# SUR QUELQUES POLYÈDRES EN GÉOMÉTRIE DES NOMBRES

par E. Ehrhart,

Dans un article paru dans *L'Enseignement mathématique*, tome X (1964), pp. 138-146, nous avons formulé une conjecture relative à un corps convexe fermé: Ses hauteurs $a$, $b$, $c$ dans les directions des axes orthonormés, son volume $V$, sa surface $S$ et le nombre $j$ de ses points entiers vérifient la relation

$$j \leqq V + \frac{S}{2} + a + b + c + 1 \; ; \tag{1}$$

l'égalité n'est atteinte que pour les parallélépipèdes entiers, dont les arêtes sont parallèles aux axes.

Nous allons démontrer cette relation pour trois classes de polyèdres convexes.

Rappelons qu'un polygone ou un polyèdre sont dits *entiers*, si les coordonnées de leurs sommets sont des nombres entiers. Le *périmètre réticulaire* d'un polygone entier s'obtient en prenant comme unité de longueur sur chaque côté la maille du réseau rectiligne de ses points entiers. De même l'*aire réticulaire* d'un polyèdre s'obtient en prenant comme unité d'aire dans chaque face la maille du réseau plan de ses points entiers.

## I. *Prisme entier*

Soit $(P)$ un prisme convexe entier fermé, $j$ le nombre de ses points entiers, $p$ le nombre de points entiers de sa surface, $V$ son volume, $S$ son aire et $S'$, $l'$, $\beta'$ respectivement les mesures réticulaires de sa surface, d'une arête latérale et du périmètre de sa base.

On sait que

$$j - \frac{p}{2} = V + l' + \frac{\beta'}{2} \, , \text{ [1]}$$

[1] *Comptes rendus de l'Acad. des sc.*, 243, 1956, p. 349 (formule 3).

ou, comme $p = S'+2$ [1]),

$$(2) \qquad j = V + \frac{S'}{2} + l' + \frac{\beta'}{2} + 1 \, .$$

Si $a$, $b$, $c$ sont les dimensions du parallélépipède $(\mathscr{P})$ circonscrit à $(P)$ parallèlement aux plans de coordonnées, $l' < c$ et $\beta' \leq \beta \leq 2\,(a+b)$, où $\beta$ est le périmètre réticulaire de la projection d'une base de $(P)$ sur le plan $XOY$. D'autre part, $S' \leq S$. Donc (2) entraîne (1), où l'égalité n'est atteinte que si $(P)$ et $(\mathscr{P})$ coïncident.

II. *Tronc de prisme entier, dont une base a un centre de symétrie*

Soit $\omega$ le centre de symétrie d'une base convexe fermée $(B')$, $j'$ le nombre de ses points entiers, $p'$ le nombre de points entiers de son contour, $s'$ son aire, $s''$ son aire réticulaire et $a', b', c'$ ses hauteurs dans les directions des axes de coordonnées. Le symétrique $(P_2)$ du tronc de prisme $(P_1)$ par rapport à $\omega$ complète $(P_1)$ à un prisme, qui vérifie (1). Comme les caractéristiques de $(P_2)$ sont les mêmes que celles de $P_1$ (dotées de l'indice 1),

$$j = 2j_1 - j' \, , \qquad S = 2S_1 - 2s' \, , \qquad V = 2V_1 \, ,$$
$$a = 2a_1 - a' \, , \qquad b = 2b_1 - b' \, , \qquad c = 2c_1 - c' \, .$$

Par ces substitutions, (1) devient

$$(3) \quad j_1 \leqq V_1 + \frac{S_1}{2} + a_1 + b_1 + c_1 + 1 + \frac{1}{2}(j' - s' - a' - b' - c' - 1) \, .$$

Or $j' = s'' + \dfrac{p'}{2} + 1$ (corollaire du théorème 1 de l'article cité au début). Mais $s'' \leqq s'$ et $p' \leqq p'' \leqq 2\,(a'+b')$, où $p''$ désigne le nombre de points entiers de la projection du contour de $(B')$ sur le plan $XOY$. (Ceci suppose que le plan de $(B')$ ne soit pas perpendiculaire à $XOY$, en quel cas on projetterait sur $XOZ$ ou sur $YOZ$.) Dans (3) l'expression entre parenthèses est donc négative ou nulle.

---

[1]) *Comptes rendus*, 242, 1956, p. 2217 (formule 1).

## III. *Une famille de pyramides*

Soit $(P)$ une pyramide dont le pied de la hauteur entière $c$ se trouve dans la base fermée convexe, qui est située dans le plan des axes $OX$, $OY$ et a pour hauteurs dans la direction de ces axes $a$ et $b$. On suppose $12c \geqq b \geqq a$.

Coupons $(P)$ par le plan $Z = c - n$. Soient $j_n$ le nombre de points entiers de la section fermée et $s_n$, $l_n$ sa surface et son périmètre. On sait (voir l'article mentionné au début) que

$$j_n < s_n + \frac{l_n}{2} + 1 .$$

Par suite

$$j = 1 + \sum_{1}^{n=c} j_n < \sum_{1}^{c} s_n + \frac{1}{2}\sum_{1}^{c} l_n + c + 1 . \tag{4}$$

Comme $s_n = n^2 \dfrac{s_c}{c^2}$,

$$\sum_{1}^{c} s_n = \frac{c(c+1)(2c+1)}{6}\frac{s_c}{c^2} = \frac{s_c}{6c}(2c^2+3c+1) = V + \frac{s_c}{2} + \frac{s_c}{6c} .$$

D'autre part, $l_n = \dfrac{n}{c} l_c$ donne

$$\sum_{1}^{c} \frac{cl_c}{2} + \frac{l_c}{2} < S' + \frac{l_c}{2} ,$$

où $S'$ désigne la surface latérale de la pyramide. De (4) on déduit alors

$$j < V + \frac{S}{2} + a + b + c + 1 + \left(\frac{s_c}{6c} + \frac{l_c}{4} - a - b\right) .$$

Il reste à montrer que l'expression entre parenthèses est négative ou nulle. Comme

$$s_c \leqq \frac{ab}{2} \qquad \text{el} \qquad l_c \leqq 2(a+b) ,$$

il suffit que

$$\frac{ab}{12c} \leqq \frac{a+b}{2},$$

qui est vérifiée car $12c \geqq b$ et $a+b \geqq 2a$.

