Voronoi's Hypothesis on Perfect Forms and L-types

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Abstract

George Voronoi (1908, 1909) introduced two important reduction methods for positive quadratic forms: the reduction with perfect forms, and the reduction with L-type domains, often called domains of Delaunay type. The ørst method is important in studies of dense lattice packings of spheres. The second method provides the key tools for ønding the least dense lattice coverings with equal spheres in lower dimensions. In his investigations Voronoi heavily relied on that in dimensions less than 6 the partition of the cone of positive quadratic forms into L-types reønes the partition of this cone into perfect domains. Voronoi conjectured implicitely and Dickson (1972) explicitely that the L-partition is always a reønement of the partition into perfect domains. This was proved for $n \leq 5$ (Voronoi, Delaunay, Ryshkov, Baranovskii). We show that Voronoi-Dickson conjecture fails already in dimension 6.

Keywords: Positive Quadratic Form, Perfect Form, Delaunay Tiling (L-partition), L-type, Repartitioning Complex, Lattice Packing, Lattice Covering, Lattice E_6, Lattice E_6^*, Gosset Polytope 2_{21}
AMS Classiøcation. Primary 11, 52 Secondary 15

1 Introduction and main result

Quadratic Forms are inportant to many areas of pure and applied mathematics, including number theory, combinatoirics, the theory of ønite groups,

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error-corresting codes, cryptography, crystallography, etc. (e.g. see Conway and Sloane (1999)). Positive (semideønite) quadratic forms (referred to as PQFs) in n indeterminates form a closed cone $\mathfrak{P}(n)$ of dimension $N = \frac{n(n+1)}{2}$ in $\mathbb{R}^N$, and this cone is the main object of study in our paper. The interior of $\mathfrak{P}(n)$ consists of quadratic forms of rank n. PQFs serve as analitic representations of point lattices. Let $\mathbf{v}_1, \ldots, \mathbf{v}_1$ be a coordinate frame in Euclidean space $\mathbb{E}^n$. A lattice of points with basis $[\mathbf{v}_1, \ldots, \mathbf{v}_n]$ is the set of all integal linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_n$. If G is the Gramm matrix of a PQF, then the basis $[\mathbf{v}_1, \ldots, \mathbf{v}_n]$ of the corresponding lattice is deøned (up to isometry) by $g_{ij} = \mathbf{v}_i \cdot \mathbf{v}_j$. The Gramm matrix of the corresponding PQF is $[\mathbf{v}_1, \ldots, \mathbf{v}_n]^T [\mathbf{v}_1, \ldots, \mathbf{v}_n]$. Thus, there is a one-to-one correspondence between congruence classes of n-dimensional lattices and integral equivalence classes (i.e. with respect to $GL(n, \mathbb{Z})$-conjugation) of positive quadratic forms. For basic results of the theory of lattices and PQFs and their applications see Ryshkov and Baranovskii (1978), Gruber and Lekkerkerker (1987), Erds, Gruber and Hammer (1989), Conway (1998), Conway and Sloane (1999).

$GL(n, \mathbb{Z})$ acts pointwise on the space of quadratic forms $\mathbb{R}^N$. The L-type and perfect partitions are important polyhedral subdivisions of $\mathfrak{P}(n)$, invariant with respect to the action of $GL(n, \mathbb{Z})$. These partitions have been intensively studied in geometry of numbers and combinatorics. In our paper we study how they are related to each other for $n = 6$.

Deønition 1 Let S be a discrete set of points in $\mathbb{E}^n$. A convex polytope P in $\mathbb{E}^n$ is called a Delaunay cell of the system of points S if:

1) all vertices of P belong to S,

2) there is a sphere circumscribed about P (called the empty sphere of P),

3) no points of S, except for the vertices of P, lie inside or on its empty sphere.

If S is not pathological, Delaunay cells form a convex face-to-face tiling of $conv\, S$ that is deøned uniquely by S. Delaunay tilings have many applications in computational geometry, mesh generation, the theory of lattices, mathematical crystallography, etc. If L is a lattice, the Delaunay tiling of L is invariant under the action of the isometry group of L.

Lattices L_1 and L_2 belong to the same L-type if their Delaunay tilings are aŒnely equivalent (the notion of L-type is, in fact, due to Voronoi (1908)). The L-type of a quadratic form is deøned as the L-type of its lattice. L-type domains are open pointed polyhedral cones in $\mathbb{R}^N$ ($N = \frac{n(n+1)}{2}$). In each dimension there are only ønitely many L-type domains (Voronoi, 1908). Each L-type domain is, of course, the union of inønitely many convex cones that are equivalent with respect to $GL(n,\mathbb{Z})$ acting pointwise on $\mathbb{R}^N$. The closures of these convex cones tile $\mathfrak{P}(n)$.

The notions of Delauny tiling and L-type are extremely important in the study of extremal and group-theoretic properties of lattices. For example, the analysis of Delaunay cells in the famous Leech lattice conducted by Conway, Sloane and Borcherds showed that 23 ȷdeep holesȷ (Delaunay cells of radius equal to the covering radius of the lattice) in the Leech lattice correspond to 23 even unimodular 24-dimensional lattices (Niemeier's list) that, in turn, give rise to 23 ȷgluingȷ constructions of the Leech lattice from root lattices. Barnes and Dickson (1967, 1968) and, later, Delaunay et al. (1969, 1970) proved that the closure of each N-dimensional L-type domain has at most one local minimum of the covering density, and if such a minimum exists and lies in the interior of the domain, the group of $GL(n,\mathbb{Z})$–automorphisms of the domain maps this form to itself.

Theorem 2 (Barnes, Dickson) The closure of any N-dimensional L-type domain contains at most one local minimum of the sphere covering density.

Using this approach Delaunay, Ryshkov and Baranovskii (1963, 1976) found the best lattice coverings in $\mathbb{E}^4$ and $\mathbb{E}^5$. The theory of L-types also has numerous connections to combinatorics and, in particular, to cuts, hypermetrics, and regular graphs (see Deza et al. (1997)).

The L-type partition of the cone of PQFs is closely related to the theory of perfect forms originated by Korkine and Zolotareœ (1873). Let $f(\mathrm{x},\mathrm{x})$ be a PQF. The arithmetic minimum of $f(\mathrm{x},\mathrm{x})$ is the minimum of this form on $\mathbb{Z}^n$. The integral vectors on which this minimum is attained are called the representations of of the minimum, or the minimal vectors of $f(\mathrm{x},\mathrm{x})$: these vectors have the minimal length among all vectors of $\mathbb{Z}^n$ when $f(\mathrm{x},\mathrm{x})$ is used as the metrical form. A form $f(\mathrm{x},\mathrm{x})$ is called perfect if it can be reconstructed from all representations of its arithmetic minimum. In other words, a form $f(\mathrm{x},\mathrm{x})$ with the arithmetic minimum m and the set of minimal vectors $\{\mathbf{v}_k | k = 1,...,2s\}$ is perfect if the system

$$\sum_{i,j=1}^{n} a_{ij}v_{ik}v_{jk} = m,$$

where $k = 1, ..., 2s$, has a unique solution (a_{ij}) in the space of symmetric matrices $\mathbb{R}^N$ (of course, there must be at least $\frac{n(n+1)}{2}$ non-collinear minimal vectors). In each dimension there are only ønitely many perfect forms up to $GL(n, \mathbb{Z})$-equivalence (Voronoi, 1908). Intuitively, perfect lattices are those that have a large supply of minimal vectors, although a perfect lattice in dimension higher than 8 is not always spanned by its minimal vectors (Ryshkov, 1973). A perfect form $f(x,x)$ can obviously be described as a hyperplane in $\mathbb{R}^N$ that contains $N + 1$ integer points whose coordinates are the images of the minimal vectors $\{\mathbf{v}_k | k = 1, ..., 2s\}$ under the Veronese mapping $V : \mathbf{v}_k \to \{v_{ik}v_{jk} | 1 \le i \le j \le n\}$. Perfect forms play an important role in lattice sphere packings. Voronoi's theorem (1908) says that a form is extreme¯i.e., a maximum of the packing density¯if and only if it is perfect and eutactic (see Coxeter (1951), Conway, Sloane (1988) for the proof). The notion of eutactic form arises in the study of the dense lattice sphere packings and is directly related to the notion of perfect form. The dual (also called reciprocal) of $f(x,x)$ is a form whose Gramm matrix is the inverse of the Gramm matrix of $f(x,x)$. The dual form is normally denoted by $f^*(\mathbf{x}, \mathbf{x})$. A form $f(\mathbf{x}, \mathbf{x})$ is called eutactic if the dual form $f^*(\mathbf{x}, \mathbf{x})$ can be written as $\sum_{k=1}^{s} \alpha_k (\mathbf{v}_k \bullet \mathbf{x})^2$, where $\{\mathbf{v}_k | k = 1, ..., s\}$ is the set of mutually non-collinear minimal vectors of $f(\mathbf{x}, \mathbf{x})$, and $\alpha_k > 0$.

Theorem 3 (Voronoi) A form $f(x,x)$ is a maximum of the sphere packing density if and only if

1) $f(x,x)$ is perfect
2) $f(x,x)$ is eutactic

A cone in $\mathbb{R}^N$ spanned by the images of the minimal vectors of a perfect form is called a perfect cone. The union of all (closed) perfect cones corresponding to forms integrally equivalent to f is called the perfect domain of f. For each perfect cone there are inønitely many $GL(n, \mathbb{Z})$-equivalent ones, so the perfect domain of f consists of inønitely many equivalent perfect cones, just like an L-type domain consists of inønitely many convex cones. A fundametal theorem of Voronoi (1908) in the interpretation of Delaunay and Ryshkov (1968) says that the cone of PQFs is tiled face-to-face by perfect cones. Voronoi proved that there are only ønitely many non-equivalent perfect forms. Therefore, there is a ønite set of perfect cones in $\mathbb{R}^N$ such that each form in n variables is equivalent to a form lying in one of these domains. Voronoi gave an algorithm ønding all perfect domains for given

n. This algorithm is known as Voronoi's reduction with perfect forms. For the computational analysis of his algorithm and its improvements see Martinet (1996). Voronoi's algorithm was sucsessfully applied by him and other researchers in searching for perfect forms in lower dimensions, i.e. $n \leq 9$ (Voronoi (1908), Barnes (1957), Stacey (1972), Martinet (1996)). Perfect forms have been completely classiøed in dimensions $n \leq 7$, but already for $n = 9$ there are billions of them (see Martinet (1996)). The theory of perfect forms was used for ønding the best lattice packings in low dimensions and for classifying ønite subgroups of $GL(n, \mathbb{Z})$ for small values of n (Ryshkov et al., 1978).

For $n = 2, 3$ the L-partition and the perfect partition of $\mathfrak{P}(n)$ coincide. The perfect domain of D_4 (the second perfect form in 4 variables) exempliøes a new pattern in the relation of these partitions. Namely, the domain of D_4 is decomposed into a number of simplicial L-type domains like a pie. These simplexes are L-type domains of two arithmetic types: type I is adjacent to the the perfect/L-type domain of A_4, type II is adacent to an arithmetically equivalent L-type domain (also type II, indeed) from the L-subdivision of the adjacent D_4 domain (the situation is depicted schematically in Fig. 1; for more details see Delaunay et al. (1963, 1968)).

Voronoi (1909) proved that for $n \leq 4$ the tiling of $\mathfrak{P}(n)$ with L-type domains reønes the partition of this cone into perfect domains. He wondered if it was just a coincidence. Ryshkov and Baranovskii (1976) proved that this reønement hypothesis is true for $n = 5$. In his paper of 1972 Dickson proved that the perfect domain of A_n, also called the ørst perfect form after Korkine and Zolotareœ (1873), is the only perfect domain that is also an L-type domain; he was also ørst to mention explicitly the common believe in Voronoi's reønement conjecture. Baranovski, Delaunay and Ryshkov also suspected that Voronoi's conjecture would hold in all dimensions (see Ryshkov and Baranovskii (1975), Delaunay et al. (1945, 1963, 1969)). Using the theory of repartitioning complexes (developed by Ryshkov and Baranovski (1976)) and the theory of dual system of vectors (introduced by Erdahl and Ryshkov (1990, 1991a,b)), we show that Voronoi's conjecture does not hold in dimension 6.

2 Perfect forms in dimension 6

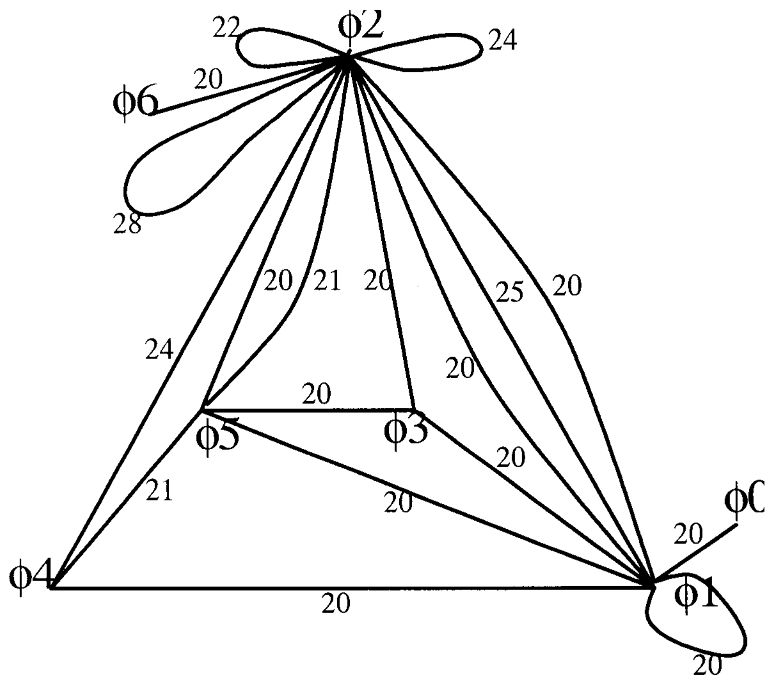

Figure 1: The incidence graph of perfect domains of $\mathfrak{P}(6)$

The incidence graph of perfect domains for n=6 will be inserted right before the submission.

All perfect forms in dimension 6 were classified by Barnes (1957). **Fig. 1** shows the incidence graph of perfect domains for $n = 6$. The nodes of the graph are non-equivalent perfect forms, the edges of the grpah are non-equivalent facets (called walls) between adjacent perfect domains. Notice that the graph has loops (two perfect cones of the same type are adjacent) and multiple edges (two perfect domains are adjacent through non-equivalent walls). Each wall is marked with the number of integral rank one forms that serve as extreme rays for this wall.

In **Fig. 1** we use Voronoi-Barnes notation for perfect forms. Below we give the full list of quadratic forms in dimension 6. Coxeter's symbols are in the second column, Korkine-Zolotareff's symbols in the third, and Conway-Sloane's in the fourth.

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ϕ_0	A_6	U_6	p_6^7	$\sum_1^6 x_i^2 + \sum_{1<i<j<6} x_i x_j$
ϕ_1	D_6	V_6	p_6^3	$\phi_0 - x_1 x_2$
ϕ_2	E_6	X	p_6^1	$\phi_0 - x_1 x_2 - x_1 x_3$
ϕ_3	$A_{6,1}$		p_6^6	$\phi_0 - \frac{1}{2}(x_1x_2 + x_3x_4 + x_5x_6)$
ϕ_4	$E_6^* = E_6^3 = A_{6,0}$		p_6^2	$\phi_0 - \frac{1}{2}(x_1x_2 + \sum_{3<i<j<6} x_i x_j)$
ϕ_5	$A_{6,2}$		p_6^4	$\phi_0 - \frac{1}{2}(x_1x_2 + x_3x_4 + x_3x_5 + x_4x_5)$
ϕ_6	A_6^2		p_6^5	$\phi_0 - \frac{1}{2}(2x_1x_2 + x_1x_3 + x_1x_6 + x_2x_5 + x_4x_6 + 2x_5x_6)$

It is known since the times of Korkine and Zolotareœ that the central ray (centroid) of the perfect domain of E_6^* is equivalent to E_6, and that the centroid of the perfect domain of E_6^* is equivalent to E_6^* (see e.g. Coxeter (1951)). Barnes (1956, 1957) showed that there is only one arithmetic type of wall between E_6 and E_6^* (see Fig. 1). In Section 6 we prove that the Delaunay type of lattice does not change in some interior points of the wall between domains E_6 and E_6^*.

