COUNTING THE INTERIOR OF A POINT CONFIGURATION

PAUL H. EDELMAN AND VICTOR REINER

Abstract. We prove a formula conjectured by Ahrens, Gordon, and McMahon for the number of interior points for a point configuration in $\mathbb{R}^d$. Our method is to show that the formula can be interpreted as a sum of Euler characteristics of certain complexes associated with the point configuration, and then compute the homology of these complexes. This method extends to other examples of convex geometries. We sketch these applications, replicating an earlier result of Gordon, and proving a new result related to ordered sets.

1. Introduction

Let $\mathcal{A}$ be a collection of points in $\mathbb{R}^d$ whose affine span is all of $\mathbb{R}^d$. Let $\mathrm{int}(\mathcal{A})$ be the set of points of $\mathcal{A}$ that lie in the interior of its convex hull, $\mathrm{conv}(\mathcal{A})$. Call a subset K of $\mathcal{A}$ *free* if $\mathrm{conv}(K) \cap \mathcal{A} = K$ and the vertices of $\mathrm{conv}(K)$ equal K. In a recent paper Ahrens, Gordon and McMahon [1, Conjecture 4.1] conjectured the following beautiful formula for $|\mathrm{int}(\mathcal{A})|$, and proved it for point configurations $\mathcal{A}$ in $\mathbb{R}^2$:

Theorem 1.1.

$$|\mathrm{int}(\mathcal{A})| = (-1)^{(d-1)} \sum_{K \,\mathrm{free}} (-1)^{|K|} |K|.$$

We prove this formula in its full generality in Section 3.

The work of Ahrens, *et. al.* is part of a larger project of extending matroid invariants to greedoids [10, 11]. Recent work has focused on generalizing Crapo's β invariant, originally defined for matroids [6], to the more general structure of a greedoid [9]. In the special case of greedoids which are convex geometries, the definition of the β invariant reduces to essentially the right-hand side of Theorem 1.1 [9]. Theorem 1.1 gives a combinatorial interpretation to the β invariant of those convex geometries which are based on Euclidean point configurations. Given this background, it is not surprising that Ahrens, *et. al.* use deletion-contraction arguments to establish their formula for $d = 2$.

Our point of view is more in the tradition of a topological approach to matroid theory [4], as we now explain. One can rewrite the right-hand side in Theorem 1.1 as follows:

$$\begin{aligned}(-1)^{(d-1)} \sum_{K \,\mathrm{free}} (-1)^{|K|} |K| &= (-1)^{(d-1)} \sum_{K \,\mathrm{free}} \sum_{a \in K} (-1)^{|K|} \\ &= (-1)^{(d-1)} \sum_{a \in \mathcal{A}} \widetilde{\chi}_a\end{aligned}$$

The second author was partially supported by a Sloan Foundation Fellowship and a University of Minnesota McKnight-Land Grant Fellowship.

where

$$\widetilde{\chi}_a = \sum_{K \text{ free}, \, a \in K} (-1)^{|K|}.$$

We will prove Theorem 1.1 by showing (see Corollaries 3.5,3.8) that

$$\widetilde{\chi}_a = \begin{cases} (-1)^{d-1} & \text{if } a \in \text{int}(\mathcal{A}) \\ 0 & \text{otherwise.} \end{cases}$$

The latter computation is achieved by interpreting $\widetilde{\chi}_a$ as the reduced Euler characteristic of a certain simplicial complex (the *link* of a in the simplicial complex of free sets of $\mathcal{A}$), and then analyzing the topology of these complexes.

The paper is organized as follows. In the next section we establish some general facts from combinatorial topology and polytope theory necessary for the proof. In Section 3 we prove Theorem 1.1. Section 4 uses similar techniques to evaluate the generalized β invariant for other convex geometries. Finally we discuss some open problems.

2. Preliminaries

In this section we establish some background results from topological combinatorics that will be used throughout the paper. For much of this material an excellent reference is [3].

Let Δ be an abstract simplicial complex on the ground set X. That is, Δ is a collection of non-empty subsets of X, called *simplices*, with the property that

$$\varnothing \neq \sigma \subseteq \tau \in \Delta$$

implies that $\sigma \in \Delta$. For $0 \leq i \leq |X| - 1$ let Δ^i be the *i-dimensional faces of* Δ, i.e.,

$$\Delta^i = \{\sigma \in \Delta \,:\, |\sigma| = i + 1\},$$

and let $f_i = |\Delta^i|$. For any $x \in X$, we define three associated subcomplexes, the *link of x in* Δ, the *star of x in* Δ, and the *deletion of x in* Δ by

$$\begin{aligned} \text{lk}_\Delta(x) &= \{\sigma - x \,|\, x \in \sigma \in \Delta\} \\ \text{st}_\Delta(x) &= \{\sigma \in \Delta \,|\, x \cup \sigma \in \Delta\} \\ \text{del}_\Delta(x) &= \{\sigma \in \Delta \,|\, x \notin \sigma\}. \end{aligned}$$

It is clear from the definitions that

$$\text{st}_\Delta(x) \cup \text{del}_\Delta(x) = \Delta \tag{1}$$

and

$$\text{del}_\Delta(x) \cap \text{st}_\Delta(x) = \text{lk}_\Delta(x). \tag{2}$$

For a given simplicial complex Δ, let $\widetilde{H}_i(\Delta) = \widetilde{H}_i(\Delta, \mathbb{Z})$ be the reduced homology groups of Δ with coefficients in $\mathbb{Z}$, for $i \geq 0$. By the universal coefficient theorem [13, Section 55], $\widetilde{H}.(\Delta)$ determines the homology of Δ with any other coefficients. The *reduced Euler characteristic* $\widetilde{\chi}(\Delta)$ is defined by

$$\widetilde{\chi}(\Delta) = \sum_{i \geq -1} (-1)^i f_i = \sum_{i \geq 0} (-1)^i \dim_{\mathbb{Q}} \widetilde{H}_i(\Delta, \mathbb{Q}).$$

where we define $f_{-1} = 1$.

