  f  0  0  0  0  b  -bf 0
0  -d 0  0  -b 0  0  bd  0
-b -e 0  0  0  0  0  0   0

----------------------------------

e
-b
-c
a
d
-g
f
0
0

----------------------------------
```

8 vertex triangulation of Klein bottle

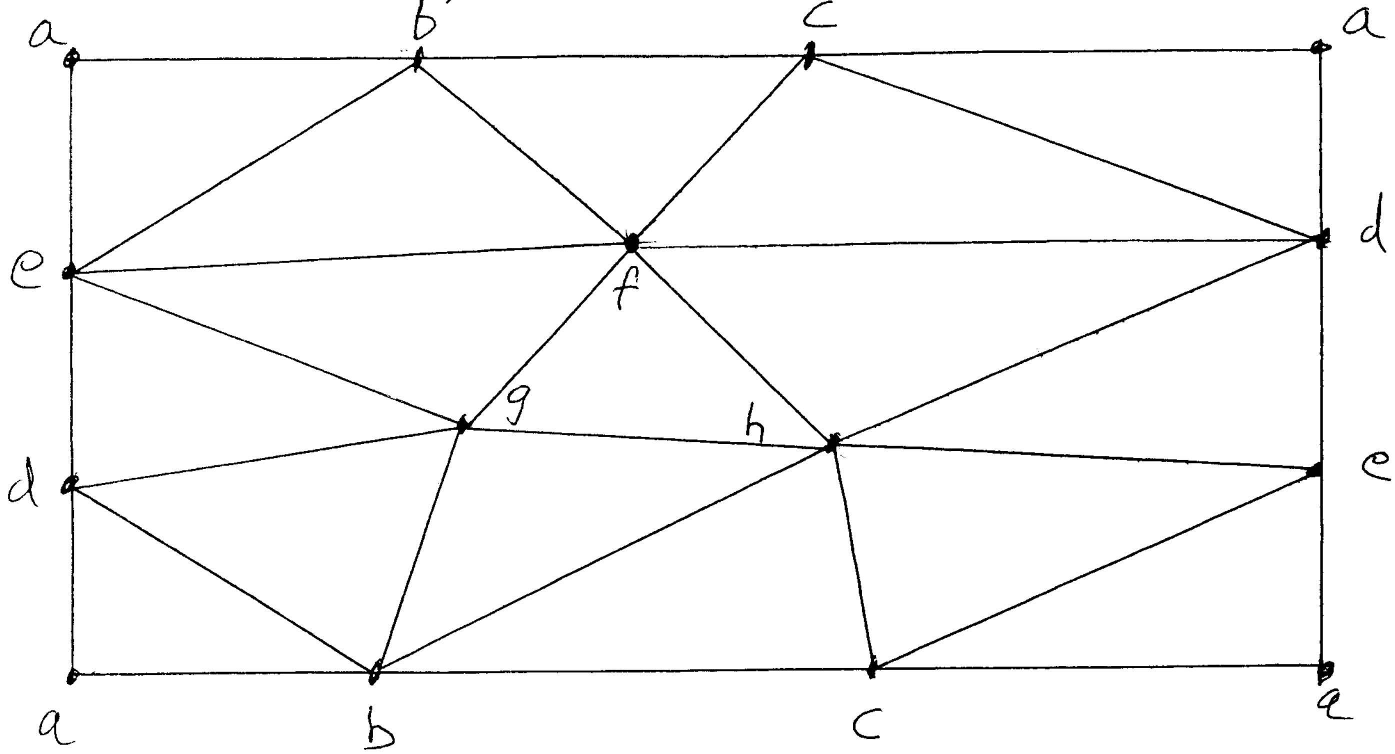


Diagram char ≠ 2:

1	16	24	8	0
				1

Diagram char = 2

1	16	24	8	1
				2

Faces: abe, bef, bcf, dfc, adc, gde, gfe, fgh
dfh, deh, abd, bdg, bgh, bhc, ceh, ace

Generators of ideal: $\left(\frac{abcdefgh}{xyz}\right)$
cdfgh, acdgh, adegh, abegh, befgh, abcfh, abcdh
abcde, abceg, abcfg, cefgh, acefh, acdef, adefg,
abdfg, bdfgh

% pres S *Macaulay output, Klein bottle, char #2*

```
-----------------------------------

cdfgh acdgh adegh abegh befgh abcfh abcdh abcde abceg abcfg cefgh acefh acdef
adefg abdfg bdfgh

