# THE HIGHER STASHEFF–TAMARI POSETS

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*Abstract.* This paper studies higher dimensional analogues of the Tamari lattice on triangulations of a convex $n$-gon, by placing a partial order on the triangulations of a cyclic $d$-polytope. Our principal results are that in dimension $d \leqslant 3$, these posets are lattices whose intervals have the homotopy type of a sphere or ball, and in dimension $d \leqslant 5$, all triangulations of a cyclic $d$-polytope are connected by bistellar operations.

§1. *Introduction.* Let $C_n$ be a convex $n$-gon with vertices labelled $\{1, 2, \ldots, n\}$ counter-clockwise. Let $T_n$ be the set of triangulations of $C_n$ partially ordered by "swapping diagonals", *i.e.*, if $t$ is a triangulation that has $ij$ as an edge and the removal of $ij$ results in a quadrangle $i<k<j<l$ then the triangulation $t'$ obtained from $t$ by replacing the triangles $ijk$, $ijl$ with $ikl$, $jkl$ covers $t$. This poset has been of interest to combinatorists, computer scientists, and geometers since it was first defined over thirty years ago [Ta].

In a series of papers in the 60's and 70's Tamari *et al.*, proved many of the structural properties of this poset, including the fact that it is a lattice [Ta], [HT]. More recently in a series of papers Pallo [Pal1]–[Pal4] has continued these investigations computing, among other things, the Möbius function on the intervals of $T_n$. In his research the poset was defined in terms of the rotation distance of binary trees. This interpretation was considered also by Sleator, Tarjan and Thurston [STT] (although they were only interested in the graph whose edges form the Hasse diagram of $T_n$.) This poset has continued to appear as an object of study in its own right [Ge] as well as an example to which one can apply new combinatorial techniques of study [BW], [Sa].

Combinatorial geometers became interested in this poset because of a conjecture of Perles that the Hasse diagram of $T_n$, when viewed as a graph, is the 1-skeleton of a convex $(n-3)$-polytope. Previously Stasheff [St] had considered this graph as the 1-skeleton of a cell complex triangulating a $(n-4)$-sphere. Perles' conjecture was proven independently by Lee [Le] and Haiman [Ha]. The work of Gel'fand, Kapranov and Zelevinsky [GKZ1, GKZ2] later showed that given any polytope the graph of the *coherent* triangulations of that polytope, where adjacency is defined by *bistellar operations* is the 1-skeleton of a polytope which they called the *secondary polytope*. Note that a corollary to this fact is that the set of coherent triangulations of a convex polytope is connected by bistellar operations. In the case of a convex $n$-gon, all triangulations are coherent, and the bistellar operations are exactly the operation of swapping diagonals mentioned above. The work of Gel'fand *et al.* was further generalized by Billera and Sturmfels in their study of fiber polytopes [BS].

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In this paper we will consider the poset $T_n$ as the 2-dimensional case in an infinite family of posets which are based on the triangulations of cyclic $d$-polytopes. Our motivation for this study is that the cyclic polytopes are well-behaved polytopes with well understood combinatorial structure. Kapranov and Voevodsky have previously considered a poset structure on the triangulations of cyclic polytopes from a category-theoretic perspective [KV], and derived a relationship between those posets and the higher Bruhat orders of Manin and Schechtman [MS]. We conjecture that their poset is equivalent to one of the ones that we consider. We will show that in the case of dimension $d=3$ our partial orders are lattices and compute the homotopy-type of the intervals. In addition we show that the graph of all triangulations of a cyclic polytope of dimension less than or equal to 5 is bistellarly connected.

This paper is structured as follows: In Section 2 we define the higher Stasheff–Tamari posets and prove some basic lemmas. We also state a series of conjectures that motivate our work. Section 3 is devoted to reproving most of the known results in the case of $d=2$. We do this to illustrate our techniques in anticipation of Section 4 where we prove these conjectures for $d=3$. The last section is devoted to remarks and open problems.

§2. *Background, set-up and conjectures.* In this section, we set the scene by defining triangulations, bistellar operations, cyclic polytopes, and two natural partial orders on the set of their triangulations. For more background on these notions, see [GKZ2, Chapter 7]. We then present a number of conjectures about these partial orders, some of which will be proven in the sequel.

We begin by defining triangulations and bistellar operations. $\mathscr{A}$ will always denote a finite set of points in $\mathbb{R}^d$. A $(d+1)$-subset $\sigma$ of $\mathscr{A}$ which is affinely independent will be identified with the $d$-simplex which is its convex hull $conv(\sigma)$. A *triangulation* $T$ of $\mathscr{A}$ is a collection $T=\{\sigma\}$ of $d$-simplices whose union is all of $conv(\mathscr{A})$, whose interiors are all disjoint, and which intersect pairwise in a lower-dimensional boundary face (possibly empty) of each. It will sometimes be convenient, in order to refer to *links* of faces, to think of $T$ as an *abstract simplicial complex* on vertex set $\mathscr{A}$ whose faces are the subsets of the maximal faces $\sigma$. If $P$ is a polytope in $\mathbb{R}^d$, we will sometimes abuse terminology by speaking of a triangulation of $P$ when we mean a triangulation of the set $\mathscr{A}=vertices(P)$.

A *bistellar operation* on a triangulation $T$ of $\mathscr{A}$ is described as follows. Assume for some $e\leqslant d$ there is an $(e+2)$-subset $\mathscr{S}\subseteq\mathscr{A}$ with the following properties.

(1) $conv(\mathscr{S})$ is an $e$-dimensional polytope inside $conv(\mathscr{A})$.
(2) $T$ restricts to a triangulation $T|_{\mathscr{S}}$ of $\mathscr{S}$, *i.e.*, the faces of the simplicial complex $T$ which only involve vertices in $\mathscr{S}$ form a triangulation of $\mathscr{S}$.
(3) All (maximal) $e$-simplices $\tau$ in $T|_{\mathscr{S}}$ have the same *link* $L$ in $T$, where the link of a face $\tau$ in a simplicial complex is the subcomplex defined by

$$link_T(\tau)=\{\alpha\in T\colon \alpha\cap\tau=\varnothing,\ \alpha\cup\tau\in T\}.$$

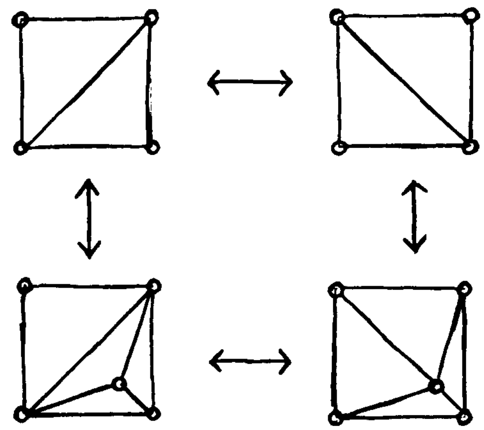

*Figure* 1. Examples of bistellar operations.

In this case, it follows easily from (1), (2) and the fact that $\mathscr{S}$ has cardinality $e+2$ that $T|_{\mathscr{S}}$ is of the form

$$Star_{\binom{\mathscr{S}}{e+1}}(\mathscr{R}) = \{\mathscr{F} \in \mathscr{S} : \#\mathscr{F} = e+1, \mathscr{R} \subseteq \mathscr{F}\},$$

for some subset $\mathscr{R} \subseteq \mathscr{S}$ (see [GKZ2, Proposition 1.2]). Then define the triangulation $T'$ by

$$T' = T - (Star_{\binom{\mathscr{S}}{e+1}}(\mathscr{R}) * L) \cup (Star_{\binom{\mathscr{S}}{e+1}}(\mathscr{S} - \mathscr{R}) * L),$$

where $*$ is the simplicial join. In this case, we say $T$ and $T'$ are related by a *bistellar operation on the set* $\mathscr{S}$. One can check that this is the same notion as the *perestroika* in [GKZ1] and the *stellar exchange* in [Pac]. Some examples are shown in Fig. 1.

We now introduce the *cyclic d-polytope* $C(n, d)$ as the convex hull of the points

$$\mathscr{A} = \{(t_i, t_i^2, \ldots, t_i^d)\}_{i=1,2,\ldots,n},$$

where $t_1 < t_2 < \ldots < t_n$ are $n$ values in $\mathbb{R}$. Some examples of $C(n, d)$ are shown in Fig. 2. Note that the projection map $\pi: \mathbb{R}^{d+1} \to \mathbb{R}^d$, which forgets the $(d+1)$-st coordinate, maps $C(n, d+1)$ onto $C(n, d)$ (*cf.* [KV], Theorem 2.5).

We will often abuse notation and refer to the point $(t_i, t_i^2, \ldots, t_i^d)$ by its index $i$, so that the entire point set $\mathscr{A}$ may be referred to as $[n] = \{1, 2, \ldots, n\}$. The polytope $C(n, d)$ and the actual coordinates of the points $i$ depend, of course, on the choice of $t_i$'s. However, it is well-known that its *combinatorial structure* does *not* depend on this choice (see [Gr], Chapter 4.7). For example, one can characterize which subsets $\mathscr{S} \subseteq [n]$ span a boundary face of $C(n, d)$. For notation, given $\mathscr{S} \subseteq [n]$, we define a *component* of $\mathscr{S}$ to be a maximal interval $[a, b] \subseteq \mathscr{S}$. A *proper component* is one that contains neither 1 nor $n$.

THEOREM 2.1 (Gale's Evenness Criterion). *An* $(e+1)$*-subset* $\mathscr{S} \subseteq [n]$ *spans a boundary e-face of* $C(n, d)$ *if, and only if,* $\mathscr{S}$ *contains at most* $d-(e+1)$ *proper components of odd cardinality.*

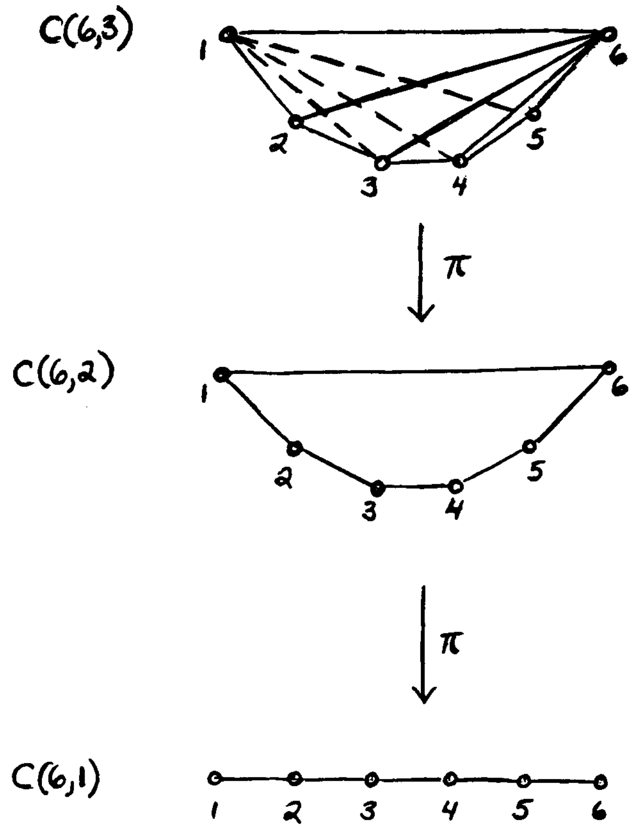


*Figure* 2. Examples of cyclic polytopes $C(n, d)$.

We leave it as an exercise for the reader to check that the question of whether a collection $T=\{\sigma\}$ of $d$-simplices forms a triangulation of $C(n, d)$ is also independent of the choice of $t_i$'s. For this reason we will often say that $T$ is a triangulation of $C(n, d)$, without referring to this choice, and denote by $S(n, d)$ the set of all triangulations of $C(n, d)$.