The following lemmas will be necessary in what follows

Lemma 2.1. *Let Δ be a contractible simplicial complex on the ground set X and $x \in X$. For all $i \geq 0$*

$$\widetilde{H}_i(\mathrm{lk}_\Delta(x)) \cong \widetilde{H}_i(\mathrm{del}_\Delta(x)).$$

Proof. Associated to (1) and (2) is a Mayer-Vietoris exact sequence [13, Theorem 25.1]

$$\cdots \to \widetilde{H}_{i+1}(\Delta) \to \widetilde{H}_i(\mathrm{lk}_\Delta(x)) \to \widetilde{H}_i(\mathrm{del}_\Delta(x)) \oplus \widetilde{H}_i(\mathrm{st}_\Delta(x)) \to \widetilde{H}_i(\Delta) \to \cdots$$

Note that both Δ and $\mathrm{st}_\Delta(x)$ are contractible, the former by hypothesis and the latter since $\mathrm{st}_\Delta(x)$ is a cone. Therefore the above sequence reduces to

$$\cdots \to 0 \to \widetilde{H}_i(\mathrm{lk}_\Delta(x)) \to \widetilde{H}_i(\mathrm{del}_\Delta(x)) \to 0 \to \cdots$$

which proves the lemma. □

Lemma 2.2. *If Δ is a contractible simplicial complex on the ground set X and $x \in X$ then*

$$\widetilde{\chi}(\mathrm{lk}_\Delta(x)) = \widetilde{\chi}(\mathrm{del}_\Delta(x)) \tag{3}$$

$$= \sum_{x \in \sigma \in \Delta} (-1)^{|\sigma|} \tag{4}$$

Proof. The first equality follows directly from Lemma 2.1. The second inequality is from the definition of the reduced Euler characteristic. □

Let P be a poset. We can associate with P its simplicial complex of chains $\Delta(P)$, i.e., the collection of totally ordered subsets of P. When we speak of P having a certain topological property, we are really referring to the topology the *geometric realization* of $\Delta(P)$. If P has both a maximum element $\hat{1}$ and a minimum element $\hat{0}$, then let $\overline{P} = P - \{\hat{0}, \hat{1}\}$. If $x \in P$ then let

$$P_{\leq x} = \{y \in P \,|\, y \leq x\}.$$

The following lemmas are essentially due to Quillen [14]. See also [3, 10.5, 10.12].

Lemma 2.3. *Suppose Q and P are posets and $f : Q \to P$ is an order-preserving map. If for all $x \in P$ the fibers $f^{-1}(P_{\leq x})$ are contractible then f induces a homotopy equivalence between Q and P.*

Lemma 2.4. *Let $f : P \to P$ be an order-preserving map such that $f(x) \geq x$ for all $x \in P$. Then f induces a homotopy equivalence between P and $f(P)$.*

In our applications of Lemma 2.3, we will often let Q be a simplicial complex Δ viewed as a partially ordered set, where the partial order on the simplices is by containment. The conclusions drawn will technically be about the chain complex of of this poset, which triangulates the *barycentric subdivision* $\mathrm{sd}(\Delta)$ of Δ. However, it is well-known [13, Section 15] that $\mathrm{sd}(\Delta)$ is homeomorphic to Δ, so the same conclusions can be drawn about Δ.

The last general facts that we will employ concern *Gale transforms* of point configurations. Let $\mathcal{A} = \{a_1, \ldots, a_n\} \subseteq \mathbb{R}^d$ be a collection of points and let $\mathrm{conv}(\mathcal{A})$ denote the polytope which is the convex hull of $\mathcal{A}$. List $a_1, \ldots, a_n$ as columns of a matrix and append a row of ones to obtain the $(d+1) \times n$ matrix $M_\mathcal{A}$. The null space of $M_\mathcal{A}$ is the set of all affine dependencies among the points in $\mathcal{A}$, i.e., vectors $(\lambda_1, \ldots, \lambda_n)$ such that $\sum_{i=1}^n \lambda_i a_i = 0$ and $\sum_{i=1}^n \lambda_i = 0$. Let A^* be an $(n-d-1) \times n$ matrix whose rows form a basis for this space and denote its columns by ${v_1}^*, \ldots, {v_n}^*$. This collection $\mathcal{A}^*$ of points in $\mathbb{R}^{n-d-1}$ is a *Gale transform* of $\mathcal{A}$. The correspondence between v_i and ${v_i}^*$ extends naturally to subsets of $\mathcal{A}$ and $\mathcal{A}^*$. For any subset $B \subseteq \mathcal{A}$, denote by $\mathrm{aff}(B)$ the affine span of B. The principal property of Gale transforms is

