ORDERING POINTS BY LINEAR FUNCTIONALS

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ABSTRACT. Given a set of points in Euclidean space, we say that two linear functionals differ on that set if they give rise to different linear orderings of the points. We investigate what the largest and smallest number of different linear functionals can be as a function of the number of points and the dimension of the space.

1. INTRODUCTION

The purpose of this paper is to investigate the number of different linear orderings of point configurations which can arise from linear functionals. This problem is related to work of Ungar [5] on the minimum number of directions that a set of points in the plane determine.

This investigation has a number of interesting features. First, it seems to be a natural question to ask. Second, it provides a framework in which to extend the work of Ungar [5], which currently is almost a curiosity (although a very beautiful one.) Our results can be applied to give bounds on the number of monotone paths on polytopes. Finally, we think that the technique used to establish the upper bound may have broader applicability and should be more widely known.

The structure of the paper is as follows. In the next section we establish our terminology and prove the fundamental connections between the various objects we study. The knowledgeable reader will recognize a number of these constructions from matroid theory, but we have deliberately kept the terminology and definitions to a minimum. In §3 we prove an upper bound on the number of different linear functionals that n points in $\mathbb{R}^d$ can support. In §4 we present a lower bound in the case of the points being in general position. We also give a conjecture for the case in which the points are in slope-general position, a variant on the notion of general position. We also discuss in this section the relationship between the work of [5] and our own. In the last section we

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how the bounds we prove apply to counting the number of monotone paths of zonotopes.

2. Preliminaries

In this section we will establish our notation and define the principal objects of study. We will also establish some base results that will be used throughout the rest of the paper.

Let $\mathcal{C}$ be a collection of n points in $\mathbb{R}^d$. For every $v \in \mathbb{R}^d$ let $l_v : \mathbb{R}^d \to \mathbb{R}$ be the linear functional such that $l_v(x) =< v, x >$ for all $x \in \mathbb{R}^d$. We also know that every linear functional l on $\mathbb{R}^d$ is of the form l_v for some $v \in \mathbb{R}^d$. We will let $H_v = \{ x \mid < v, x >= 0\}$.

We will say that a linear functional l is *generic with respect to* $\mathcal{C}$ (just generic if $\mathcal{C}$ is clearly understood) if $l(x) \neq l(y)$ for all distinct $x, y \in \mathcal{C}$. A generic linear functional l gives rise to a linear ordering of the points of $\mathcal{C}$, $\sigma_l = x_1 \dots x_n$ defined by $l(x_1) < l(x_2) < \cdots < l(x_n)$. This allows us to define an equivalence relation on generic linear functionals by

$$l \sim m \Longleftrightarrow \sigma_l = \sigma_m.$$

Let $f(\mathcal{C})$ equal the number of equivalence classes of linear functionals that are generic with respect to $\mathcal{C}$. For brevity's sake we will refer to $f(\mathcal{C})$ as the number of linear functionals on $\mathcal{C}$. Note that around every generic linear functional there is an open neighborhood of $\mathbb{R}^d$ consisting of equivalent functionals.

We will analyze the structure of the inequivalent linear functionals by means of matroids and hyperplane arrangements. We will briefly review the facts we need from these two areas.

Let $V = \{v_i \mid i \in E\}$ be a collection of vectors that span $\mathbb{R}^d$ indexed by a set E. A subset $C \subseteq E$ is called a *circuit* if the set of vectors $V(C) = \{v_j \mid j \in C\}$ is a minimal dependent set. We denote the set of circuits of V by $\mathbf{C}(V)$. A subset $F \subseteq E$ is called a *flat* if it has the property that if v_j is in the linear span of $V(F)$ then $j \in F$. We can partially order the flats of V to get a geometric lattice $\mathbf{L}(V)$ of rank d.