Gale's Evenness Criterion has the following strong consequence for bistellar operations on triangulations of $C(n, d)$.

PROPOSITION 2.2. *Let $T$ and $T'$ be triangulations of $C(n, d)$ which are related by a bistellar operation on the set $\mathscr{S}=\{i_1<i_2<\ldots<i_{e+2}\}$, and $\mathscr{R}$ and $L$ as in the definition of a bistellar operation. Then $e=d$ so that the link $L$ is empty, and $\mathscr{R}$ and $\mathscr{S}-\mathscr{R}$ are of the form*

$$\{i_1, i_3, i_5 \ldots\}, \{i_2, i_4, i_6 \ldots\}$$

*or vice-versa.*

*Proof.* Since $C(n, d)$ is a *simplicial* polytope, *i.e.*, all of its boundary faces are simplices (see [Gr, Section 4.7]), it follows that any triangulation of $C(n, d)$ must restrict to the same triangulation of the *boundary complex* $\partial C(n, d)$. In particular, $T$ and $T'$ must agree on the boundary, so we cannot have either $\mathscr{R}$ or $\mathscr{S}-\mathscr{R}$ entirely contained in boundary faces. But $C(n, d)$ is also $(\lfloor d/2 \rfloor - 1)$-*neighbourly*, *i.e.*, every $\lfloor d/2 \rfloor$-subset of $[n]$ forms a boundary face by the Evenness Criterion (see [Gr]). Thus

$$\#\mathscr{R},\ \#(\mathscr{S}-\mathscr{R}) > \lfloor d/2 \rfloor,$$

and

$$\#\mathscr{R} + \#(\mathscr{S}-\mathscr{R}) = \#\mathscr{S} = e+2 \leqslant d+2,$$

force $e=d$ so $L=\varnothing$, and without loss of generality

$$\#\mathscr{R} = \lfloor d/2 \rfloor + 1, \qquad \#(\mathscr{S}-\mathscr{R}) = \lceil d/2 \rceil + 1.$$

We can now restrict attention to the cyclic $d$-polytope $C(d+2, d)$ spanned by $\mathscr{S}$, and again we must have that neither $\#R$ nor $\#(\mathscr{S}-\mathscr{R})$ is contained in a boundary face. But one can check that this implies by the Evenness Criterion that $\mathscr{R}$ and $\mathscr{S}-\mathscr{R}$ have the form asserted in the Proposition.

Figure 3 shows the prototypical bistellar operations on triangulations of $C(n, d)$ described in the proposition for $d \leqslant 3$.

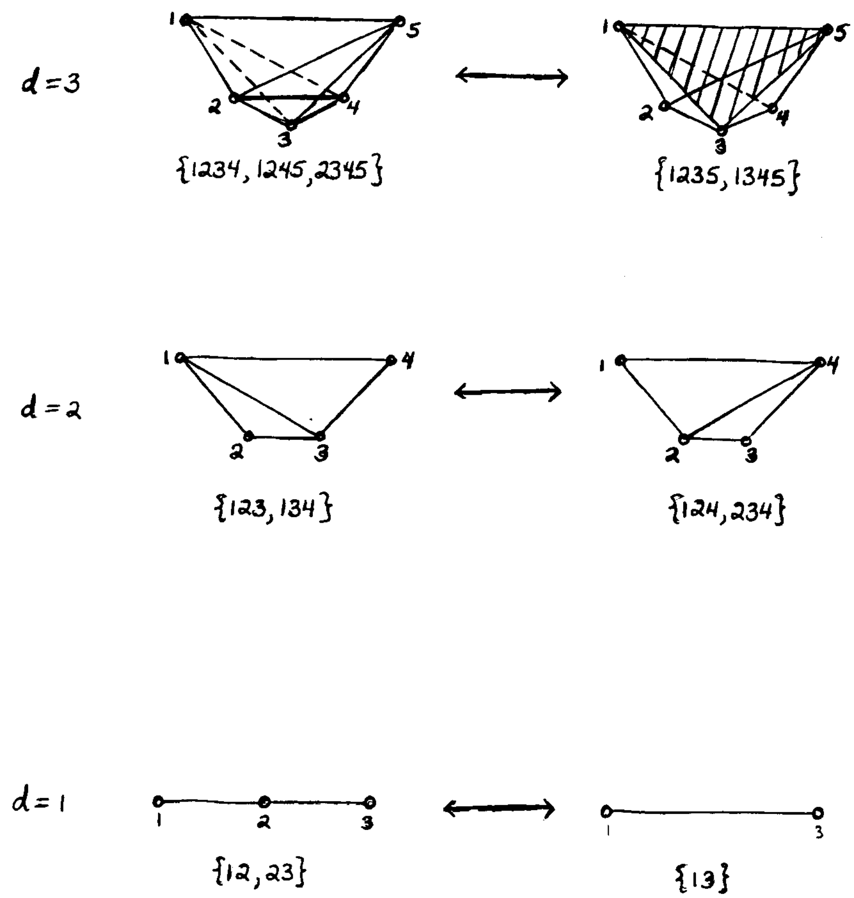


*Figure* 3. Prototypical bistellar operations on triangulations of $C(n, d)$.

The previous proposition suggests a natural partial order on the set $S(n, d)$ of triangulations of $C(n, d)$. Namely, take the transitive closure of the relation $T <_1 T'$ in which $T$ and $T'$ are related by a bistellar operation on the set $\mathscr{S} = \{i_1 < i_2 < \ldots < i_{d+2}\}$, and $T$ contains the $d$-simplex $i_1 i_2 \ldots i_{d+1}$. To check that the transitive closure of this relation does not create any cycles, note that the quantity given by summing over all simplices in $T$ the sum of the indices of its vertices is smaller than the same quantity for $T'$ when $d$ is even, and the sum of the simplices in $T$ is bigger than in $T'$ when $d$ is odd.

On the other hand, the projection map $\pi: C(n, d+1) \to C(n, d)$ suggests a second natural partial order on $S(n, d)$. To define it, note that if $T$ is a triangulation of $C(n, d)$, then $T$ defines a unique *section* $s_T$ of the projection $\pi: \mathbb{R}^{d+1} \to \mathbb{R}^d$ which lifts $i = (t_i, t_i^2, \ldots, t_i^d)$ to $\hat{i} = (t_i, t_i^2, \ldots, t_i^d, t_i^{d+1})$ and is extended *piecewise-linearly* over the simplices of $T$. Therefore given two triangulations $T$ and $T'$ of $C(n, d)$, we define $T \leqslant_2 T'$ if,

$$s_T(x)_{d+1} \leqslant s_{T'}(x)_{d+1}$$

for all points $x \in C(n, d)$, *i.e.*, if $s_T$ lifts all points of $C(n, d)$ *weakly lower* than $s_{T'}$ with respect to the $(d+1)$-coordinate.

Note that this partial order $\leqslant_2$ has an obvious top element $\hat{1}$ and bottom element $\hat{0}$. Specifically $\hat{1}_{n,d}$ is the triangulation of $C(n, d)$ consisting of all $\pi(\hat{\sigma})$ as $\hat{\sigma}$ runs over the *top* boundary $d$-faces of $C(n, d+1)$, where "top" means "visible from a point on the $x_{d+1}$-axis with very large $(d+1)$-coordinate". Similarly $\hat{0}_{n,d}$ is defined by replacing *top* with *bottom*. For the sequel we will need a more explicit description of these two triangulations. For integers $a < b$ and a positive integer $k$ say that $A \subseteq [a, b]$ is of *k-domino type* if $A$ can be written as the disjoint union of $k$ pairs $\{i, i+1\}$ where $a \leqslant i \leqslant b-1$. We will let $D(k, [a, b])$ be the set,

$$D(k, [a, b]) = \{A \subseteq [a, b] : A \text{ is of } k\text{-domino type}\}.$$

LEMMA 2.3. *The triangulations $\hat{1}_{n,d}$ and $\hat{0}_{n,d}$ are given explicitly as*

$$\hat{1}_{n,d} = \begin{cases} \{n\} * D\left(\frac{d}{2}, [1, n-1]\right), & d \text{ even}, \\ \{1, n\} * D\left(\frac{d-1}{2}, [2, n-1]\right), & d \text{ odd}; \end{cases}$$

$$\hat{0}_{n,d} = \begin{cases} \{1\} * D\left(\frac{d}{2}, [2, n]\right), & d \text{ even}, \\ D\left(\frac{d+1}{2}, [1, n]\right), & d \text{ odd}. \end{cases}$$

*Proof.* As mentioned previously, the triangulation $\hat{1}_{n,d}$ consists of the *top* boundary $d$-faces of $C(n, d+1)$, where "top" means "visible from a point on the $x_{d+1}$-axis with very large $(d+1)$-coordinate". This set of faces is equivalent to the set of faces visible from a point $n+1$ on the $(d+1)$-dimensional moment curve, and hence $\hat{1}_{n,d}$ is the set of facets of $C(n, d+1)$ which are not facets of

$C(n+1, d+1)$. It then follows from Theorem 2.1 that $\hat{1}_{n,d}$ has the description above.

The triangulation $\hat{0}_{n,d}$ is equivalent to the bottom facets of $C(n, d+1)$. It is clear that this set of facets is just the complement of $\hat{1}_{n,d}$. Applying Theorem 2.1 again results in the description above.

COROLLARY 2.4. *The link of* $\{n\}$ *in* $\hat{0}_{n,d}$ *is the triangulation* $\hat{1}_{n-1,d-1}$. *The triangulation* $\hat{0}_{n,d}$ *restricts to the triangulation* $\hat{0}_{n-1,d}$.

*Proof.* Both of these assertions follow easily from the explicit descriptions given in Lemma 2.3.

PROPOSITION 2.5. $\leqslant_1$ *is a weaker partial order than* $\leqslant_2$, *i.e.*,

$$T \leqslant_1 T' \Rightarrow T \leqslant_2 T'.$$

*Proof.* It suffices to show that if $T <_1 T'$ is a covering relation defining $\leqslant_1$ then $T \leqslant_2 T'$. Since $T$ and $T'$ differ by a bistellar operation on the $(d+2)$-set $\mathscr{S}$, the sections $s_T$ and $s_{T'}$ will agree on all points except for those inside the cyclic $d$-polytope $C(d+2, d)$ spanned by $\mathscr{S}$. Therefore, it suffices to assume $n = d+2$, and $T$ and $T'$ are the two possible triangulations of $C(d+2, d)$. However in this case, it easy to check by Lemma 2.3 that $T = \hat{0}$ and $T' = \hat{1}$ in the partial order $\leqslant_2$, so $T \leqslant_2 T'$.

We are now in a position to discuss a sequence of successively weaker conjectures, all implying that the set $S(n, d)$ of triangulations of $C(n, d)$ are connected by bistellar operations.

CONJECTURE 2.6 (True for $d \leqslant 3$). *The partial orders* $\leqslant_1$ *and* $\leqslant_2$ *coincide.*

In the case $d = 2$, the partial order $\leqslant_1$ is well-known as the *Tamari lattice* ([Ge], [HT], [Ta].) We conjecture that $\leqslant_1$ is equivalent to what Kapranov and Voevodsky call the *higher Stasheff order* $S(n, d)$ [KV, Definition 3.3]. For this reason we have opted to call the posets $\leqslant_1$ and $\leqslant_2$ the *higher Stasheff–Tamari orders* on $S(n, d)$.