Lemma 2.5. [12, Section 5.5] *Let F be a proper subset of $\mathcal{A}$. Then*

$$\mathrm{aff}(F) \cap \mathrm{conv}(\mathcal{A} - F) = \varnothing$$

if and only if

$$0 \in \mathrm{relint}(\mathrm{conv}(\mathcal{A} - F)^*).$$

That is, F is exactly the set of points of $\mathcal{A}$ that lie on a face of $\mathrm{conv}(\mathcal{A})$ *if and only if $(\mathcal{A} - F)^*$ has 0 in its relative interior.*

One should observe that any set of n points in $\mathbb{R}^{n-d-1}$ that has 0 in its relative interior is the Gale transform of some set of n points in $\mathbb{R}^d$. For an interesting survey of the uses of Gale transforms, see [2].

Lemma 2.6. *Let $\mathcal{A}$ be a point configuration in $\mathbb{R}^d$, $\mathbf{P} = \mathrm{conv}(\mathcal{A})$, and $\mathcal{A}^*$ its Gale diagram. Let N be the poset of proper subsets A of $\mathcal{A}^*$, ordered by containment, such that*

$$0 \notin \mathrm{relint}(\mathrm{conv}(A)).$$

Then N has the same integral homology as the $(n-d-2)$-sphere, and hence

$$\widetilde{\chi}(N) = (-1)^{n-d-2}.$$

Proof. Let $2^{\mathcal{A}^*}$ be the collection of all subsets of $\mathcal{A}^*$. Considering $\overline{2^{\mathcal{A}^*}}$ as partially ordered by inclusion, its order complex is the barycentric subdivision of the boundary of an $(n-1)$-simplex, and hence is homeomorphic to an $(n-2)$-sphere. Lemma 2.5 implies that $2^{\mathcal{A}^*} - N$ is isomorphic as a poset to the opposite of the face lattice of $\mathbf{P}$. Thus the order complex of $\overline{2^{\mathcal{A}^*} - N}$ is the barycentric subdivision of the boundary complex of $\mathbf{P}$, and hence is homeomorphic to a $(d-1)$-sphere. It follows from [5, Lemma 4.7.27] that N is a deformation retract of the space $\overline{2^{\mathcal{A}^*} - N}$, which is to say the complement of a $(d-1)$-sphere in an $(n-2)$-sphere. It then follows from Alexander duality [13, Theorem 71.1] that N has the homology of an $(n-d-2)$-sphere. $\square$

3. Proof of Theorem 1.1

In this section we prove the conjecture of Ahrens, *et. al.*. Our discussion in this section will be only in terms of the geometric structure of point configurations in $\mathbb{R}^d$, although there are natural generalizations to the more abstract situation of convex geometries. We take up those more general questions in the next section.

Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ be a point configuration in $\mathbb{R}^d$. Given a subset A of $\mathcal{A}$ we define the *closure of* A, A^c, to be

$$A^c = \mathrm{conv}(A) \cap \mathcal{A}.$$

It is clear that this is a true closure operator on the subsets of $\mathcal{A}$, that is, it is idempotent, inclusion-preserving, and $A \subseteq A^c$. The closed sets under this operator will be called *the convex sets of* $\mathcal{A}$. Let $\mathbf{L} = \mathbf{L}(\mathcal{A})$ be the lattice of convex sets of $\mathcal{A}$ ordered by inclusion.

If C is a convex set of $\mathcal{A}$, a point $a \in C$ is called *an extreme point of* C if a is vertex of the polytope $\mathrm{conv}(C)$. Let $\mathrm{ex}(C)$ be the set of extreme points of C. A convex set C is said to be *free* if $\mathrm{ex}(C) = C$. Let $\mathrm{Free} = \mathrm{Free}(\mathcal{A})$ be the collection of free sets of $\mathcal{A}$, and note that Free is a simplicial complex.

Lemma 3.1. *Let $\mathcal{A}$ be a point configuration and $\mathbf{L}(\mathcal{A})$ be its lattice of convex sets. Then $\overline{\mathbf{L}}$ triangulates an*

$$\begin{cases} (|\mathcal{A}|-2)-\ \textit{ball} & \textit{if } \mathcal{A} \neq \mathrm{ex}(\mathcal{A}) \\ (|\mathcal{A}|-2)-\ \textit{sphere} & \textit{if } \mathcal{A} = \mathrm{ex}(\mathcal{A}) \end{cases}$$

Proof. Apply [7, Theorem 4.10] to [7, Example 1, page 252]. □

Lemma 3.2. *Let $\mathcal{A}$ be a point configuration and* Free *its simplicial complex of free sets. Then* Free *is contractible.*

Proof. If $\mathrm{ex}(\mathcal{A}) = \mathcal{A}$, then Free is just a simplex and hence is contractible. Thus, we may assume that $\mathrm{ex}(\mathcal{A}) \neq \mathcal{A}$, so that the inclusion map ι is a map from Free to $\overline{\mathbf{L}}$. To apply Lemma 2.3, we need to analyze the structure of the fibers $\iota^{-1}(\mathbf{L}_{\leq C})$ for each $C \in \overline{\mathbf{L}}$. It is easy to check that $\iota^{-1}(L_{\leq C}) = \mathrm{Free}(C)$ (where here C is considered as a point configuration), and hence is contractible by induction on the number of points. Hence, by Lemma 2.3, Free has the same homotopy-type as $\overline{\mathbf{L}}$. By Lemma 3.1, $\overline{\mathbf{L}}$ is contractible if $\mathcal{A} \neq \mathrm{ex}(\mathcal{A})$, and so the lemma is proven. □

Let

$$\beta(\mathcal{A}) = \sum_{K \in \mathrm{Free}} (-1)^{|K|}|K|.$$

Our definition of $\beta(\mathcal{A})$ differs from that of Ahrens, *et. al.* by a minus sign [1, Definition 2.2]. For our purposes this is the more convenient convention.