Associated with the lattice $\mathbf{L} = \mathbf{L}(V)$ is its *characteristic polynomial*, $\chi(\mathbf{L}, t)$ defined by

$$\chi(\mathbf{L}, t) = \sum_{F \in \mathbf{L}} \mu(\hat{0}, F)\, t^{d - rank\, F} \tag{1}$$

$$= \sum_{k=0}^{d} (-1)^k\, w_k\, t^{d-k} \tag{2}$$

where μ is the Möbius function of $\mathbf{L}$ and the numbers $\{w_k\}$ are positive numbers call *the Whitney numbers of the first kind.*

Given the geometric lattice $\mathbf{L} = \mathbf{L}(V)$ we define the *truncation of* $\mathbf{L}$, $\mathrm{Trunc}(\mathbf{L})$, to be the geometric lattice obtained by removing all of the flats of co-rank 1 from $\mathbf{L}$. Let $\mathrm{Trunc}_k(\mathbf{L})$ be the result of iterating the truncation operator k times.

Lemma 2.1. *Let $V = \{v_i \mid i \in E\}$ be a set of vectors that span $\mathbb{R}^d$ and let $w \in \mathbb{R}^d$ be a vector that is not in any hyperplane spanned by elements of V. If $V' = \{v'_i \mid i \in E\}$ is the set of vectors obtained by projecting the vectors of V into the hyperplane orthogonal to w, then $Trunc(\mathbf{L}(V)) = \mathbf{L}(V')$. Moreover, we have that a set $C \in E$ is a circuit of V' if and only if either C is a circuit of V or $V(C)$ is a basis for $\mathbb{R}^d$.*

Proof. This is the standard construction for truncations. See [2, Proposition 7.4.9]. □

It is not hard to see how the truncation operator affects the characteristic polynomial.

Lemma 2.2. *If $\mathbf{L}$ is a geometric lattice and*

$$\chi(\mathbf{L}, t) = \sum_{k=0}^{d} (-1)^k \, w_k \, t^{d-k}$$

then

$$\chi(Trunc(\mathbf{L}), t) = (-1)^{d-1} w'_{d-1} + \sum_{k=0}^{d-2} (-1)^k \, w_k \, t^{d-1-k}$$

where

$$(-1)^{d-1} w'_{d-1} = \sum_{j=0}^{d-2} (-1)^{j+1} w_j.$$

Proof. Since the lattice $\mathrm{Trunc}(\mathbf{L})$ is the same as $\mathbf{L}$ up through rank $d-2$ the characteristic polynomials must agree through those terms. The coefficient $(-1)^{d-1} w'_{d-1}$ is determined by the condition that $\chi(\mathrm{Trunc}(\mathbf{L}), 1) = 0$. □

Given two sets of vectors $V = \{v_i \mid i \in E\}$ and $V' = \{v'_i \mid i \in E\}$ indexed by the same set E, we say that there is a *weak map* from V to V' if for every subset $D \subseteq E$ we have that $V(D)$ is dependent implies that $V'(D)$ is dependent. Equivalently, we we have that $V'(C)$ is dependent for every circuit $C \in C(V)$. The significance for us of the existence of a weak map is given in the following lemma.

Lemma 2.3. *Let $V = \{v_i \mid i \in E\}$ and $V' = \{v'_i \mid i \in E\}$ be two sets of vectors both indexed by the same set E and both of which span $\mathbb{R}^d$. Let*

$$\chi(\mathbf{L}(V), t) = \sum_{i=0}^{d} (-1)^i \, w_k \, t^{d-k}$$

and

$$\chi(\mathbf{L}(V'), t) = \sum_{i=0}^{d} (-1)^i \, w'_k \, t^{d-k}.$$

If there is a weak map from V to V' then $w_k \geq w'_k$ for all $1 \leq k \leq d$.

