LATTICE POINTS IN A TETRAHEDRON AND GENERALIZED DEDEKIND SUMS

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Let p, q be two positive integers prime to each other. One form of the reciprocity formula for the so-called Dedekind sums is given by the

THEOREM

$$\sum_{x=1}^{p} \frac{x}{p}\left(\left(\frac{qx}{p}\right)\right)+\sum_{x=1}^{q} \frac{x}{q}\left(\left(\frac{px}{q}\right)\right)=\frac{1}{12}\left(\frac{p}{q}+\frac{q}{p}+\frac{1}{pq}-3\right). \quad (1)$$

Here $((X)) = X-[X]-\frac{1}{2}$, X not an integer,

$= 0$ $\qquad X$ an integer.

Various proofs have been given by Rademacher, Rademacher and Whiteman, Re'dei and myself. For references see[1]. I have shown that the theorem is the particular case $f(x) = x$ in the evaluation of

$$\sum f\left(\frac{x}{p}+\frac{y}{q}\right), \quad (2)$$

where f is a polynomial and the summation is extended over those integer sets (x, y), i.e. lattice points, lying in the triangle

$$0<x<p,\ 0<y<q,\ \frac{x}{p}+\frac{y}{q}<1. \quad (3)$$

This method suggests the extension of the formula (1) to a set of n positive integers $p, q, r, s, \ldots$ no two of which have a common factor. The results, however, now take a different form. Take $n=3$ and write

$$S_3(p, q, r) = \sum_{x=1}^{p-1} \frac{x}{p}\left(\left(\frac{qrx}{p}\right)\right)+\sum_{x=1}^{q-1} \frac{x}{q}\left(\left(\frac{rpx}{q}\right)\right) + \sum_{x=1}^{r-1} \frac{x}{r}\left(\left(\frac{pqx}{r}\right)\right). \quad (4)$$

Denote by $N_3(p, q, r)$ the number of lattice points in the tetrahedron

$$0 \leqslant x<p,\ 0 \leqslant y<q,\ 0 \leqslant z<r,\ 0<\frac{x}{p}+\frac{y}{q}+\frac{z}{r}<1. \quad (5)$$

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Write for shortness S_3, N_3. Then we have the

THEOREM.

$$S_3+N_3 = \tfrac{1}{6}pqr+\tfrac{1}{4}\sum qr+\tfrac{1}{4}\sum p+\frac{1}{12}\sum\frac{qr}{p}+\frac{1}{12\,pqr}-2, \qquad (6)$$

the summation referring to p, q, r.

A formula will also be found for $n = 4$.

1. The method of proof shows that for $n > 4$, the formula for S_n will depend upon the number of lattice points in sections of an n-dimensional tetrahedron defined by

$$\lambda < \Sigma\, x/p < \lambda+1$$

for a number of values of $\lambda \leqslant n-1$.

The function $((X))$ has some well-known properties. It is an odd periodic function of X, and so

$$((-X)) = -((X)),\ ((X+1)) = ((X)).$$

Also

$$((X))+((X+1/p))+\dots+((X+(p-1)/p)) = ((pX)). \qquad (7)$$

2. We require a formula for N_3. One is given by

$$2N_3 = \sum_{x,\,y,\,z}{}' \left(\left[\frac{x}{p}+\frac{y}{q}+\frac{z}{r}\right]-1\right)\left(\left[\frac{x}{p}+\frac{y}{q}+\frac{z}{r}\right]-2\right), \qquad (8)$$

where the summation is taken over

$$0 \leqslant x < p,\ 0 \leqslant y < q,\ 0 \leqslant z < r, \qquad (9)$$

and the accent denotes the omission of the term $x = y = z = 0$.

For from (9), $0 < \frac{x}{p}+\frac{y}{q}+\frac{z}{r} < 3$, and those lattice points for which

$$0 < \frac{x}{p}+\frac{y}{q}+\frac{z}{r} < 1,$$

contribute each 2 to the sum, and each of those for which

$$1 < \frac{x}{p}+\frac{y}{q}+\frac{z}{r} < 3$$

contribute zero. There are no lattice points with

$$\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=1 \text{ or } 2.$$

We write (8) as

$$2N_3=\sum_{x,y,z}{}' \,(E-3/2-((E)))\,(E-5/2-((E))), \qquad (10)$$

where $E=\frac{x}{p}+\frac{y}{q}+\frac{z}{r}$. Hence

$$2N_3=A+B+C,$$

say, where

$$A=\sum_{x,y,z}{}'(E-3/2)(E-5/2),\quad B=-2\sum_{x,y,z}{}'(E-2)((E)),$$
$$C=\sum_{x,y,z}{}'((E))^2 \qquad (11)$$

summed over (9).