Lemma 3.3. *If $\mathcal{A}$ is a point configuration then*

$$\beta(\mathcal{A}) = \sum_{a \in \mathcal{A}} \widetilde{\chi}(\mathrm{del}_{\mathrm{Free}}(a)).$$

Proof. As noted in Lemma 2.2

$$\widetilde{\chi}(\mathrm{del}_{\mathrm{Free}}(a)) = \sum_{a \in K \in \mathrm{Free}} (-1)^{|K|}$$

since Free is contractible by Lemma 3.2. Thus

$$\begin{aligned} \sum_{a \in \mathcal{A}} \widetilde{\chi}(\mathrm{del}_{\mathrm{Free}}(a)) &= \sum_{a \in \mathcal{A}} \sum_{a \in K \in \mathrm{Free}} (-1)^{|K|} \\ &= \sum_{K \in \mathrm{Free}} (-1)^{|K|}|K| \\ &= \beta(\mathcal{A}). \end{aligned}$$

□

From Lemma 3.3, we see that it is sufficient to understand the topology of each deletion $\mathrm{del}_{\mathrm{Free}}(a)$ in order to evaluate $\beta(\mathcal{A})$. That is what we endeavor to do in the rest of this section. Let $\mathbf{P} = \mathrm{conv}(\mathcal{A})$. We will assume that $\dim(\mathbf{P}) = d$ and thus the interior of $\mathbf{P}$ is the same as its relative interior. Let $\mathrm{bdy}(\mathbf{P})$ denote the points of $\mathcal{A}$ lying in the boundary of $\mathrm{conv}(\mathcal{A})$, that is, $\mathrm{bdy}(\mathbf{P}) := \mathcal{A} - \mathrm{int}(\mathcal{A})$.

Lemma 3.4. *Let $x \in \mathcal{A}$ and suppose that $x \in \operatorname{bdy}(\mathbf{P})$. Then $\operatorname{del}_{\operatorname{Free}}(x)$ is contractible.*

Proof. If x is on the boundary of $\mathbf{P}$ then it is contained in the relative interior of a unique face $\mathbf{F}$ of $\mathbf{P}$. Let $F = \mathbf{F} \cap \mathcal{A}$. Define a new point configuration $\mathcal{B}$ by

$$\mathcal{B} = F \cup \{t\}$$

where t is a new point in $\mathbb{R}^d$ that is affinely independent of the points in F. The map f from $\mathcal{A}$ to $\mathcal{B}$ defined by

$$f(a) = \begin{cases} a & \text{if } a \in F \\ t & \text{otherwise.} \end{cases}$$

induces a map $\hat{f}$ from subsets of $\mathcal{A}$ to subsets of $\mathcal{B}$ by $\hat{f}(A) = \{f(a) \,|\, a \in A\}$. It is easy to check that $\hat{f}$ also maps $\operatorname{del}_{\operatorname{Free}(\mathcal{A})}(x)$ to $\operatorname{del}_{\operatorname{Free}(\mathcal{B})}(x)$

We now analyze the fibers of the map $\hat{f}$. For notational convenience, let

$$\Gamma = \operatorname{del}_{\operatorname{Free}(\mathcal{A})}(x)$$
$$\Gamma' = \operatorname{del}_{\operatorname{Free}(\mathcal{B})}(x).$$

Suppose that $K \in \Gamma'$, and we consider two cases in analyzing the fiber $\hat{f}^{-1}(\Gamma'_{\leq K})$.

If $t \notin K$, then

$$\hat{f}^{-1}(\Gamma'_{\leq K}) = \{C \subseteq K \,|\, C \in \Gamma\},$$

which has K as a maximum element, since $K \in \Gamma'$ and $t \notin K$ implies that $K \in \Gamma$. Hence this fiber is contractible.

If $t \in K$, then

$$\begin{aligned} \hat{f}^{-1}(\Gamma'_{\leq K}) &= \{C \in \Gamma \,|\, C \cap F \subseteq K\} \\ &= \operatorname{Free}((K - \{t\}) \cup (\mathcal{A} - F)), \end{aligned}$$

(here $(K - \{t\}) \cup (\mathcal{A} - F)$ is considered as a point configuration on its own). This fiber is then contractible by Lemma 3.2.