Remarque. — *L'inégalité (1) est donc en particulier vérifiée par tout tétraèdre entier* O (o, o, o) A (a, o, o) B (o, b, o) C (o, o, c).

*(reçu le 30 janvier 1964)*

E. Ehrhart
11, rue de Bruges
Strasbourg

# POLYNOMIALS ASSOCIATED WITH FINITE CELL-COMPLEXES

I. G. MACDONALD

## *Introduction*

Let $X$ be a finite simplicial complex whose underlying topological space $|X|$ is a $d$-dimensional manifold, possibly with boundary, and let $\partial X$ be the boundary sub-complex of $X$. Let $\tilde{P}(X, t)$ denote the polynomial

$$1-\alpha_0 t+\alpha_1 t^2-\ldots+(-1)^{d+1} \alpha_d t^{d+1},$$

where $\alpha_r$ is the number of $r$-simplexes in $X$, for $r = 0, 1, \ldots, d$. Define the polynomial $\tilde{P}(\partial X, t)$ in the same way, and put

$$f(t) = \tilde{P}(X, t)-\tfrac{1}{2}\tilde{P}(\partial X, t).$$

Then one of our results (Theorem (2.1)) is that the polynomial $f(t)$ satisfies a functional equation of the form

(0.1) $$f(1-t)+(-1)^d f(t) = \text{constant}.$$

In particular, if $|X|$ is a sphere, the constant on the right hand side of (0.1) is zero, and the equation (0.1) is a concise way of writing the Dehn–Sommerville equations ([3], chapter 9) relating the numbers of faces of a simplicial polytope.

For a finite cubical complex $X$ (i.e. a finite cell-complex whose cells are combinatorial cubes) such that $|X|$ is a manifold of dimension $d$, there is an analogous result. This time let $P(X, t)$ denote the polynomial

$$\alpha_0-\alpha_1 t+\ldots+(-1)^d \alpha_d t^d,$$

where $\alpha_r$ is the number of $r$-cubes in $X$, and put

$$f(t) = P(X, t)-\tfrac{1}{2}P(\partial X, t).$$

Then (Theorem (3.1)) the polynomial $f(t)$ satisfies the functional equation

(0.2) $$f(2-t) = (-1)^d f(t).$$

Next, let $L$ be the lattice of points with integer coordinates in $\mathbf{R}^N$, and let $X$ be a lattice polyhedron in $\mathbf{R}^N$, that is to say a finite rectilinear simplicial complex all of whose vertices belong to $L$. As before, we suppose that $|X|$ is a manifold of dimension $d$. For each integer $n > 0$, let $n^{-1} L$ be the lattice of points $x \in \mathbf{R}^N$ such that $nx \in L$ and put

$$L(X, n) = \text{card}\,(X \cap n^{-1} L).$$

Then $L(X, n)$ is known to be a polynomial function of $n$ (Ehrhart [1]). Denote the corresponding polynomial by $L(X, t)$, and put

$$f(t) = L(X, t)-\tfrac{1}{2}L(\partial X, t).$$

---

Received 21 October, 1967; revised 29 December, 1970.

[J. London Math. Soc. (2), 4 (1971), 181–192]

We shall show (Theorem (4.7)) that the polynomial $f(t)$ satisfies the functional equation

$$f(-t) = (-1)^d f(t). \tag{0.3}$$

This result, or an equivalent form of it, was conjectured by Ehrhart in 1959 (see **[1]**, where it is called the reciprocity law).

We also introduce polynomials connected with angle-sums for the three types of cell-complex (simplicial, cubical and lattice) and we show that they satisfy functional equations of the same type as (0.1)—(0.3). These functional equations include as special cases all the linear relations between angle-sums in convex polytopes in Euclidean or spherical space which have been found at various times by various authors.

§1 of this paper is devoted to establishing some general results for finite cell-complexes. In the succeeding sections these results are applied to simplicial complexes (§2), cubical complexes (§3) and lattice polyhedra (§4).

## 1. *Finite cell-complexes*

Let $X$ be a finite cell-complex. We think of $X$ as the set of its open cells (including the empty cell, denoted by $\varnothing$). If $\sigma$, $\tau$ are cells of $X$, the notation $\tau \leqslant \sigma$ or $\sigma \geqslant \tau$ means that $\tau$ is a face of $\sigma$. The topological space underlying $X$ is $|X| = \bigcup_{\sigma \in X} \sigma$. Throughout, we shall suppose that $|X|$ is a manifold, with or without boundary, not necessarily connected, but everywhere of the same dimension $d$. Let $\partial X$ denote the boundary subcomplex of $X$.

Let $V$ be a real vector space (in the applications, $V$ will always be the space $\mathbf{R}[t]$ of real polynomials in one variable) and let $\phi$ be any function on $X$ with values in $V$. For any subset $Y$ of $X$ we define

$$S(Y, \phi) = \sum_{\sigma \in Y} (-1)^{1+\dim \sigma} \phi(\sigma)$$

(here and throughout, the dimension of the empty cell $\varnothing$ is $-1$).