*Remark.* This situation of two related partial orders is reminiscent of the *higher Bruhat orders* $B(n, d)$ introduced by Manin and Schechtman [MS] and further studied by Kapranov and Voevodsky [KV] and Ziegler [Zi]. In particular, [MS] defines the higher Bruhat order $B(n, d)$ by *single-step inclusion* on inversion sets as the transitive closure of a certain covering relation, analogous to the $\leqslant_1$ on $S(n, d)$. Ziegler also considers a related partial order $B_{\subseteq}(n, d)$ based on *inclusion* of inversion sets, analogous to the order $\leqslant_2$. It is well-known that these two orders coincide for $d = 1$, and proven for $d = 2$ in [ER], but Ziegler shows that they do *not* coincide already for $d = 3$ ([Zi], Theorem 4.5). Thus this situation contrasts with Conjecture 2.6.

The next two conjectures are a symmetric pair of weakenings of Conjecture 2.6.

CONJECTURE 2.7a (True for $d \leqslant 5$). *Let $T$ be a triangulation of $C(n, d)$. If $T \neq \hat{0}$ then there exists a covering relation $T' \lessdot_1 T$.*

CONJECTURE 2.7b (True for $d \leqslant 4$). *Let $T$ be a triangulation of $C(n, d)$. If $T \neq \hat{1}$ then there exists a covering relation $T' \gtrdot_1 T$.*

We note that if the partial order defined by Kapranov and Voevodsky is indeed equivalent to $\leqslant_1$, then their Theorem 4.10 [KV] would imply both Conjecture 2.7a and 2.7b.

Lastly, we have.

CONJECTURE 2.8 (True for $d \leqslant 5$). *Any two triangulations $T$ and $T'$ of $C(n, d)$ are connected by a sequence of bistellar operations.*

This last conjecture is a drastic weakening of a special case of the generalized Baues conjecture (see [BKS]). The truth of the Baues conjecture in this case would imply that the graph structure on $S(n, d)$ with edges given by bistellar operations is not just connected, but is furthermore the 1-skeleton of a simplicial complex homotopy equivalent to an $(n-d-1)$-sphere.

PROPOSITION 2.9. *Conjecture* 2.6 $\Rightarrow$ *Conjecture* 2.7a *or* 2.7b $\Rightarrow$ *Conjecture* 2.8.

*Proof.* Conjecture 2.6 $\Rightarrow$ Conjecture 2.7a: Assuming $T \neq \hat{0}$, then since $\hat{0} <_2 T$ there must be some covering relation $T' \lessdot_2 T$. But by Conjecture 2.6, this is also a covering relation $T' \lessdot_1 T$. The proof that Conjecture 2.6 implies Conjecture 2.7b is symmetric.

Conjecture 2.7a $\Rightarrow$ Conjecture 2.8: Using induction, one sees that every triangulation $T$ is connected by a chain of covering relations in $\leqslant_1$ to $\hat{0}$. But a covering relation in $\leqslant_1$ is always a bistellar operation, so every triangulation is connected to $\hat{0}$ by bistellar operations, and hence any two triangulations are connected. The proof that Conjecture 2.7b implies Conjecture 2.8 is symmetric.

PROPOSITION 2.10. *Conjecture* 2.7b *for* $d \Rightarrow$ *Conjecture* 2.7a *for* $d+1$.

*Proof.* We sketch a proof using induction on $n$. Assume Conjecture 2.7b for $d$, and let $T$ be a triangulation of $C(n, d+1)$ with $T \neq \hat{0}_{n,d+1}$ (here the subscripts $(n, d+1)$ indicate we are referring to the bottom element of $\leqslant_2$ on $S(n, d+1)$.) Consider the link $L$ of the vertex $n$ in the simplicial complex $T$, which triangulates the *vertex-figure* (see [Gr, page 49]) of $n$ in the polytope $C(n, d+1)$. It follows from Gale's Evenness Criterion (or see [BP]) that this vertex-figure is isomorphic to the cyclic polytope $C(n-1, d)$, and hence $L$ triangulates $C(n-1, d)$. From Corollary 2.4 we know that the link of the vertex $n$ in the triangulation $\hat{0}_{n,d+1}$ is the triangulation $\hat{1}_{n-1,d}$ of $C(n-1, d)$, and that this triangulation $\hat{0}_{n,d+1}$ restricts to the triangulation $\hat{0}_{n-1,d+1}$ of $C(n-1, d+1)$.

*Case* 1. $L=\hat{1}_{n-1,d}$. In this case, $T$ and $\hat{0}_{n,d+1}$ both restrict to triangulations of $C(n-1, d)$, where they must disagree, since $T\neq\hat{0}_{n,d+1}$. But then by induction on $n$ there is a covering relation $T' <_1 T|_{[n-1]}$ since $T|_{[n-1]}\neq\hat{0}_{n-1,d+1}$, and this gives a covering relation $T' * \{n\} <_1 T$ as desired.

*Case* 2. $L\neq\hat{1}_{n-1,d}$. In this case, by Conjecture 2.7b for $d$, there is a covering relation $L <_1 L'$ in $S(n-1, d)$. One can check that this always "lifts" to a covering relation $T' <_1 T$ in $S(n, d+1)$, in the sense that there is a covering relation $T' <_1 T$ where the link of $n$ in $T'$ is $L'$ and $L <_1 L'$. (We thank B. Sturmfels for pointing out the subtlety that the symmetric statement to this is **not** true, *i.e.*, there are covering relations $L >_1 L'$ in $S(n-1, d)$ which do not have a lift to a covering relation $T' >_1 T$ in $S(n, d+1)$.)

At this point it is useful to point out a symmetry in $S(n, d)$ which we have neglected so far. If we embed $C(n, d)$ symmetrically with respect to the $x_1$-axis by choosing $t_i=-t_{n+1-i}$, then the map,

$$i \overset{\alpha}{\leftrightarrow} n+1-i$$

on vertices induces an involutive symmetry of the polytope $C(n, d)$ and its set of triangulations $S(n, d)$. Note that this combinatorial symmetry is present regardless of the embedding of $C(n, d)$.

PROPOSITION 2.11. *The map $\alpha$ induces an order-preserving (resp. order-reversing) map on $\leqslant_1$ and $\leqslant_2$ for $d$ odd (resp. $d$ even). Hence both higher Stasheff–Tamari orders on $S(n, d)$ are self-dual for $d$ even.*

*Proof.* The assertion for $\leqslant_1$ follows immediately by checking that the map $\alpha$ *reverses* all the covering relations which define the order. The assertion for $\leqslant_2$ follows by noting that the map $t_i\leftrightarrow -t_i$ either inverts or preserves the $(d+1)$-st coordinate $t_i^{d+1}$ of $\hat{i}$ depending on the parity of $d$, and hence does the same for the $(d+1)$-st-coordinate of $s_T(x)$ for any $x\in C(n, d)$.

Figures 4(a) and (b) depict the posets $S(6, 2)$ and $S(7, 3)$. Figure 4(a) is self-explanatory. In Fig. 4(b) we have identified a triangulation with a list of 4-sets which are its tetrahedra, and the covering relations in $\leqslant_1$ are labelled by the 5-set on which the bistellar operation is based.

COROLLARY 2.12. *Assuming Conjecture* 2.6 *for a fixed odd value $d$ implies Conjecture* 2.7b *is true for $d+1$ and Conjectures* 2.7a *and* 2.8 *are true for $d+2$.*

*Proof.* Assume that Conjecture 2.6 is true for an odd value $d$. By Proposition 2.9 this implies Conjectures 2.7a and 7b for $d$. Then Proposition 2.10 implies Conjecture 2.7a for $d+1$. By the symmetry $\alpha$, Proposition 2.11 then implies Conjecture 2.7b for $d+1$. Applying Proposition 2.10 once more implies Conjecture 2.7a for $d+2$, and then Proposition 2.9 implies Conjecture 2.8 for $d+2$.

(a)

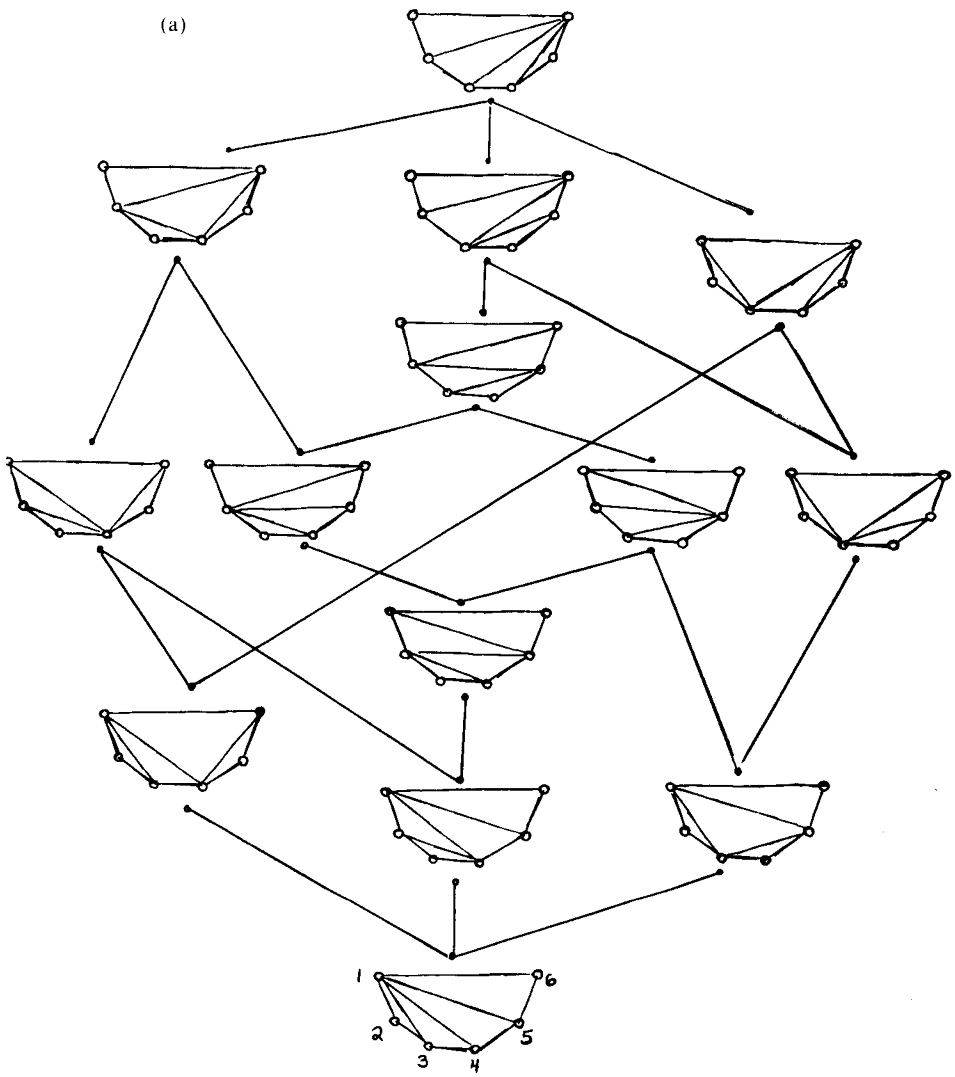


*Figure* 4. (a) The higher Stasheff–Tamari poset $S(6, 2)$.

Conjecture 2.6 for $d \leqslant 1$ is nearly trivial, and for $d = 2$ was (essentially) proven by Huang and Tamari [HT], who gave a very useful encoding of the Tamari lattice $S(n, 2)$. We will re-prove their result using our language in the next section. We will also prove Conjecture 2.6 for $d = 3$ in Section 4 by giving a very similar encoding for the higher Stasheff–Tamari poset $S(n, 3)$. There are a few other corollaries to this encoding, which we state here as conjectures for all $d$.