Proof. See [4, Corollary 9.3.7]. □

Finally, let $\mathcal{A} = \mathcal{A}(V)$ be the set of hyperplanes in $\mathbb{R}^d$ given by

$$\mathcal{A}(V) = \{H_v \mid v \in V\}$$

where $H_v = \{x \in \mathbb{R}^d \mid <x, v>= 0\}$. Let $\mathcal{T}(\mathcal{A})$, *the topes of* $\mathcal{A}$ be the set of connected components of $\mathbb{R}^d - \mathcal{A}$. A fundamental property of hyperplane arrangements is

Lemma 2.4. *The cardinality of $\mathcal{T}(\mathcal{A})$ is equal to*

$$(-1)^d \chi(\mathbf{L}(V), -1) = \sum_{k=0}^{d} w_k.$$

Proof. See [6]. □

We are now ready to see how these facts from matroid theory can be used to analyze the structure of linear functionals on point configurations. Let $\mathcal{C} = \{x_1, \ldots, x_n\}$ be a point configuration whose affine span is all of $\mathbb{R}^d$. Let the *difference set of* $\mathcal{C}$, $\mathcal{D}(\mathcal{C})$ be the collection of vectors

$$\mathcal{D}(\mathcal{C}) = \{x_{\{i,j\}} = x_i - x_j \mid 1 \leq i < j \leq n\}$$

which is indexed by the set of ordered pair $E = \{\{i, j\}\}$. Note that $\mathcal{D}(\mathcal{C})$ could, in fact, be a multi-set. Let $\mathcal{A} = \mathcal{A}(\mathcal{D})$ be the arrangement related to $\mathcal{D} = \mathcal{D}(\mathcal{C})$. The size of $\mathcal{A}$ will, in general, be less than $\binom{n}{2}$ since some of the vectors in $\mathcal{D}(\mathcal{C})$ may either be the same or scalar multiples of each other.

Theorem 2.5. *A vector v is generic with respect to $\mathcal{C}$ if and only if it lies in a tope of $\mathcal{A}$. Two vectors v and w are in the same tope of $\mathcal{A}$ if and only if $l_v \sim l_w$.*

Proof. A vector v is in a tope of $\mathcal{A}$ if and only if it does not lie on any hyperplane $H_{x_{\{i,j\}}}$ and thus $< v, x_i - x_j > \neq 0$ for any $x_i, x_j \in \mathcal{C}$. Hence l_v must be generic with respect to $\mathcal{C}$. Moreover, if v and w are in the same tope then $< v, x_i - x_j >< 0$ if and only if $< w, x_i - x_j >< 0$ for all $x_i, x_j \in \mathcal{C}$, and so $l_v \sim l_w$. □

Corollary 2.6. *The number of inequivalent linear functionals for a point configuration $\mathcal{C}$ that spans $\mathbb{R}^d$ is*

$$f(\mathcal{C}) = (-1)^d \chi(\mathbf{L}(\mathcal{D}(\mathcal{C})), -1).$$

Proof. It follows from Lemma 2.5 that $f(\mathcal{C})$ is equal to the cardinality of $\mathcal{T}(\mathcal{A})$, which is, by Lemma 2.4 equal to $(-1)^d \chi(\mathbf{L}(\mathcal{D}(\mathcal{C})), -1)$. □

In the subsequent sections we will put bounds on the size of $f(\mathcal{C})$.

3. Upper Bounds

In this section we will present upper bounds on the number of inequivalent linear functionals on n points in $\mathbb{R}^d$. Our technique will be to produce explicit point configurations for which we can compute the number $f(\mathcal{C})$ and then exhibit maps from these point configurations to general configurations which induce weak maps on the related difference sets.

Our fundamental point configuration will be

$$\Delta_n = \{e_1, e_2, \ldots, e_n\}$$

where e_i is the i^{th} standard basis vector in $\mathbb{R}^n$. Note that Δ_n has affine dimension $n-1$.