3 L-types of lattices

Ryshkov and Baranovskii (1976) showed that for $n = 5$ there are 221 general (simplicial) L-types. The total number of L-types for $n = 5$ is not known. Recently, P. Engel (1998) has made important steps towards a complete desrciption of the combinatorics of L-types of 5-dimensional lattices. In particuar, he classiøed all extreme L-types, i.e. L-types that serve as extreme rays of L-type domains (see Engel (1998), Engel and Grishukhin (2000)). A complete classiøcation of L-types in the 6-dimensional case seems to be out of reach in the near future, although the L-types of the perfect lattices have been completely classiøed by Baranovski (1991). Below we give a short description of Delaunay tilings of E_6 and E_6^*.

3.1 L-partition of E_6

In this subsection we discuss Delaunay tilings of lattices lying in a small neigbourhood of E_6 in the space of parameters. More speciøcally, we study the L-partition of $\mathfrak{P}(n)$ near the ray corresponding of E_6. The Delaunay tiling of lattice E_6 is formed by congruent copies of the Gosset polytope

(2_{21} in Coxeter's notation), which is the convex hull of a unique two- distance spherical set in $\mathbb{E}^6$. We refer to the Gosset polytope as G-tope. The G-topes of the Delaunay tiling of E_6 fall into two translation classes. The star of a lattice point is formed by 54 G-topes, 27 in each translation class.

The G-tope is quite remarkable. It has 27 vertices, 216 edges, 72 regular simplicial facets, and 36 regular cross-polytopal facets (e.g. Coxeter (1995)). Thus, the vertices of the G-tope form a spherical two distance set. Polytopes whose vertices form a spherical two distance set are very interesting combinatorial objects (see Deza and Laurent (1997), Deza, Grishukhin, and Laurent (1992)). In the case of G-tope the two distance structure is realized so that for each vertex $\mathbf{v}$ of the G-tope there is a vector $\mathbf{p}_v$ such that the vertex set of G-tope can be represented as $\mathbf{v} \cup V_1 \cup V_2$, where $V_1 = \{\mathbf{u} \in S \mid (\mathbf{u} - \mathbf{v}) \bullet \mathbf{p} = 1\}$, and $V_2 = \{\mathbf{u} \in S \mid (\mathbf{u} - \mathbf{v}) \bullet \mathbf{p} = 2\}$ (see Fig. ??). For a detailed description of geometric and group theoretic properties of the G-tope see (Coxeter (1973, 1995)).

Fig. 2: The three-Æoor structure of the Gosset polytope

Below, we show that for every subset of vertices of a Dealunay cell of E_6, E_6 can be perturbed so that this subset becomes a Delaunay cell for the perturbed lattice. In particular, this implies that there are perturbations of E_6 having a Delaunay simplex of volume 3, the maximal relative volume of a Delaunay lattice simplex in R^6. Delaunay (1937) was the ørst to ask about possible volumes of Delaunay simplexes. Ryshkov (1973) showed that in every dimension $2r+1$ there is a lattice with a Delaunay simplex of relative volume r. Namely, Ryshkov proved that a Coxeter-Barnes lattice A_n^k, where $n \geq 2k+1$, has a Delaunay simplex of volume k. Ryshkov also noticed that in the case of A_n^k the existence of non-fundamental Delaunay simplexes is directly related to another interesting phenomenon: for $n \geq 9$ perfect lattice A_n^k is not generated by its shortest vectors (Coxeter (1951), Ryshkov (1973)). Our approach to the Delaunay structure of E_6 gives us as a by-product an inønite series of lattices with a Delaunay simplex of relative volume $n-3$.

Proposition 4 For every convex polytope D whose vertex set is a subset of the vertex set of the G-tope there is a perturbation of E_6 making D a Delaunay polytope for the perturbed lattice.

Proof. Denote by $\phi_{E_6}(x)$ an inhomogenious quadratic function whose quadratic part is E_6, and such that $\phi_{E_6}(x) = 0$ is an ellipsoid circumscribing the G-tope. For $\alpha > 0$ consider quadratic function

Figure 1:

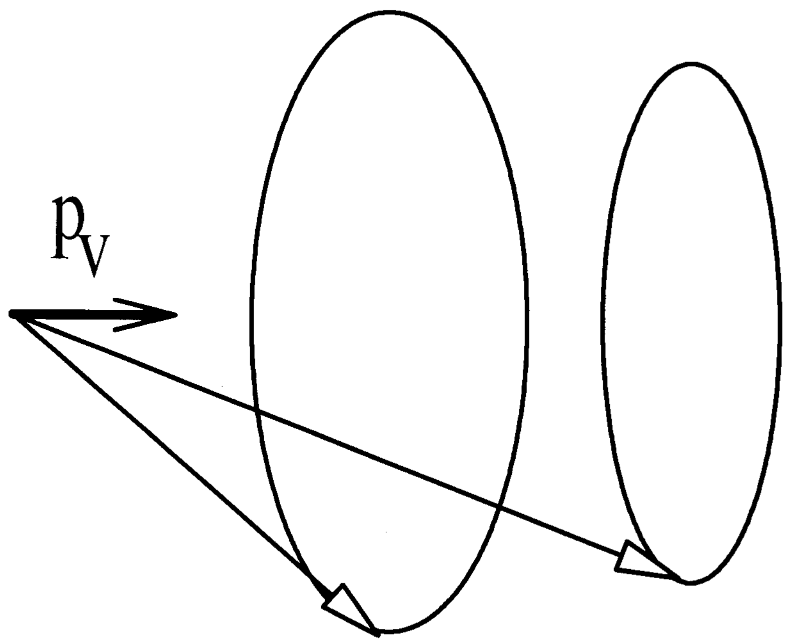

Figure 1:

$$\phi(\mathbf{x}) = \phi_{E_6}(\mathbf{x}) + \alpha \sum_{\mathbf{v} \notin D} (\mathbf{p_v} \bullet \mathbf{x} - 1)(\mathbf{p_v} \bullet \mathbf{x} - 2).$$

When α is suŒciently small the quadratic part of $\phi_{E_6}(x)$ is close to E_6 in the space of parameters. The ellipsoid $\phi_{E_6} = 0$ circumscribes D, since forms $\alpha(\mathbf{p_v} \bullet \mathbf{x} - 1)(\mathbf{p_v} \bullet \mathbf{x} - 2)$, $\mathbf{v} \notin D$ guarantee that all vertices of the G-tope that are not in D lie outside of $\phi_{E_6} = 0$. □

3.2 Delaunay tiling of E_6^*

The Delaunay tiling of E_6^* consists of polytopes congruent to a 9-vertex diagonal-free polytope T which is the convex hull of three regular triangles with common center situated in three mutually orthogonal 2-planes. We shall refer to this polytope as T-tope. Two Delaunay cells are said to belong to the same homology class if one of them can be obtained from the other by a lattice translation or by the superposition of a lattice translation and an inversion. The Delaunay cells of E_6^* fall into 40 homology classes; the Delaunay star of a point of E_6^* has 720 polytopes congruent to T-tope, 18 in each homology class (Baranovskii, 1992).

I the following sections we give a description of some L-types having E_6 or E_6^* as extreme rays. In our description we use the language of commensurate Delaunay tilings.

4 Commensurate Delaunay Tilings

Remark 5 Instead of varying the parameters of the lattice one can set $L = \mathbb{Z}^n$ and vary the parameters of the metrical form. In this model points in the space of parameters are interpreted as metrics, and all Delaunay polytopes have integer coordinates. In the context of øxed lattice $\mathbb{Z}^n$ we refer to the quadratic form of a lattice W as the metrical form of W. We will use this model for the rest of the paper.

Deønition 6 Let T_1 T_2 be two lattice tilings of $\mathbb{Z}^n$. They are called commenurate if the vertex set of their intersection tiling is $\mathbb{Z}^n$.

Proposition 7 Delaunay tilings $D(f)$ and $D(g)$ of $\mathbb{Z}^n$ for metrical forms f and g are commensurate if and only if their intersection tiling is Delaunay for $\alpha f + \beta g$, $\alpha, \beta > 0$.

Proof.

5 Commensurate and incommensurate tilings

Let P and Q be two lattice polytopes. We do not require that they have full dimension, although that is the most signiøcant case for our considerations, when they are tiles in distinct Delaunay tilings. Any integer point belonging to both is a vertex of the intersection polytope $P \cap Q$. There is, of course, the possibility that this intersection has vertices that do not belong to $\mathbb{Z}^d$.

Deønition 8 We will say the lattice polytopes P and Q are commensurate if the vertices of the intersection polytope $P \cap Q$ belong to $\mathbb{Z}^d$.

The following lemma characterizes the vertices of $P \cap Q$ that do not belong to $\mathbb{Z}^d$.

Proposition 9 Assume that $\mathbf{v} \notin \mathbb{Z}^d$. Then $\mathbf{v}$ is a vertex of $P \cap Q$ if and only if it can be represented $\mathbf{v} = relint(F_P) \cap relint(F_Q)$, where F_P is a face of P and F_Q is a face of Q.

As we will see, this Proposition gives a useful test to determine whether lattice polytopes are commensurate. SUDA □

The remaining sections will be devoted to the proof of the following theorem.

Theorem 10 Let $\phi(E_6)$ and $\phi(E_6^*)$ be two forms of types E_6 and E_6^* respectively, that have the same determinant and are the centroids of two adjacent perfect domains of types E_6^* and E_6. The segment with the end points $\phi(E_6)$ and $\phi(E_6^*)$ has forms of 5 distinct L-types. The points where L-type changes are $\phi(E_6)$, $\phi(E_6^*)$, and $\frac{2}{3}\phi(E_6) + \frac{1}{3}\phi(E_6^*)$. The wall between the perfect domains crosses this segment at point $\frac{2}{5}\phi(E_6) + \frac{3}{5}\phi(E_6^*)$.

6 Metrical forms for the lattices E_6 and E_6^*

Consider the following three symetric sets of vectors in $\mathbb{Z}^6$.

$$\mathcal{P}_1 = \{\pm[-3, 2; 2^4], \pm[1, 0; (-1)^4], \pm[2, -2; -1^4]\}$$

$$\mathcal{P}_2 = \{\pm[-2, 1; 2, 1^3] \times 4, \pm[-1, 1; 0, 1^3] \times 4, \pm[1, 0; 0, -1^3] \times 4, \pm[0, 0; -1, 0^3] \times 4$$
$$\pm [1, -1; -1^2, 0^2] \times 6, \pm[2, -1; -1^4], \pm[0, -1; 0^4]\}$$

$$\mathcal{P}_3 = \{\pm[0, 0; 1, -1, 0^2] \times 6, \pm[1, 0; -1^2, 0^2] \times 6\}$$

A compact notation is used where, for example, $\pm[0,0;1,-1,0^2]\times 6$ denotes the 12 vectors generated by permuting the last four components of $[0,0;1,-1,0,0]$. Let $\mathcal{P}_{E_6} = \mathcal{P}_2 \cup \mathcal{P}_3$, and let $\mathcal{P}_{E_6^*} = \mathcal{P}_1 \cup \mathcal{P}_2$. As indicated by the notation, these are sets of minimal vectors for metrical forms for the root lattice E_6 and its dual E_6^*. More precisely, if π_{E_6}, $\pi_{E_6^*}$ are the metrical forms with coeŒcient matrices

$$\mathbf{P}_{E_6} = \frac{m}{2}\begin{bmatrix} 8 & 1 & 3 & 3 & 3 & 3 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 2 & 1 & 1 & 1 \\ 3 & 0 & 1 & 2 & 1 & 1 \\ 3 & 0 & 1 & 1 & 2 & 1 \\ 3 & 0 & 1 & 1 & 1 & 2 \end{bmatrix}, \qquad \mathbf{P}_{E_6^*} = \frac{m}{4}\begin{bmatrix} 16 & 5 & 5 & 5 & 5 & 5 \\ 5 & 4 & 1 & 1 & 1 & 1 \\ 5 & 1 & 4 & 1 & 1 & 1 \\ 5 & 1 & 1 & 4 & 1 & 1 \\ 5 & 1 & 1 & 1 & 4 & 1 \\ 5 & 1 & 1 & 1 & 1 & 4 \end{bmatrix},$$

then, if $\mathbf{z} \in \mathbb{Z}^6$ is non-zero, $\pi_{E_6^*}(\mathbf{z}) \geq m$ with equality if and only if $\mathbf{z} \in \mathcal{P}_{E_6^*}$, and, $\pi_{E_6}(\mathbf{z}) \geq m$ with equality if and only if $\mathbf{z} \in \mathcal{P}_{E_6}$. That is, $\mathcal{P}_{E_6}$ is the set of minimal vectors for π_{E_6}, and, $\mathcal{P}_{E_6^*}$ is the set of minimal vectors for $\pi_{E_6^*}$. That π_{E_6}, $\pi_{E_6^*}$ are metrical forms for the root lattices E_6, E_6^* is conørmed by the respective numbers of minimal vectors: $|\mathcal{P}_{E_6}| = 72$, $|\mathcal{P}_{E_6^*}| = 54$. Both π_{E_6} and $\pi_{E_6^*}$ are perfect forms. The arithmetic minimum for each of these forms, the minimum value on non-zero integer vectors, is equal to m. As shown below, if this scale parameter is set equal to $\sqrt{8/3}$, then, the geometric lattices corresponding to π_{E_6}, $\pi_{E_6^*}$ are dual lattices ˘ up to a scale factor, the lattices E_6 and E_6^*.

Minimal vectors and perfect domains. The perfect domains corresponding to π_{E_6}, $\pi_{E_6^*}$ are given by

$$\Phi_{E_6} = \{ \sum_{\mathbf{p} \in \mathcal{P}^+_{E_6}} \omega_{\mathbf{p}} (\mathbf{p} \cdot \mathbf{x})^2 \mid \omega_{\mathbf{p}} \geq 0\}, \quad \Phi_{E_6^*} = \{ \sum_{\mathbf{p} \in \mathcal{P}^+_{E_6^*}} \omega_{\mathbf{p}} (\mathbf{p} \cdot \mathbf{x})^2 \mid \omega_{\mathbf{p}} \geq 0\}.$$

As indicated by the superscript $+$, these summations are over oriented subsets of vectors ˘ one vector from each pair of opposites in the symmetric sets $\mathcal{P}_{E_6}$, $\mathcal{P}_{E_6^*}$; thus, the summation in the expression for $\Phi_{E_6^*}$ is over 27 vectors. These domains are independent of the orientation that is chosen because the sign of each minimal vector is absorbed by the square in the formula $(\mathbf{p} \cdot \mathbf{x})^2$.

Perfection requires that the minimal vectors be suŒciently numerous that the closed polyhedral cones Φ_{E_6}, $\Phi_{E_6^*}$ have the full dimension 21 in the linear space of metrical forms.

The two sets of minimal vectors $\mathcal{P}_{E_6}$, $\mathcal{P}_{E_6^*}$ have an overlap given by $\mathcal{P}_{E_6} \cap \mathcal{P}_{E_6^*} = \mathcal{P}_2$, which is a symmetric set of 48 vectors. This corresponds to an overlap

$$\Phi_{E_6} \cap \Phi_{E_6^*} = \{ \sum_{\mathbf{p} \in \mathcal{P}_2^+} \omega_{\mathbf{p}} (\mathbf{p} \cdot \mathbf{x})^2 \mid \omega_{\mathbf{p}} \geq 0 \}$$

for the corresponding perfect domains, where the summation is over an oriented subset of 24 elements. As can be checked, the elements of $\Phi_{E_6} \cap \Phi_{E_6^*}$ generate a linear sub-space of co-dimension one, so the perfect domains $\Phi_{E_6^*}$, Φ_{E_6} share a facet and are adjacent.