CONJECTURE 2.13 (True for $d \leqslant 3$). *The higher Stasheff–Tamari posets $S(n, d)$ are lattices.*

(b)

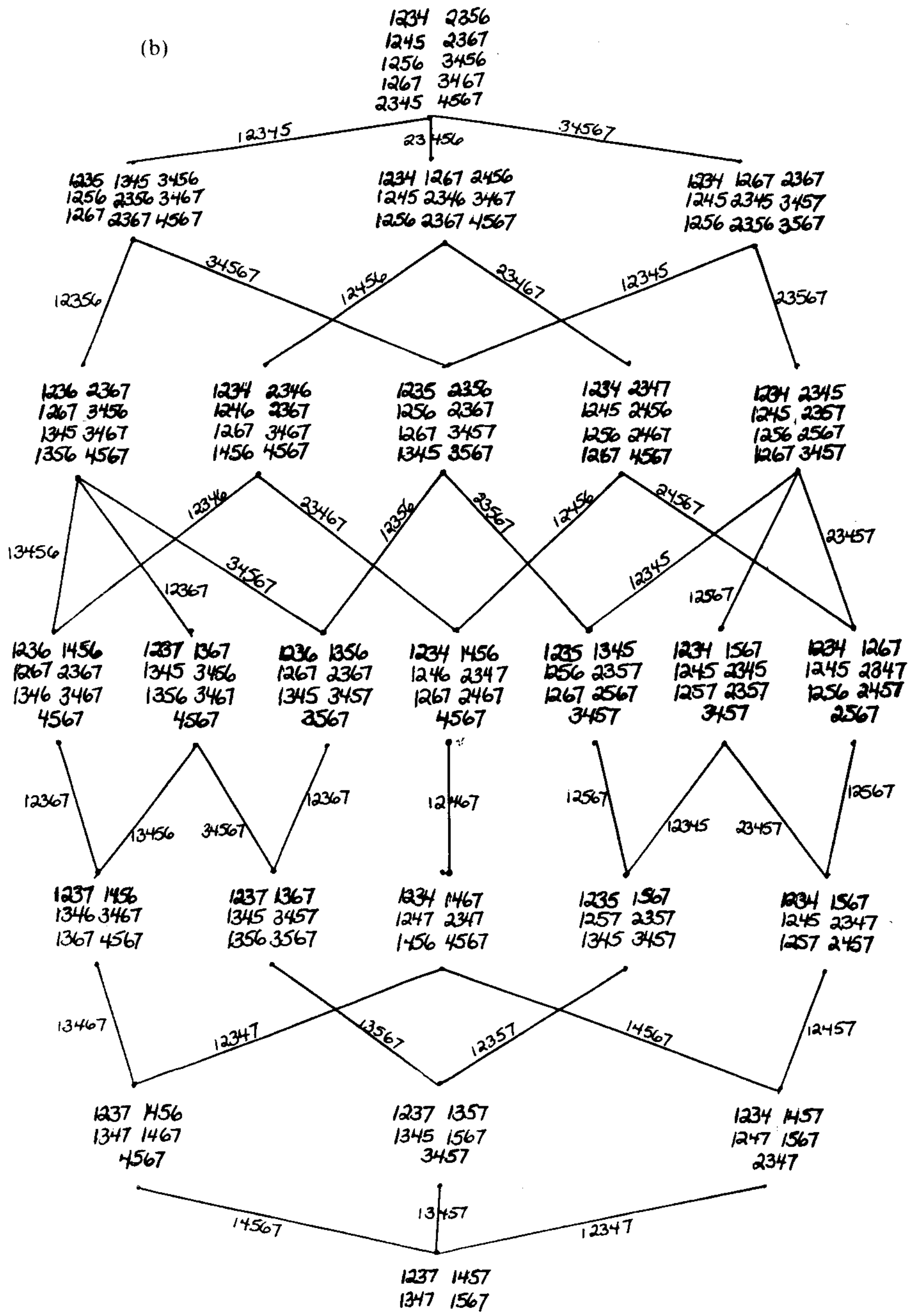


*Figure* 4. (b) The poset dual of S(7, 3).

CONJECTURE 2.14 (True for $d \leqslant 3$). *In any interval* $[x, y]$ *of* $S(n, d)$, *distinct subsets of the coatoms have distinct meets.*

The last conjecture was proven for $d=2$ by Pallo [Pal4]. A consequence of this conjecture is that the homotopy type of the simplicial complex of chains in any open interval $(x, y)$ in $S(n, d)$ is either contractible or spherical, and the Möbius function $\mu(x, y)$ is either 0, +1, or −1 [Bj, Theorem 2.1].

Before closing this section, we introduce a combinatorial way of encoding the partial order $\leqslant_2$ on $S(n, d)$ for general $d$, which leads to the simpler encodings for $d=2$ and 3. Given an $i$-simplex $\sigma$ spanned by some $(i+1)$-subset (which is also denoted $\sigma$) of the vertices of $C(n, d)$, say that $\sigma$ is *submerged* by the triangulation $T$ of $C(n, d)$ if the restriction of the piecewise linear section $s_T$ to the simplex $\sigma$ has the property that,

$$s_\sigma(x)_{d+1} \leqslant s_T(x)_{d+1}$$

for all points $x \in \sigma$, where $s_\sigma$ is the unique section of the map $\pi$ defined only over $\sigma$ by setting $s_\sigma(i) = \hat{i}$ and then extending piecewise-linearly over the rest of $\sigma$. For a triangulation $T$ of $C(n, d)$ define the *i-submersion set*, $sub_i(T)$, to be the set of $i$-simplices $\sigma$ which are submerged by $T$. It turns out that for appropriate values of $i$, the partial order $\leqslant_2$ is characterized by *inclusion* of $i$-submersion sets. Specifically, we have

PROPOSITION 2.15.

$$T_1 \leqslant_2 T_2 \quad \Leftrightarrow \quad sub_{\lceil \frac{1}{2}d \rceil}(T_1) \subseteq sub_{\lceil \frac{1}{2}d \rceil}(T_2).$$

*Proof.* The forward implication is trivial from the definitions of $\leqslant_2$ and submersion in terms of the sections $s_{T_1}$, $s_{T_2}$ and $s_\sigma$.

For the reverse implication, assume $sub_{\lceil \frac{1}{2}d \rceil}(T_1) \subseteq sub_{\lceil \frac{1}{2}d \rceil}(T_2)$, and choose any point $x \in C(n, d)$. We must show that $s_{T_1}(x)_{d+1} \leqslant s_{T_2}(x)_{d+1}$.

For $j = 1, 2$, let $\sigma_j$ be a $d$-simplex of the triangulation $T_j$ which contains the point $x$. Then $x \in \sigma_1 \cap \sigma_2 = P$, which is a convex polytope lying inside $C(n, d)$. Hence it is possible to express $x$ as,

$$x = \sum_{v \in vertices(P)} c_v v$$

with $0 \leqslant c_v \leqslant 1$ for all $v$ and $\sum_v c_v = 1$. Since both $s_{T_1}$ and $s_{T_2}$ are *linear* functions when restricted to $P = \sigma_1 \cap \sigma_2$, it suffices to show that $s_{T_1}(v)_{d+1} \leqslant s_{T_2}(v)_{d+1}$ for all vertices $v$ of $P$.

Now each vertex $v$ of $P$ is an intersection $F_1 \cap F_2$ where $F_j$ is some $d_j$-face of $\sigma_j$ for $j = 1, 2$, and because $v$ is 0-dimensional, we must have $d_1 + d_2 \leqslant d$.

*Case* 1. One of the $d_i$'s, say $d_1$, is less than $\lfloor \frac{1}{2}d \rfloor$. Then $F_1$ is a boundary $(\lfloor \frac{1}{2}d \rfloor - 1)$-face of $C(n, d)$ by the Evenness Criterion. Since the triangulations $T_1$ and $T_2$ must agree on the simplicial boundary $\partial C(n, d)$, it follows that $s_{T_1}$ and $s_{T_2}$ agree on the boundary, so they will agree on the point $v \in F_1$, which implies what we wanted.

*Case* 2. Both $d_i$ are at least $\lfloor \frac{1}{2}d \rfloor$. Then since their sum is at most $d$, they are both at most $\lceil \frac{1}{2}d \rceil$. In this case, we have that $v \in F_1$ which is a $d_1$-simplex of $T_1$, so $v$ lies in some $\lceil \frac{1}{2}d \rceil$-simplex $\sigma$ of $T_1$. But any $\lceil \frac{1}{2}d \rceil$-simplex of $T_1$ clearly lies in $sub_{\lceil \frac{1}{2}d \rceil}(T_1)$, and hence also in $sub_{\lceil \frac{1}{2}d \rceil}(T_2)$ by the original hypothesis of the proof. Therefore, we have,

$$s_{T_1}(v)_{d+1} = s_\sigma(v)_{d+1} \leqslant s_{T_2}(v)_{d+1}$$

as we wanted.

*Remark.* We will not need it in what follows, but one can easily deduce from Proposition 2.15 the following. For any $i \geqslant \lceil \frac{1}{2}d \rceil$ we have,

$$T_1 \leqslant_2 T_2 \Leftrightarrow sub_i(T_1) \subseteq sub_i(T_2).$$

In fact, it appears to be true that the same holds for any $i \geqslant \lfloor \frac{1}{2}d \rfloor$, although we have not proven this.

§3. *The case* $d=2$. In this section, we prove the main encoding result (Theorem 3.6) for the higher Stasheff–Tamari poset $S(n, 2)$, showing that it is the lattice of closed sets for a certain closure relation. From this we will deduce all of the conjectures of the previous section in the case $d=2$. This encoding turns out to be equivalent to that of Huang and Tamari [HT] for the Tamari posets, but is phrased in different language in their paper. For this reason, we have chosen to re-prove their results in this section using our language, and also because exactly analogous methods will be used in the next section for $d=3$.

We will need to describe the submersion set $sub_{\lceil \frac{1}{2}d \rceil}(T)$ *combinatorially* in the case $d=2$. To do this we begin with a lemma.

LEMMA 3.1. *Let* $e=ij$ *and* $e'=kl$ *be* 1*-simplices inside* $C(n, 2)$ *with* $k<i<l<j$. *If* $x=e \cap e'$ *then* $s_{e'}(x)_3 < s_e(x)_3$.

*Proof.* By choosing coordinates for $i$, $j$, $k$ and $l$ one can solve explicitly for $s_{e'}(x)_3$ and $s_e(x)_3$ and see that the inequality holds. The details are left to the reader.

PROPOSITION 3.2. *Let* $T$ *be a triangulation of* $C(n, 2)$ *and* $e=ij$ *a* 1*-simplex (or edge) inside* $C(n, 2)$. *Then* $ij$ *is in* $sub_1(T)$ *if, and only if, there does not exist an edge* $e'=kl$ *of* $T$ *with* $k<i<l<j$.

*Sketch of proof.* Using a case analysis and techniques similar to those used to prove Proposition 2.15 one can show the following: checking whether $ij \in sub_1(T)$ is equivalent to checking whether any edge $e'$ in $T$ has $x=e \cap e'$ with $s_{e'}(x)_3 < s_e(x)_3$. Applying Lemma 3.1 finishes the proof.

Given this proposition, there are a number of necessary combinatorial conditions on the set $sub_1(T)$.