If $\sigma$ is a cell of $X$, its closure $\bar{\sigma}$ in $X$ is the subcomplex consisting of all $\tau \in X$ such that $\tau \leqslant \sigma$. We define $\phi^* : X \to V$ by

$$\phi^*(\sigma) = S(\bar{\sigma}, \phi) = \sum_{\tau \leqslant \sigma} (-1)^{1+\dim \tau} \phi(\tau).$$

The *augmented Euler characteristic* of $X$ is

$$\tilde{\chi}(X) = \sum_{\sigma \in X} (-1)^{1+\dim \sigma}$$

$$= 1 - \chi(X),$$

where $\chi(X)$ is the ordinary Euler characteristic. Finally, let $e$ denote $\phi(\varnothing)$. Then we have

PROPOSITION (1.1). $S(X, \phi^*) + (-1)^d S(X - \partial X, \phi) = \tilde{\chi}(X)\, e.$

*Proof.* From the definitions,

$$\begin{aligned} S(X,\phi^*) &= \sum_{\sigma\in X}(-1)^{1+\dim\sigma}\phi^*(\sigma) \\ &= \sum_{\sigma\in X}(-1)^{1+\dim\sigma}\sum_{\tau\leqslant\sigma}(-1)^{1+\dim\tau}\phi(\tau) \\ &= \sum_{\tau\in X}\phi(\tau)\sum_{\sigma\geqslant\tau}(-1)^{\dim\sigma-\dim\tau}. \end{aligned}$$

Consider the last sum. It is the augmented Euler characteristic of the linked complex [3; p. 86] of $\tau$ in $X$, if $\tau\neq\varnothing$. Hence, since $|X|$ is a manifold, we have

$$(1.2)\qquad \sum_{\sigma\geqslant\tau}(-1)^{\dim\sigma-\dim\tau} = \begin{cases} (-1)^{d-\dim\tau} & \text{if } \tau\notin\partial X, \\ 0 & \text{if } \tau\in\partial X \text{ and } \tau\neq\varnothing, \\ \tilde\chi(X) & \text{if } \tau=\varnothing. \end{cases}$$

Therefore

$$\begin{aligned} S(X,\phi^*) &= \sum_{\tau\notin\partial X}(-1)^{d-\dim\tau}\phi(\tau)+\tilde\chi(X)\,\phi(\varnothing), \\ &= (-1)^{d+1}S(X-\partial X,\phi)+\tilde\chi(X)\,e, \end{aligned}$$

and the proposition is proved.

COROLLARY (1.3). $\phi^{**}=\phi$.

*Proof.* Take $X$ to be the closure $\bar\sigma$ of a cell $\sigma$. Then from (1.1) we have

$$\phi^{**}(\sigma) = S(\bar\sigma,\phi^*) = (-1)^{1+\dim\sigma}S(\sigma,\phi) = \phi(\sigma),$$

since $\bar\sigma-\partial\bar\sigma=\sigma$ and $\tilde\chi(\bar\sigma)=0$.

It follows from (1.3) that (1.1) remains true with $\phi$ and $\phi^*$ interchanged, since $\phi^*(\varnothing)=e$:

COROLLARY (1.4). $S(X-\partial X,\phi^*)+(-1)^d S(X,\phi) = (-1)^d\tilde\chi(X)\,e.$

Now write

$$S(X-\tfrac12\partial X,\phi) = S(X,\phi)-\tfrac12 S(\partial X,\phi)$$

and likewise

$$\tilde\chi(X-\tfrac12\partial X) = \tilde\chi(X)-\tfrac12\tilde\chi(\partial X).$$

COROLLARY (1.5).

$$S(X-\tfrac12\partial X,\phi^*)+(-1)^d S(X-\tfrac12\partial X,\phi) = \left(\tilde\chi(X-\tfrac12\partial X)+\tfrac12(-1)^d\right)e.$$

*If $d$ is odd, the right hand side is* 0. *If $d$ is even, it is equal to* $\tilde\chi(X)\,e$.

*Proof.* By applying (1.1) to the boundary complex $\partial X$, we obtain

$$S(\partial X,\phi^*)+(-1)^{d-1}\left(S(\partial X,\phi)-e\right) = \tilde\chi(\partial X)\,e$$

since $\partial(\partial X)=\varnothing$ and $S(\varnothing,\phi)=\phi(\varnothing)=e$. From this equation and (1.1) we get (1.5). On the other hand, from (1.1) and (1.4) we obtain directly

$$S(X-\tfrac12\partial X,\phi^*)+(-1)^d S(X-\tfrac12\partial X,\phi) = \begin{cases} 0 & \text{if } d \text{ is odd}, \\ \tilde\chi(X)\,e & \text{if } d \text{ is even}. \end{cases}$$

COROLLARY (1.6). *If $d$ is odd, then $\chi(X-\frac{1}{2}\partial X) = 0$; if $d$ is even, $\chi(\partial X) = 0$.*

*Proof.* From (1.5), choosing $\phi$ so that $e \neq 0$, we have

$$\tilde{\chi}(X-\tfrac{1}{2}\partial X) = \begin{cases} \frac{1}{2} & (d \text{ odd}), \\ \tilde{\chi}(X)-\frac{1}{2} & (d \text{ even}), \end{cases}$$

from which (1.6) follows directly, since $\chi = 1-\tilde{\chi}$.