A scalar product on forms can be introduced as follows. If forms π, φ have coeŒcient matrices $\mathbf{P}$, $\mathbf{F}$, deøne $\langle \pi, \varphi \rangle = trace(\mathbf{PF})$. Consider now the form $\pi_{E_6^*} - \pi_{E_6}$ and the rank one form $\varphi_{\mathbf{p}}(\mathbf{x}) = (\mathbf{p} \cdot \mathbf{x})^2$. Then, $\langle \pi^*_{E_6} - \pi_{E_6}, \varphi_{\mathbf{p}} \rangle = trace(\mathbf{P}_{E_6^*} - \mathbf{P}_{E_6})\mathbf{p}\mathbf{p}^T = \mathbf{p}^T(\mathbf{P}_{E_6^*} - \mathbf{P}_{E_6})\mathbf{p} = \pi_{E_6^*}(\mathbf{p}) - \pi_{E_6}(\mathbf{p})$. Easy calculations show that: if $\mathbf{p} \in \mathcal{P}_1$, then $\pi_{E_6^*}(\mathbf{p}) - \pi_{E_6}(\mathbf{p}) = 1 - \frac{3}{2} = -\frac{1}{2}$; if $\mathbf{p} \in \mathcal{P}_2$, then $\pi_{E_6^*}(\mathbf{p}) - \pi_{E_6}(\mathbf{p}) = 1 - 1 = 0$; if $\mathbf{p} \in \mathcal{P}_3$, then $\pi_{E_6^*}(\mathbf{p}) - \pi_{E_6}(\mathbf{p}) = 2 - 1 = 1$. From this data it follows that the intersecion domain $\Phi_{E_6} \cap \Phi_{E_6^*}$ lies on the hyperplane with equation $\langle \pi_{E_6^*} - \pi_{E_6}, \varphi \rangle = 0$, that the domain Φ_{E_6} lies in the positive half-space determined by this hyperplane, and the domain $\Phi_{E_6^*}$ lies in the negative half-space.

Minimal vectors and eutactic forms. The forms

$$\varphi_{E_6}(\mathbf{x}) = \frac{m}{12} \sum_{\mathbf{p} \in \mathcal{P}^+_{E_6^*}} (\mathbf{p} \cdot \mathbf{x})^2, \qquad \varphi_{E_6^*}(\mathbf{x}) = \frac{m}{16} \sum_{\mathbf{p} \in \mathcal{P}^+_{E_6}} (\mathbf{p} \cdot \mathbf{x})^2,$$

lie on the central axes of the perfect domains $\Phi_{E_6^*}$, Φ_{E_6}, and have coeŒcient matrices given by

$$\mathbf{F}_{E_6} = \frac{m}{2} \begin{bmatrix} 8 & -5 & -5 & -5 & -5 & -5 \\ -5 & 4 & 3 & 3 & 3 & 3 \\ -5 & 3 & 4 & 3 & 3 & 3 \\ -5 & 3 & 3 & 4 & 3 & 3 \\ -5 & 3 & 3 & 3 & 4 & 3 \\ -5 & 3 & 3 & 3 & 3 & 4 \end{bmatrix}, \mathbf{F}_{E_6^*} = \frac{m}{4} \begin{bmatrix} 10 & -5 & -8 & -8 & -8 & -8 \\ -5 & 4 & 3 & 3 & 3 & 3 \\ -8 & 3 & 6 & 3 & 3 & 3 \\ -8 & 3 & 3 & 6 & 3 & 3 \\ -8 & 3 & 3 & 3 & 6 & 3 \\ -8 & 3 & 3 & 3 & 3 & 6 \end{bmatrix};$$

m is the scale parameter introduced above. These forms are related to the original π_{E_6}, $\pi_{E_6^*}$ by the formulas $\varphi_{E_6}(\mathbf{x}) = \pi_{E_6}(\mathbf{Ux})$, $\varphi_{E_6^*}(\mathbf{x}) = \pi_{E_6^*}(\mathbf{Ux})$, where $\mathbf{U} \in GL(6, \mathbb{Z})$ is given by

$$\mathbf{U} = \begin{bmatrix} 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}. \qquad \mathbf{U}^{-1} = \begin{bmatrix} 4 & 3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Being arithmetically equivalent, φ_{E_6}, π_{E_6} are alternate metrical forms for the same geometric lattice, and similarly for $\varphi_{E_6^*}$, $\pi_{E_6^*}$. The minimal vectors for these forms are related accordingly: $\mathcal{S}_{E_6} = \mathcal{F}_2 \cup \mathcal{F}_3$, $\mathcal{S}_{E_6^*} = \mathcal{F}_1 \cup \mathcal{F}_2$, where $\mathcal{F}_1 = \mathbf{U}^{-1}(\mathcal{P}_1)$, $\mathcal{F}_2 = \mathbf{U}^{-1}(\mathcal{P}_2)$, $\mathcal{F}_3 = \mathbf{U}^{-1}(\mathcal{P}_3)$. Explicit form for the sets $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ can be obtained by a direct calculation.

$$\begin{aligned} \mathcal{F}_1 &= \{\pm[2,-1;1^4], \pm[0,1;0^4], \pm[-2,0;-1^4]\} \\ \mathcal{F}_2 &= \{\pm[0,-1;1,0^3] \times 4, \pm[2,0;0,1^3] \times 4, \pm[1,1;1,0^3] \times 4, \pm[-1,0;-1,0^3] \times 4, \\ &\quad \pm[-1,0;-1^2,0^2] \times 6, \pm[1,1;0^4], \pm[-3,-1;-1^4]\} \\ \mathcal{F}_3 &= \{\pm[0,0;1,-1,0^2] \times 6, \pm[2,1;1^2,0^2] \times 6\} \end{aligned}$$

The various coeŒcient matrices satisfy the relations $\mathbf{F}_{E_6}\mathbf{P}_{E_6^*} = \mathbf{F}_{E_6^*}\mathbf{P}_{E_6} = \frac{3}{8}m^2\mathbf{I}$, so that when $m = \sqrt{8/3}$, $\mathbf{F}_{E_6} = (\mathbf{P}_{E_6^*})^{-1}$, $\mathbf{F}_{E_6^*} = (\mathbf{P}_{E_6})^{-1}$. Under these circumstances the pair of forms φ_{E_6}, $\pi_{E_6^*}$, and the pair $\varphi_{E_6^*}$, π_{E_6}, are in duality: $\varphi_{E_6} = \pi^{\circ}_{E_6^*}$, $\pi_{E_6^*} = \varphi^{\circ}_{E_6}$, $\varphi_{E_6^*} = \pi^{\circ}_{E_6}$, $\pi_{E_6} = \varphi^{\circ}_{E_6^*}$. Dual forms correspond to dual lattices, so the single geometric lattice corresponding to the forms φ_{E_6}, π_{E_6} is dual to the geometric lattice corresponding to $\varphi_{E_6^*}$, $\pi_{E_6^*}$. This pair of geometric lattices is the root lattice E_6 and its dual E_6^*.

The above representations for φ_{E_6}, $\varphi_{E_6^*}$ can equally well be considered as representations for the dual forms $\pi^{\circ}_{E_6^*} = \varphi_{E_6}$, $\pi^{\circ}_{E_6} = \varphi_{E_6^*}$, which shows that $\pi^{\circ}_{E_6^*}$, $\pi^{\circ}_{E_6}$ lie interior to the domains $\Phi_{E_6^*}$, Φ_{E_6} determined, respectively, by $\pi_{E_6^*}$, π_{E_6}. Forms with this property are said to be eutactic. Perfect forms are frequently eutactic, and accompanied by such eutaxy representations for dual forms. The forms $\pi^{\circ}_{E_6^*}$, $\pi^{\circ}_{E_6}$ lie on the central axes of their respective domains, but this is not a requirement of eutaxy. Since $\pi_{E_6} = \varphi^{\circ}_{E_6^*}$, $\pi_{E_6^*} = \varphi^{\circ}_{E_6}$, the original forms have eutaxy representations identicle to those for φ_{E_6}, $\varphi_{E_6^*}$,

$$\pi_{E_6}(\mathbf{x}) = \frac{m}{12} \sum_{\mathbf{s}\in\mathcal{S}^+_{E_6^*}} (\mathbf{s}\cdot\mathbf{x})^2, \qquad \pi_{E_6^*}(\mathbf{x}) = \frac{m}{16} \sum_{\mathbf{s}\in\mathcal{S}^+_{E_6}} (\mathbf{s}\cdot\mathbf{x})^2;$$

the summation again is over an oriented subset of minimal vectors for the reciprocal forms, respectively, the forms $\varphi_{E_6^*}$, φ_{E_6}.

The symmetrical conøguration provided by the adjacent perfect domains, $\Phi_{E_6^*}$, Φ_{E_6}, is remarkable. It is the fact that these domains have central forms corresponding to dual geometric lattices that distinguishes this from other conøgurations formed from adjacent perfect domains. While the lattices are dual, the forms φ_{E_6}, $\varphi_{E_6^*}$ are not ˘ the form $\varphi_{E_6^*}$ is arithmetically equivalent to $\pi_{E_6^*}$, which is turn is dual to φ_{E_6}.

The line between φ_{E6} and $\varphi_{E_6^*}$. The line segment $\varphi_t = (1-t)\varphi_{E_6} + t\varphi_{E_6^*}$, $0 \le t \le 1$, runs between the central axes of the perfect domains $\Phi_{E_6^*}$, Φ_{E_6}, and each form on this segment has arithmetic minimum equal to m. At the end points φ_{E_6}, $\varphi_{E_6^*}$, the minimal vectors are respectively $\mathcal{S}_{E_6}$ and $\mathcal{S}_{E_6^*}$, but at intermediary points, where $0 < t < 1$, the set of minimal vectors is the intersection $\mathcal{S}_{E_6} \cap \mathcal{S}_{E_6^*} = \mathcal{F}_2$.

Proposition 11 The line segment $\varphi_t = (1-t)\varphi_{E_6} + t\varphi_{E_6^*}$, $0 \le t \le 1$, pierces the facet $\Phi_{E_6} \cap \Phi_{E_6^*}$ along the central axis when $t = \frac{2}{5}$.

The short, long and perfect vectors for E_6 and E_6^*. We will consider $\mathcal{S}_{E_6}$, $\mathcal{P}_{E_6^*}$ as dual sets of vectors ˘ the minimal vectors for φ_{E_6}, and for the dual form $\pi_{E_6^*}$. In order to make sharp distinctions we will refer to $\mathcal{S}_{E_6}$ as the short vectors for E_6, and to $\mathcal{P}_{E_6^*}$ as the perfect vectors for E_6. The long vectors for E_6 is deøned by

$$\mathcal{L}_{E_6} = \{\mathbf{z} \in \mathbb{Z}^6 | \varphi_{E_6}(\mathbf{z}) = 2m\},$$

where $2m$ is the second minimum for φ_{E_6}, the minimal value assumed on the non-zero elements of $\mathbb{Z}^6$ not belonging to $\mathcal{S}_{E_6}$.

The vectors in $\mathcal{S}_{E_6}$ are partitioned into 36 parity classes ˘ equivalence classes determined by the quotient $2\mathbb{Z}^6/\mathbb{Z}^6$. Each short class contains a pair of opposite vectors, accounting for the total number 72 of short vectors. The remaining $27 = 63 - 36 = (2^6 - 1) - 36$ non-zero parity classes are long, each containing ten vectors from $\mathcal{L}_{E_6}$. Thus there are $10 \times 27 = 270$ long vectors for E_6, given by $\mathcal{L}_{E_6} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \cup \mathcal{L}_4 \cup \mathcal{L}_5 \cup \mathcal{L}_6$, where

$$\begin{aligned}
\mathcal{L}_1 &= \{\pm[-1;-2,0^4] \times 5\}; \mathcal{L}_2 = \{\pm[2;-1,1^4] \times 5\};\\
\mathcal{L}_3 &= \{\pm[3,0;1^4], \pm[-1,0;1,-1^3] \times 4\} \times 5;\\
\mathcal{L}_4 &= \{\pm[-2,0;-1^4], \pm[4,2;1^4], \pm[0^2;-1^2,1^2]\} \times 5\\
\mathcal{L}_5 &= \{\pm[0,1;0^4], \pm[2,1;2,0^3] \times 4\} \times 5;\\
\mathcal{L}_6 &= \{\pm[-3,-2,0;-1^3], \pm[-3,0,-2;-1^3], \pm[1,0^2;-1,1^2] \times 2\} \times 10.
\end{aligned}$$

Each bracket includes 10 long vectors in the same parity class, and the notation indicates how these elements are generated by permuting coordinates. The sets $\mathcal{L}_1$, $\mathcal{L}_2$ each include a single parity class, accounting for 2 of the 27 parity classes. The sets $\mathcal{L}_3$, $\mathcal{L}_4$, $\mathcal{L}_5$ include øve parity classes, accounting for a further 15 classes. This is indicated by the $\times 5$; in each case the øve parity classes are generated by permuting the last øve coordinates (the form φ_{E_6} is invariant with respect to permutation of these øve coordinates). The ønal set $\mathcal{L}_6$ completes the list of 27 parity classes since it contains 10, as indicated by the $\times 10$; these are generated by permuting the last øve coordinates of the single class given.

Similarly, we will refer to $\mathcal{S}_{E_6^*}$, $\mathcal{P}_{E_6}$ as the short and perfect vectors for E_6^*. The long vectors for E_6^* are given by

$$\mathcal{L}_{E_6^*} = \{\mathbf{z} \in \mathbb{Z}^6 | \varphi_{E_6^*}(\mathbf{z}) = \frac{3}{2}m\},$$

where $\frac{3}{2}m$ is the second minimum for $\varphi_{E_6^*}$.

The set $\mathcal{S}_{E_6^*}$ contains 27 short parity classes each with a pair of opposite vectors, accounting for the total number 54. The remaining 36 parity classes are long, each containing a pair of opposite vectors. The 72 long vectors for E_6^* are given by $\mathcal{L}_{E_6^*} = \mathcal{F}_3 \cup \mathcal{F}_4$, where

$$\begin{aligned}
\mathcal{F}_4 = \{&\pm[0,0;1,0^3] \times 4, \pm[2,1;0,1^3] \times 4, \pm[-1,1;0,-1^3] \times 4, \pm[-3,0;-2,-1^3] \times 4\\
&\pm [1,-1;1^2,0^2] \times 6, \pm[3,0;1^4], \pm[-1,-2;0^4]\}.
\end{aligned}$$

These long vectors are paritioned into 36 parity classes, each containing a pair of opposite vectors.

In summary, $\mathcal{S}_{E_6} = \mathcal{F}_2 \cup \mathcal{F}_3$, $\mathcal{L}_{E_6} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \cup \mathcal{L}_4 \cup \mathcal{L}_5 \cup \mathcal{L}_6$, $\mathcal{S}_{E_6^*} = \mathcal{F}_1 \cup \mathcal{F}_2$, and $\mathcal{L}_{E_6^*} = \mathcal{F}_3 \cup \mathcal{F}_4$. The sets $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$ are partitioned, respectively, into 3, 24, 12, and 24 parity classes, each with a pair of opposite vectors. Notice that $\mathcal{F}_1 \cup \mathcal{F}_4 \subset \mathcal{L}_{E_6}$, so that passage from φ_{E_6} to the form $\varphi_{E_6^*}$ is accompanied by the three parity classes of $\mathcal{F}_1$ going from long to short, and the twelve parity classes of $\mathcal{F}_3$ going from short to long. The 24 parity classes of $\mathcal{F}_2 = \mathcal{S}_{E_6} \cap \mathcal{S}_{E_6^*}$ are always short, and the 24 parity classes of $\mathcal{F}_4 = \mathcal{L}_{E_6} \cap \mathcal{L}_{E_6^*}$ are always long.Short-perfect and long-perfect duality.