The difference set

$$\mathcal{D}_n = \mathcal{D}(\Delta_n) = \{e_{\{i,j\}} \mid 1 \leq i < j \leq n\}$$

is the set of positive roots for the classical root system $\mathbf{A}_{n-1}$. It is well known that the lattice $\mathbf{L}(\mathcal{D}_n)$ is isomorphic to the lattice of partitions of a set of n elements. Hence

Lemma 3.1. *The characteristic polynomial of $\mathbf{L}(\mathcal{D}_n)$ is given by*

$$\chi(\mathbf{L}(\mathcal{D}_n), t) = \sum_{i=0}^{n-1} (-1)^i \, c(n, n-i) \, t^{n-1-i},$$

where $\{c(n, n-i)\}$ are the unsigned Stirling numbers of the first kind.

Let Δ_n^k be a configuration of points in $\mathbb{R}^{n-1-k}$ obtained by k successive projections as in Corollary 2.2. We will let $\mathcal{D}_n^k = \mathcal{D}(\Delta_n^k)$.

Corollary 3.2. *The characteristic polynomial* $\mathbf{L}(\mathcal{D}_n^k)$ *is given by*

$$\chi(\mathbf{L}(\mathcal{D}_n^k), t) = (-1)^{n-k-1}\hat{c}(n,k) + \sum_{i=0}^{n-k-2} (-1)^i c(n, n-i)\, t^{n-k-1-i}$$

where $\hat{c}(n,k)$ *is chosen so that* $\chi(\mathbf{L}(\mathcal{D}_n^k), 1) = 0$. *In particular*

$$f(\Delta_n^k) = \hat{c}(n,k) + \sum_{i=0}^{n-k-2} c(n, n-i) \tag{3}$$

$$= \begin{cases} \sum_{i=0}^{\frac{n-k-2}{2}} 2\,c(n, n-2i) & \text{if } n-k \text{ even} \\ \sum_{i=0}^{\lfloor\frac{n-k-2}{2}\rfloor} 2\,c(n, n-2i-1) & \text{if } n-k \text{ odd} \end{cases} \tag{4}$$

Proof. The proof follows from repeated applications of Lemma 2.2 □

The configurations Δ_n^k will be the fundamental objects in this section. In the rest of this section we will show that they have largest number of inequivalent linear functionals among all configurations with n points in $\mathbb{R}^{n-1-k}$. To do this we will show that there is a weak map from $\mathcal{D}_n^k$ to any point configuration $\mathcal{C}$ of n points that spans $\mathbb{R}^{n-k-1}$. To do this we need to understand the circuits $\mathbf{C}(\mathcal{D}_n^k)$.

Lemma 3.3. *The circuits* $\mathbf{C}(\mathcal{D}_n)$ *are all of the form*

$$\{\{i,j\},\{j,k\},\{k,l\},\ldots,\{m,i\}\}.$$

Proof. □

Corollary 3.4. *If* $C \in \mathbf{C}(\mathcal{D}_n^k)$ *and* $|C| \leq n-k-1$ *then it has the form*

$$\{\{i,j\},\{j,k\},\{k,l\},\ldots,\{m,i\}\}.$$

Proof. This follows from Lemma 3.3 and Lemma 2.1. □

We can now prove our upper bound theorem for linear functionals

Theorem 3.5. *If* $\mathcal{C}$ *is a point configuration of* n *points in* $\mathbb{R}^d$, *then*

$$f(\mathcal{C}) \leq f(\Delta_n^{n-d-1}) = \sum_{i=0}^{\lfloor n-d-1/2\rfloor} 2\,c(n, n-2i).$$

Proof. We will show that there is a weak map from $\mathcal{D}_n^{n-d-1}$ to $\mathcal{D}(\mathcal{C})$. The result then follows from Lemma 2.3, Corollary 2.6 and Corollary 3.2.