Now let $\omega$ be a real-valued function on $X$ such that

(1.7) $$\sum_{\sigma \geqslant \tau} (-1)^{\dim \sigma} \omega(\sigma) = (-1)^d \omega(\tau)$$

for every cell $\tau \in X$. For any function $\phi : X \to V$ let $\omega\phi$ denote the function $\sigma \mapsto \omega(\sigma)\phi(\sigma)$ from $X$ to $V$. Then

PROPOSITION (1.8). $S(X, \omega\phi^*) = (-1)^{d+1} S(X, \omega\phi)$.

*Proof.* This is a direct consequence of our definitions:

$$\begin{aligned} S(X, \omega\phi^*) &= \sum_{\sigma \in X} (-1)^{1+\dim \sigma} \omega(\sigma)\phi^*(\sigma) \\ &= \sum_{\sigma \in X} (-1)^{1+\dim \sigma} \sum_{\tau \leqslant \sigma} (-1)^{1+\dim \tau} \omega(\sigma)\phi(\tau) \\ &= \sum_{\tau \in X} (-1)^{\dim \tau} \Big( \sum_{\sigma \geqslant \tau} (-1)^{\dim \sigma} \omega(\sigma) \Big) \phi(\tau) \\ &= (-1)^d \sum_{\tau \in X} (-1)^{\dim \tau} \omega(\tau)\phi(\tau) \qquad \text{(by (1.7))} \\ &= (-1)^{d+1} S(X, \omega\phi). \end{aligned}$$

We shall be interested in one particular choice of the function $\omega$. To give it a name, we shall call it the angle-function. From now until the end of this section, we suppose that $X$ is a finite cell-complex in a Euclidean or spherical space $E^d$ of dimension $d = \dim X$, and that the cells of $X$ are convex polytopes. For each non-empty cell $\sigma \in X$, let $\omega(\sigma)$ be the angle "subtended by $X$ at $\sigma$". Precisely, $\omega(\sigma)$ is defined as follows. Take a point $x \in \sigma$ and draw a small $(d-1)$-sphere $\Sigma$ with centre $x$. Let $\mu$ be Lebesgue measure on $\Sigma$, normalised so that $\mu(\Sigma) = 1$. Then

$$\omega(\sigma) = \mu(|X| \cap \Sigma).$$

It is clear that $\omega(\sigma)$ does not depend on the radius of $\Sigma$, provided that this radius is small enough, nor on the choice of $x \in \sigma$. If $\sigma \notin \partial X$, obviously $\omega(\sigma) = 1$. For the empty cell, we define $\omega(\varnothing)$ to be 0 in the Euclidean case, and to be equal to the ratio of the Lebesgue measure of $|X|$ to that of $E^d$ in the spherical case. By Gram's theorem [5] we have

(1.9) $$\sum_{\sigma \in X} (-1)^{\dim \sigma} \omega(\sigma) = (-1)^d \omega(\varnothing)$$

in both cases.

PROPOSITION (1.10). *The angle-function $\omega$ defined above satisfies* (1.7).

*Proof.* When $\tau = \varnothing$, the assertion is just (1.9). Next, let $\tau \notin \partial X$. Then for every $\sigma \geqslant \tau$ we have $\sigma \notin \partial X$ and therefore $\omega(\sigma) = 1$, as remarked above. Hence

$$\sum_{\sigma \geqslant \tau} (-1)^{\dim \sigma} \omega(\sigma) = \sum_{\sigma \geqslant \tau} (-1)^{\dim \sigma} = (-1)^d$$

by (1.2), and therefore (1.7) is satisfied when $\tau \notin \partial X$.

Finally, suppose $\tau \in \partial X$ and $\tau \neq \varnothing$. Choose a point $x \in \tau$ and draw a small $(d-1)$-sphere $\Sigma$ with centre $x$, as above. If $\dim \tau = r$, the plane in which $\tau$ lies is a diametral $r$-plane of $\Sigma$ (in the spherical case, "plane" means great sphere in $E^d$). Let $\Sigma'$ be the section of $\Sigma$ by the diametral $(d-r)$-plane orthogonal to $\tau$, so that $\Sigma'$ is a $(d-r-1)$-sphere. The cells $\sigma$ of $X$ which contain $\tau$ cut out a cell-complex $X'$ on $\Sigma'$. If $\sigma' = \sigma \cap \Sigma'$, and if $\omega'(\sigma')$ is the angle subtended by $X'$ at $\sigma'$, then $\omega(\sigma) = \omega'(\sigma')$ (and in particular $\omega(\tau) = \omega'(\varnothing) = \mu(|X'|)$). Hence, applying (1.9) to the complex $X'$ in $\Sigma'$, we have

$$\sum_{\sigma' \in X'} (-1)^{\dim \sigma'} \omega'(\sigma') = (-1)^{d-r-1} \omega'(\varnothing)$$

or, since $\dim \sigma' = \dim \sigma - r - 1$,

$$\sum_{\sigma \geqslant \tau} (-1)^{\dim \sigma} \omega(\sigma) = (-1)^d \omega(\tau),$$

as required. This completes the proof of (1.10).

It follows that (1.8) is valid for the angle-function $\omega$ and any function $\phi$ on $X$ with values in a real vector space $V$. In fact (1.8) now takes the following form:

PROPOSITION (1.11).

$$S(\partial X, \omega\phi^*) + (-1)^d S(\partial X, \omega\phi) = S(\partial X, \phi^*) - \tilde{\chi}(X)\, e.$$

*Proof.* We have already observed that $\omega(\sigma) = 1$ whenever $\sigma \notin \partial X$, from which observation it follows that

$$S(X, \omega\phi) = S(\partial X, \omega\phi) + S(X - \partial X, \phi).$$

Hence, from (1.8),

$$S(\partial X, \omega\phi^*) + (-1)^d S(\partial X, \omega\phi) + S(X - \partial X, \phi^*) + (-1)^d S(X - \partial X, \phi) = 0,$$

and now (1.11) follows from this equation and (1.1).