7 Short triangles

The point group for the root lattice E_6 is the product of two element group generated by central inversion and the reÆection group E_6; it has order $2^8 3^4 5$. Naturally, there are four separate representations in $GL(6, \mathbb{Z})$ that are the invariance groups of the forms φ_{E_6}, $\pi_{E_6^*}$, $\varphi_{E_6^*}$, π_{E_6}. Let $\mathcal{G}_{E_6} \subset GL(6, \mathbb{Z})$ be the representation that leaves φ_{E_6} invariant, so that if $\mathbf{g} \in \mathcal{G}_{E_6}$, then $\varphi_{E_6}(\mathbf{gx}) = \varphi_{E_6}(\mathbf{x})$. The dual representation, which is given by $\mathcal{G}^\circ_{E_6} = \{\mathbf{g}^\circ = (\mathbf{g}^T)^{-1} | \mathbf{g} \in \mathcal{G}_{E_6}\}$, is the invariance group of the dual form $\pi_{E_6^*}$. The full invariance groups of the forms $\varphi_{E_6^*}$, π_{E_6} can also be expressed in terms of $\mathcal{G}_{E_6}$ using arithmetic equivalence. Using the formula $\varphi_{E_6^*}(\mathbf{x}) = \pi_{E_6^*}(\mathbf{Ux})$, where $\mathbf{U}$ is described above, it follows that if $\mathbf{g} \in \mathcal{G}^\circ_{E_6}$, then $\varphi_{E_6^*}(\mathbf{x}) = \varphi_{E_6^*}(\mathbf{U}^{-1}\mathbf{gUx}) = \pi_{E_6}(\mathbf{x})$. Therefore, $\mathbf{U}^{-1}\mathcal{G}^\circ_{\mathbf{E_6}}\mathbf{U}$ is the invariance group of $\varphi_{E_6^*}$. The representation dual to this, which is given by $(\mathbf{U}^{-1}\mathcal{G}^\circ_{\mathbf{E_6}}\mathbf{U})^\circ = \mathbf{U}^T\mathcal{G}_{\mathbf{E_6}}(\mathbf{U}^T)$, is the invariance group of π_{E_6}. It will be convenient to use the symbol $\mathcal{G}_{\mathbf{E_6^*}}$ for the representation $\mathbf{U}^{-1}\mathcal{G}^\circ_{\mathbf{E_6}}\mathbf{U}$, and the symbol $\mathcal{G}^\circ_{\mathbf{E_6^*}}$ for the representation $\mathbf{U}^T\mathcal{G}_{\mathbf{E_6}}(\mathbf{U}^T)^{-1}$, which is dual to $\mathcal{G}_{\mathbf{E_6^*}}$. (Note ˇIt would be nice to have a symmetric U. Check notes)

Using eutaxy representations for the forms it is easy to show that the groups $\mathcal{G}_{\mathbf{E_6}}$, $\mathcal{G}^\circ_{\mathbf{E_6}}$, $\mathcal{G}_{\mathbf{E_6^*}}$, $\mathcal{G}^\circ_{\mathbf{E_6^*}}$ are the full invariance groups the corresponding sets of minimal vectors, which are given respectively by $\mathcal{S}_{E_6}$, $\mathcal{P}_{E_6^*}$, $\mathcal{S}_{E_6^*}$, $\mathcal{P}_{E_6}$. It is well-known that these invariance groups act transitively on corresponding sets of minimal vectors, and we will make use of this fact below.

Short-perfect duality. Consider a symmetric set of vectors $X \subset \mathbb{Z}^d$. Then, the dual set X° is deøned by the formula

$$X^\circ = \{\mathbf{z} \in \mathbb{Z}^{\mathbf{d}} | \mathbf{z} \cdot \mathbf{x} \in \{0, \pm 1\}, \mathbf{x} \in X\}.$$

If sets $X, Y \subset \mathbb{Z}^d$ satisfy the relations $Y = X^\circ, X = Y^\circ$ we will say they are in duality, and refer to X, Y as dual sets of integer vectors.

Direct calculations show that $\mathcal{S}_{E_6} = (\mathcal{P}_{E_6^*})^\circ$, $\mathcal{P}_{E_6^*} = (\mathcal{S}_{E_6})^\circ$, so that $\mathcal{S}_{E_6}$, $\mathcal{P}_{E_6^*}$ are dual sets of integer vectors. Through short-perfect duality, either of the sets $\mathcal{S}_{E_6}$, $\mathcal{P}_{E_6^*}$ is determined by its partner. Similarly, the sets $\mathcal{S}_{E_6^*}$, $\mathcal{P}_{E_6}$ follow such a duality law.

Consider a lattice polytope P with vertices in $\mathbb{Z}^d$, and a symmetric set of vectors $X \subset \mathbb{Z}^d$. We will say that P is an $X-$tope if the vectors running between vertices belong to X. That is, all the edges and diagonals of P belong to X.

Proposition 12 Assume that $X, Y \subset \mathbb{Z}^d$ are in duality. Assume also that P is an $X-$tope, and let $\varphi(\mathbf{x}) = \sum_{\mathbf{y} \in Y^+} \omega_{\mathbf{y}} (\mathbf{y} \cdot \mathbf{x})^2$, where $\omega_{\mathbf{y}} > 0$ and Y^+ is an orientation for Y. Then P is a Delaunay cell in $\mathcal{D}_\varphi$.

Corollary 13 If P is an $\mathcal{S}_{E_6}-$tope, then P is Delaunay with respect to the form φ_{E_6}; if P is an $\mathcal{S}_{E_6^*}-$tope, then P is Delaunay with respect to the form $\varphi_{E_6^*}$.

Corollary 14 Any pair of $\mathcal{S}_{E_6}-$topes are commensurate; any pair of $\mathcal{S}_{E_6^*}-$topes are commensurate.

Short and perfect triangles. If distinct non-zero vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{Z}^d$, satisfy the equation $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$, then, by arranging them head-to-tail in the order given, and starting at the origin, they trace out a lattice triangle. The vertices are given by $\{\mathbf{0}, \mathbf{a}, \mathbf{a} + \mathbf{b}\}$, and the edge set is given by $\{\pm\mathbf{a}, \pm\mathbf{b}. \pm\mathbf{c}\}$.

Deønition 15 Let $T \subset \mathbb{Z}^d$ be a symmetric set of six vectors. Then, T is triangular if and only if there are vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in T$, distinct and non-zero, so that $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$.

For a triangular set there are twelve head-to-tail arrangements of three vectors each, which trace out triangles at the origin. However, there are only six triangles; each triangle is traced twice, once in the clockwise, and once in the counter-clockwise direction. These six tiangles øt together edge-to-edge to tile the hexagon with vertex set $\{\pm\mathbf{a}, \pm\mathbf{b}, \pm\mathbf{c}\}$. These triangles are homologous.

The ten subsets

$$\{\pm[3;1,1,1,1,1], \pm[-2;-1,-1,-1,0,0], \pm[-1;0,0,0,-1,-1]\} \times 10 \subset \mathcal{S}_{E_6}$$

are triangular; as is indicated by the notation, these subsets are generated by permuting the last øve coordinates of the representative, which is displayed. This is the collection of all triangular subsets that contain the co-linear pair $\pm[3;1,1,1,1,1]$, which can easily be veriøed by refering to the list of vectors in $\mathcal{S}_{E_6}$. We will refer to this collection as the triangular star, or more simply, the T-star of $\{\pm[3;1,1,1,1,1]\}$. More formally, if $\mathcal{T}(\mathcal{S}_{E_6})$ is the collection of all triangular sets in $\mathcal{S}_{E_6}$, then

$$\mathcal{T}(\mathcal{S}_{E_6};\pm[3;1,1,1,1,1]) = \{T \in \mathcal{T}(\mathcal{S}_{E_6}) | \pm[3;1,1,1,1,1] \in T\}$$

is the T-star of $\{\pm[3;1,1,1,1,1]\}$. The permutation subgroup $\mathcal{A}_5 \subset \mathcal{G}_{E_6}$ øxes this star, and acts transitively on the triangular subsets of this star. By the observation that the $\mathcal{G}_{E_6}$−action is transitive on the 36 co-linear pairs $\{\pm\mathbf{a}\} \in \mathcal{S}_{E_6}$, it follows that: for each pair $\{\pm\mathbf{a}\} \in \mathcal{S}_{E_6}$, the stability group of the T-star $\mathcal{T}(\mathcal{S}_{E_6};\pm\mathbf{a})$ acts transitively on the triangular subsets of the star. The following secondary conclusions are immediate: (1) the T-stars $\mathcal{T}(\mathcal{S}_{E_6};\pm\mathbf{a})$ are $\mathcal{G}_{E_6}$−equivalent; (2) there are $(36 \times 10)/3 = 120$ triangular sets in $\mathcal{S}_{E_6}$, which are $\mathcal{G}_{E_6}$−equivalent; (3) there are 120×6 $\mathcal{S}_{E_6}$−triangles at the origin, and, 120 homology classes of $\mathcal{S}_{E_6}$−triangles.

Similarly, the øve subsets

$$\{\pm[-3;2,2,2,2,2], \pm[2;-2,-1,-1,-1,-1], \pm[1;0,-1,-1,-1,-1]\} \times 5 \subset \mathcal{P}_{E_6^\star}$$

are triangular; this collection is the T-star, $\mathcal{T}(\mathcal{P}_{E_6^\star};\pm[-3;2,2,2,2,2])$, of the co-linear pair $\pm[-3;2,2,2,2,2]$. The subgroup $\mathcal{A}_5 \subset \mathcal{G}^\circ_{E_6}$ leaves this star invariant. and acts transitively on its triangular subsets. As before, this leads to the primary conclusion: for each pair $\{\pm\mathbf{a}\} \in \mathcal{P}_{E_6^\star}$, the stability group of the T-star $\mathcal{T}(\mathcal{P}_{E_6^\star};\pm\mathbf{a})$ acts transitively on the triangular subsets of the star. The secondary conclusions are: (1) the T-stars $\mathcal{T}(\mathcal{P}_{E_6^\star};\pm\mathbf{a})$ are $\mathcal{G}^\circ_{E_6}$−equivalent; (2) there are $(27 \times 5)/3 = 45$ triangular sets in $\mathcal{P}_{E_6^\star}$, which are $\mathcal{G}^\circ_{E_6}$−equivalent; (3) there are 45×6 $\mathcal{P}_{E_6^\star}$−triangles at the origin, and, 45 homology classes of $\mathcal{P}_{E_6^\star}$−triangles.

If $\mathcal{T}^c(X)$ is a subclass of triangular subsets for some symmetric set $X \subset \mathbb{Z}^d$, it is natural to consider the corresponding subclass of triangular stars, $\mathcal{T}^c(X;\mathbf{a}) = \{T \in \mathcal{T}^c(X) | \mathbf{a} \in T\}$, where $\mathbf{a} \in X$. In this setting, it is also natural to consider the subclass of X^c−triangles, where the edge sets are triangular subsets of $\mathcal{T}^c(X)$.

Lemma 16 Assume that the elements of the symmetric set $X \subset \mathbb{Z}^d$ are $\mathcal{G}-$equivalent, where $\mathcal{G} \subset GL(d,\mathbb{Z})$ is ønite. Let $\mathcal{T}^c(X)$ be a subclass of triangular subsets, with the property that the stability group of each T-star in the subclass, $\mathcal{T}^c(X;\pm\mathbf{a}), \{\pm\mathbf{a}\} \in X$, acts transitively on the triangular subsets of the star. Then, the subclass of X^c-triangles are $\mathcal{G}-$equivalent.

Lemma 17 Assume that the elements of the symmetric set $X \subset \mathbb{Z}^d$ are $\mathcal{G}-$equivalent, where $\mathcal{G} \subset GL(d,\mathbb{Z})$ contains the central inversion $\mathbf{i}$, and is ønite. Also assume that $\mathcal{T}^c(X)$ is a $\mathcal{G}-$orbit of triangular subsets, and that $\mathbf{x} \in X$ has the property that the stability group $\mathcal{G}(\mathbf{x})$ acts transitively on the triangular sets in $\mathcal{T}^c(X;\mathbf{x})$. Then, the subclass of X^c-triangles are $\mathcal{G}-$equivalent.

Proof. Let $T=\{\pm\mathbf{a},\pm\mathbf{b},\pm\mathbf{c}\} \in \mathcal{T}^c(X)$ be arbitrary, and choose $\mathbf{g} \in \mathcal{G}$ so that $\mathbf{ga} = \mathbf{b}$. Then $\mathbf{b} \in \mathbf{g}T \in \mathcal{T}^c(X;\mathbf{b})$. By assumption, there is an element $\mathbf{h} \in \mathcal{G}(\mathbf{b})$ so that $\mathbf{hg}T = T$, and $\mathbf{hga} = \mathbf{b}$. Since we can assume that $\mathbf{a}, \mathbf{b}$ were chosen arbitrarily, it follows that the stability group $\mathcal{G}(T)$ acts transitively on the elements of T. This being the case, $\mathcal{G}(T)$ must be either the dihedral group D, or the reÆection group f $\mathbf{hb} = -\mathbf{b}$, replace $\mathbf{h}$ by $\mathbf{ih}$, where i is the central inversion. in the stability group of $inthe$ ■

Proposition 18 There are 120×6 $\mathcal{S}_{E_6}-$triangles at the origin; these are $\mathcal{G}_{E_6}-$equivalent, mutually commensurate, and Delaunay with respect to the form φ_{E_6}. There are $45{\times}6$ $\mathcal{S}_{E_6^*}-$triangles at the origin; these are $\mathcal{G}_{E_6^*}-$equivalent, mutually commensurate, and Delaunay with respect to the form $\varphi_{E_6^*}$.

8 Long triangles, G-topes and T-topes

Consider the triangular set $\{\pm[2;-1,1,1,1,1], \pm[-2;0,-1,-1,-1,-1], \pm[0;1,0,0,0,0]\} \subset \mathcal{L}_{E_6}$; $\mathcal{L}_{E_6}$ is the set of long vectors for φ_{E_6}, deøned by

$$\mathcal{L}_{E_6} = \{\mathbf{z} \in \mathbb{Z}^6 | \varphi_{E_6}(\mathbf{z}) = 2m\}.$$

The value $2m$ is the second minimum for φ_{E_6}, which is the minimal value assumed on the non-zero elements of $\mathbb{Z}^6$ not belonging to $\mathcal{S}_{E_6}$. $|\mathcal{L}_{E_6}| = 270$.

The reference G-tope, $G_{\mathbf{p}_1}$, is the convex hull of 45 long triangles, each with center of gravity $\mathbf{c}_G = \frac{1}{3}[2,1,1,1,1,1]$, which is the center of gravity of

$G_{\mathbf{p}_1}$ itself. Representatives from each A_5-class of long triangles are given by

$$\begin{aligned}
\Delta^1_{\mathbf{p}_1} &= conv\{[0;0,0,0,0,0],[0;1,0,0,0,0],[2;0,1,1,1,1]\} \times 5,\\
\Delta^6_{\mathbf{p}_1} &= conv\{[3;1,1,1,1,1],[-1;-1,0,0,0,0],[0;1,0,0,0,0]\} \times 5,\\
\Delta^{11}_{\mathbf{p}_1} &= conv\{[1;0,1,1,0,0],[1;0,0,0,1,1],[0;1,0,0,0,0]\} \times 15,\\
\Delta^{26}_{\mathbf{p}_1} &= conv\{[1;1,1,0,0,0],[-1;0,-1,0,0,0],[2;0,1,1,1,1]\} \times 20;
\end{aligned}$$

the notation indicates how many triangles there are in each class (permutation of the last øve components generates the class). There are øve long triangles attached to each vertex of $G_{\mathbf{p}_1}$.

These triangles are long because their sets of edge vectors, given respectively by

$$\begin{aligned}
&\{\pm[0;1,0,0,0,0],\pm[2;-1,1,1,1,1],\pm[-2;0,-1,-1,-1,-1]\} \times 5,\\
&\{\pm[-4;-2,-1,-1,-1,-1],\pm[1;2,0,0,0,0],\pm[3;0,1,1,1,1]\} \times 5,\\
&\{\pm[0;0,-1,-1,1,1]],\pm[-1;1,0,0,-1,-1],\pm[1;-1,1,1,0,0]\} \times 15,\\
&\{\pm[-2;-1,-2,0,0,0],\pm[3;0,2,1,1,1],\pm[-1;1,0,-1,-1,-1]\} \times 20,
\end{aligned}$$

are long vectors for φ_{E_6}. These 45 sets of edge vectors give a second geometric description of the 270 long vectors for φ_{E_6}.

The reference T-tope also has a representation in terms of long triangles; T_{Δ_1} is the convex hull of the three long triangles

$$\begin{aligned}
\Delta^1_{\Delta_1} &= conv\{[0,0;0,0,0,0],[0,0;0,0,1,0],[0,0;0,0,0,1]\},\\
\Delta^2_{\Delta_1} &= conv\{[-1,0;-1,0,0,0],[-1,0;0,-1,0,0],[2,0;1,1,1,1]\},\\
\Delta^3_{\Delta_1} &= conv\{[1,0;0,0,1,1],[-1,-1;0,0,0,0],[0,1;0,0,0,0]\}.
\end{aligned}$$

These triangles have center of gravity $\mathbf{c}_{\Delta_1} = \frac{1}{3}[0,0,0,0,1,1]$ equal to the center of gravity of T_{Δ_1}, and lie in complementary $2-$spaces. They are long because their edge vectors

$$\begin{aligned}
&\{\pm[0,0;0,0,1,0],\pm[0,0;0,0,-1,1],\pm[0,0;0,0,0,-1]\},\\
&\{\pm[0,0;1,-1,0,0],\pm[3,0;1,2,1,1],\pm[-3,0;-2,-1,-1,-1]\},\\
&\{\pm[-2,-1;0,0,-1,-1],\pm[1,2;0,0,0,0],\pm[1,-1;0,0,1,1]\},
\end{aligned}$$

are long vectors for $\varphi_{E_6^*}$.