PROPOSITION 3.3. *If $I$ is a collection of 2-subsets of $[n]$ with $I = sub_1(T)$ for some triangulation $T$ of $C(2, n)$, then $I$ satisfies the following conditions.*

D1. *$I$ contains all the edges of the boundary $\partial C(n, 2)$.*

D2. *If $ij$ is in $I$ and $i < j' < j$, then $ij'$ is in $I$.*

D3. *If $ik$, $jl$ are in $I$ with $i < j < k < l$, then $il$ is in $I$.*

*Proof.* Assume $I = sub_1(T)$ for some triangulation $T$ of $C(n, 2)$. Property D1 follows either from the definition of submersion, which implies that all boundary faces of $C(n, d)$ are submerged for any triangulation, or from Proposition 3.2 once it is observed that the boundary edges of $C(n, 2)$ are exactly 12, 23, 34, ..., $(n-1)n$ and $1n$.

We now prove the contrapositive of D2. If $i, j, j'$ satisfy $i < j' < j$, and $ij'$ is not in $I = sub_1(T)$, then by Proposition 3.2, there must be some edge $kl$ of $T$ with $k < i < l < j'$. But then this implies $k < i < l < j$, so $ij$ is also not in $I$.

To prove property D3, again work with the contrapositive. If $i$, $j$, $k$, $l$ satisfy $i < j < k < l$ and $il$ is not in $I = sub_1(T)$, then by Proposition 3.2, there must be some $ab$ of $T$ with $a < i < b < l$. We either have $b < k$, in which case $a < i < b < k$ so that $ik$ is not in $I$, or we have $b \geqslant k$, in which case $a < j < b < l$ so that $jl$ is not in $I$.

We will call a collection $I$ of 2-subsets of $[n]$ *diagonally closed* if it satisfies conditions D1, D2 and D3. Our goal now is to show the converse of Proposition 3.3, *i.e.*, if $I$ is diagonally closed then $I = sub_1(T)$ for some triangulation $T$ of $C(n, 2)$. Therefore beginning with a diagonally closed $I$ we must first exhibit a candidate triangulation $T$, and then check that it satisfies $sub_1(T) = I$. We first describe our candidate for the triangulation $T$.

Let $M$ be the set of "maximal edges" in the following sense: $M$ is the subset of edges $ij$ in $I$ for which there does not exist $kl$ in $I$ satisfying $i < k < j < l$. The set $M$ is our candidate for the edges of triangles in the triangulation $T$, and therefore define $T$ to be the set of triangles $ijk$ for which $ij$, $ik$, $jk$ are all in $M$.

THEOREM 3.4. *With $I$, $M$ and $T$ as above, $T$ is a triangulation of $C(n, 2)$, having edge set $M$.*

*Proof.* Note first that $M$ contains every edge of the boundary $\partial C(n, 2)$, since $I$ contains them by D1, and they are all maximal. Also note that $T$ cannot contain two triangles with intersecting interiors, since in all cases this would force two edges $e$, $e'$ in $M$ with intersecting interiors, and then one would not be maximal.

Given these two observations, we claim that it suffices to show that for every edge $e$ in $M$, either $e$ is contained in at least two triangles $t$, $t'$ of $T$ if $e$ is not in the boundary $\partial C(n, 2)$, or $e$ is contained in at least one triangle if $e$ is in $\partial C(n, 2)$. To prove this claim, we first argue that the triangles in $T$ must cover $C(n, 2)$. For if they missed some point $x$ of $C(n, d)$, one could "walk" from $x$ in a generic direction toward the boundary $\partial C(n, 2)$, and then the first edge $e$ of a triangle of $T$ encountered along the way would contradict one of the observations in the first paragraph. Since the triangles of $T$ were already observed to have disjoint interiors, they must form a triangulation of $C(n, 2)$.

So choose an edge $e=ij$ in $M$ which is not in $\partial C(n,2)$, and we will show that it lies in at least two triangles $t$, $t'$ of $T$ (the argument for why $e$ lies in at least one triangle $t$ of $T$ if $e$ is in $\partial C(n,2)$ is very similar but easier, and will be omitted).

To obtain the first triangle $t$, let $a$ be the smallest value in the open interval $(i,j)$ with $aj$ in $I$ (there exist such values since $j-1$ is one of them by D1), and let $t=iaj$. We now start to reason using D2, D3 and the maximal property of edges in $M$. We must have $aj \in M$, since any edge $bc$ with $a<c<j<b$ would have $i<c<j<b$, contradicting $ij$ in $M$. Also $ia$ is in $I$ by D2 since $ij$ is in $I$. We further claim that $ia$ is in $M$, because if $bc$ in $I$ satisfies $i<b<a<c$ then there are three cases for $c$ which all lead to contradictions: (1) if $c<j$ then $bc$, $aj$ in $I$ imply using D3 that $bj$ is in $I$, which contradicts the choice of $a$, (2) if $c=j$ then again $bc=bj$ is in $I$ contradicting the choice of $a$, or (3) if $c>j$, then $i<b<j<c$ contradicts $ij$ being in $M$. Hence we conclude that $ia$, $ij$, $aj$ are all in $M$, so $t=iaj$ is a triangle of $T$ containing $e=ij$.

To obtain the second triangle $t'$, we consider two cases.

*Case* 1. There exists $b>j$ such that $ib$ is in $I$. If so, let $b$ be the largest value with $b>j$ such that $ib$, $jb$ are both in $I$. Such a value $b$ exists, since $j+1$ is an example because the hypothesis that $ib$ is in $I$ for some $b>j$ implies by D2 that $i(j+1)$ is in $I$, and $j(j+1)$ is in $I$ by D1. Let $t'=ijb$. We claim that $jb$ must be in $M$, since otherwise there is some $cd$ in $I$ with $j<c<b<d$, which would imply using D3 on $ib$, $cd$ and on $jb$, $cd$ that $id$, $jd$ were both in $I$, contradicting the choice of $b$. We further claim that $ib$ is in $M$, because if $cd$ in $I$ satisfies $i<c<b<d$ then there are three cases for $c$ which all lead to contradictions: (1) if $c>j$ then we have already seen the contradiction, (2) if $c=j$ then applying D3 to $cd=jd$ and $ib$ gives that $id$ is in $I$, and then $id$, $jd$ being in $I$ again contradicts the choice of $b$, or (3) if $c<j$ then $i<c<j<d$ contradicts $ij$ being in $M$. Hence in this case, $t'=ijb$ has $ij$, $jb$, $ib$ all in $M$, so $t'$ is a second triangle in $T$ containing $e=ij$.

*Case* 2. There is no $b>j$ with $ib$ in $I$. If so, then let $b$ be the largest value less than $i$ such that $bi$, $bj$ are both in $I$. Such a value $b$ exists, since 1 is an example (using D1 and D2), unless $i=1$ in which case $1n$ is in $I$ so we must actually be in Case 1! Let $t'=bij$. We claim that $bi$ must be in $M$, since otherwise there is some $cd$ in $I$ with $b<c<i<d$, which implies either by using D2 if $d \geqslant j$ or by applying D3 to $cd$, $ij$ if $d<j$, that $cj$ is in $I$ and hence by D2 that $ci$ is in $I$ contradicting the choice of $b$. We further claim that $bj$ is in $M$, because if $cd$ in $I$ satisfies $b<c<j<d$ then there are three cases for $c$ which all lead to contradictions: (1) if $c<i$ then we arrive at the same contradiction with $b<c<i<d$ as before, (2) if $c=i$ then $id$ being in $I$ would contradict the assumption of Case 2, (3) if $c>i$, then $i<c<j<d$ contradicts $ij$ being in $M$. Hence in this case, $t'=bij$ has $bi$, $bj$, $ij$ all in $M$, so $t'$ is a second triangle in $T$ containing $e=ij$.

It now remains to show.

PROPOSITION 3.5. *With $I$, $M$ and $T$ as above, $I=sub_1(T)$.*

*Proof.* We first show the forward inclusion $I \subseteq sub_1(T)$. Note that for any fixed $i$, if $j$ is the largest value such that $ij$ is in $I$, then $ij$ is in $M$, since otherwise there is some $kl$ in $I$ with $i<k<j<l$ and then by D3 one would have $il$ in $I$, contradicting the choice of $j$. Therefore the closure of $M$ under property D2 generates all of $I$. But since every edge in $M$ is also an edge of the triangulation $T$, it follows that $M \subseteq sub_1(T)$ (any edge of a triangulation $T$ is always submerged by that triangulation). Since $sub_1(T)$ is closed under property D2 by Proposition 3.3, we conclude that $I \subseteq sub_1(T)$.

To show the reverse inclusion $sub_1(T) \subseteq I$, assume $e$ is an edge not contained in $I$, and we will show that it is not contained in $sub_1(T)$. Since all edges of $T$ are in $M \subseteq I$, it must be that $e$ is not an edge of $T$. Let $e = ij$ with $i<j$. Since $T$ is a triangulation which covers $C(n, 2)$, there must be a triangle $t$ of $T$ which intersects the interior of $e$ non-trivially and also contains one of the endpoints $i$ or $j$. If $i$ is in $t$, then $t$ must either have the form $t = ikl$ with some $i<k<j<l$ or $t = kil$ with $k<i<l<j$. In the first case, $il$ being in $M \subseteq I$ implies by D2 that $ij$ is in $I$, a contradiction. The second case implies that $ij$ is not in $sub_1(T)$.

The last three results together prove the main encoding theorem for $d=2$.

THEOREM 3.6 (*cf.* [HT, Proposition 2]). *The map $T \mapsto sub_1(T)$ is an isomorphism between the higher Stasheff–Tamari order $\leqslant_2$ on $S(n, 2)$ and the lattice of diagonal-closed families $I$ in $\binom{[n]}{2}$ ordered by inclusion.*

*Proof.* The only thing which remains to be proven is that the poset of diagonally closed families $I$ really *is* a lattice. Clearly the intersection of any two diagonally closed sets $I_1$ and $I_2$ will also be diagonally closed, so the lattice meet $I_1 \wedge I_2$ is the intersection $I_1 \cap I_2$. Since the poset has the top element $\hat{1}$, it follows that the join is also defined, and hence it is a lattice.

The last result also proves Conjecture 2.13 for the partial order $\leqslant_2$ in the case $d=2$. For future use, we record here the connection of Theorem 3.6 to Huang and Tamari's *bracketing functions.* Given a triangulation $T$ of $C(n, 2)$, define a function $f_T\colon [2, n-2] \to [3, n]$ by

$$f_T(i) = max\{j\colon ij \text{ is an edge of } T\}$$
$$= max\{j\colon ij \in sub_1(T)\}.$$

This function $f_T$ is (up to a slight translation) the *bracketing function* of [HT].

PROPOSITION 3.7. *The bracketing function $f_T$ satisfies these axioms:*
B1. *$f_T(i) > i$ for all $i$;*
B2. *if $i<j<f_T(i)$ then $f_T(j) \leqslant f_T(i)$;*
*and these axioms characterize the functions $f\colon [2, n-2] \to [3, n]$ which are bracketing functions $f_T$ for some triangulation. Furthermore, the map $T \mapsto f_T$ is an isomorphism from the higher Stasheff–Tamari order $S(n, 2)$ to the set of bracketing functions ordered pointwise.*

*Proof.* Property D2 for submersion sets shows that $sub_1(T)$ is completely characterized by knowing the values of the bracketing function $f_T$, and furthermore that inclusion of submersion sets corresponds to the pointwise partial order on bracketing functions. Axiom D3 then translates into axiom B2 for bracketing functions.

The next theorem proves Conjecture 2.6 for $d=2$ and finishes the proof of Conjecture 2.13 for $d=2$.

THEOREM 3.8 (*cf.* [HT, Corollaries 2 and 3]). *The higher Stasheff–Tamari orders $\leqslant_1$ and $\leqslant_2$ coincide for $d=2$.*

*Proof.* We already know that $T \leqslant_1 T'$ implies $T \leqslant_2 T'$ from Proposition 2.5. So assume that $T \leqslant_2 T'$ and we will show $T \leqslant_1 T'$.