## 2. *Simplicial complexes*

For a first application of the formulas of §1, we take $X$ to be a finite simplicial complex and $V$ to be the vector space $\mathbf{R}[t]$ of real polynomials in one variable. For each simplex $\sigma$ of $X$ take

$$\phi(\sigma) = t^{1+\dim \sigma}$$

(so that $e = \phi(\varnothing) = 1$). Then

$$S(X, \phi) = \sum_{\sigma \in X} (-t)^{1+\dim \sigma} = \tilde{P}(X, t), \text{ say,}$$

$$= 1 - \alpha_0 t + \alpha_1 t^2 - \ldots + (-1)^{d+1} \alpha_d t^{d+1},$$

where $\alpha_r$ is the number of $r$-simplexes in $X$, for $r = 0, 1, \ldots, d$. Also we have

$$\phi^*(\sigma) = \sum_{\tau \leqslant \sigma} (-t)^{1+\dim \tau} = (1-t)^{1+\dim \sigma},$$

so that $S(X, \phi^*) = \tilde{P}(X, 1-t)$. Hence from (1.5) we have

THEOREM (2.1). *The polynomial* $\tilde{P}(X-\frac{1}{2}\partial X, t) = \tilde{P}(X, t)-\frac{1}{2}\tilde{P}(\partial X, t)$ *satisfies the functional equation*

$$\tilde{P}(X-\tfrac{1}{2}\partial X, 1-t)+(-1)^d\,\tilde{P}(X-\tfrac{1}{2}\partial X, t) = \begin{cases} 0 \text{ if } d \text{ is odd}, \\ \tilde{\chi}(X) \text{ if } d \text{ is even}. \end{cases}$$

In particular, if $|X|$ is homeomorphic to a sphere—for example if $|X|$ is a convex polytope in Euclidean space—then $\partial X = \varnothing$ and $\tilde{\chi}(X) = (-1)^{d+1}$, so that in this case (2.1) reduces to

$$\tilde{P}(X, 1-t) = (-1)^{d+1}\,\tilde{P}(X, t). \tag{2.2}$$

This polynomial functional equation is one way of writing the Dehn–Sommerville equations relating the numbers of faces of various dimensions of a convex simplicial polytope [**2**; chapter 9].

We shall now find all polynomial solutions of the functional equation of Theorem (2.1.) First consider the case where $d$ is odd, say $d = 2m-1$. Let

$$f(t) = \sum_{r=0}^{2m} a_r t^r \tag{2.3}$$

be such that $f(t) = f(1-t)$. By equating coefficients of $t^{2m-1}$ on either side, we obtain

$$a_{2m-1}+ma_{2m} = 0. \tag{2.4}$$

Now consider the polynomial

$$g(t) = f(t)-a_{2m}t^m(t-1)^m.$$

By (2.4) its degree is at most $2m-2$, and clearly it satifies the same functional equation $g(t) = g(1-t)$. We therefore conclude, by induction on $m$, that

PROPOSITION (2.5). *If* $d = \dim X$ *is odd, the polynomial* $f(t) = \tilde{P}(X-\frac{1}{2}\partial X, t)$ *is of the form*

$$F(u) = b_0+b_1u+\ldots+b_m u^m,$$

*where* $u = t(1-t)$ *and* $m = \frac{1}{2}(d+1)$.

Comparison of (2.3) and (2.5) shows that

$$a_r = b_r-\binom{r-1}{1}b_{r-1}+\binom{r-2}{2}b_{r-2}-\ldots \tag{2.6}$$

for $0 \leqslant r \leqslant 2m$, with the convention that $b_r = 0$ if $r > m$. These equations show that $b_0, b_1, \ldots, b_m$ are uniquely determined by $a_0, \ldots, a_m$ ,and consequently that $a_{m+1}, \ldots a_{2m}$ are uniquely determined by $a_0, \ldots, a_m$. The explicit formulas can of course be beaten out by hand, but the following device saves hard work. We have

$$b_r = \text{residue at } u = 0 \text{ of } F(u)/u^{r+1}$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{F(u)}{u^{r+1}}\,du,$$

where $\gamma$ is a small circle with centre at the origin in the $u$-plane. Changing the variable in the integral by means of $u = t(1-t)$, we have $du = (1-2t)\,dt$, and hence

$$b_r = \frac{1}{2\pi i}\int_{\gamma'} \frac{f(t)(1-2t)}{(t(1-t))^{r+1}}\,dt$$

$$= \text{residue at } t = 0 \text{ of } \frac{f(t)(1-2t)}{(t(1-t))^{r+1}}$$

$$= \left.\frac{f(t)(1-2t)}{(1-t)^{r+1}}\right]_{t^r}$$

where in general the notation $A(t)]_{t^r}$ means the coefficient of $t^r$ in the power series expansion of $A(t)$.