The long triangles in $\mathcal{L}_{E_6}$. The long vector $[0,1;0^4]$ participates in the following 17 tripples of vectors in $\mathcal{L}_{E_6}$, which sum to zero.

$$\{[0,1;0^4],[2,-1;1^4],[-2,0;-1^4]\}$$
$$\{[0,1;0^4],[3,1;0,1,1,1],[-3,-2;0,-1,-1,-1]\}\times 4$$
$$\{[0,1;0^4],[-1,-1;0,1,-1,-1],[1,0;0,-1,1,1]\}\times 12$$

Since the $\mathcal{G}_{E_6}$−action is transitive on the long vectors, this action generates $(270\times17)/3$ such tripples, which account for all tripples of vectors in $\mathcal{L}_{E_6}$ that sum to zero. These occur in pairs of opposites, and each pair of opposites corresponds to a single homology class of six long triangles at the origin. Consequently, there are 45×17 homology classes of long triangles, and $45\times 17\times 6$ long triangles at the origin.

The homology classes of long triangles can be classiøed using the $\mathcal{G}_{E_6}$−action, and using the notion of c−homology classes.

Deønition 19 We will say that two long triangles are $c-$ homologous if their centers are homologous. A set of triangles belongs to the same c−class if the triangles are mutually c−homologous.

Two triangles that are homologous are neccessarily c−homologous, but c−homologous triangles need not be homologous. For example, the 45 triangles $\Delta^1_{\mathbf{p}_1}$, $\Delta^2_{\mathbf{p}_1},\ldots,\Delta^{45}_{\mathbf{p}_1}$, are c−homologous because they have a common center of gravity, but are inhomologous. As a second example, consider the vectors in the ørst tripple, which are edge vectors for the triangle $\Delta^1_{\mathbf{p}_1}$ with vertex set $\{[0;0,0,0,0,0],\ [0;1,0,0,0,0],\ [2;0,1,1,1,1]\}$, and, the vectors in the second tripple, which are edge vectors for the triangle Δ' with vertex set $\{[0,0;0^4],[0,1;0^4],[3,2;0,1,1,1]\}$. The two centers, $\mathbf{c}_G=\frac{1}{3}[2;1,1,1,1,1]$, $\mathbf{c}_{\Delta'}=\frac{1}{3}[3,3;0,1,1,1]$, are inhomologous, so the triangles themselves are inhomologous and c−inhomologous.

Proposition 20 There are $45x17$ homology classes of long triangles at the origin. These belong to two $\mathcal{G}_{E_6}$−classes and 121 c−classes. The ørst $\mathcal{G}_{E_6}$−class contains 45 homology classes, including the homology class of the triangle $\Delta^1_{\mathbf{p}_1}$. These 45 homology classes are c−homologous, and are commensurate with the star of G-topes at the origin. The second $\mathcal{G}_{E_6}$−class contains the remaining $45x16=720$ homology classes, including the homology class of the triangle Δ'. This $\mathcal{G}_{E_6}$−class is further reøned into 120 c−classes of 6 homology classes each, which are incommensurate with the star of G-topes at the origin; the $\mathcal{G}_{E_6}$−action is transitive on these 120 c−classes.

Proof. The ørst assertion was established in the counting at the beginning of this section.

The øve triangles $\Delta^1_{\mathbf{p}_1}$, $\Delta^2_{\mathbf{p}_1}$,...,$\Delta^5_{\mathbf{p}_1}$ are $\mathcal{G}_{E_6}(G_{\mathbf{p}_1})$−equivalent, and therefore $\mathcal{G}_{E_6}$−equivalent. The $\mathcal{G}_{E_6}$−action on $\Delta^1_{\mathbf{p}_1}$ therefore generates $5x54 = 270 = 45x6$ long triangles. These long triangles are commensurate with the star of G-topes at the origin.

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The long triangles in $\mathcal{L}_{E_6^*}$.

The Delaunay tilings $\mathcal{D}_{E_6}$ and $\mathcal{D}_{E_6^*}$.

9 G-Topes, T-Topes, and incidence relations

The G-tope $G_{\mathbf{p}_1}$ is the convex hull of the origin, $\mathbf{0} = [0^6]$, and the two layers of vertices

$$\begin{aligned} \mathcal{S}_{\mathbf{p}_1} &= \{\mathbf{s} \in \mathcal{S}_{E_6} | \mathbf{p}_1 \cdot \mathbf{s} = 1\} = \{[3;1^5], [-1;-1,0^4] \times 5, [1;1^2,0^3] \times 10\}, \\ \mathcal{L}_{\mathbf{p}_1} &= \{\mathbf{l} \in \mathcal{L}_{E_6} | \mathbf{p}_1 \cdot \mathbf{l} = 2\} = \{[0;1,0^4] \times 5, [2;0,1^4] \times 5\}. \end{aligned}$$

The perfect vector $\mathbf{p}_1 = [-3,2,2,2,2,2] \in \mathcal{P}_{E_6^*}$ appears in the incidence relations that characterize these layers, and, also as a subscript. The sixteen lattice points in the short layer, and the ten in the long layer, are found using the incidence relations, and the list of short vectors for φ_{E_6} above, and the list of long vectors below. This reference polytope, with 27 vertices, is frequently referred to as a Gosset polytope since X. X. Gosset was the ørst to make an extensive investigation of its properties [cite.].

The long vectors for φ_{E_6} are deøned by

$$\mathcal{L}_{E_6} = \{\mathbf{z} \in \mathbb{Z}^6 | \varphi_{E_6}(\mathbf{z}) = 2m\},$$

where $2m$ is the second minimum for φ_{E_6}, which is the minimal value assumed on the non-zero elements of $\mathbb{Z}^6$ not belonging to $\mathcal{S}_{E_6}$. $|\mathcal{L}_{E_6}| = 270$.

A second reference polytope is the T-tope T_{Δ_1}, which is the convex hull of the origin and the following two layers of vertices.

$$\begin{aligned} \mathcal{S}_{\Delta_1} &= \{\mathbf{s} \in \mathcal{S}_{E_6^*} | \boldsymbol{\alpha}_{\Delta_1} \cdot \mathbf{s} = 1\} = \{[2,0,1^4], [-1,0,-1,0^3], [-1,0^2,-1,0^2], [0,1,0^4], [-1^2,0^4], [1,0^3,1 \\ \mathcal{L}_{\Delta_1} &= \{\mathbf{l} \in \mathcal{L}_{E_6^*} | \boldsymbol{\alpha}_{\Delta_1} \cdot \mathbf{l} = \frac{3}{2}\} = \{[0^4,1,0], [0^5,1]\} \end{aligned}$$

The subscript refers to the perfect triangle $\Delta_1 = conv\{\mathbf{0}, \mathbf{p}_1, \mathbf{p}_2\}$, where $\mathbf{p}_1 = [-2,1,1,1,2,1]$, $\mathbf{p}_2 = [-2,1,1,1,1,2]$; we say that Δ_1 is a perfect triangle in $\mathcal{P}_{E_6}$ because the edge vectors $\{\mathbf{p}_1, \mathbf{p}_2 - \mathbf{p}_1, -\mathbf{p}_2\} \subset \mathcal{P}_{E_6}$. The incidence relations that characterize the short and long layers are formulated in terms of the vector $\boldsymbol{\alpha}_{\Delta_1} = \frac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2)$.

The long vectors for $\varphi_{E_6^*}$ are deøned by

$$\mathcal{L}_{E_6} = \{\mathbf{z} \in \mathbb{Z}^6 | \varphi_{E_6^*}(\mathbf{z}) = \frac{3}{2}m\};$$

where $\frac{3}{2}m$ is the second minimum for $\varphi_{E_6^*}$. $|\mathcal{L}_{E_6}| = 72$.

(At the moment we are unconconcerned whether the reference G-tope and reference T-tope are lattice polytopes. That is, we have not yet excluded the possiblility that there are integer points interior to either of these polytopes, or in the relative interior of one of the faces.)

Equivalent G-topes and T-topes. More generally each perfect vector $\mathbf{p} \in \mathcal{P}_{E_6^*}$ determines a short and long layer by the incidence relations $\mathcal{S}_{\mathbf{p}} = \{\mathbf{s} \in \mathcal{S}_{E_6} | \mathbf{p} \cdot \mathbf{s} = 1\}$, $\mathcal{L}_{\mathbf{p}} = \{\mathbf{l} \in \mathcal{L}_{E_6} | \mathbf{p} \cdot \mathbf{l} = 2\}$, and, a corresponding G-tope $G_{\mathbf{p}} = conv\{\mathbf{0}, \mathcal{S}_{\mathbf{p}}, \mathcal{L}_{\mathbf{p}}\}$. Each perfect triangle $\Delta \in \mathcal{P}_{E_6}$ at the origin determines a short and long layer, $\mathcal{S}_\Delta = \{\mathbf{s} \in \mathcal{S}_{E_6^*} | \boldsymbol{\alpha}_\Delta \cdot \mathbf{s} = 1\}$, $\mathcal{L}_\Delta = \{\mathbf{l} \in \mathcal{L}_{E_6^*} | \boldsymbol{\alpha}_\Delta \cdot \mathbf{l} = \frac{3}{2}\}$, and, a corresponding T-tope $T_\Delta = conv\{\mathbf{0}, \mathcal{S}_\Delta, \mathcal{L}_\Delta\}$; $\boldsymbol{\alpha}_\Delta = \frac{1}{2}(\mathbf{p}_i + \mathbf{p}_j)$, where $\mathbf{p}_i, \mathbf{p}_j \in \mathcal{P}_{E_6}$ are the two vertices of Δ not equal to $\mathbf{0}$ ($\Delta = conv\{\mathbf{0}, \mathbf{p}_i, \mathbf{p}_j\}$).

Theorem 21 The 54 G-topes given by $G_{\mathbf{p}}, \mathbf{p} \in \mathcal{P}_{E_6^*}$, are $\mathcal{G}_{E_6}$−equivalent. The 720 T-topes given by T_Δ, $\Delta \in \mathcal{P}_{E_6}$, are $\mathcal{G}_{E_6^*}$−equivalent (Δ is required to have a vertex at the origin).

Proof. The ørst assertion of the Theorem follows from the fact that $\mathcal{G}^\circ_{E_6}$ acts transitively on the perfect vectors $\mathcal{P}_{E_6^*}$. The second assertion follows from the fact that $\mathcal{G}^\circ_{\mathbf{E_6^*}}$ act transitively on the perfect triangles $\Delta \in \mathcal{P}_{E_6}$ that have a vertex at the origin (established in Theorem XX below).
The proof is completed by showing that the number of perfect triangles $\Delta \in \mathcal{P}_{E_6}$, with a vertex at the origin, is 720. The perfect vector $[2,-1;-1,-1,-1,-1] \in \mathcal{P}_{E_6}$ belongs to the two tripples $E_1 = \{[2,-1;-1,-1,-1,-1],[-2,1;2,1,1,1],[0,0;-1,0,0,0]\}$, $E_2 = \{[2,-1;-1,-1,-1,-1],[-1,1;1,1,0,0],[-1,0;0,0,1,1]\} \subset \mathcal{P}_{E_6}$. Since the vectors of each tripple sum to zero, each is a tripple of edge vectors for a perfect triangle. By permuting the last four entries, three additional tripples in $\mathcal{P}_{E_6}$ are generated from E_1, and nine additional tripples are generated

from E_2. Altogether there are ten, and these are the only tripples in $\mathcal{P}_{E_6}$ that contain the perfect vector $[2,-1;-1,-1,-1,-1]$; this can easily be checked. Since the dual action of $\mathcal{G}_{E_6^*}$ is transitive on $\mathcal{P}_{E_6}$, the total number of tripples of edge vectors is $(|\mathcal{P}_{E_6}|\times 10)/3 = (72\times 10)/3 = 240$. These tripples occur in positive and negative pairs, each pair of opposites corresponding to a single homology class of perfect triangles at the origin. As a consequence there are 120 homology classes of perfect triangles at the origin, and $6 \times 120 = 720$ perfect triangles.

■

The incidence relations for G-topes and T-topes invlove the following values for scalar products: $\mathbf{p}\cdot\mathbf{s} =1$, $\mathbf{p}\cdot\mathbf{l} =2$, $\boldsymbol{\alpha}_\Delta\cdot\mathbf{s} =1$, $\boldsymbol{\alpha}_\Delta\cdot\mathbf{l} =\frac{3}{2}$. As shown in the following Lemma, these are critical values.

Lemma 22 For $\mathbf{p} \in\mathcal{P}_{E_6^*}$, $\max_{\mathbf{s}\in\mathcal{S}_{E_6}} \mathbf{p}\cdot\mathbf{s} =1$ and $\mathbf{p}\max_{\mathbf{l}\in\mathcal{L}_{E_6}} \mathbf{p}\cdot\mathbf{l} =2$. For $\Delta = conv\{\mathbf{0},\mathbf{p}_i,\mathbf{p}_j\} \in \mathcal{P}_{E_6}$, $\boldsymbol{\alpha}_\Delta = \frac{1}{2}(\mathbf{p}_i + \mathbf{p}_j)$, $\max_{\mathbf{s}\in\mathcal{S}_{E_6^*}} \boldsymbol{\alpha}_\Delta\cdot\mathbf{s} =1$ and $\max_{\mathbf{l}\in\mathcal{L}_{E_6^*}} \boldsymbol{\alpha}_\Delta\cdot\mathbf{l} =\frac{3}{2}$.

Proof. The ørst assertion follows from the fact that for $\mathbf{p} \in\mathcal{P}_{E_6^*}$, $\mathbf{s} \in\mathcal{S}_{E_6}$, $\mathbf{l} \in\mathcal{L}_{E_6}$, $\mathbf{p}\cdot\mathbf{s} \in \{0,\pm1\}$, $\mathbf{p}\cdot\mathbf{l}\in \{0,\pm1,\pm2\}$. Similarly, for $\mathbf{p} \in\mathcal{P}_{E_6}$, $\mathbf{s} \in\mathcal{S}_{E_6^*}$, $\mathbf{l} \in\mathcal{L}_{E_6^*}$, $\mathbf{p}\cdot\mathbf{s} \in \{0,\pm1\}$, $\mathbf{p}\cdot\mathbf{l}\in \{0,\pm1,\pm2\}$. Proof of the second assertion also requires the fact that for each perfect vector $\mathbf{p} \in\mathcal{P}_{E_6}$ there is a unique long vector $\mathbf{l}_\mathbf{p}\in\mathcal{L}_{E_6^*}$ with the property that $\mathbf{p}\cdot\mathbf{l}_\mathbf{p} = 2$, and the mapping $\mathbf{p}\to\mathbf{l}_\mathbf{p}$ establishes a one-to-one correspondence between the elements of $\mathcal{P}_{E_6}$ and $\mathcal{L}_{E_6^*}$ (this may be established by examining the list of vectors in $\mathcal{P}_{E_6}$ and $\mathcal{L}_{E_6^*}$). ■

Proposition 23 For arbitrary G-topes $G_{\mathbf{p}_i}, G_{\mathbf{p}_j}$, $\mathbf{p}_i,\mathbf{p}_j \in \mathcal{P}_{E_6^*}$, $G_{\mathbf{p}_i}\cap G_{\mathbf{p}_j}$ is a proper face of both $G_{\mathbf{p}_i}$ and $G_{\mathbf{p}_j}$, with vertex set $\{\mathbf{0},\mathcal{S}_{\mathbf{p}_i}\cap \mathcal{S}_{\mathbf{p}_j},\mathcal{L}_{\mathbf{p}_i}\cap \mathcal{L}_{\mathbf{p}_j}\}$. For arbitrary T-topes $T_{\Delta_i}, T_{\Delta_j}$, $\Delta_i,\Delta_j \in \mathcal{P}_{E_6}$, $T_{\Delta_i}\cap T_{\Delta_j}$ is a proper face of both T_{Δ_i} and T_{Δ_j}, with vertex set $\{\mathbf{0},\mathcal{S}_{\Delta i}\cap \mathcal{S}_{\Delta_j},\mathcal{L}_{\Delta i}\cap \mathcal{L}_{\Delta_j}\}$.