By induction on the length of the longest chain between $T$ and $T'$, it suffices to show there exists $\tilde{T}$ with $T \leqslant_2 \tilde{T} \lessdot_1 T'$, where $\lessdot_1$ denotes a covering relation in the order $\leqslant_1$. To construct $\tilde{T}$, note that by Proposition 3.7 we have $f_T < f_{T'}$, so there must be some largest value $i$ for which $f_T(i) < f_{T'}(i)$. Let $k = f_{T'}(i)$ and thus $ik$ is an edge of $T'$. Since $ik$ is not an edge of $sub_1(T)$ it cannot lie on the boundary $\partial C(n, 2)$, and so there must be a triangle $ijk$ of $T'$ with $i<j<k$ containing $ik$. Since $i>1$, the edge $ik$ must lie in exactly one other triangle $hik$ of $T'$ with $h<i<k$. Let $\tilde{T}$ be obtained from $T'$ by a bistellar operation on the set $hijk$. One can check that this does not affect the value of the bracketing function except at $i$, *i.e.*, $f_{\tilde{T}}(m) = f_{T'}(m)$ for all $m$ except $i$, and $f_{\tilde{T}}(i) \geqslant j$. So the proof would be done if we could show $j \geqslant f_T(i)$, since this would imply $f_T \leqslant f_{\tilde{T}}$ as we want.

So assume not, *i.e.*, $j < f_T(i)$. Then $i<j<f_T(i)$ implies by B2 that $f_T(j) \leqslant f_T(i)$. But then we have,

$$f_T(j) \leqslant f_T(i) < f_{T'}(i) = k \leqslant f_{T'}(j),$$

the last inequality following from the fact that $ijk$ is a triangle in $T'$. This contradicts the choice of $i$ and finishes the proof.

Lastly we prove Pallo's result (Conjecture 2.14 for $d=2$).

THEOREM 3.9 (*cf.* [Pal4, Lemma 4.1]). *In any interval $[x, y]$ of $S(n, 2)$, distinct subsets of the coatoms have distinct meets.*

*Proof.* Let $T \leqslant T'$ in $S(n, 2)$, and let $T_1, T_2, \ldots, T_r$ be the coatoms of the interval $[T, T']$. Then since each $T_m$ is covered by $T'$, there is some bistellar operation by which it differs from $T'$. Since the bistellar operations for $d=2$ are *diagonal flips* in which a quadrangle $ijkl$ exchanges one of its diagonal edges $jl$ for the other diagonal $ik$, there must exist for each $m$ an edge $i_m j_m$ which is present in the triangulation $T'$ but absent in $T$. Notice also that $i_m j_m$ will be an edge in all the other $T_p$ with $p \neq m$ since they differ from $T'$ by a *different* diagonal flip. This implies that for each $m$, the edge $i_m j_m$ is an element of $sub_1(T_p)$ for all $p \neq m$, but not an element of $sub_1(T_m)$. Since the meet operation in $S(n, 2)$ corresponds to intersection of 1-submersion sets, this implies that distinct subsets of $T_1, T_2, \ldots, T_r$ will have distinct meets.

The significance of the previous result for the homotopy type and Möbius function of intervals in $S(n, 2)$ was discussed in the previous section.

§4. *The case* $d=3$. The development of this section will exactly parallel the previous one. We prove the main encoding result (Theorem 4.9) for the higher Stasheff–Tamari poset $S(n, 3)$, showing that the partial order $\leqslant_2$ is the lattice of closed sets for a certain closure relation. We will also prove that the two partial orders $\leqslant_1$ and $\leqslant_2$ are equivalent. From these facts we can establish all of the conjectures of Section 2 in the case $d=3$.

As before, we begin with a simple proposition (whose proof is similar to that of Proposition 3.2 and is left to the reader) which describes the submersion set $sub_{\lceil \frac{1}{2}d \rceil}(T)$ *combinatorially* in the case $d=3$.

PROPOSITION 4.1. *Let T be a triangulation of* $C(n, 3)$ *and* $t=ijk$ *a 2-simplex (or triangle) inside* $C(n, 3)$. *Then ijk is in* $sub_2(T)$ *if, and only if, there does not exist an edge* $e=ab$ *of T which "intertwines" ijk in the sense that* $i<a<j<b<k$.

One may interpret the condition in the proposition geometrically as saying that there does not exist an edge $e$ of $T$ which pierces the interior of the triangle $t$, *i.e.*, which intersects $t$ transversally in one interior point. In fact this geometric condition is equivalent to the combinatorial one, a fact which will be of use in the sequel.

Given this proposition, there are a number of necessary combinatorial conditions on the set $sub_2(T)$.

PROPOSITION 4.2. *If I is a collection of* 3-*subsets of* $[n]$ *with* $I=sub_2(T)$ *for some triangulation T of* $C(3, n)$, *then I satisfies the following conditions.*

T1. *I contains all the edges of the boundary* $\partial C(n, 3)$.

T2. *If ijk is in I and* $j<k'<k$, *then ijk' is in I. Similarly, if* $i<i'<j$ *then i'jk is in I.*

T3. *If ijk and abc are in I with* $a<i<b<j<c<k$, *then abk and ajk are in I.*

*Proof.* Assume $I=sub_2(T)$ for some triangulation $T$ of $C(n, 2)$. Property T1 follows either from the definition of submersion sets, which implies that all boundary faces of $C(n, d)$ are submerged for any triangulation, or from Proposition 4.1 once it is observed (*e.g.*, *via* the Evenness Criterion) that the boundary triangles of $C(n, 3)$ are exactly,

$$123, 134, 145, \ldots, 1(n-1)n$$

$$\text{and } 12n, 23n, 34n, \ldots, (n-2)(n-1)n.$$

To show property T2, we prove the contrapositive. If $i, j, k$ and $k'$ satisfy $i<j<k'<k$, and $ijk'$ is not in $I=sub_2(T)$, then by Proposition 4.1, there must

be some edge $ab$ of $T$ with $i<a<j<b<k'$. But then this implies $i<a<j<b<k$, so $ijk$ is also not in $I$. The argument for the other case is symmetric.

To prove property T3, we need the following lemma.

LEMMA 4.3. *If $ik$ is an edge of a triangulation $T$ of $C(n, 3)$ and $k>i+1$, then there exists some $j$ with $i<j<k$ such that $ij$ and $jk$ are also both edges of $T$.*

*Proof.* Assume $ik$ is an edge of $T$ and $k>i+1$. Let $C$ be the strictly smaller cyclic polytope $C(d, n')$ obtained by taking the convex hull of the points $[1, i] \cup [k, n]$. Starting at the midpoint of the edge $ik$, move in any direction *out of* $C$ by a small distance $\varepsilon$. If $\varepsilon$ is chosen small enough, then one will stay inside of $C(n, d)$, and hence one can choose $\varepsilon$ even smaller so that during the entire process one never leaves a certain 3-simplex (tetrahedron) $\sigma$ of the triangulation $T$. But then this tetrahedron $\sigma$ must have the edge $ik$ in its boundary, and it must also contain some vertex $j$ with $i<j<k$ or else it would be a tetrahedron contained in $C$. Hence, $\sigma$ contains the edges $ij$, $jk$, and therefore they are edges of $T$, as we wanted.

Given this lemma, to prove T3 we again work with the contrapositive. Assume $a<i<b<j<c<k$ and $abk$ is not in $I$ (the argument will be symmetric if $ajk$ is not in $I$). This implies by Proposition 4.1 that there is some edge $de$ of $T$ which intertwines $abk$ *i.e.*, $a<d<b<e<k$. If $d>i$ then $de$ intertwines $ijk$ or $abc$ depending on the location of $e$, so either $ijk$ or $abc$ is not in $I$. If $e<c$ then $de$ intertwines $abc$ so $abc$ is not in $I$. If neither of these hold, so that $d\leqslant i$ and $e\geqslant c$, one can repeatedly apply Lemma 4.3 to produce another edge $d'e'$ in $T$ which is one of the previous cases and hence intertwines $abc$ or $ijk$, so that one of these two triangles is not in $I$.

Call a collection $I$ of 3-subsets of $[n]$ *triangle closed* if it satisfies conditions T1, T2 and T3. Our goal is then to show the converse of Proposition 4.2, *i.e.*, if $I$ is triangle closed then $I=sub_2(T)$ for some triangulation $T$ of $C(n, 3)$. Beginning with a triangle closed set $I$ we must first exhibit a candidate triangulation $T$, and then check that it satisfies $sub_2(T)=I$. We first describe our candidate for the triangulation $T$.

Let $M$ be the set of "maximal triangles" in the following sense: $M$ is the subset of triangles $ijk$ in $I$ for which none of the three edges $ij$, $ik$, $jk$ intertwine any triangle in $I$. The set $M$ is our candidate for the boundary triangles of tetrahedra in the triangulation $T$, and therefore define $T$ to be the set of tetrahedra $ijkl$ for which $ijk$, $ijl$, $ikl$, $jkl$ are all in $M$.

THEOREM 4.4. *With $I$, $M$ and $T$ as above, $T$ is a triangulation of $C(n, 3)$.*

*Proof.* Note first that $M$ contains every triangle of the boundary $\partial C(n, 3)$, since $I$ contains them by T1, and they are all maximal. Also note that $T$ cannot contain two tetrahedra with intersecting interiors, since one can check that in all cases this would force one of them to have an edge which intertwined a triangle of the other.

Given these two observations, we claim that it suffices to show that for every triangle $t$ in $M$, either $t$ is contained in at least two tetrahedra $\sigma$, $\sigma'$ of $T$ if $t$ is not in the boundary $\partial C(n, 3)$, or $t$ is contained in at least one tetrahedron if $t$ is in $\partial C(n, 3)$. The proof of this claim is essentially identical to the analogous statement in the proof of Theorem 3.4.

So choose a triangle $t = ijk$ in $M$ which is not in $\partial C(n, 3)$, and we will show that it lies in at least two tetrahedra $\sigma$ and $\sigma'$ of $T$ (the argument for why $t$ lies in at least one tetrahedron $\sigma$ of $T$ if $t$ is in $\partial C(n, 3)$ is very similar but easier, and will be omitted).

The proof will follow from two lemmas regarding the location of candidates for the fourth vertex in a tetrahedron which contains $ijk$.

LEMMA 4.5. If there exists a value a > k with ija in I, then there is such a value for which the tetrahedron ijka is in T.

LEMMA 4.6. If there does not exist a value b < i with bjk in I, then there is a value c with j < c < k such that the tetrahedron ijck is in T.

Before proving these lemmas, let us see how they imply the theorem. If there exists $a > k$ with $ija$ in $I$, then Lemma 4.5 produces the first tetrahedron $\sigma = ijka$ containing $ijk$ that we want. If there also exists $b < i$ with $bjk$ in $I$, then applying the symmetry $\alpha: i \leftrightarrow n+1-i$ of Proposition 2.11 to Lemma 4.5 produces a second tetrahedron $\sigma' = bijk$ containing $ijk$, and we are done. If such a $b$ does not exist, then Lemma 4.6 produces a second tetrahedron $\sigma' = ijck$ containing $ijk$, and we are done. On the other hand, if there does not exist $a > k$ with $ija$ in $I$, then applying the symmetry $\alpha$ to Lemma 4.6 produces the first tetrahedron $\sigma = icjk$ containing $ijk$. If there also does not exist $b < i$ with $bjk$ in $I$, applying Lemma 4.6 produces the second tetrahedron $\sigma = ijc'k$ containing $ijk$, and we are done. If such a $b$ does exist, then applying the symmetry $\alpha$ to Lemma 4.5 produces the second tetrahedron $\sigma' = bijk$ containing $ijk$, and we are done. This completes the proof, granting Lemmas 4.5 and 4.6.