Hence

$$A+15/4=\sum_{x,y,z}(E-3/2)(E-5/2).$$

Multiplying out and summing Σx^2, Σxy, Σx, we have, Σ now denoting $\sum_{p,q,r}$,

$$A+\frac{15}{4}=\sum\frac{(p-1)(p)(2p-1)}{6p^2}qr+\sum\frac{2}{4pq}(p)(p-1)(q)(q-1)r$$
$$-4/2\,\Sigma(p)\,(p-1)q/p+15\,pqr/4=\Sigma\tfrac{1}{3}pqr-\Sigma\tfrac{1}{2}qr+\Sigma\tfrac{1}{6}qr/p$$
$$+\Sigma\tfrac{1}{2}pqr-\Sigma\tfrac{1}{2}(p+q)r+\Sigma\tfrac{1}{2}(r)-2\Sigma pqr+2\Sigma qr+15/4pqr,$$

and so

$$A=\tfrac{1}{4}pqr+\tfrac{1}{2}\Sigma qr+\tfrac{1}{2}\Sigma p+\tfrac{1}{6}\Sigma\, qr/p-15/4. \qquad (12)$$

Clearly we can include $x=y=z=0$ in the summations for B and C in (11).

Next for B. From (7) on summing for y, z in turn, we have

$$\sum_{x,y,z}\frac{x}{p}\left(\left(\frac{x}{p}+\frac{y}{q}+\frac{z}{r}\right)\right)=\sum_{x,y}\frac{x}{p}\left(\left(\frac{rx}{p}+\frac{ry}{q}\right)\right)=\sum_{x}\frac{x}{p}\left(\left(\frac{qrx}{p}\right)\right)$$

since ry runs through a complete set of residues mod q.

Also $$\sum_{x,y,z}\left(\left(\frac{x}{p}+\frac{y}{q}+\frac{z}{r}\right)\right)=\sum_{x=0}^{pqr-1}\left(\left(\frac{x}{pqr}\right)\right),$$

since $qrx+rpy+pqz$ runs through a complete set of residues mod pqr. The sum is

$$\sum_{x=1}^{pqr-1}\left(\frac{x}{pqr}-\tfrac{1}{2}\right)=0.$$

Hence $B=-2\,S_3$. (13)

Finally

$$C=\sum_{x=0}^{pqr-1}\left(\left(\frac{x}{pqr}\right)\right)^2=\sum_{x=1}^{pqr-1}\left(\frac{x}{pqr}-\tfrac{1}{2}\right)^2$$

$$=\frac{1}{6p^2q^2r^2}(pqr-1)(pqr)(2pqr-1)-\frac{1}{2pqr}(pqr-1)(pqr)+\tfrac{1}{4}(pqr-1)$$

$$=\frac{(pqr-1)(2pqr-1)}{6pqr}-\frac{(pqr-1)}{2}+\tfrac{1}{4}(pqr-1)$$

$$=\frac{1}{12}pqr-\tfrac{1}{4}+\frac{1}{6pqr}.$$

Hence we have the required formula

$$2\,N_3=\frac{1}{3}pqr+\tfrac{1}{2}\Sigma\,qr+\tfrac{1}{2}\Sigma\,p+\frac{1}{6}\Sigma\frac{qr}{p}+\frac{1}{6pqr}-4-2\,S_3,$$

3. For the four dimensional result, let N_4 denote the number of lattice points in

$$0\leqslant x<p,\ 0\leqslant y<q,\ 0\leqslant z<r,\ 0\leqslant w<s,$$

$$0<\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+\frac{w}{s}<1,$$

where no two of p, q, r, s have a common factor. Write

$$S_4=\sum_{p,q,r,s,}\ \sum_{x=1}^{p-1}\frac{x}{p}\left(\left(\frac{qrsx}{p}\right)\right).$$

We consider now

$$S=\sum_{x,y,z,w}{}'\left(\left[\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+\frac{w}{s}\right]-1\right)\left(\left[\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+\frac{w}{s}\right]-2\right),$$

where the summation is taken over the lattice points L given by

$$0\leqslant x<p,\ 0\leqslant y<q,\ 0\leqslant z<r,\ 0\leqslant w<s$$

with the exclusion of $x=y=z=w=0$.