By Corollary 3.4 if C is a circuit of $\mathcal{D}_n^{n-d-1}$ for which $|C| \leq d-1$ then it has the form

$$C = \{\{i,j\},\{j,k\},\{k,l\},\ldots,\{m,i\}\}.$$

But it is clear that

$$\{x_{\{i,j\}}, x_{\{j,k\}}, x_{\{k,l\}}, \ldots, x_{\{m,i\}}\}$$

is a dependent set in $\mathcal{D}(\mathcal{C})$. If $|C| \geq d$ then $\{x_{\{i,j\}} \mid \{i,j\} \in C\}$ is dependent since $\mathcal{C} \subseteq \mathbb{R}^d$. Hence every circuit in $\mathbf{C}(\mathcal{D}_n^{n-d-1})$ gives rise to a dependent set of $\mathcal{D}(\mathcal{C})$ and hence there is a weak map from $\mathcal{D}_n^{n-d-1}$ to $\mathcal{D}(\mathcal{C})$. □

The argument in Theorem 3.5 is a special case of the following more general framework. Suppose that $V = \{v_i \mid i \in E\} \subseteq \mathbb{R}^d$ and $V' = \{v_1' \mid i \in E\} \subseteq \mathbb{R}^{d'}$ are two sets of vectors that each span their ambient space with $d \geq d'$. If there is a linear map A such that $A\,v_i = v_i'$ for all $1 \leq i \leq n$, then there is a weak map from V to V'. It follows from [4, Lemma 9.3.1] that there is a set of vectors $\hat{V} = \{\hat{v}_1, \ldots, \hat{v}_n\}$ that spans $\mathbb{R}^{d'}$ such that $\mathrm{Trunc}_{d-d'}(\mathbf{L}(V)) = \mathbf{L}(\hat{V})$. Thus

$$(-1)^{d'}\,\chi(\mathbf{L}(\hat{V}), -1) \geq (-1)^{d'}\,\chi(\mathbf{L}(V'), -1).$$

4. Lower Bounds

In this section we will prove some lower bounds for the number of inequivalent linear functionals on n points in $\mathbb{R}^d$. The main theorem in this section only applies to point configurations in general position. Lower bounds for general configurations seem quite difficult. Indeed even in the case $d = 2$ complications arise. We begin with that discussion.

Let $\mathcal{C}$ be a point configuration in $\mathbb{R}^2$. It is easy to see that if the difference set $\mathcal{D}(\mathcal{C})$ has cardinality k then the related arrangement $\mathcal{A}$ of lines in the plane has size k and that $f(\mathcal{C}) = \mathcal{T}(\mathcal{A}) = 2k$. So the question of finding a lower bound on $f(\mathcal{C})$ for a point configuration in $\mathbb{R}^2$ is equivalent to asking what is the minimum size of the underlying set to $\mathcal{D}(\mathcal{C})$ for a 2-dimensional point configuration.

Two vectors $x_{\{i,j\}}, x_{\{k,l\}} \in \mathcal{D}(\mathcal{C})$ give rise to different hyperplanes $H_{x_{\{i,j\}}}$ and $H_{x_{\{k,l\}}}$ if and only if their slopes are different. Thus to minimize $f(\mathcal{C})$ we must minimize the number of different slopes that a set of lines in the plane can determine. Thankfully this job was done by Ungar [5].

Theorem 4.1. *Every set of n points in the plane determines at least $2\lfloor \frac{n}{2} \rfloor$ different slopes.*

Corollary 4.2. *If $\mathcal{C}$ is a point configuration that spans $\mathbb{R}^2$ with n points then $f(\mathcal{C}) \geq 4\lfloor \frac{n}{2} \rfloor$.*

Ungar's proof, a true gem, relies on purely combinatorial arguments. Jamison [3] has catalogued the extremal configurations. The fact that for $d = 2$ we already are exhibiting quasi-polynomial behavior is a hint that a closed form for a lower bound might be difficult.

On the other hand. If we make some assumptions to eliminate degeneracies the problem becomes tractable. We will say that a set of points $\mathcal{C} \in \mathbb{R}^d$ is in *general position* if no point is in any hyperplane spanned by the rest. Of course, in $\mathbb{R}^2$, this condition is equivalent to there being no 3 point lines.