Thus the fibers $\hat{f}^{-1}(\Gamma'_{\leq K})$ are always contractible, and so by Lemma 2.3, we know that $\operatorname{del}_{\operatorname{Free}(\mathcal{A})}(x)$ has the same homotopy-type as $\operatorname{del}_{\operatorname{Free}(\mathcal{B})}(x)$. Since the point t was affinely independent of all of the other points in $\mathcal{B}$, we see that $\operatorname{del}_{\operatorname{Free}(\mathcal{B})}(x)$ has t as a cone point, and hence is contractible. Thus $\operatorname{del}_{\operatorname{Free}(\mathcal{A})}(x)$ is contractible and the lemma is proven. $\square$

Corollary 3.5. *If $x \in \mathcal{A}$ and $x \in \operatorname{bdy}(\mathbf{P})$, then $\widetilde{\chi}(\operatorname{del}_{\operatorname{Free}}(x)) = 0$.*

Proof. By Lemma 3.4, the complex $\operatorname{del}_{\operatorname{Free}}(x)$ is contractible, so its reduced Euler characteristic vanishes. $\square$

We have left to consider the topology of $\operatorname{del}_{\operatorname{Free}}(x)$ in the instance when x is in the interior of $\mathbf{P}$.

Lemma 3.6. *Let $x \in \mathcal{A}$ and suppose that $x \in \operatorname{int}(\mathbf{P})$. Then $\operatorname{del}_{\operatorname{Free}}(x)$ has the integral homology of a $(d-1)$-sphere.*

Proof. Suppose that $x \in \operatorname{int}(\mathbf{P})$. Define the poset $CN(x)$ by

$$CN(x) = \{K \in \mathbf{L} \,|\, x \notin \operatorname{relint}(\operatorname{conv}(K))\}.$$

Note that $\operatorname{del}_{\operatorname{Free}}(x) \subseteq CN(x)$ and so we can consider the inclusion map

$$\iota : \operatorname{del}_{\operatorname{Free}}(x) \to CN(x).$$

Claim 3.7. *The fibers $\iota^{-1}(CN(x)_{\leq K})$ are contractible for every $K \in CN(x)$.*

Proof of claim. There are two cases to consider. First, suppose that $x \notin K$. Then we see that

$$\iota^{-1}(CN(x)_{\leq K}) = \mathrm{Free}(K).$$

This follows since $K \in \mathbf{L}$ implies that any free subset of K is also a free subset of $\mathcal{A}$, and $x \notin K$ implies that it is not in any subset of K. By Lemma 3.2, Free(K) is contractible and so we are done.

The second case is that $x \in K$. Then by the definition of $CN(x)$, we have $x \in \mathrm{bdy}(\mathrm{conv}(K))$. From this we conclude that

$$\begin{aligned} \iota^{-1}(CN(x)_{\leq K}) &= \{C \in \mathrm{Free}(K) \,|\, x \notin C\} \\ &= \mathrm{del}_{\mathrm{Free}(K)}(x). \end{aligned}$$

which is contractible by Lemma 3.4. □

Continuing the proof of Lemma 3.6, since the fibers of ι are all contractible, Lemma 2.3 implies that $\mathrm{del}_{\mathrm{Free}}(x)$ has the same homotopy-type as $CN(x)$. Consider another poset $N(x)$ defined by

$$N(x) = \{B \subseteq \mathcal{A} \,|\, x \notin \mathrm{relint}(\mathrm{conv}(B))\}.$$

The difference between $N(x)$ and $CN(x)$ is that the subsets in $N(x)$ need not be convex. The closure operator $f(B) = B^c$ gives a map f from $N(x)$ to itself, since

$$x \notin \mathrm{relint}(\mathrm{conv}(B)) \quad \Rightarrow \quad x \notin \mathrm{relint}(\mathrm{conv}(B^c)).$$

The image of $N(x)$ under the map f is exactly $CN(x)$. Thus by Lemma 2.4 we have that $N(x)$ has the same homotopy-type as $CN(x)$, and thus the same homotopy-type as $\mathrm{del}_{\mathrm{Free}}(x)$. It only remains to analyze the homology of $N(x)$.

Translate the point configuration $\mathcal{A}$ so that $x = 0$. As observed at the end of last section, we can view $\mathcal{A}$ as the Gale transform of some point configuration $\mathcal{A}^*$ in $\mathbb{R}^{n-d-1}$, since 0 is in the relative interior of $\mathcal{A}$. Also note that our poset $N(x)$ is now the same as the one denoted N in Lemma 2.6. Thus by Lemma 2.6 we conclude that $N(x)$ has the same homology as a $(d-1)$-sphere. Hence $\mathrm{del}_{\mathrm{Free}}(x)$ has the homology of a $(d-1)$-sphere, and the proof of Lemma 3.6 is complete. □

Corollary 3.8. *If $x \in \mathcal{A}$ and $x \in \mathrm{int}(\mathbf{P})$, then $\widetilde{\chi}(\mathrm{del}_{\mathrm{Free}}(x) = (-1)^{d-1}$.*

Proof. By Lemma 3.6, $\mathrm{del}_{\mathrm{Free}}(x)$ has the same homology as a $(d-1)$-sphere and hence $\widetilde{\chi}(\mathrm{del}_{\mathrm{Free}}(x) = (-1)^{d-1}$. □

Theorem 1.1 [see [1, Conjecture 4.1]] *If $\mathcal{A}$ is a point configuration in $\mathbb{R}^d$ then*

$$\beta(\mathcal{A}) = \sum_{K \in \mathrm{Free}} (-1)^{|K|}|K| = (-1)^{d-1}|\mathrm{int}(\mathcal{A})|,$$

i.e., $\beta(\mathcal{A})$ equals the number of points in $\mathcal{A}$ which are in the interior of the convex hull of $\mathcal{A}$.