*Proof of Lemma* 4.5. Assume that there exists a value $a > k$ with $ija$ in $I$. Let $a$ be the largest value such that $ija$, $ika$, $jka$ are all in $I$. Such values exist since $k+1$ is one of them: $ik(k+1)$ and $jk(k+1)$ are in $I$ by T1, and $ija$ being in $I$ implies by T2 that $ij(k+1)$ is in $I$. We proceed to show that $ijka$ is a tetrahedron of $T$, *i.e.*, the other three triangles $ija$, $ika$, $jka$ are all in $M$, or equivalently that the other three edges $ia$, $ja$, $ka$ do not intertwine any triangles of $I$.

We first show $ia$ does not intertwine any triangles of $I$. Suppose it did, *i.e.*, $x < i < y < a < z$ for some triangle $xyz$ of $I$. This implies $k \leqslant y < a$ or else $ij$ or $ik$ or $jk$ would intertwine $xyz$, contradicting $ijk$ being in $M$. Now applying T2 to $xyz$, we can produce $x'yz$ and $x''yz$ in $I$ with $i < x' < j$ and $j < x'' < k$. In the generic case where $k < y < a$, applying T3 three times as follows,

$$x'yz, ija \in I \Rightarrow ijz \in I,$$

$$x'yz, ika \in I \Rightarrow ikz \in I,$$

$$x''yz, jka \in I \Rightarrow jkz \in I,$$

yields a contradiction to the choice of $a$. In the special case where $y=k$, the first implication of the three still holds, and one can deduce the other two elements $iyz$ and $jyz$ are in $I$ from T2 and the fact that $xyz$ is in $I$. This again contradicts the choice of $a$.

Next we show that $ka$ does not intertwine any triangles of $I$. Suppose it did, *i.e.*, $x<k<y<a<z$ for some triangle $xyz$ of $I$. We may use T2 to assume without loss of generality that $j<x<k$. Applying T3 twice gives,

$$xyz, ika \in I \Rightarrow ikz \in I,$$

$$xyz, jka \in I \Rightarrow jkz \in I.$$

We also can apply T2 to the fact that $ika$ is in $I$ to produce an $i'ka$ in $I$ with $i<i'<j$, and then apply T3 twice to give,

$$xyz, i'ka \in I \Rightarrow i'kz \in I,$$

$$i'kz, ija \in I \Rightarrow ijz \in I.$$

But then the implications that $ikz$, $jkz$ and $ijz$ are all in $I$ again contradict the choice of $a$.

Lastly we show that $ja$ does not intertwine any triangles of $I$. Suppose it did, *i.e.*, $x<j<y<a<z$ for some triangle $xyz$ of $I$. Using T2, we may assume without loss of generality that $i<x$. This implies $k\leqslant y<a$ or else $ij$ or $il$ or $jk$ would intertwine $xyz$. Also if $k<y<a$ then $ka$ also intertwines $xyz$, so we are in a previous case. Thus we may assume $y=k$. Then T3 applied to $ija$ and $xkz$ in $I$ implies $ijz$ and $ikz$ are in $I$, and T2 applied to $xkz$ in $I$ implies $jkz$ is in $I$. This again contradicts the choice of $a$. This finishes the proof of Lemma 4.5.

*Proof of Lemma* 4.6. Assume that there does not exist a value $b<i$ with $bjk$ in $I$. This implies that $j\neq k-1$ since $1(k-1)k\in I$. Let $c$ be the smallest value with $j<c<k$ such that $ick$ is in $I$. Such values exist since $k-1$ is one of them. Note that $ick$ and $ijk$ being in $I$ imply by T2 that $jck$ and $ijc$ are in $I$. We proceed to show that $ijck$ is a tetrahedron of $T$, *i.e.*, the other three triangles $ijc$, $ick$, $jck$ are all in $M$, or equivalently that the three edges $ic$, $jc$, $ck$ do not intertwine any triangles of $I$.

We first show $ic$ does not intertwine any triangles of $I$. Suppose it did, *i.e.*, $x<i<y<c<z$ for some triangle $xyz$ of $I$. We can use T2 to assume without loss of generality that $c<z\leqslant k$. There are three cases for the location of $y$. If $i<y<j$ then $ij$ would also intertwine $xyz$, contradicting $ijk$ being in $M$. If $y=j$ then either $z=k$ and then $xyz=xjk$ being in $I$ contradicts the assumption of the lemma, or $c<z<k$ and then applying T3 to $xjz$, $ick$ in $I$ gives $xjk$ in $I$, which again contradicts the assumption of the lemma. If $j<y<c$, one can use T2 to assume without loss of generality that $i<x<j$, and then applying T3 three times as follows,

$$xyz, jck \in I \Rightarrow xck \in I,$$

$$xyz, ijc \in I \Rightarrow iyz \in I,$$

$$xck, iyz \in I \Rightarrow iyk \in I.$$

The last implication is a contradiction to the choice of $c$.

Next we show that $jc$ does not intertwine any triangles of $I$. Suppose it did, *i.e.*, $x<j<y<c<z$ for some triangle $xyz$ of $I$. We may use T2 to assume without loss of generality that,

$$i<x<j<y<c<z\leqslant k.$$

If $z=k$, then $xyz=xyk$ and $ijc$ being in $I$ implies by T3 that $iyk$ is in $I$, contradicting the choice of $c$. If $z<k$, then applying T3 three times gives,

$$\begin{aligned} xyz,\, ijc\in I &\Rightarrow ijz\in I,\\ xyz,\, jck\in I &\Rightarrow xyk\in I,\\ xyk,\, ijz\in I &\Rightarrow iyk\in I, \end{aligned}$$

and the last implication again contradicts the choice of $c$.

Lastly we show that $ck$ does not intertwine any triangles of $I$. Suppose it did, *i.e.*, $x<c<y<k<z$ for some triangle $xyz$ of $I$. We may use T2 to assume without loss of generality that,

$$i<j\leqslant x<c<y<k<z\leqslant k.$$

Then $xyz$, $ick$ being in $I$ imply by T3 that $iyz$ is in $I$. However $iyz$ is intertwined by $jk$, contradicting the fact that $ijk$ is in $M$. This finishes the proof of Lemma 4.6.

The proof of Theorem 4.4 is now complete.

It now remains to show

THEOREM 4.7. *With $I$, $M$ and $T$ as above, $I=sub_2(T)$.*

*Proof.* To show the forward inclusion $\subseteq$, note that $sub_2(T)$ is closed under T2 by Proposition 4.2, and $M$ is contained in $sub_2(T)$ since $M$ is the set of triangles which are 2-faces in the triangulation $T$, and hence are all submerged by $T$. Therefore it would suffice to show that the closure of $M$ under T2 is all of $I$. This is equivalent to the following lemma.

LEMMA 4.8. *If $ijk$ in $I$ is maximal with respect to T2 in the sense that there does not exist $i'jk'$ in $I$ having $[i,k]\subsetneq[i',k']$, then $ijk$ is in $M$.*

*Proof.* Assume $ijk$ is not in $M$, and we will get a contradiction to $ijk$ being maximal with respect to T2. Since $ijk$ is not in $M$, either $ij$ or $ik$ or $jk$ intertwines some triple $xyz$ in $I$. Assuming for the moment that it is $ij$, then using T2 we may assume without loss of generality that either $x<i<y<j<z<k$ or that $x<i<y<j<j+1=z=k$. In the first instance, applying T3 to $xyz$, $ijk$ in $I$ we get $xjk$ in $I$, contradicting the maximality of $ijk$. In the second case, $xj(j+1)=xjk$ is in $I$, which contradicts the maximality of $ijk$. By a symmetric argument, it cannot be $jk$ which intertwines some triple $xyz$ in $I$. This only leaves the possibility that $ik$ intertwines some $xyz$ in $I$, and then $x<i<j<k<z$ with $i<y<k$. There are three cases for the location of $y$. If $i<y<j$ then $ij$ also intertwines $xyz$, contrary to what we have already proved, and similarly if

$j<y<k$ then $jk$ also intertwines $xyz$. In the third case, $y=j$ and then $xyz=xjz$ being in $I$ contradicts the maximality of $ijk$.

Continuing the proof of Theorem 4.7, we must show the reverse inclusion $sub_2(T) \subseteq I$. So assume $ijk$ is a triangle not in $I$, and we will show that it is not in $sub_2(T)$. First of all, we claim that none of the three edges $ij$, $ik$, $jk$ can *transversally* intersect any triangle $xyz$ of the triangulation $T$, *i.e.*, they can neither intersect $xyz$ in a single interior point nor in a single point interior to an edge of $xyz$. To prove this claim, note that if one of the edges, say $ij$, intersected one of the edges of $xyz$, say $xy$, in a single interior point, then $i$, $j$, $x$, $y$ would all be coplanar which is impossible since the points lie on the moment curve. And if $ij$, $ik$, or $jk$ intersected $xyz$ in an interior point of the whole triangle, this would imply that it intertwined $xyz$, so that $ijk$ is not in $M$. Then the contrapositive of Lemma 4.8 shows that there is some other triangle $t$ in $I$ which would imply by T2 that $ijk$ is in $I$.

Therefore, since $ij$, $ik$, $jk$ all lie in $C(n, 3)$ and $T$ is a triangulation of $C(n, 3)$, it must be that all three lie inside the 2-skeleton of $T$, *i.e.*, they each lie in some triangle of $T$. But an edge like $ij$ cannot lie inside a triangle $xyz$ unless it is one of the edges of this triangle, since otherwise one would find at least four coplanar points among $i$, $j$, $x$, $y$, $z$, which is impossible. Hence each of the edges $ij$, $ik$, $jk$ is actually an edge of the triangulation $T$. But the triangle $ijk$ is not in $T$, otherwise it would be in $M \subseteq I$, so since $T$ triangulates $C(n, 3)$ there must be some edge $ab$ of $T$ which transversally intersects $ijk$ in an interior point. This implies $i<a<j<b<k$ (see the comment after Proposition 4.1), so $ijk$ is not in $sub_2(T)$, as we wanted.

Proposition 4.2, Theorem 4.4 and Theorem 4.7 together prove the main encoding theorem for $d=3$.

THEOREM 4.9. *The map* $T \mapsto sub_2(T)$ *is an isomorphism between the higher Stasheff–Tamari order* $\leqslant_2$ *on* $S(n, 3)$ *and the lattice of triangle-closed families* $I$ *in* $\binom{[n]}{3}$ *ordered by inclusion.*

*Proof.* The only thing remaining to prove is that the poset of triangle closed families $I$ *is* a lattice, but this follows exactly as in the proof of Theorem 3.6.

Our next goal is to prove Conjecture 2.6 for $d=3$, but to do this we need a little bit of terminology. Say a collection of edges $E$ is *supported* if whenever $ik$ is in $E$ with $k>i+1$, there is some $j$ with $i<j<k$ such that $ij$, $jk$ are also in $E$. Note that Lemma 4.3 may be rephrased to say that $sub_2(T)$ is supported for any triangulation $T$ of $C(n, 3)$. Also we shall say $ik$ is *maximal* in the set $E$ if there does not exist $i'k'$ in $E$ with $[i, k] \subsetneq [i'k']$.

THEOREM 4.10. *The higher Stasheff–Tamari orders* $\leqslant_1$ *and* $\leqslant_2$ *coincide for* $d=3$.