We will require the following lemma

Lemma 4.3. *If $\mathcal{C}$ is collection of n affinely independent points in $\mathbb{R}^d$ then $f(\mathcal{C}) = n!$.*

Proof. Without loss of generality we will assume that $n = d+1$ and let $\mathcal{C} = \{a_0, a_2, \ldots, a_d\}$. If $\Delta_d = \{0 = e_0, e_1, \ldots, e_d\}$ is a standard simplex, where e_i is the i^{th} standard basis vector, then there is an invertible affine map $T(x) = A\,x + a_0$ that takes $e_i \to a_i$ for all $0 \le i \le d$.

Given any permutation π of the set $\{0, 1, \ldots, d\}$ there is a linear functional l that gives rise to that permutation of Δ_d, namely the linear functional $l = (l_1, l_2, ..., l_d)$ where

$$l_i = \begin{cases} -(d+1-\pi^{-1}(i), & \text{if } \pi^{-1}(i) < \pi^{-1}(0) \\ i, & \text{if } \pi^{-1}(i) > \pi^{-1}(0). \end{cases}$$

It then follows that the linear functional $(A^{-1})^{\mathrm{T}}\,l$ will give rise to the related ordering on $\mathcal{C}$. □

In what follows n_k is the falling factorial $n\,n-1\,\ldots n-k+1$.

Theorem 4.4. *Let $n \ge d+1 \ge 2$ and suppose that $\mathcal{C}$ is a point configuration in $\mathbb{R}^d$ in general position. Then $f(\mathcal{C}) \ge 2(n_{d-1})$.*

Proof. For each subset $X = \{x_1, x_2, \ldots, x_d\}$ of $\mathcal{C}$ we can produce $d!$ inequivalent linear functionals by doing the following: Suppose H is the hyperplane spanned by X. By the assumption of general position, no other point of $\mathcal{C}$ is on X. By adding a suitably small multiple of the linear functionals guaranteed by Lemma 4.3 we can construct $2(d!)$ linear functionals on all of $\mathcal{C}$

1. that are generic with respect to $\mathcal{C}$, and
2. have the points of X appearing consecutively

We can get $2(d!)$ different ones because we can always reverse the direction of the normal to get a new ordering. If we enumerate the set of ordered pairs $O = \{(\pi, X)\}$ where π is one of the permutations of

$\mathcal{C}$ arising from this construction associated with the subset X we see that $|O| = 2\,d!\,\binom{n}{d} = 2\,n_d$.

Let Π be the set of permutations of $\mathcal{C}$ arising from the above procedure. Given any $\pi \in \Pi$, we see that π can appear in an ordered pair of O at most $n-d+1$ times, since that is the number of different consecutive strings of length d in π. Thus we have that

$$|\Pi|\,(n-d+1) \geq |O| = 2\,n_d$$

and hence $|\Pi| \geq 2\,n_{d-1}$. □

Note that if $d=2$ the lower bound is $2n$ which is sharp for n even as Ungar's theorem demonstrates.

There is another natural way to eliminate degeneracies in this problem. We will say that a point configuration $\mathcal{C}$ is in *slope-general position* if no pair of the vectors $x_{\{i,j\}}, x_{\{k,l\}} \in \mathcal{D}(\mathcal{C})$ are linearly dependent. Equivalently, $\mathcal{C}$ is in slope-general position if $|\mathcal{A}| = \binom{n}{2}$. If $d=2$ then slope-general position implies general position but not vice-versa. On the other hand, if $d \geq 3$ then general position implies slope-general position but not vice-versa.

Conjecture 4.5. *Let $n \geq d+1 \geq 2$ and suppose that $\mathcal{C}$ is a point configuration in $\mathbb{R}^d$ in slope-general position. Then $f(\mathcal{C}) \geq n_d$.*

5. Monotone paths on zonotopes

Theorem 3.5 and Theorem 4.4 can be applied to get bounds on the number of monotome paths on zonotopes. In this section we will describe that application. We will employ the theory of fiber polytopes as described in [1].