Proof. By Lemma 3.3 we have that

$$\beta(\mathcal{A}) = \sum_{a \in \mathcal{A}} \widetilde{\chi}(\mathrm{del}_{\mathrm{Free}}(a)).$$

From Corollary 3.5 and Corollary 3.8 we have that

$$\widetilde{\chi}(\mathrm{del}_{\mathrm{Free}}(a)) = \begin{cases} 0, & a \in \mathrm{bdy}(\mathbf{P}) \\ (-1)^{d-1} & a \in \mathrm{int}(\mathbf{P}). \end{cases}$$

and so

$$\sum_{a \in \mathcal{A}} \widetilde{\chi}(\mathrm{del}_{\mathrm{Free}}(a)) = (-1)^{d-1} |\mathcal{A} \cap \mathrm{int}(\mathbf{P})|.$$

□

4. Convex geometries

In this section we discuss generalizations of Theorem 1.1 to more general convex geometries. Most of the framework discussed in Section 3 goes through for general convex geometries, but we lack at the moment completely general versions of Lemma 3.4 or Lemma 3.6. We can, however, prove analogues of these lemmas for other specific examples of convex geometries.

We begin with a brief description of the theory of convex geometries. For a more detailed introduction see [7], whose notation we follow. Let X be a finite set, and $\mathcal{L}$ a collection of subsets of X that contains $\varnothing$ and X and is closed under intersection. We can alternatively think of $\mathcal{L}$ as a closure operator on X defined by

$$\mathcal{L}(A) = \bigcap_{\{C \in \mathcal{L} \mid C \supseteq A\}} C.$$

The subsets in $\mathcal{L}$ or, equivalently, those subsets A of X such that $\mathcal{L}(A) = A$ will be called *convex sets*. We will say that $\mathcal{L}$ is *anti-exchange* if given any convex set C, and two unequal points p and q in X, neither in C, one has that

$$q \in \mathcal{L}(K \cup p) \quad \Rightarrow \quad p \notin \mathcal{L}(C \cup q).$$

A collection $\mathcal{L}$ of convex sets that is anti-exchange will be called a *convex geometry*. Let $\mathbf{L}(\mathcal{L})$ be the lattice of convex sets of $\mathcal{L}$ ordered by containment. Some examples of convex geometries are

Point configurations: The collection of *convex subsets* of a point set $\mathcal{A}$ in $\mathbb{R}^d$ (as described in the previous section) is a convex geometry, with $X = \mathcal{A}$.

Order convex sets: Let Q be a poset, and for any subset $A \subseteq Q$ define

$$\mathcal{C}_Q(A) = \{q \in Q \mid \exists\, a, a' \in A,\ a \leq q \leq a'\}.$$

The closure $\mathcal{C}_Q$ is anti-exchange, and we call the associated convex geometry with $X = Q$ the *order convex sets of* Q, denoted $\mathcal{C}_Q$.

Chordal graphs: Let G be a chordal graph on a vertex set V. That is, G is a graph such that every cycle of length longer than 4 has a chord. A subset C of V will be called *m-convex* if it contains every vertex on every chordless path between vertices in C. The collection $\mathcal{L}(G)$ of m-convex sets of a chordal graph is a convex geometry, with $X = V$ [8].

If C is a convex set in a convex geometry $\mathcal{L}$, a point $x \in C$ is called *an extreme point of* C if $x \notin \mathcal{L}(C - x)$. We denote the set of extreme points of C by $ex(C)$. We will call a convex set *free* if $ex(C) = C$. The set of free sets of a convex geometry form a simplicial complex which we denote $\mathrm{Free}(\mathcal{L})$. The following lemmas are direct generalizations of Lemma 3.1 and Lemma 3.2.

Lemma 4.1. *Let $\mathcal{L}$ be a convex geometry on a ground set X and $\mathbf{L} = \mathbf{L}(\mathcal{L})$ be its lattice of convex sets. Then $\overline{\mathbf{L}}$ triangulates an*

$$\begin{cases} (|X|-2)-\textit{ball} & \textit{if } X \neq \mathrm{ex}(X) \\ (|X|-2)-\textit{sphere} & \textit{if } X = \mathrm{ex}(X) \end{cases}$$

Proof. See [7, Theorem 4.10]. □

Lemma 4.2. *Let $\mathcal{L}$ be a convex geometry and* Free *its simplicial complex of free sets. Then* Free *is contractible.*

Proof. The proof follows exactly the argument for Lemma 3.2. □

As before we define

$$\beta(\mathcal{L}) = \sum_{K \in \mathrm{Free}(\mathcal{L})} (-1)^{|K|}|K|.$$

and note that because of Lemma 4.2, Lemma 3.3 still holds, i.e.,

$$\beta(\mathcal{L}) = \sum_{x \in X} \widetilde{\chi}(\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)). \tag{5}$$

Although we do not have a uniform way of computing $\beta(\mathcal{L})$ for all convex geometries, we can use the framework we have established to evaluate it for the examples of convex geometries we described above. We sketch those arguments here.