Since $1-2t = (1-t)-t$, we have proved

PROPOSITION (2.7). *The solution of the equations* (2.6) *is*

$$b_r = \left.\frac{f(t)}{(1-t)^r}\right]_{t^r} - \left.\frac{f(t)}{(1-t)^{r+1}}\right]_{t^{r-1}}$$

*or explicitly*

$$b_r = \sum_{i=0}^{r}\left\{\binom{r+i-1}{i} - \binom{r+i-1}{i-1}\right\}a_{r-i}$$

$$= a_r + \sum_{i=1}^{r}\frac{r-i}{i}\binom{r+i-1}{i-1}a_{r-i}.$$

As an example, suppose that $X$ is a "neighbourly polytope" [**2**; Chapter 8] in $\mathbf{R}^{2m}$ with $v$ vertices. For our purposes, $X$ may be defined to be a simplicial polytope (that is, a triangulation of the $(2m-1)$-sphere) in which the number of $r$-simplexes is equal to $\binom{v}{r+1}$ for $r < m$, so that

$$\tilde{P}(x, t) \equiv (1-t)^v \,(\text{mod. } t^{m+1}).$$

By (2.7) it follows that $\tilde{P}(X, t) = \sum_{r=0}^{m} b_r(-u)^r$, where

$$b_r = \left.\frac{\tilde{P}(X, t)}{(1-t)^r}\right]_{t^r} - \left.\frac{\tilde{P}(X, t)}{(1-t)^{r+1}}\right]_{t^{r-1}}$$

$$= \left.(1-t)^{v-r}\right]_{t^r} - \left.(1-t)^{v-r-1}\right]_{t^{r-1}},$$

so that

$$(-1)^r b_r = \binom{v-r}{r} + \binom{v-r-1}{r-1} = \frac{v}{r}\binom{v-r-1}{r-1}.$$

Hence

$$\tilde{P}(X,t) = 1 + \sum_{r=1}^{m} \frac{v}{r} \binom{v-r-1}{r-1} t^r(t-1)^r.$$

From this the number of $q$-simplexes in $X (0 \leqslant q \leqslant 2m-1)$ can be picked out as the coefficient of $(-t)^{q+1}$. The answer agrees with [**2**; p. 166].

Next consider the solutions of (2.1) when $d$ is even, say $d = 2m$. This time, we have to solve the equation

$$f(t)+f(1-t) = \tilde{\chi}(X).$$

If we put

$$f_1(t) = f(t) - \tfrac{1}{2}\tilde{\chi}(X),$$

then $f_1(t)+f_1(1-t) = 0$. Consequently $f_1(t)$ has a zero at $t = \frac{1}{2}$, and so has $1-2t$ as a factor: say

$$f_1(t) = (1-2t)f_2(t).$$

Then $f_2(t) = f_2(1-t)$, and so by our previous analysis $f_2(t)$ is a polynomial of degree $m$ in $u = t(1-t)$. Consequently

PROPOSITION (2.8). *If $d = \dim X$ is even, then the polynomial $\tilde{P}(X-\frac{1}{2}\partial X, t)$ is of the form*

$$\tilde{P}(X-\tfrac{1}{2}\partial X, t) = \tfrac{1}{2}\tilde{\chi}(X) + (1-2t)(b_0 + b_1 u + \ldots + b_m u^m),$$

*where $u = t(1-t)$ and $m = \frac{1}{2}d$.*

In place of (2.7) we find

$$b_r = \left.\frac{f_1(t)}{(1-t)^{r+1}}\right]_{t^r} \tag{2.9}$$

by the same sort of residue argument as before.

For the rest of this section, let $X$ be a finite rectilinear simplicial complex of dimension $d$ in a Euclidean or spherical space of dimension $d$. For each simplex $\sigma$ of $X$, let $\omega(\sigma)$ denote the angle subtended by $X$ at $\sigma$, as in §1. We take $\phi(\sigma) = t^{1+\dim\sigma}$ as before. Then

$$S(X, \omega\phi) = \sum_{\sigma \in X} \omega(\sigma)(-t)^{1+\dim\sigma} = \tilde{\Omega}(X, t), \text{ say,}$$

and since $\phi^*(\sigma) = (1-t)^{1+\dim\sigma}$, we have

$$S(X, \omega\phi^*) = \tilde{\Omega}(X, 1-t).$$

Hence, from (1.8) and (1.10),

PROPOSITION (2.10). $\tilde{\Omega}(X, 1-t) = (-1)^{d+1}\tilde{\Omega}(X, t).$

In particular, when $|X|$ is a simplex the equation (2.10) is equivalent to Poincaré's equations between the angle-sums of a (spherical or Euclidean) simplex: see [**2**; chapter 14].

With $X$ as above let

$$\tilde{\Omega}(\partial X, t) = \sum_{\sigma \in \partial X} \omega(\sigma)(-t)^{1+\dim\sigma} = S(\partial X, \omega\phi).$$

Then from (1.11) we have

PROPOSITION (2.11). $\tilde{\Omega}(\partial X, t)+(-1)^d\tilde{\Omega}(\partial X, 1-t) = \tilde{P}(\partial X, t)-\tilde{\chi}(X)$.

In particular, if $\partial X$ is a simplicial polytope, then $|X|$ is a closed ball, so that $\tilde{\chi}(X) = 0$; in this case (2.11) is equivalent to the set of relations between the angle-sums of a simplicial polytope, due to Perles and quoted on p. 307 of [**2**].

## 3. *Cubical complexes*

A cell-complex $X$ is *cubical* if all its cells are combinatorial cubes. For cubical complexes there are results analogous to those of §2. As before, we take the vector space $V$ to be the space $\mathbf{R}[t]$ of real polynomials in one variable, but this time we define

$$\phi(\sigma) = t^{\dim \sigma} \text{ if } \sigma \neq \varnothing,$$
$$\phi(\varnothing) = 0.$$