Let V be a set of vectors $V = \{v_1, v_2, \ldots, v_n\} \subseteq \mathbb{R}^d$ which spans all of $\mathbb{R}^d$. The zonotope $Z(V)$ defined by V is the convex polytope defined by

$$Z(V) = \{x = \sum \alpha_v\, v \;:\; |\alpha_v| \leq 1,\ v \in V, \}.$$

It is well-known that the number of vertices of Z is equal to $|\mathcal{T}(\mathcal{A}(V))|$ [6]

Let π be a linear functional on $\mathbb{R}^d$ and then

$$\pi : Z \to Q = \{\pi(x) \,|\, x \in Z\} \subseteq \mathbb{R}^1$$

is a projection of Z to the 1-dimensional polytope $Q = [a,b]$ where $a = min\,\{\pi(x)\,|x \in P\}$ and $b = max\,\{\pi(x)\,|x \in P\}$. Associated with this map is the *fiber polytope* $\Sigma(Z,Q)$ [1], which in this case is called the *monotone path polytope* of Z and π. The vertices of this polytope are in bijection with the *coherent monotone paths* of Z with respect to π, certain paths on the boundary of Z that are monotone (strictly increasing) with respect to the linear functional π. For more details see [1, Section 5].

It turns out that the monotone path polytope of a zonotope is itself a zonotope which has an easy description. Let D be the collection of vectors

$$D = \{\pi(w)\, v - \pi(v)\, w \mid v, w \in V\}.$$

Then from [1, Lemma 2.3] and [1, Theorem 4.1] we can conclude that, up to a multiplicative constant,

$$\Sigma(Z, Q) = Z(D),$$

and the number of coherent monotone paths is the same as $|\mathcal{T}(\mathcal{A}(D))|$. The combinatorial type of $Z(D)$ is independent of the lengths of the vectors that define it. For each $v \in V$ let $\hat{v} = \frac{v}{\pi(v)}$. Then if we replace D by the set

$$\hat{D} = \{\hat{v} - \hat{w} \mid v, w \in V\}$$

we see that $Z(\hat{D})$ has the same combinatorial type as $Z(D)$. But $\hat{D} = \mathcal{D}(\mathcal{C})$ where $\mathcal{C} = \{\hat{v} \mid v \in V\}$ and thus

Theorem 5.1. *With the notation as above, the number of coherent monotone paths $Z(V)$ with respect to π is equal to the number of inequivalent linear functionals $f(\mathcal{C})$.*

Thus the bounds of Theorem 3.5 and Theoreom 4.4 apply to the number of coherent monotone paths on a zonotope.

References

[1] L. J. Billera and B. Sturmfels, *Fiber polytopes*, Ann. of Math., **135** (1992), 527–549.

[2] T. Brylawski, *Constructions*, Theory of Matroids (N. White, ed.), Encyclopedia of Mathematics, vol. 26, Cambridge University Press, 1986.

[3] R. Jamison, *A survey of the slope problem*, Discrete Geometry and Convexity (J.E. Goodman, E.Lutwak, J. Malkevitch, R. Pollack, eds.), Annals of the New York Academy of Sciences **440**, 34–51.

[4] J. P. S. Kung and H. Q. Nguyen, *Weak Maps*, Theory of Matroids (N. White, ed.), Encyclopedia of Mathematics, vol. 26, Cambridge University Press, 1986.

[5] P. Ungar, *2N noncollinear points determine at least 2N directions*, J. Combin. Th., Ser. A, **33** (1982), 343–347.

[6] T. Zaslavsky, *Facing up to arrangements: face-count formulas for partitions of space by hyperplanes*, Memoirs of the Amer. Math. Soc., no. 154, 1975.

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