4.1. **Order convex sets.** Let Q be a poset and $\mathcal{C}_Q$ be the collection of order convex sets of Q. We call an element $q \in Q$ a *bottleneck* if q is comparable to every element of Q, but q is neither a maximum nor a minimum element.

Lemma 4.3. *Let* $\mathrm{Free} = \mathrm{Free}(\mathcal{C}_Q)$ *be the free sets of the convex geometry of order convex sets of Q. Then $\mathrm{del}_{\mathrm{Free}}(q)$ has the homotopy type of the 0-sphere if q is a bottleneck, and contractible otherwise.*

Sketch of Proof. If q is a bottleneck define $Q_{>q}$ (resp. $Q_{<q}$) to be the points in Q strictly bigger than (resp. less than) q. It is easy to check that $\mathrm{lk}_{\mathrm{Free}}(q)$ is the disjoint union of the two (non-empty) complexes $\mathrm{Free}(\mathcal{C}_{Q_{>q}})$ and $\mathrm{Free}(\mathcal{C}_{Q_{<q}})$, each of which is contractible by Lemma 3.2. Hence $\mathrm{del}_{\mathrm{Free}}(q)$ has the homotopy-type of the 0-sphere.

If q is not a bottleneck, then there are two possibilities. On the one hand, q might be a maximum or minimum element, in which case it is easy to see that $\mathrm{del}_{\mathrm{Free}}(q)$ is the same as $\mathrm{Free}(\mathcal{C}_{Q-\{q\}})$, and hence contractible by Lemma 3.2.

If q is not a bottleneck and is also neither a maximum nor a minimum element of Q, then there exist elements of Q which are bigger than q, smaller than q, and incomparable to Q. Define a new four-element poset $Q' = \{b, s, i, q\}$ (b, s, i for "bigger","smaller", "incomparable" to q) in which $b \geq q \geq s$ and i is incomparable to any of the other three. One can easily check that $\mathrm{del}_{\mathrm{Free}(\mathcal{C}_{Q'})}$ is contractible since it is a cone with cone point i. The map $f : Q \to Q'$ given by

$$f(x) = \begin{cases} b & \text{if } x > q \\ q & \text{if } x = q \\ s & \text{if } x < q \\ i & \text{if } x \text{ is incomparable to } q. \end{cases}$$

induces a map from $\mathrm{del}_{\mathrm{Free}(\mathcal{C}_Q)}(q)$ to $\mathrm{del}_{\mathrm{Free}(\mathcal{C}_{Q'})}(q)$. One can check the relevant fibers of f^{-1} to see that they are all contractible, and thus $\mathrm{del}_{\mathrm{Free}(\mathcal{C}_Q)}(q)$ has the homotopy-type of $\mathrm{del}_{\mathrm{Free}(\mathcal{C}_{Q'})}$, and hence is contractible. □

Corollary 4.4. *Let Q be a poset and $\mathcal{C}_Q$ be the collection of order convex sets of Q. Then $\beta(\mathcal{C}_Q)$ is equal to the number of bottlenecks in Q.*

Proof. From the previous lemma we see that

$$\widetilde{\chi}(\mathrm{del}_{\mathrm{Free}}(q)) = \begin{cases} 1 & \text{if } q \text{ is a bottleneck} \\ 0 & \text{otherwise.} \end{cases}$$

Apply this formula to (5). □

4.2. **Chordal graphs.** Gordon has previously interpreted $\beta(\mathcal{L}(G))$ for a chordal graph G in [9]. He did the computation by means of a deletion-contraction argument. Our topological approach gives a more detailed result.

Lemma 4.5. *Let G be a connected chordal graph. A subset K of V is free in $\mathcal{L}(G)$ if and only if it induces a clique in G.*

Proof. See [9, Lemma 5.1]. □

For each vertex v in G let $c(v)$ be the number of connected components of $G - v$.

Lemma 4.6. *Let G be a connected chordal graph and* $\mathrm{Free} = \mathrm{Free}(\mathcal{L}(G))$. *For each vertex v in G, the complex* $\mathrm{del}_{\mathrm{Free}}(v)$ *has the homotopy-type of $c(v)$ disjoint points.*

Sketch of Proof. It is easy to show that $\mathrm{del}_{\mathrm{Free}}(v)$ is just the disjoint union of $\mathrm{Free}(\mathcal{L}(G'))$ for each connected component G' of $G - v$. By Lemma 4.2 these are all contractible so the lemma follows. □

Recall that a *block* of a graph is a maximal subgraph which contains no cut-vertex. We denote by $b(G)$ the number of blocks in the graph G.

Corollary 4.7. [9, Theorem 5.1] *Let G be a connected chordal graph. Then*

$$\beta(\mathcal{L}(G)) = b(G) - 1.$$

Proof. From the previous lemma we see that

$$\widetilde{\chi}(\mathrm{del}_{\mathrm{Free}}(v)) = c(v) - 1,$$

and so

$$\beta(\mathcal{L}(G)) = \sum_{v \in G} (c(v) - 1).$$

It is then an elementary graph-theoretic fact [16, Problem 4.1.28] that

$$\sum_{v \in G} (c(v) - 1) = b(G) - 1.$$

□

5. Open problems

In this section we discuss some open problems suggested by our work. The first is to generalize Theorem 1.1 to all convex geometries. The second is to strengthen the results in Lemma 2.6.