-----------------------------------

0  b  0  0  0  0  0  0  0  0  0  0  -e 0  0  0  0  0  0  0  0  0  0  a
0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  b  0  0  -e -f
0  0  0  0  0  0  f  0  0  0  0  0  0  0  0  0  0  0  0  0  0  b  c  0
0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  c  0  0  0  -f -d 0  0
0  0  d  0  0  0  0  0  0  0  0  -c 0  0  0  0  0  0  0  0  a  0  0  0
0  0  0  0  0  0  0  0  0  -e 0  0  0  0  g  0  0  0  d  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  e  -f -g 0  0  0  0
0  0  0  0  0  0  0  -f 0  0  0  0  0  0  0  -g 0  -h 0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0  0  0  0  -f 0  d  -h 0  0  0  0  0  0  0
0  0  0  -d 0  0  0  0  0  0  0  0  0  e  -h 0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0  0  a  b  d  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  d  b  -g 0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  -g 0  b  -h 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  b  c  -h 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
-h 0  0  c  -e 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
a  -c -e 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0

-----------------------------------

0  e  c  0  0  0  0  0  0
0  0  a  0  e  0  0  0  0
0  a  0  0  -c 0  0  0  0
0  0  h  0  0  0  0  -e 0
0  -h 0  0  0  0  0  -c 0
0  0  0  h  0  0  0  b  0
0  -b 0  c  0  0  0  0  0
0  0  0  0  0  0  h  g  0
0  0  0  -g 0  0  b  0  0
0  0  0  0  0  -g -d 0  0
0  0  0  -d 0  -b 0  0  0
0  0  0  0  -d a  0  0  0
0  0  0  a  b  0  0  0  0
0  0  0  0  0  -h 0  -d dh
0  0  -d 0  0  -e 0  0  de
-h 0  0  0  0  0  0  -f fh
-d 0  0  0  0  f  0  0  0
g  0  0  0  0  0  -f 0  -fg
0  0  g  0  0  0  -e 0  -eg
e  0  -f 0  0  0  0  0  0
0  -d 0  0  0  c  0  0  0
-c f  0  0  0  0  0  0  0
b  0  0  -f 0  0  0  0  0
0  0  -b e  0  0  0  0  0

-----------------------------------
```

```
pres T
```

Macaulay output, Klein bottle, char=2

```
-----------------------------------

cdfgh acdgh adegh abegh befgh abcfh abcdh abcde abceg abcfg cefgh acefh acdef

adefg abdfg bdfgh

-----------------------------------

0 b 0 0 0 0 0 0 0 0 0 0 e 0 0 0 0 0 0 0 0 0 0 a
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 b 0 0 e f
0 0 0 0 0 0 f 0 0 0 0 0 0 0 0 0 0 0 0 0 0 b c 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 c 0 0 0 f d 0 0
0 0 d 0 0 0 0 0 0 0 0 c 0 0 0 0 0 0 0 0 a 0 0 0
0 0 0 0 0 0 0 0 0 e 0 0 0 0 g 0 0 0 d 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 e f g 0 0 0 0
0 0 0 0 0 0 0 f 0 0 0 0 0 0 0 g 0 h 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 f 0 d h 0 0 0 0 0 0 0
0 0 0 d 0 0 0 0 0 0 0 0 0 e h 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 a b d 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 d b g 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 g 0 b h 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 b c h 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
h 0 0 c e 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
a c e 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

-----------------------------------

0 e c 0 0 0 0 0 0  0
0 0 a 0 e 0 0 0 0  0
0 a 0 0 c 0 0 0 0  0
0 0 h 0 0 0 0 e eh 0
0 h 0 0 0 0 0 c ch 0
0 0 0 h 0 0 0 b 0  0
0 b 0 c 0 0 0 0 bc 0
0 0 0 0 0 0 h g 0  0
0 0 0 g 0 0 b 0 0  0
0 0 0 0 0 g d 0 0  0
0 0 0 d 0 b 0 0 0  0
0 0 0 0 d a 0 0 0  0
0 0 0 a b 0 0 0 0  0
0 0 0 0 0 h 0 d dh dh
0 0 d 0 0 e 0 0 0  de
h 0 0 0 0 0 0 f 0  fh
d 0 0 0 0 f 0 0 df 0
g 0 0 0 0 0 f 0 0  fg
0 0 g 0 0 0 e 0 0  eg
e 0 f 0 0 0 0 0 0  0
0 d 0 0 0 c 0 0 0  0
c f 0 0 0 0 0 0 cf 0
b 0 0 f 0 0 0 0 0  0
0 0 b e 0 0 0 0 0  0

-----------------------------------

f
c
e
b
a
d
g
h
0
0

-----------------------------------
```