Then

$$-S(X, \phi) = \sum_{\sigma \in X}{}' (-t)^{\dim \sigma} = P(X, t), \text{ say,}$$
$$= \alpha_0 - \alpha_1 t + \ldots + (-1)^d \alpha_d t^d,$$

where $\alpha_r$ is the number of $r$-cubes in $X$, for $r = 0, 1, \ldots, d$, and the symbol $\sum'$ indicates that the empty cell is to be omitted from the summation. We have

$$\phi^*(\sigma) = -\sum_{\tau \leqslant \sigma}{}' (-t)^{\dim \tau} = -(2-t)^{\dim \sigma},$$

since an $r$-cube has $2^{r-q}$ faces of dimension $q$, for $0 \leqslant q \leqslant r$. Hence

$$S(X, \phi^*) = P(X, 2-t)$$

and therefore we deduce from (1.5) that (since here $e = \phi(\varnothing) = 0$)

THEOREM (3.1). *The polynomial* $P(X-\tfrac{1}{2}\partial X, t) = P(X, t)-\tfrac{1}{2} P(\partial X, t)$ *satisfies the functional equation*

$$P(X-\tfrac{1}{2}\partial X, 2-t) = (-1)^d P(X-\tfrac{1}{2}\partial X, t).$$

In particular, if $X$ has no boundary this becomes simply

$$P(X, 2-t) = (-1)^d P(X, t). \tag{3.2}$$

The polynomial solutions of this functional equation can be derived from the results of §2 by putting $f(t) = P(X-\tfrac{1}{2}\partial X, 2t)$, for then we have $f(t) = \pm f(1-t)$. When $X$ is a convex cubical polytope, the equation (3.2) is equivalent to the set of equations on p. 156 of [**2**].

Next we shall apply the results of §1 to obtain angle-relations for rectilinear cubical complexes in Euclidean space. With $\phi$ as above and $\omega$ as in §1, we have

$$-S(X, \omega\phi) = \sum_{\sigma \in X}{}' \omega(\sigma)(-t)^{\dim \sigma} = \Omega(X, t), \text{ say,}$$

and

$$S(X, \omega\phi^*) = \Omega(X, 2-t).$$

Hence, from (1.8) and (1.10),

PROPOSITION (3.3). $\Omega(X, 2-t) = (-1)^d \Omega(X, t)$,

which is the analogue of (2.10) for a cubical complex of dimension $d$.

Let
$$\Omega(\partial X, t) = \sum_{\sigma \in \partial X}' \omega(\sigma)(-t)^{\dim \sigma},$$

the empty cell being excluded from the summation. Then from (1.11) we have

PROPOSITION (3.4). $\Omega(\partial X, t) + (-t)^{d+1} \Omega(\partial X, 2-t) = P(\partial X, t)$.

For the case where $\partial X$ is a cubical polytope (i.e., $|X|$ is a closed ball), (3.4) is equivalent to the set of relations found by Perles and Shephard [5].

## 4. *Lattice polyhedra*

Let $L$ be the lattice of points with integer coordinates in $\mathbf{R}^N$, where $N$ is a positive integer, and for each positive integer $n$ let $n^{-1} L$ denote the lattice consisting of all $x \in \mathbf{R}^N$ such that $nx \in L$. Let $X$ be a finite rectilinear simplicial complex in $\mathbf{R}^N$, all of whose vertices belong to the lattice $L$. As before, we shall suppose that $|X|$ is a manifold of dimension $d$.

Let $\sigma$ be a simplex belonging to $X$, and for each integer $n > 0$ let $L(\sigma, n)$ be the number of points of $n^{-1} L$ which belong to $\sigma$. Then $L(\sigma, n)$ is a polynomial function of $n$ (see below) of degree equal to $\dim \sigma$. Let $L(\sigma, t)$ be the corresponding polynomial, and define $\phi$ on $X$ with values in $\mathbf{R}[t]$ by

$$\phi(\sigma) = (-1)^{1+\dim \sigma} L(\sigma, t).$$

Then
$$S(X, \phi) = \sum_{\sigma \in X} L(\sigma, t) = L(X, t), \text{ say,}$$

is a polynomial with the property that, for each integer $n > 0$, the number of points of $n^{-1} L$ which lie in $X$ is equal to $L(X, n)$. In particular,

$$\phi^*(\sigma) = S(\bar{\sigma}, \phi) = L(\bar{\sigma}, t).$$

PROPOSITION (4.1). $L(\bar{\sigma}, t) = (-1)^{\dim \sigma} L(\sigma, -t)$.

*Proof.* Since the intersection of the lattice $L$ with the affine subspace spanned by $\sigma$ is a sublattice of $L$, we may as well assume that $\dim \sigma = N$, the dimension of the ambient space. The proof to be given involves a certain amount of repetition of [4], but this seems desirable in order to present a self-contained account.

Let $e_1, \ldots, e_N$ be the standard basis of $L$, and embed $\mathbf{R}^N$ as the $(N+1)$th coordinate hyperplane in $\mathbf{R}^{N+1}$. Then we have an integer lattice $L'$ in $\mathbf{R}^{N+1}$ generated by $e_1, \ldots, e_N$ and, say, $e_0$. Let $u_0, \ldots, u_N$ be the vertices of the simplex $\sigma$, and let $v_i = e_0 + u_i$ $(0 \leqslant i \leqslant N)$. Then the points $v_i$ are the vertices of a simplex $\sigma'$ congruent to $\sigma$, and they generate a sublattice $M$ of finite index in $L'$. The set $T$ of points $x \in L'$ of the form

$$x = \sum_{i=0}^{N} \mu_i v_i \text{ with } 0 \leqslant \mu_i < 1 \text{ for } i = 0, 1, \ldots, N$$

is a complete set of representatives for $M$ in $L'$. So also is the set $T'$ of points $x \in L'$ of the form

$$x' = \sum_{i=0}^{N} \mu_i' v_i \text{ with } 0 < \mu_i' \leqslant 1 \text{ for } i = 0, 1, \ldots, N.$$