As noted in Section 4, we do not yet have a uniform generalization of Theorem 1.1 that applies to all convex geometries. As a first step in this direction we formulate a conjecture that generalizes Lemma 3.4 for all convex geometries. Let $\mathcal{L}$ be a convex geometry on the ground set X. A subset A of X will be called *independent* if $a \notin \mathcal{L}(A-a)$ for all $a \in A$. We will say that x *depends on* y if there exists an independent set A such that $y \in A$, $x \in \mathcal{L}(A)$ but $x \notin \mathcal{L}(A-y)$. Let $Dep(x)$ be the set of all points y such that x depends on y. The situation in which $Dep(x) = X$ includes the following as special cases

- $x \in \mathrm{int}(\mathcal{A})$ for a point configuration,
- x is a bottleneck of a poset,
- x is a vertex for which $c(v) = 1$ in a chordal graph.

Conjecture 5.1 (cf. Lemmas 3.4, 4.3, and 4.6)**.** *For a convex geometry $\mathcal{L}$ on the ground set X, the complex $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ is contractible unless $Dep(x) = X$.*

The question of what happens if $Dep(x) = X$ is much less clear, as the diversity of outcomes in Lemmas 3.6, 4.3, and 4.6 indicate. In this direction, we pose the following question.

Question 5.2. *For any convex geometry $\mathcal{L}$ on the ground set X, and any point x with $Dep(x) = X$, does the complex $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ have same integral homology as a bouquet of equidimensional spheres? Is it homotopy equivalent to a bouquet of equidimensional spheres?*

Note that we do not even know the answer to the second question for the case of point configurations.

The last open problem is along the same lines and concerns a strengthening of Lemma 2.6. For the sake of clarity, we rephrase that lemma without the use of Gale diagrams.

Let $\mathcal{A}$ be a point configuration in $\mathbb{R}^d$ and $\mathbf{P} = \mathrm{conv}(\mathcal{A})$. Consider the poset N of *non-faces* of $\mathbf{P}$ ordered by inclusion, that is, N is the poset consisting of those subsets B of $\mathcal{A}$ for which there is no face $\mathbf{F}$ of $\mathbf{P}$ having $B = \mathbf{F} \cap \mathcal{A}$. Lemma 2.6 essentially shows that N has the same integral homology as an $(n-d-2)$ sphere.

Question 5.3. *Does the poset of non-faces N for a configuration of n points which affinely span $\mathbb{R}^d$ always have the homotopy type of an $(n-d-2)$-sphere?*

It can be shown that N is *stably homotopy equivalent* to an $(n-d-2)$-sphere, so this is essentially the question of triviality of the fundamental group $\pi_1(N)$ (see [15, Lemma 3] for more discussion).

References

[1] C. Ahrens, G. Gordon and E. McMahon, *Convexity and the beta invariant*, preprint, 1998.

[2] M. Bayer and C. Lee, *Combinatorial aspects of convex polytopes*, in Handbook on Convex Geometry (P. Gruber and J. Wills, eds.), North-Holland, Amsterdam, 1993, pp. 485–534.

[3] A. Björner, *Topological methods*, Handbook of Combinatorics (R. Graham, M. Grötschel, and L. Lovász, eds.), North Holland, Amsterdam, 1995, pp. 1819-1872.

[4] A. Björner, *Homology and shellability of matroids and geometric lattices*, in Matroid Applications (N. White, ed.), Encyclopedia of Mathematics and its Applications, Vol. 40, Cambridge Univ. Press, Cambridge, 1992, pp. 226-283.
[5] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. Ziegler, Oriented Matroids, Encyclopedia of Mathematics and its Applications, vol. 46, Cambridge University Press, Cambridge, 1993.
[6] H. Crapo, *A higher invariant for matroids*, J. Comb. Th.,**2** (1967), 406–417.
[7] P. H. Edelman and R. Jamison, *The theory of convex geometries*, Geom. Ded.,**19** (1985), 247–270.
[8] M. Farber and R. Jamison, *On local convexity in graphs*, Discrete Math., **66** (1987), 231–247.
[9] G. Gordon, *A β invariant for greedoids and antimatroids*, Elect. J. Combin.,**4** (1997), #R13.
[10] G. Gordon and E. McMahon, *A greedoid polynomial which distinguishes rooted arborescences*, Proc. Amer. Math. Soc. **107** (1989), 287–298.
[11] G. Gordon and E. McMahon, *A greedoid characteristic polynomial*, Contemp. Math. **197** (1996), 343-351.
[12] B. Grünbaum, Convex Polytopes, Interscience, London, 1967.
[13] J. Munkres, Elements of Algebraic Topology, Addison-Wesley, Menlo Park, 1984.
[14] D. Quillen, *Homotopy properties of the poset of non-trivial p-subgroups of a group*, Adv. in Math.,**28** (1978), 101-128.
[15] V. Reiner, *On some instances of the generalized Baues problem*, preprint, 1998.
[16] D. B. West, Introduction to Graph Theory, Prentice-Hall, Upper Saddle River, 1996.

School of Mathematics, University of Minnesota, Minneapolis, MN 55455 USA
E-mail address, P. H. Edelman: `edelman@math.umn.edu`

E-mail address, V. Reiner: `reiner@math.umn.edu`