Here $$0<\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+\frac{w}{s}<4.$$

The points L with

$$1<\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+\frac{w}{s}<3$$

contribute zero to S, while each of the points L with

$$0<\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+\frac{w}{s}<1$$

say N_4 in number, and each of those with

$$3<\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+\frac{w}{s}<4$$

say, N' in number, contribute 2 to S. Hence

$$S=2\,(N_4+N').$$

Let N'' be the number of the points L satisfying

$$0<\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+\frac{w}{s}<1,\ xyzw=0.$$

Then excluding these, we have a 1-1 correspondence between the remaining ones in N_4 and those in N' given by

$$x+x'=p,\ y+y'=q,\ z+z'=r,\ w+w'=s.$$

Hence $N_4=N'+N''$ and $S=4N_4-2N''$.

Now $$N''=N_4''+N_3''+N_2''+N_1'',$$

where N''_r denotes the number of the lattice points L when exactly r of the variables equal zero. Clearly

$$N_3''=\sum_{p,\,q,\,r,\,s}(p-1),\ N_4''=0,$$

also

$$N_2''=\sum_{p,\,q,\,r,\,s}(p-1)\,(q-1)/2,$$

since there are $(p-1)(q-1)$ solutions of

$$0<x<p,\ 0<y<q,\ x/p+y/q<2$$

and there is a 1-1 correspondence between those in $x/p+y/q<1$ and those in $x/p+y/q>1$ given by $x+x'=p, y+y'=q$. Finally $N_1''=\sum_{p,\,q,\,r,\,s}N_3(p,\,q,\,r)$, and so $S=4N_4$
$-2\sum_{p,\,q,\,r\,s}(p-1)-\sum_{p,\,q,\,r,\,s}(p-1)\,(q-1)-2\sum_{p,\,q,\,r,\,s}N_3(p,q,r)$. (14)

Next we split S into three sums, say A', B', C', corresponding to A, B, C but now a fourth variable w/s also occurs. We find $A'+15/4$

$$=\sum\frac{(p-1)(p)(2p-1)qrs}{6p^2}+\sum\frac{2}{4pq}(p)(p-1)(q)(q-1)rs$$

$$-\frac{4}{2}\sum\frac{(p)(p-1)qrs}{p}+\frac{15}{4}pqrs=\sum\frac{(p-1)(2p-1)qrs}{6p}$$

$$+\Sigma\tfrac{1}{2}(p-1)(q-1)rs-2\Sigma(p-1)qrs$$

$$+15\,pqrs/4=pqrs/12+\Sigma\tfrac{1}{2}rs+\Sigma qrs/(6p).$$

Next $$B'=-2S_4.$$

Finally

$$C'=\sum_{x=1}^{pqrs-1}(x-\tfrac{1}{2})^2=\frac{1}{12}pqrs-\tfrac{1}{4}+\frac{1}{6pqrs}.$$

This gives

$$S=\frac{1}{6}pqrs+\tfrac{1}{2}\Sigma pq-4+\sum\frac{qrs}{p}+\frac{1}{6pqrs}-2S_4.$$

Hence on substituting for N_3 from (6) in (14),

$$4N_4+2S_4+2\sum_{p,q,r,s}S_3(p,q,r)=2\Sigma(p-1)+\Sigma(p-1)(q-1)$$

$$+\frac{1}{3}\Sigma pqr+\tfrac{1}{2}\Sigma(qr+rp+pq)+\tfrac{1}{2}\Sigma(p+q+r)$$

$$+\frac{1}{6}\sum\left(\frac{qr}{p}+\frac{rp}{q}+\frac{pq}{r}\right)+\frac{1}{6}\sum\frac{1}{pqr}-16$$

$$+\frac{1}{6}pqrs+\tfrac{1}{2}\Sigma pq+\frac{1}{6}\sum\frac{qrs}{p}-4+\frac{1}{6pqrs}$$

$$=\frac{1}{6}pqrs+2\tfrac{1}{2}\Sigma pq+\tfrac{1}{2}\Sigma p-22+\frac{1}{3}\Sigma pqr$$

$$+\frac{1}{6}\sum\left(\frac{qr}{p}+\frac{rp}{q}+\frac{pq}{r}\right)+\frac{1}{6}\sum\frac{qrs}{p}+\frac{1}{6}\sum\frac{1}{pqr}+\frac{1}{6pqrs}.$$

REFERENCE

1. L. J. Mordell: On the reciprocity formula for the Dedekind sums, *Amer. J. Math.*, (in course of publication).

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