Proof. Consider $\mathbf{p}_i,\mathbf{p}_j \in \mathcal{P}_{E_6^*}$, and the hyperplane with equation $(\mathbf{p}_i - \mathbf{p}_j)\cdot\mathbf{x} =0$. By Lemma 22, $\mathcal{S}_{\mathbf{p}_\mathrm{i}}$ and $\mathcal{L}_{\mathbf{p}_\mathrm{i}}$ are contained in the positive half-space determined by this hyperplane, and, $\mathcal{S}_{\mathbf{p}_j}$ and $\mathcal{L}_{\mathbf{p}_j}$ are contained in the negative half-space. Therefore, the hyperplane with equation $(\mathbf{p}_i-\mathbf{p}_j)\cdot\mathbf{x} =0$ separates $G_{\mathbf{p}_i}$ and $G_{\mathbf{p}_j}$.

Any short vertex $\mathbf{s} \in G_{\mathbf{p}_i}\cup G_{\mathbf{p}_j}$ lying on the separating hyperplane must satisfy the equalities $\mathbf{p}_i\cdot\mathbf{s} = \mathbf{p}_j\cdot\mathbf{s} =1$, and therefore belong to the intersection $\mathcal{S}_{\mathbf{p}_i}\cap \mathcal{S}_{\mathbf{p}_j} \subset G_{\mathbf{p}_i}\cap G_{\mathbf{p}_j}$. Similarly, any long vertex $\mathbf{l} \in G_{\mathbf{p}_i}\cup G_{\mathbf{p}_j}$ on the separating hyperplane must satisfy the equalities $\mathbf{p}_i \cdot \mathbf{l} = \mathbf{p}_j \cdot \mathbf{l} = 2$, and belong to the intersection $\mathcal{L}_{\mathbf{p}_i} \cap \mathcal{L}_{\mathbf{p}_j} \subset G_{\mathbf{p}_i} \cap G_{\mathbf{p}_j}$. Therefore, the vertices of $G_{\mathbf{p}_i} \cap G_{\mathbf{p}_j}$ are given by $\{\mathbf{0}, \mathcal{S}_{\mathbf{p}_i} \cap \mathcal{S}_{\mathbf{p}_j}, \mathcal{L}_{\mathbf{p}_i} \cap \mathcal{L}_{\mathbf{p}_j}\}$, and determine a proper face of both $G_{\mathbf{p}_i}$ and $G_{\mathbf{p}_j}$.

An identicle argument establishes the second assertion.

■

This proposition is easily generalized to cover the case of an arbitrary intersection of G-topes or T-topes: the intersection $G_{\mathbf{p}_i} \cap G_{\mathbf{p}_j} \cap ... \cap G_{\mathbf{p}_k}$ is a face of each of the intersection G-topes, with vertices given by $\{\mathbf{0}, \mathcal{S}_{\mathbf{p}_i} \cap \mathcal{S}_{\mathbf{p}_j} \cap ... \cap \mathcal{S}_{\mathbf{p}_k}, \mathcal{L}_{\mathbf{p}_i} \cap \mathcal{L}_{\mathbf{p}_j} \cap ... \cap \mathcal{L}_{\mathbf{p}_k}\}$; the intersection $T_{\Delta_i} \cap T_{\Delta_j} \cap ... \cap T_{\Delta_k}$ is a face of each of the intersection T-topes, with vertices given by $\{\mathbf{0}, \mathcal{S}_{\Delta_i} \cap \mathcal{S}_{\Delta_j} \cap ... \cap \mathcal{S}_{\Delta_k}, \mathcal{L}_{\Delta_i} \cap \mathcal{L}_{\Delta_j} \cap ... \cap \mathcal{L}_{\Delta_k}\}$.

The star of G-topes. $\mathcal{L}_{\mathbf{p}_1}$ is the vertex set for a cross polytope, which is a facet of $G_{\mathbf{p}_1}$. The subsets $\{\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_5\} = \{[0; 1, 0^4] \times 5\}$, $\{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_5\} = \{[2; 0, 1^4] \times 5\} \subset \mathcal{L}_{\mathbf{p}_1}$ determine opposite facets, which we will refer to as the red and blue 4-simplexes. These simplexes are short because their edge vectors, which are generated by permuting the last øve entries of $[0; 1, -1, 0, 0, 0]$, are short. The diagonals intersect at the point $\frac{1}{2}[2, 1, 1, 1, 1, 1]$, which is the centroid of $\mathcal{L}_{\mathbf{p}_1}$. The diagonal vectors, given by $\{\pm(\mathbf{b}_1 - \mathbf{r}_1), \pm(\mathbf{b}_2 - \mathbf{r}_2), ..., \pm(\mathbf{b}_5 - \mathbf{r}_5)\} = \{\pm[2; -1, 1, 1, 1, 1] \times 5\}$, are long.

The dual for the stability group for $G_{\mathbf{p}_1}$, which is the subgroup $\mathcal{G}_{E_6}(G_{\mathbf{p}_1}) \subset \mathcal{G}_{E_6}$, is the stability group for the perfect vector $\mathbf{p}_1$, which is the subgroup $\mathcal{G}^{\circ}_{E_6}(\mathbf{p}_1) \subset \mathcal{G}^{\circ}_{E_6}$. $|\mathcal{G}_{E_6}(G_{\mathbf{p}_1})| = |\mathcal{G}^{\circ}_{E_6}(\mathbf{p}_1)|| = |\mathcal{G}^{\circ}_{E_6}|/|\mathcal{P}_{E_6^*}| = 2^8 3^4 5/54 = 5! \times 2^4$.

The stability group for the red simplex (or the blue simplex) is the permutation group $\mathcal{A}_5$ acting on the last øve coordinates, and has order $|\mathcal{A}_5| = 5!$. Since $\mathcal{A}^{\circ}_5 \subset \mathcal{G}^{\circ}_{E_6}(\mathbf{p}_1)$, it follows that $\mathcal{A}_5 \subset \mathcal{G}_{E_6}(G_{\mathbf{p}_1})$.

The matrix

$$\mathbf{I}_{12} = \begin{bmatrix} -1 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix},$$

induces an inversion of the ørst two axes of $conv(\mathcal{L}_{\mathbf{p}_1})$, and maps the red and blue vertex sets onto vertex sets for other facets of $conv(\mathcal{L}_{\mathbf{p}_1})$: $\mathbf{I}_{12}\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4, \mathbf{r}_5\} =$

$\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{r}_3, \mathbf{r}_4, \mathbf{r}_5\}$; $\mathbf{I}_{12}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4, \mathbf{b}_5\} = \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{b}_3, \mathbf{b}_4, \mathbf{b}_5\}$. It is easy to check that $\mathbf{I}_{12}^{\circ} \in \mathcal{G}_{E_6}^{\circ}(\mathbf{p}_1)$, so $\mathbf{I}_{12} \in \mathcal{G}_{E_6}(G_{\mathbf{p}_1})$. Similarly, there are matrices $\mathbf{I}_{13}$, $\mathbf{I}_{14}$, $\mathbf{I}_{15}$ that induce inversions of the pairs of axes indicated by the subscripts. Each is an element of $\mathcal{G}_{E_6}(G_{\mathbf{p}_1})$, and the group $\mathcal{I}_5$ generated by these actions is a subgroup of $\mathcal{G}_{E_6}(G_{\mathbf{p}_1})$. An arbitrary element $\mathbf{I} \in \mathcal{I}_5$ inverts an even number of axes of the cross-polytope, and $\mathcal{I}_5$ is the group of all such even inversions. The $\mathcal{I}_5-$orbit of the red simplex is the set of all simplicial facets with an odd number of red vertices, and the $\mathcal{I}_5-$orbit of the blue simplex is the set with an even number of red vertices. The facets in each orbit alternate on the boundary of $conv(\mathcal{L}_{\mathbf{p}_1})$, and are short. The order of $\mathcal{I}_5$ is equal to the size of each of these orbits, which is half the number of facets 2^4.

Clearly, $\mathcal{I}_5 \times \mathcal{A}_5 \subseteq \mathcal{G}_{E_6}(G_{\mathbf{p}_1})$. Since $|\mathcal{I}_5 \times \mathcal{A}_5| = |\mathcal{I}_5||\mathcal{A}_5| = 5! \times 2^4 = |\mathcal{G}_{E_6}(G_{\mathbf{p}_1})|$, it follows that $\mathcal{I}_5 \times \mathcal{A}_5 = \mathcal{G}_{E_6}(G_{\mathbf{p}_1})$.

Theorem 24 $\mathcal{G}_{E_6}(G_{\mathbf{p}_1}) = \mathcal{I}_5 \times \mathcal{A}_5$. Alternate facets of $conv(\mathcal{L}_{\mathbf{p}_1})$ are $\mathcal{G}_{E_6}(G_{\mathbf{p}_1})-$equivalent, each class containing 2^4 short $4-$simplexes. The $k-$faces, for $0 \leq k \leq 3$, are short simplexes, and are $\mathcal{G}_{E_6}(G_{\mathbf{p}_1})-$equivalent.

Proof. The ørst two assertions were established in the discussion before the statement of the Theorem. Assuming that alternate facets are equivalent, each $k-$face, for $0 \leq k \leq 3$, belongs to a pair of alternate facets, and is therefore $\mathcal{G}_{E_6}(G_{\mathbf{p}_1})-$equivalent to a $k-$face of the red simplex. Since the $k-$faces of the red simplex are $\mathcal{A}_5-$equivalent, and since $\mathcal{A}_5 \subset \mathcal{G}_{E_6}(G_{\mathbf{p}_1})$, the second assertion of the Theorem follows.

■

The diagonal vectors for $conv(\mathcal{L}_{\mathbf{p}_1})$, given by $\mathcal{L}_1 = \{\pm[2; -1, 1, 1, 1, 1] \times 5\} \subset \mathcal{L}_{E_6}$, are also the diagonal vectors for the opposite cross-polytope $conv(\mathcal{L}_{-\mathbf{p}_1})$. Since the 27 pairs $\{\mathbf{p}_i, -\mathbf{p}_i\} \subset \mathcal{P}_{E_6^*}$ are $\mathcal{G}_{E_6}^{\circ}-$equivalent, there are 27 sets of diagonals $\mathcal{L}_i$ that are $\mathcal{G}_{E_6}-$equivalent. Since the diagonals $\mathcal{L}_1 \subset \mathcal{L}_{E_6}$ belong to a single parity class, and are orthogonal to the subset $\{\pm\mathbf{p}_1\}$, similar statements hold for each of the diagonal sets $\mathcal{L}_i \subset \mathcal{L}_{E_6}$. The

27 $\mathcal{G}_{E_6}$−equivalent long parity classes are given by:

$$\{\pm[-3, 2^5]\} \perp \mathcal{L}_1 = \{\pm[2; -1, 1^4] \times 5\};$$
$$\{\pm[-2; 1^5]\} \perp \mathcal{L}_2 = \{\pm[-1; -2, 0^4] \times 5\};$$
$$\{\pm[0; 1, 0^4]\} \perp \mathcal{L}_3 = \{\pm[3, 0; 1^4], \pm[-1, 0; 1, -1^3] \times 4\} \times 5;$$
$$\{\pm[-1; 0^2, 1^3]\} \perp \mathcal{L}_8 = \{\pm[-3, -2, 0; -1^3], \pm[-3, 0, -2; -1^3], \pm[1, 0^2; -1, 1^2] \times 3\} \times 10;$$
$$\{\pm[-2; 2, 1^4]\} \perp \mathcal{L}_{18} = \{\pm[-2, 0, -1; -1^3], \pm[4, 2, 1; 1^3], \pm[0^2; -1; -1, 1^2] \times 3\} \times 5$$
$$\{\pm[-1; 0, 1^4]\} \perp \mathcal{L}_{23} = \{\pm[0, 1; 0^4], \pm[2, 1; 2, 0^3] \times 4\} \times 5.$$

These 27 classes account for the 270 long vectors of φ_{E_6}.

Corollary 25 $\mathcal{L}_{E_6} = \cup_{1 \le i \le 27} \mathcal{L}_i$. The parity class $\mathcal{L}_i$ is characterized by the equalities $\mathcal{L}_i = \{\pm \mathbf{p}_i\}^\perp \cap \mathcal{L}_{E_6}$, $\{\pm \mathbf{p}_i\} = \mathcal{L}_i^\perp \cap \mathcal{P}_{E_6^*}$, and has stability group $\mathcal{G}_{E_6}(\mathcal{L}_i) = \mathcal{I} \times \mathcal{G}_{E_6}(G_{\mathbf{p}_i})$, where $\mathcal{I}$ is the two element subgroup with the central inversion.

Proof. The øfirst assertion follows from the above discussion. The second assertion can be veriøed for the case of $\mathcal{L}_1$, then extended to arbitrary $\mathcal{L}_i$ by the $\mathcal{G}_{E_6}$ action. It is apparent that the stability group for $\{\mathbf{p}_i, -\mathbf{p}_i\}$ is given by $\mathcal{I} \times \mathcal{G}^\circ_{E_6}(\mathbf{p}_i)$, from which it immediately follows that the stability group for $\mathcal{L}_i$ is given by $\mathcal{I} \times \mathcal{G}_{E_6}(G_{\mathbf{p}_i})$.

■

Consider the G-topes $G_{\mathbf{p}_1}$ and $G_{\mathbf{p}_2}$, where $\mathbf{p}_1 = [-3, 2, 2, 2, 2, 2]$, $\mathbf{p}_2 = [-2, 1, 1, 1, 1, 1] \in \mathcal{P}_{E_6^*}$. A straight forward calculation shows that

$$\mathcal{S}_{\mathbf{p}_1} \cap \mathcal{S}_{\mathbf{p}_2} = \{[-1; -1, 0, 0, 0, 0] \times 5\}$$
$$\mathcal{L}_{\mathbf{p}_1} \cap \mathcal{L}_{\mathbf{p}_2} = \mathit{empty}.$$

By Proposition 23 $G_{\mathbf{p}_1} \cap G_{\mathbf{p}_2} = conv\{\mathbf{0}, \mathcal{S}_{\mathbf{p}_1} \cap \mathcal{S}_{\mathbf{p}_2}\} = S$, which is a facet of both $G_{\mathbf{p}_1}$ and $G_{\mathbf{p}_2}$; S is a short 5−simplex.

Another example is the cross-polytope facet that results when the perfect vectors $\mathbf{p}_1 = [-3, 2, 2, 2, 2, 2]$, $\mathbf{p}_{18} = [-2, 2, 1, 1, 1, 1] \in \mathcal{P}_{E_6^*}$ are chosen. In this case

$$\mathcal{S}_{\mathbf{p}_1} \cap \mathcal{S}_{\mathbf{p}_{18}} = \{[-1, 0; -1, 0, 0, 0] \times 4, [1, 1; 1, 0, 0, 0] \times 4\}$$
$$\mathcal{L}_{\mathbf{p}_1} \cap \mathcal{L}_{\mathbf{p}_{18}} = \{[0, 1; 0, 0, 0, 0]\},$$

and $C = conv\{\mathbf{0}, \mathcal{S}_{\mathbf{p}_1} \cap \mathcal{S}_{\mathbf{p}_{18}}, \mathcal{L}_{\mathbf{p}_1} \cap \mathcal{L}_{\mathbf{p}_{18}}\}$ is a 5−dimensional cross-polytope, which is a facet of both $G_{\mathbf{p}_1}$ and $G_{\mathbf{p}_{18}}$; the centroid is $\mathbf{c}_C = [0, \frac{1}{2}; 0, 0, 0, 0]$, and $\mathcal{L}_{23} = \{\pm[0, 1; 0, 0, 0, 0], \pm[2, 1; 2, 0, 0, 0] \times 4\}$ is the set of diagonal vectors.

Corollary 26 $G_{\mathbf{p}_1}$ has two $\mathcal{G}_{E_6}(G_{\mathbf{p}_1})$−classes of facets with vertices at the origin: 16 simplicial facets $G_{\mathbf{p}_1} \cap G_{\mathbf{p}_i}, 2 \leq i \leq 17$, which are short; 10 cross-polytopes facets $G_{\mathbf{p}_1} \cap G_{\mathbf{p}_j}, 18 \leq j \leq 27$. $G_{\mathbf{p}_1}$ has two $\mathcal{G}_{E_6}(G_{\mathbf{p}_1})$−classes of 4−faces with vertices at the origin: 80 short 4−simplexes, which separate simplicial and cross-polytope facets; 40 short 4−simplexes, which separate a pair of cross-polytope facets.