*Proof.* As in the proof of Theorem 3.8, given $T$ and $T'$ with $T \leqslant_2 T'$, it suffices to show there exists $\tilde{T}$ with $T \leqslant_2 \tilde{T} \lessdot_1 T'$, where $\lessdot_1$ denotes a covering relation in the order $\leqslant_1$.

So assume $T \leqslant_2 T'$. To construct $\tilde{T}$, we must produce an appropriate set $\mathscr{S} = \{i < x < j < y < z\}$ which supports the bistellar operation between $T'$ and $\tilde{T}$. To this end, note that by the previous theorem we have $sub_2(T) \subsetneq sub_2(T')$, and hence by Proposition 4.1 we have $edges(T') \subsetneq edges(T)$. Now since both $edges(T')$ and $edges(T)$ are supported (by Lemma 4.3), we can choose a maximal edge $xy$ in the non-empty set $E$ defined by,

$$E = \{xy \in edges(T) - edges(T') : \{xy\} \cup edges(T') \text{ is supported}\}.$$

Since $xy$ is not an edge of $T'$, it must in fact intersect some triangle $ijk$ of $T'$ transversally in an interior point of $ijk$. This follows since all of the vertices lie on the moment curve and so no pair of edges are coplanar. This transverse intersection implies $xy$ intertwines $ijk$, *i.e.*, $i < x < j < y < k$ since those two conditions are equivalent (see the comment after Proposition 4.1). Without loss of generality, we may choose $ijk$ so that $ik$ is maximal among the set of $ik$ for which $ijk$ is in $sub_2(T')$, so $ijk$ is in $M$ and is hence a triangle in $T$ by Lemma 4.8.

Our goal is to show that this choice of $xy$ and $ijk$ lead to the correct set $\mathscr{S}$ for our bistellar operation, and we claim that it only remains to show that the link $link_T(ijk)$ is the two-vertex set $\{x, y\}$. Assuming this is true for the moment, define $\tilde{T}$ by doing the bistellar operation on $\mathscr{S}$ to $T'$ which removes the tetrahedra $ixjk$, $ijyk$ and replaces them by $ixjy$, $ixyk$, $xjyk$. This means that $\tilde{T} <_1 T'$. Furthermore

$$edges(\tilde{T}) = \{xy\} \cup edges(T') \subseteq edges(T)$$

and hence by Proposition 4.1 that $sub_2(T) \subseteq sub_2(\tilde{T})$ so $T \leqslant_2 \tilde{T}$ as desired.

The rest of the proof consists of showing that $link_T(ijk) = \{x, y\}$. So assume $link_T(ijk) = \{x', y'\}$. We know that $x'y'$ intertwines $ijk$ since otherwise we would contradict $ik$'s maximality among the set of $ik$ for which $ijk$ is in $sub_2(T')$.

We can say more about the location of $x'$ and $y'$ by noting the following. Since $\{xy\} \cup edges(T')$ is supported, there must be some $j'$ with $x < j' < y$ and $xj'$, $yj'$ both edges of $T'$. But if $j' < j$ then we would get the contradiction that the edge $xj'$ of $T'$ intertwines the triangle $ijk$ in $T'$, and similarly if $j' > j$. Hence $j' = j$, so $xj$, $jy$ are edges of $T'$. Now this implies $x' \leqslant x$, since otherwise $x'ik$ would be a triangle of $T'$ intertwined by the edge $xj$ in $T'$, and similarly we must have $y' \geqslant y$. Thus,

$$i < x' \leqslant x < j < y \leqslant y' < k.$$

Since we are trying to show $x'y' = xy$, assume not, *i.e.*, either $x' < x$ or $y' > y$. If $x'y'$ were an edge of $T$, we would have a contradiction to the choice of $xy$, since $\{x', y'\} \cup edges(T')$ is supported (by the existence of $x'j$, $jy'$ as edges of $T'$). Therefore $x'y'$ is not an edge of $T$, so there must be some triangle $abc$ of $T$ which it intertwines. But this means $abc$ is in $sub_2(T)$, so $abc$ cannot be intertwined by $xy$, implying either $x' < b \leqslant x$ or $y \leqslant b < y'$. Assume without loss of generality (by symmetry) that $x' < b \leqslant x$ so we have,

$$a < x' < b \leqslant x < j < y \leqslant y' < c.$$

At this point, we try to contradict the fact that $x'$ must be the largest value in $(i, j)$ for which $ix'k$ is in $sub_2(T')$ (this fact comes from the symmetric version of Lemma 4.6, since $ix'jk$ is a tetrahedron of $T'$). There are four cases depending upon the locations of $a$ and $c$. If $a \leqslant i$ and $c \geqslant k$, then $abc$ in $sub_2(T')$ implies by T2 that $ibk$ is in $sub_2(T')$, giving the contradiction. If $i < a < x'$ and $c \geqslant k$, then $abc$, $ix'j$ in $sub_2(T')$ imply by T3 that $ibc$ is in $sub_2(T')$, which then implies by T2 that $ibk$ is in $sub_2(T')$, giving the contradiction. If $a \leqslant i$ and $y' < c < k$, then $abc$, $xjk$ in $sub_2(T')$ imply by T3 that $abk$ is in $sub_2(T')$, which then implies by T2 that $ibk$ is in $sub_2(T')$, giving the contradiction. If $i < a < x'$ and $y' < c < k$, then $abc$, $ix'j$ in $sub_2(T')$ imply by T3 that $ibc$ is in $sub_2(T')$, and $ibc$, $x'jk$ in $sub_2(T')$ imply by T3 that $ibk$ is in $sub_2(T')$, giving the contradiction. This completes the proof of the theorem.

Lastly we prove Conjecture 2.14 for $d = 3$.

THEOREM 4.11. *In any interval* $[x, y]$ *of* $S(n, 3)$, *distinct subsets of the coatoms have distinct meets.*

*Proof.* As in the proof of Theorem 3.9, it suffices to show that if $T \leqslant T'$ in $S(n, 3)$, and $T_1, T_2, \ldots, T_r$ are the coatoms of the interval $[T, T']$, then for each $m$ there is a triangle $i_m j_m k_m$ which is not in $sub_2(T_m)$ but which is in $sub_2(T_p)$ for all $p \neq m$.

Since each $T_m$ is covered by $T'$, there is some bistellar operation by which it differs from $T'$. Since the bistellar operations for $d = 3$ are of the form

$$\{i_1 i_2 i_3 i_5,\ i_1 i_3 i_4 i_5\} \leftrightarrow \{i_1 i_2 i_3 i_4,\ i_1 i_2 i_4 i_5,\ i_2 i_3 i_4 i_5\}$$

with $i_1 < i_2 < i_3 < i_4 < i_5$, note that the top triangulation loses exactly one triangle $i_1 i_3 i_5$ (while gaining some others) and gains exactly one edge $i_2 i_4$. It therefore follows that for each $m$, there must exist a triangle $i_m j_m k_m$ which is present in the triangulation $T'$ but absent in $T$. Notice also that $i_m j_m k_m$ will be present in all the other $T_p$ with $p \neq m$ since they can only miss one triangle present in $T'$ and are already missing their own $i_p j_p k_p$. This implies that for each $m$, the triangle $i_m j_m k_m$ is an element of $sub_2(T_p)$ for all $p \neq m$, but not an element of $sub_2(T_m)$.

Again, the consequences of this theorem for homotopy type and Möbius functions of intervals of $S(n, d)$ were discussed in Section 2.

## §5. *Remarks and open questions.*

*Remark* 1. It would be natural to attempt to prove the conjectures of Section 2 for $d > 3$ by proving an encoding of $S(n, d)$ using a closure operator similar to those used for $d = 2, 3$. However, one can check (*e.g.*, for $d = 4, 5$ and $n = d + 3$) that the $\lceil \frac{1}{2} d \rceil$-submersion sets are not the closed sets for any closure relation on the $\lceil \frac{1}{2} d \rceil$-subsets of $[n]$.

*Remark* 2. One can easily obtain some asymptotic bounds on the cardinality $\#S(n, 3)$, namely for some constants $c_1, c_2$ we have,

$$2^{c_1 n \log n} \leqslant \#S(n, 3) \leqslant 2^{c_2 n^2}.$$

The lower bound follows from an explicit construction which shows the recursive bound,

$$\#S(n, 3) \geqslant (n-3)\#S(n-2, 3).$$

The upper bound follows from the inequality,

$$\#S(n, d) \leqslant \#S(n-1, d-1)\#S(n-1, d).$$

To derive this inequality, note that a triangulation $T$ of $C(n, d)$ is completely determined by two pieces of information: the link $lk_T(n)$ of the vertex $n$ which triangulates $C(n-1, d-1)$, and the triangulation $T'$ of $C(n-1, d)$ obtained from $T$ by coalescing the vertices $n-1$ and $n$ into a single vertex. Using the above inequality, and the fact that $\#S(n, 2)$ is the Catalan number,

$$\frac{1}{n-1}\binom{2(n-2)}{n-2},$$

one gets the asserted asymptotic upper bound.

We suspect that the asymptotic behaviour of $\#S(n, 3)$ is closer to the lower bound given above.

*Remark* 3. Rather than the poset $S(n, 3)$ of all triangulations, one might examine more closely the subposet $S^{coh}(n, 3)$ of *coherent* triangulations of $C(n, d)$ [GKZ, Chapter 7]. Already for $n=9$, $d=3$ there will exist at least one incoherent triangulation of $C(n, d)$, although the set of incoherent triangulations can depend upon the choice of points on the moment curve generation $C(n, d)$ even for $d=3$. Sturmfels (personal communication) has shown that the number of incoherent triangulations of $C(12, 8)$ can depend on the choice of points on the moment curve. Is there a simple test for the coherence of a triangulation $T$, given this choice? Can one show that,

$$\lim_{n\to\infty} \frac{\#S^{coh}(n, 3)}{\#S(n, 3)} = 0?$$

We also note that the subposet $S^{coh}(n, 3)$ is not a sublattice of $S(n, 3)$ already for $n=9$.

*Remark* 4. Ignoring the poset structures on $S(n, d)$, one can ask questions about the underlying graph of its Hasse diagram. In the case of $d=2$, this was studied in [STT] who showed that this graph has diameter $2n-10$. What is the diameter in general? What is its edge- and vertex-connectivity? It follows from the theory of *secondary polytopes* [GKZ1, Chapter 7] that the subgraph of coherent triangulations has vertex-connectivity $n-d-1$. However, there are no known lower bounds on the degree of an incoherent triangulation. It may or may not be relevant that de Loera and Santos (personal communication) have constructed a point configuration in $\mathbb{R}^3$ with $n=13$ points in which there is an incoherent triangulation with 6 bistellar neighbours which is smaller than the lower bound $n-d-1=9$ for neighbors of coherent triangulations.

A more detailed understanding of this graph might lead to good algorithms for constructing random triangulations of $C(n, d)$ using random walks in the graph.

*Remark* 5. Conjecture 2.6 suggests the following construction. Given $\mathscr{A}$ a finite set of points in $\mathbb{R}^d$ and generic heights $h_1, \ldots, h_n$ used to lift the points of $\mathscr{A}$ to a configuration $\tilde{\mathscr{A}}$ in $\mathbb{R}^{d+1}$, one can define two partial orders $\leqslant_1$, $\leqslant_2$ on the set of triangulations of $\mathscr{A}$, as in Section 2. One might ask whether these two partial orders always coincide (as they do for $C(n, 3)$). However, one can construct examples which show that they do not coincide even for $\mathscr{A}$ in $\mathbb{R}^2$.

*Note added in proof.* Rambau has recently settled conjectures 2.7a, 2.7b and 2.8 in the affirmative. See J. Rambau, Triangulations of cyclic polytopes and higher Bruhat orders, *Preprint*, 1996.

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