Now $L(\bar{\sigma}, n)$ is equal to the number of points $y \in L$ which lie in the simplex $n\bar{\sigma}'$ with vertices $nv_0, \ldots, nv_N$. Each such point $y$ is congruent mod. $M$ to exactly one point $x$ of $T$, so that we have

$$y = x + \sum_{i=0}^{N} m_i v_i$$

for suitable integers $m_i \geqslant 0$. From the $e_0$-coordinates of either side of this equation we get

$$n = x_0 + \sum_{i=0}^{N} m_i, \tag{4.2}$$

where $x_0$ is the $e_0$-coordinate of $x \in T$; and conversely each solution of (4.2) in non-negative integers $m_i$ gives rise to a point $y \in L$ such that $y \equiv x \pmod M$ and $y \in n\bar{\sigma}'$. Hence the number of such points is equal to the number of solutions of (4.2), which in turn is equal to the coefficient of $u^n$ in

$$u^{x_0}(1+u+u^2+\ldots)^{N+1} = u^{x_0}/(1-u)^{N+1}$$

and is therefore equal to $\binom{n+N-x_0}{N}$. Hence we have

$$L(\bar{\sigma}, n) = \sum_{x \in T} \binom{n+N-x_0}{N}, \tag{4.3}$$

from which it is clear that $L(\bar{\sigma}, n)$ is a polynomial in $n$. (This proof that $L(\bar{\sigma}, n)$ is a polynomial in $n$ is due to Ehrhart [**1**]).

Exactly the same argument, with $T'$ in place of $T$ and $\sigma$ in place of $\bar{\sigma}$, shows that

$$L(\sigma, n) = \sum_{x' \in T'} \binom{n+N-x_0'}{N}. \tag{4.4}$$

Now the mapping $x \to x' = v_0 + \ldots + v_N - x$ interchanges the sets $T$ and $T'$. The $e_0$-coordinates satisfy

$$x_0' = N+1-x_0$$

and therefore from (4.4)

$$L(\sigma, n) = \sum_{x \in T} \binom{n-1+x_0}{N}.$$

Hence finally

$$L(\sigma, -n) = \sum_{x \in T} \binom{-n-1+x_0}{N} = (-1)^N \sum_{x \in T} \binom{n+N-x_0}{N}$$
$$= (-1)^N L(\bar{\sigma}, n).$$

This completes the proof of (4.1).

From (4.1) it follows that

$$\phi^*(\sigma) = L(\bar{\sigma}, t) = (-1)^{\dim \sigma} L(\sigma, -t) \tag{4.5}$$

and therefore

$$S(X, \phi^*) = \sum_{\sigma \in X} -L(\sigma, -t) = -L(X, -t).$$

Hence from (1.1) we have

THEOREM (4.6). $L(X, -t) = (-1)^d L(X-\partial X, t)$,
since in the present situation $e = \phi(\varnothing) = 0$. This functional equation (4.6) was conjectured in 1959 by Ehrhart, who calls it the reciprocity law, and we refer to **[1]** for various consequences of it. We remark also that it remains valid, when suitably reformulated, under the more general hypothesis that the vertices of $X$ belong not to the lattice $L$ but to some "fractional" lattice $m^{-1}L$, where $m$ is a positive integer. In this case, $L(X, n)$ is no longer a polynomial function of $n$ but is what Ehrhart calls a "polynôme mixte" **[1]**.

Also, from (1.4) (or from (4.6)) we deduce

COROLLARY (4.7). $L(X-\frac{1}{2}\partial X, -t) = (-1)^d L(X-\frac{1}{2}\partial X, t)$,
where of course $L(X-\frac{1}{2}\partial X, t)$ means $L(X, t)-\frac{1}{2}L(\partial X, t)$. Hence the polynomial $L(X-\frac{1}{2}\partial X, t)$ is of the form

$$a_0 t^d + a_1 t^{d-2} + a_2 t^{d-4} + \dots .$$

In fact, the leading coefficient $a_0$ is equal to the Euclidean volume of $X$ (see **[4]**, where the fact that the coefficient of $t^{d-1}$ is zero was proved directly), and the constant term is $\chi(X-\frac{1}{2}\partial X)$.

Finally, consider the polynomial $A(X,n)$ of **[4**, §5**]**, which is defined as follows:

$$A(X, n) = \sum_{nx \in L} \omega(x),$$

where $\omega(x)$ is the angle subtended by $X$ at $x$, in the sense of §1 (so that $\omega(x) = 0$ unless $x \in X$, and therefore the sum above is finite). Clearly

$$A(X, n) = \sum_{\sigma \in X} \omega(\sigma) L(\sigma, n),$$

so that $A(X, n)$ is the value at $n$ of the polynomial

$$A(X, t) = \sum_{\sigma \in X} \omega(\sigma) L(\sigma, t) = S(X, \omega\phi).$$

Hence from (1.8), (1.10) and (4.5) we have

THEOREM (4.8). $A(X, -t) = (-1)^d A(X, t)$ *for a lattice polyhedron of dimension d in Euclidean d-space.*

## *References*

1. E. Ehrhart, "Sur un problème de géométrie diophantienne linéaire I, II", *Crelle* 226 (1967), 1–29; 227 (1967), 25–49.
2. B. Grünbaum, *Convex polytopes* (Interscience, 1967).
3. P. J. Hilton and S. Wylie, *Homology theory* (Cambridge, 1960).
4. I. G. Macdonald, "The volume of a lattice polyhedron", *Proc. Cambridge Philos. Soc.*, 59 (1963), 719–726.
5. M. A. Perles and G. C. Shephard, "Angle sums of convex polytopes", *Math. Scand.*, 21 (1967), 199–218.

Magdalen College,
Oxford.