Proof. It can easily be checked that $\mathcal{G}^{\circ}_{E_6}(\mathbf{p}_1)$ acts transitively on the following two sets of perfect vectors.

$$\{\mathbf{p}_2, \mathbf{p}_3, ..., \mathbf{p}_{17}\} = \{[-2; 1^5], [0; 1, 0^4] \times 5, [-1; 0^2, 1^3] \times 10\}$$
$$\{\mathbf{p}_{18}, \mathbf{p}_{19}, ..., \mathbf{p}_{27}\} = \{[-2; 2, 1^4] \times 5, [-1; 0, 1^4] \times 5\}$$

It immediately follows that $G_{\mathbf{p}_1} \cap G_{\mathbf{p}_i}, 3 \leq i \leq 17$ are simplicial facets $\mathcal{G}_{E_6}(G_{\mathbf{p}_1})$−equivalent to the reference S, which is short, and, that $G_{\mathbf{p}_1} \cap G_{\mathbf{p}_j}, 19 \leq j \leq 27$ are cross-polytope facets $\mathcal{G}_{E_6}(G_{\mathbf{p}_1})$−equivalent to the reference C.

By Proposition 23 the intersection $S_4 = S \cap C$ is given by

$$\begin{aligned} G_{\mathbf{p}_1} \cap G_{\mathbf{p}_2} \cap G_{\mathbf{p}_{18}} &= conv\{\mathbf{0}, \mathcal{S}_{\mathbf{p}_1} \cap \mathcal{S}_{\mathbf{p}_2} \cap \mathcal{S}_{\mathbf{p}_{18}}\} \\ &= \{\mathbf{0}, [-1, 0; -1, 0, 0, 0] \times 4\}, \end{aligned}$$

which is a simplicial 4−face of both S and C, and is short. The øve 4−faces of S with a vertex at the origin are $\mathcal{A}_5$− equivalent, and, since $\mathcal{A}_5 \subset \mathcal{G}_{E_6}(G_{\mathbf{p}_1})$, are $\mathcal{G}_{E_6}(G_{\mathbf{p}_1})$−equivalent. Since the 16 simplicial facets are $\mathcal{G}_{E_6}(G_{\mathbf{p}_1})$−equivalent, there are $16 \times 5 = 80$ $\mathcal{G}_{E_6}(G_{\mathbf{p}_1})$−equivalent 4−faces of this type, which separate simplicial and cross-polytope facets.

By Corollary 25 the subgroup $\mathcal{G}_{E_6}(G_{\mathbf{p}_{23}})$ acts eœectively on $\mathcal{L}_{23} = \{\pm[0, 1; 0, 0, 0, 0], \pm[2, 1; 2, ($ 4}, which is the set of diagonal vectors of C. The subgroup of $\mathcal{G}_{E_6}(G_{\mathbf{p}_{23}})$ that øxes the diagonal $[0, 1; 0, 0, 0, 0]$ is also a subgroup of $\mathcal{G}_{E_6}(G_{\mathbf{p}_1})$, so is equal to $\mathcal{G}_{E_6}(G_{\mathbf{p}_{23}}) \cap \mathcal{G}_{E_6}(G_{\mathbf{p}_1})$. By the geometric description of the $\mathcal{G}_{E_6}(G_{\mathbf{p}_1})$−action, alternate 4−faces of C, attached to the origin, are $\mathcal{G}_{E_6}(G_{\mathbf{p}_{23}}) \cap \mathcal{G}_{E_6}(G_{\mathbf{p}_1})$−equivalent, therefore $\mathcal{G}_{E_6}(G_{\mathbf{p}_1})$−equivalent. Therefore, C has 8 equivalent 4−faces, which separate C from simplicial facets equivalent to S. Since there are 10 cross-polytope facets, there are $10 \times 8 = 80$ equivalent 4−faces of this type, which gives a second accounting for the 4−faces that separate simplicial and cross-polytope facets.

Let $X = G_{\mathbf{p}_1} \cap G_{\mathbf{p}_{19}}$, which by the ørst assertion is a cross-polytope facet $\mathcal{G}_{E_6}(G_{\mathbf{p}_1})$−equivalent to C. By Proposition 23, the intersection $X_4 = X \cap C$

is given by

$$G_{\mathbf{p}_1} \cap G_{\mathbf{p}_{18}} \cap G_{\mathbf{p}_{19}} = conv\{\mathbf{0}, \mathcal{S}_{\mathbf{p}_1} \cap \mathcal{S}_{\mathbf{p}_{18}} \cap \mathcal{S}_{\mathbf{p}_{19}}\}$$
$$= conv\{\mathbf{0}. [-1,0,0,-1,0,0], [-1,0,0,0,-1,0], [-1,0,0,0,0,-1], [1,1,1,0,0,0]\},$$

which is a simplicial 4−face of X and C, and is $\mathcal{G}_{E_6}(G_{\mathbf{p}_1})$−inequivalent to S_4. Therefore, C has 8 equivalent 4−faces, which separate C from cross-polytope facets equivalent to C. Consequently, there are $(10 \times 8)/2 = 40$ equivalent 4−faces, which separate cross-polytopes equivalent to C.

Since the 4−faces enumerated account for all 4−faces of simplicial or cross-polytope facets, which have a vertex at the origin, the lists of facets and 4−faces are complete. ■

Theorem 27 The 54 G-topes $G_{\mathbf{p}}, \mathbf{p} \in \mathcal{P}_{E_6^*}$, øt together facet-to-facet to form the star of G-topes at the origin. There are 432 $\mathcal{G}_{E_6}$−equivalent facets with a vertex at the origin that are simplexes, and 270 $\mathcal{G}_{E_6}$−equivalent facets that are cross-polytopes. For each dimension $0 \leq k \leq 4$ the faces are simplicial, and $\mathcal{G}_{E_6}$−equivalent.

Proof. The ørst two assertions are immediate consequences of Corollary 26.

Consider the 4−faces S_4, X_4, which were described in the proof of Corollary 26, and are representatives of the two $\mathcal{G}_{E_6}(G_{\mathbf{p}_1})$−classes of 4−faces described in that Corollary. The face X_4 is the intersection of two cross-polytope facets X and C, and the simplicial facet Y, which has the following description.

$$Y = G_{\mathbf{p}_{18}} \cap G_{\mathbf{p}_{19}} = conv\{\mathbf{0}, \mathcal{S}_{\mathbf{p}_{18}} \cap \mathcal{S}_{\mathbf{p}_{19}}\}$$
$$= conv\{\mathbf{0}, [-1,0,0;-1,0,0] \times 3, [1^3;0^3], [-2,0^2;-1^3]\}$$

Since Y is $\mathcal{G}_{E_6}$−equivalent to S, X_4 is $\mathcal{G}_{E_6}$−equivalent to a 4−face of S. Since S_4 is also a 4−face of S, and since the 4−faces of S, with a vertex at the origin, are $\mathcal{A}_5$−equivalent, X_4 is $\mathcal{G}_{E_6}$−equivalent to S_4. It follows that there is a single $\mathcal{G}_{E_6}$−class of 4−faces with a vertex at the origin. An extension of this argument shows that all faces of each lower dimension, with a vertex at the origin, are simplicial, and $\mathcal{G}_{E_6}$−equivalent. ■

The star of T-topes.

Theorem 28 The 720 T-topes $T_\Delta, \Delta \in \mathcal{P}_{E_6}$ øt together facet-to-facet to form the star of T-topes at the origin. There are 6400 $\mathcal{G}_{E_6^*}$−equivalent facets that separate these T-topes, and have a vertex at the origin.Long triangles

10 Commensurate and incommensurate T−topes

The lattice polytope T is contained in G, and is therefore commensurate with G. A second reference T-tope is $Q = conv\{\Delta_Q^1, \Delta_Q^2, \Delta_Q^3\}$, where

$$\begin{aligned}\Delta_Q^1 &= conv\{[0^3,1,0^2],[0^4,1,0],[0^5,1]\},\\ \Delta_Q^2 &= conv\{[-1,0,-1,0^3],[2,0,1^4],[-1,0^5]\},\\ \Delta_Q^3 &= conv\{[-1,-1,0^4],[0,1,0^4],[1,0^2,1^3]\}.\end{aligned}$$

The edge vectors of each of these triangles belong to $\mathcal{L}_{E_6^*}$, so these triangles are long with respect to the form $\varphi_{E_6^*}$; they have a common centroid equal to the centroid of Q, which is given by $\mathbf{c}_Q=\frac{1}{3}[0^2;0,1^3]$.

The simplex $S_Q = conv\{[2,0;1^4],[-1,0;0^4],[-1,-1;0^4],[0,1;0^4]\}$ has two vertices from each of the triangles Δ_Q^2, Δ_Q^3, so is a simplicial 3−face of Q. The centroid $\frac{1}{4}[0,0;1,1,1,1]$ is also the centroid of the simplex $S_{G_{\mathbf{p}_1}} = conv\{[0^2;1,0^3],[0^2;0,1,0^2],[0^2;0^2,1,0],[0^2;0^3,1]\}$, which is a 3−face of $G_{\mathbf{p}_1}$. These faces of Q and $G_{\mathbf{p}_1}$ lie in a complementary 3−spaces, so satisfy the condition that $relint(S_Q) \cap relint(S_{G_{\mathbf{p}_1}}) = \frac{1}{4}[0,0;1,1,1,1]$. It follows from Proposition 9 that $\frac{1}{4}[0,0;1,1,1,1]$ is a vertex of the intersection polytope $Q \cap G_{\mathbf{p}_1}$, and that Q is incommensurate with $G_{\mathbf{p}_1}$.

The long triangles.

Equivalence along the line of centers. The forms along the line of centers, $(1-t)\varphi_{E_6}+t\varphi_{E_6^*}$, are invariant with respect to the action of the group $\mathcal{G}_{E_6\cap E_6^*} = \mathcal{G}_{E_6} \cap \mathcal{G}_{E_6^*}$, which leaves both φ_{E_6} and $\varphi_{E_6^*}$ invariant. Accordingly, both the short and long vectors of E_6 and E_6^* are left invariant, and in particular the subset $\mathcal{L}_{E_6} \cap \mathcal{S}_{E_6^*}$ is left invariant.

We ørst consider the stability group $\mathcal{G}_{E_6}(\mathcal{L}_{E_6} \cap \mathcal{S}_{E_6^*}) \subset \mathcal{G}_{E_6}$ of the set $\mathcal{L}_{E_6} \cap \mathcal{S}_{E_6^*} = \{\pm[2,-1,1,1,1,1], \pm[0,1,0,0,0,0], \pm[-2,0,-1,-1,-1,-1]\}$, which must contain $\mathcal{G}_{E_6\cap E_6^*}$. Since $\mathcal{L}_{E_6} \cap \mathcal{S}_{E_6^*}$ is the set of edge vectors for a single homology class of long triangles in $\mathcal{L}_{E_6}$, the action of the elements of the group $\mathcal{G}_{E_6}(\mathcal{L}_{E_6} \cap \mathcal{S}_{E_6^*})$ must permute the six members of this homology class. We take $\Delta = conv\{\mathbf{0}, [0,1,0,0,0,0], [-2,0,-1,-1,-1,-1]\}$ to be a reference in this homology class, and consider the its stability group $\mathcal{G}_{E_6}(\Delta)$. The triangle $\Delta \subset G$, so Δ is commensurate with G. Since $\mathcal{G}_{E_6}$ acts transitively on the commensurate long triangles in $\mathcal{L}_{E_6}$, $|\mathcal{G}_{E_6}(\mathcal{L}_{E_6} \cap \mathcal{S}_{E_6^*})| = 3!|\ \mathcal{G}_{E_6}(\Delta)|$.

It is geometrically obvious that The action of $\mathcal{G}_{E_6 \cap E_6^*}$ must leave invariant the short and long vectors of both E_6 and E_6^*, so must leave invariant each of the sets $\mathcal{S}_{E_6} \cap \mathcal{S}_{E_6^*}$, $\mathcal{S}_{E_6} \cap \mathcal{L}_{E_6^*}$, $\mathcal{L}_{E_6} \cap \mathcal{S}_{E_6^*}$, $\mathcal{L}_{E_6} \cap \mathcal{L}_{E_6^*}$.

Consider the øve-dimensional cross-polytope opposite the origin in G, with vertices given by

$$[\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3, \mathbf{l}_4, \mathbf{l}_5; \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4, \mathbf{r}_5] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

The four-dimensional cross-polytope with diagonal vectors $\mathbf{l}_i - \mathbf{r}_i, 2 \leq i \leq 5 i = 2, 3$.that is equatorial to the ørst diagonal diagonal and the diagonal running between the opposite vertices $\mathbf{l}_1 = [0,1;0,0,0,0]$, $\mathbf{r}_1 = [2,0;1,1,1,1]$. The four-dimensional cross-polytope that is equatorial to this diagonal has vertices given by

of the cross polytope opposite the originPermutation of the last four coordinates leaves each of these sets invariant, so these actions belong to $\mathcal{G}_{E_6 \cap E_6^*}$.

Classiøcation of T-topes. The long triangle Δ_Q^1 is a $2-$face of the reference polytope $G_{\mathbf{p}_1}$, so the edge vectors of this triangle are short with respect to the form φ_{E_6}. The edges vectors of the other two triangles are long with respect to the form φ_{E_6}.

$$edges(\Delta_Q^1) = \{\pm[0^3,1,-1,0], \pm[0^4,1,-1], \pm[0^3,1,0,-1]\} \in \mathcal{L}_{E_6^*} \cap \mathcal{S}_{E_6}$$
$$edges(\Delta_Q^2) = \{\pm[3,0,2,1^3], \pm[-3,0,-1^4], \pm[0^2,-1,0^3]\} \in \mathcal{L}_{E_6^*} \cap \mathcal{L}_{E_6}$$
$$edges(\Delta_Q^3) = \{\pm[1,2,0^4], \pm[1,-1,0,1^3], \pm[-2,-1,0,-1^3]\} \in \mathcal{L}_{E_6^*} \cap \mathcal{L}_{E_6}.$$

11 R-topes and the intersection tiling $\mathcal{D}_{E_6} \cap \mathcal{D}_{E_6^*}$.

12 Proof of Main Theorem.

We will refer to $R = conv(S_Q \cup S_G)$ as an R-tope. Like any other repartitioning complex it can be triangulated in precisely two ways. Deleting successively the vertices of S_G gives four simplexes that tile R, each with S_Q as a 3−face. This is the S_Q−tiling. Deleting successively the vertices of S_Q gives another four simplexes that tile R, the S_G−tiling. R has sixteen simplicial facets, each with three vertices from S_Q and three vertices from S_G. The centroid $\mathbf{c}_R = \frac{1}{4}[0,0;1,1,1,1]$ is a vertex of the barycentric subdivision of R, each tile of the subdivision formed by taking the convex hull of the centroid and a facet. Each of the sixteen simplicial tiles can be represented as the intersection of a simplex in the S_Q−tiling with a simplex in the S_G−tiling.

The T−star of S_Q is the collection of T−topes that have S_Q as a 3−face. The vertices of S_Q are invariant with respect to permuatation of the last four components, but under these actions three additional copies of Q are produced, and, these four T−topes form the T−star of S_Q. The tiles in the intersection with this T−star with R are the intersections of the individual T−topes with R. Similarly, by Corollary xx below, the G−star of S_G includes four G−topes. Correspondingly, the intersection of this G−star with R includes four tiles.

Proposition 29 The intersection of R with the T−star of S_Q is the S_Q−tiling of R: the intersection of R with the G−star of S_G is the S_G−tiling of R. The intersection of an arbitrary G−tope in the T−star of S_G, with an arbitrary T−tope in the G−star of S_G, is one of the simplexes in the barycentric subdivision of R.

By this Proposition R is commensurate with both Q and G, and therefore is commensurate with the two Delaunay tilings $\mathcal{D}_{E_6}$, $\mathcal{D}_{E_6^*}$. Also by this Proposition the intersection polytope $Q \cap G$ is a simplex in the barycentric subdivision of R, once again showing that Q and G are incommensurate. This explicit description shows that $\frac{1}{4}[0,0;1,1,1,1]$ is the only non-integer vertex of the intersection polytope $Q \cap G$.

The triangle Δ_2^1 is a 2−face of the cross-polytope opposite the origin in G, so $\mathbf{c}_{\Delta_2^1} = \mathbf{c}_{T_2} \in \partial G$, and, $T_2 \notin G$. This is an E_6-short triangle because

$edges(\Delta_2^1) = \{\pm[0^2, 1, -1, 0^2], \pm[0^2, 0, 1, -1, 0], \pm[0^2, -1, 0, 1, 0]\} \subset \mathcal{S}_{E_6}$. The triangles Δ_2^2, Δ_2^3 are E_6-long because $edges(\Delta_2^2) = \{\pm[3, 0, 1^3, 2], \pm[-3, 0, -1^4], \pm[0^5, -1]\} \subset \mathcal{L}_{E_6}$, $edges(\Delta_2^3) = \{\pm[1, 2, 0^4], \pm[1, -1, 1^3, 0], \pm[-2, -1^4, 0]\} \subset \mathcal{L}_{E_6}$. It turns out that the T_2 is incommensurate with$G$ and therefore is incommensurate with the Delaunay tiling $\mathcal{D}_{E_6}$. The determining characteristic for such incommensurate T-topes is the product of one E_6-short triangle with two E_6-long triangles.

Equivalence along the line of centers. The forms along the line of centers, $(1-t)\varphi_{E_6} + t\varphi_{E_6^*}$, are invariant with respect to the action of the group $\mathcal{G}_{E_6 \cap E_6^*} = \mathcal{G}_{E_6} \cap \mathcal{G}_{E_6^*}$. Accordingly, the intersection tiling $\mathcal{D}_{E_6} \cap \mathcal{D}_{E_6^*}$ is also invariant with respect to this action. In order to describe this tiling, we must ørst describe the orbits of the $G-$topes, the $T-$topes and the $R-$topes with respect to this action.

The action of $\mathcal{G}_{E_6 \cap E_6^*}$ must leave invariant the short and long vectors of both E_6 and E_6^*, so must leave invariant each of the sets $\mathcal{F}_1 = \mathcal{L}_{E_6} \cap \mathcal{S}_{E_6^*}$, $\mathcal{F}_2 = \mathcal{S}_{E_6} \cap \mathcal{S}_{E_6^*}$, $\mathcal{F}_3 = \mathcal{S}_{E_6} \cap \mathcal{L}_{E_6^*}$, $\mathcal{F}_4 = \mathcal{L}_{E_6} \cap \mathcal{L}_{E_6^*}$. Similarly, the dual action must leave invariant, separately, each of the subsets $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$.

Consider the øve-dimensional cross-polytope opposite the origin in G, with vertices given by

$$[\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3, \mathbf{l}_4, \mathbf{l}_5; \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4, \mathbf{r}_5] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

The four-dimensional cross-polytope with diagonal vectors $\mathbf{l}_i - \mathbf{r}_i, 2 \le i \le 5 i = 2, 3.$that is equatorial to the ørst diagonal diagonaland the diagonal running between the opposite vertices $\mathbf{l}_1 = [0, 1; 0, 0, 0, 0]$, $\mathbf{r}_1 = [2, 0; 1, 1, 1, 1]$. The four-dimensional cross-polytope that is equatorial to this diagonal has vertices given by

of the cross polytope opposite the originPermutation of the last four coordinates leaves each of these sets invariant, so these actions belong to $\mathcal{G}_{E_6 \cap E_6^*}$.

The short complex.

A commensurate tiling with R-topes.

13 R-topes and proof of Main Theorem

14 Scrap Text

Dual forms correspond to dual lattices, so the forms $\vartheta_G^\circ, \vartheta_T^\circ$ that are dual respectively to ϑ_G, ϑ_T, are metrical forms for the lattices E_6, E_6^*. Since the time of (***Kostya) it has been known that the dual forms $\vartheta_G^\circ, \vartheta_T^\circ$ lie on the central axes of the corresponding perfect domains $\mathcal{D}_G$, $\mathcal{D}_T$ [cit.]. The form $\varphi_G(\mathbf{x}) = \frac{1}{24}\sum_{\mathbf{p}\in\mathcal{P}_G}(\mathbf{p}\cdot\mathbf{x})^2$ lies on the central axis of $\mathcal{D}_G$, so must be proportional to ϑ_G°, and therefore is a metrical form for the lattice E_6. The weights $\frac{1}{24}$ are chosen so that φ_G has minimum one, and therefore φ_G is arithmetically equivalent to ϑ_T. That is, there is a unimodular integer matrix $\mathbf{U} \in GL(6,\mathbb{Z})$ so that $\varphi_G(\mathbf{x}) = \vartheta_T(\mathbf{U}\mathbf{x})$. Similarly, $\varphi_T(\mathbf{x}) = \frac{1}{32}\sum_{\mathbf{p}\in\mathcal{P}_T}(\mathbf{p}\cdot\mathbf{x})^2$ lies on the central axis of $\mathcal{D}_T$, is proportional to ϑ_T°, and is a metrical form for the lattice E_6^*. The weights are again chosen so that φ_T has arithmetic minimum equal to one, making φ_T arithmetically equivalent to ϑ_G. In fact, we will show below that $\varphi_T(\mathbf{x}) = \vartheta_G(\mathbf{U}^T\mathbf{x})$.

By direct calculation, the matrices for the forms φ_G, φ_T are given respectively by

$$\frac{1}{2}\begin{bmatrix} 8 & -5 & -5 & -5 & -5 & -5 \\ -5 & 4 & 3 & 3 & 3 & 3 \\ -5 & 3 & 4 & 3 & 3 & 3 \\ -5 & 3 & 3 & 4 & 3 & 3 \\ -5 & 3 & 3 & 3 & 4 & 3 \\ -5 & 3 & 3 & 3 & 3 & 4 \end{bmatrix}, \quad \frac{1}{4}\begin{bmatrix} 16 & -9 & -11 & -11 & -11 & -11 \\ -9 & 6 & 6 & 6 & 6 & 6 \\ -11 & 6 & 10 & 7 & 7 & 7 \\ -11 & 6 & 7 & 10 & 7 & 7 \\ -11 & 6 & 7 & 7 & 10 & 7 \\ -11 & 6 & 7 & 7 & 7 & 10 \end{bmatrix}.$$

If we denote these two matrices by $\mathbf{F}_G, \mathbf{F}_T$, then $\mathbf{F}_G\mathbf{T}_G = \mathbf{F}_T\mathbf{T}_T = \frac{3}{8}\mathbf{I}$, so that $\varphi_G = \frac{3}{8}\vartheta_G^\circ$, and $\varphi_T = \frac{3}{8}\vartheta_T^\circ$. If $\mathbf{U} \in GL(6,\mathbb{Z})$ satisøes the equality $\varphi_G(\mathbf{x}) = \vartheta_T(\mathbf{U}\mathbf{x})$, then $\mathbf{F}_G = \mathbf{U}^T\mathbf{T}_T\mathbf{U}$; an easy calculation shows that $\mathbf{F}_T = \mathbf{U}\mathbf{T}_G\mathbf{U}^T$.

The tiling $\mathcal{T}_G$ and its commensurable long triangles.. Let G be the convex hull of the origin $[0,0,0,0,0,0]$ and the following sets of 16 and 10 lattice points in $\mathbb{Z}^6$

$$\mathcal{S}_G(\mathbf{p}_G) = \{[3;1^5], [-1;-1,0^4]\times 5, [1;1^2,0^3]\times 10\},$$
$$\mathcal{L}_G(\mathbf{p}_G) = \{[0;1,0^4]\times 5, [2;0,1^4]\times 5\},$$

which we associate with the perfect vector $\mathbf{p}_G = [-3,2,2,2,2,2] \in \mathcal{P}_G$. The coordinate vectors $\mathbf{s} \in \mathcal{S}_G(\mathbf{p}_G)$ satisfy the equality $\mathbf{p}_G \cdot \mathbf{s} = 1$, and are short because $\varphi_G(\mathbf{s}) = 1$, which is the minimal possible value for non-zero integer vectors. Also, the coordinate vectors $\mathbf{l} \in \mathcal{L}_G(\mathbf{p}_G)$ satisfy the equality $\mathbf{p}_G \cdot \mathbf{l} = 2$, and are long because $\varphi_G(\mathbf{l}) = 2$, which is the minimal possible value for non-zero integer vectors which are not short. If $\mathbf{s} \in \mathcal{S}_G(\mathbf{p}_G)$ and $\mathbf{p} \in \mathcal{P}_G$, then $\mathbf{p}\cdot\mathbf{s} \in \{0,\pm 1\}$, and if $\mathbf{l} \in \mathcal{L}_G(\mathbf{p}_G)$ and $\mathbf{p} \in \mathcal{P}_G$, then $\mathbf{p}\cdot\mathbf{l} \in \{0,\pm1,\pm2\}$. The following table gives frequency data on these scalar products, for a particular short vector $\mathbf{s} \in \mathcal{S}_G$ and a particular long vector $\mathbf{l} \in \mathcal{L}_G$, as $\mathbf{p}$ ranges over the 54 elements of $\mathcal{P}_G$.

value	−2	−1	0	1	2	total
$\mathbf{p}\cdot\mathbf{s}$	0	12	30	12	0	54
$\mathbf{p}\cdot\mathbf{l}$	2	16	18	16	2	54

By symmetry, this distribution of scalar products holds for any of the 72 short vectors $\mathcal{S}_G$ of the form φ_G, or any of the 270 long vectors $\mathcal{L}_G$. Turning things around, each perfect vector has a scalar product of one with $(12\times72)/54 = 16$ short vectors, and a scalar product of two with $(2\times270)/54 = 10$ long vectors. Hence $\mathcal{S}_G(\mathbf{p}_G) = \{\mathbf{s} \in \mathcal{S}_G | \mathbf{p}_G\cdot\mathbf{s} = 1\}$, and $\mathcal{L}_G(\mathbf{p}_G) = \{\mathbf{l} \in \mathcal{L}_G | \mathbf{p}_G\cdot\mathbf{l} = 2\}$. Similarly, for each perfect vector $\mathbf{p} \in \mathcal{P}_G$ there is a set of 16 short vectors $\mathcal{S}_G(\mathbf{p})$, and 10 long vectors $\mathcal{L}_G(\mathbf{p})$, which together with the origin are the vertices of a copy of G. These 54 polytopes glue together along facets, which are either simplexes or cross polytopes, to form the Delaunay star polytopes at the origin for φ_G.

The equation $f(\mathbf{x}) = -\mathbf{p}_G\cdot\mathbf{x} + \varphi_G(\mathbf{x}) = 0$ determines an ellipsoid $\mathcal{E}_G$ which contains the origin. As can easily be checked, this ellipsoid also contains the other vertices of G since $f(\mathbf{v}) = -\mathbf{p}_G\cdot\mathbf{v} + \varphi_G(\mathbf{v}) = 0$, when $\mathbf{v} \in \mathcal{S}_G(\mathbf{p}_G) \cup \mathcal{L}_G(\mathbf{p}_G)$. Moreover, $\mathcal{E}_G$ is empty, and this supplies a formal proof that G is Delaunay. If $\mathbf{U} \in \mathcal{G}_G$, the empty ellipsoid with equation $f_{\mathbf{U}}(\mathbf{x}) = -(\mathbf{U}^\circ\mathbf{p}_G)\cdot\mathbf{x} + \varphi_G(\mathbf{x}) = 0$ circumscribes $\mathbf{U}G$ ($\mathbf{U}^\circ = (\mathbf{U}^{-1})^T$ is the matrix dual to $\mathbf{U}$).

From the frequency data, each short vector attached to the origin belongs to twelve G—topes in the star. Since only six of these can be translationally equivalent to G, each short lattice vector has precisely six lattice translates that øt inside G. These six positions correspond to six parallel edges of G. Because opposite short vectors correspond to the same set of six parallel edges, G has a total of $6 \times 36 = 216$ edges. It also follows from the frequency data that each long vector at the origin belongs to two G—topes in the star. Since these must be in distinct translation classes, each long vector has only a single translate that øts into G. This position corresponds to the diagonal of a cross-polytope facet. Since the 270 long vectors account for 135 such diagonals, there are a total of $135/5 = 27$ such facets. The edges of these cross-polytopes are short, as are the edges of the 72 simplicial facets.

The 27 cross-polytope facets can be easily located on the reference polytope G. The ten vertices belonging to $\mathcal{L}_G$ are the vertices of such a cross polytope, with center $\frac{1}{2}[2,1,1,1,1,1]$, and with axes parallel to the øve long vectors $\{[2;-1,1,1,1,1] \times 5\}$ (generated by permuting the last øve components). Opposite to each of the other vertices of G is such a cross polytope, which can similarly be located using perfect vectors.

The long vectors also appear as edge vectors of 45 long triangles that are commensurable with the tiling $\mathcal{T}_G$. To be commensurable, a triangle must øt entirely within a G—tope. The triangle Δ, with vertices $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{[0,0,0,0,0,0],[0,1,0,0,0,0],[2,0,1,1,1,1]\}$, is commensurable with $\mathcal{T}_G$, since Δ øts inside G and the edge vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \{[2,-1,1,1,1,1],[-2,0,-1,-1,-1,-1],[0,1,0$ are all long. Moreover, the center of gravity of the triangle Δ is $\frac{1}{3}[2,1,1,1,1,1]$, which is also the center of gravity of G. The trangle Δ corresponds to the two vertices $[0,1,0,0,0,0],[2,0,1,1,1,1]$ that are opposite ends of a diagonal of the cross polytope opposite to the origin, and there are four others corresponding to the other diagonals. Since there are øve attached to the origin, and there are 27 vertices, the total number of long triangles in G is equal to $(27 \times 5)/3 = 45$. These triangles have a common center of gravity, equal to the center of gravity of G, and the convex hull of these triangles is equal to G.

Another interesting point is that these commensurable long triangles are equitorial. That is, for each triangle there is an aŒne map of rank two so that the image of the triangle is equal to the image of G itself. Consider for example, the triangle Δ and the associated perfect vectors $\mathcal{P}_\Delta = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} = \{[3,-2,-2,-2,-2,-2],[-2,2,1,1,1,1],[-1,0,1,1,1,1]\}$. These perfect vectors sum to zero, and each provides two supporting hyperplanes for G. That is, if $\mathbf{x} \in G$ the following inequalities hold, and are sharp.

$$0 \leq 2 + \mathbf{p}_1 \cdot \mathbf{x} \leq 2, \quad 0 \leq \mathbf{p}_2 \cdot \mathbf{x} \leq 2, \quad 0 \leq \mathbf{p}_3 \cdot \mathbf{x} \leq 2$$

Let $\mathbf{A}_\Delta : \mathbb{E}^6 \longrightarrow \mathbb{E}^2$ be the aŒne map given by the formula $\mathbf{A}_\Delta(\mathbf{x}) = [2 + \mathbf{p}_1 \cdot \mathbf{x}, \mathbf{p}_2 \cdot \mathbf{x}, \mathbf{p}_3 \cdot \mathbf{x}]$. A triangular coordinate system is used for the two dimensional image (since $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = 0$ the three coordinates sum to 2). The image of Δ is the triangle with vertices given by $\mathbf{A}_\Delta(\mathbf{v}_1) = [2, 0, 0]$, $\mathbf{A}_\Delta(\mathbf{v}_2) = [0, 2, 0]$, $\mathbf{A}_\Delta(\mathbf{v}_3) = [0, 0, 2]$. The other 24 vertices of G map to the mid-points of the edges of this image triangle, with coordinate vectors given by $[1, 1, 0], [1, 0, 1]$, or $[0, 1, 1]$; each mid-point is the image of eight distinct vertices of G.

Figure?The image $\mathbf{A}_\Delta(G)$

The tiling $\mathcal{T}_T$ and its commensurable long triangles.. Let T be the convex hull of the origin $[0, 0, 0, 0, 0, 0]$ and the following sets of 6 and 2 lattice points in $\mathbb{Z}^6$

$$\begin{aligned} \mathcal{S}_T(\mathbf{p}_{T1}, \mathbf{p}_{T2}) = \{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4, \mathbf{s}_5, \mathbf{s}_6\} =& \{[3, 1, 1, 1, 1, 1], [-1, 0, -1, 0, 0, 0], [-1, 0, 0, -1, 0, 0], \\ & [1, 1, 0, 0, 1, 0], [1, 1, 0, 0, 0, 1], [-1, -1, 0, 0, 0, 0]\} \\ \mathcal{L}_T(\mathbf{p}_{T1}, \mathbf{p}_{T2}) = \{\mathbf{l}_1, \mathbf{l}_2\} =& \{[0, 1, 0, 0, 0, 0], [1, 0, 0, 0, 1, 1]\}, \end{aligned}$$

associated with the two perfect vectors $\mathbf{p}_{T1} = [-3, 2, 2, 2, 2, 2], \mathbf{p}_{T1} = [1, 0, 0, 0, 1, 1]$ in $\mathcal{P}_T$. The coordinate vectors $\mathbf{s} \in \mathcal{S}_T(\mathbf{p}_{T1}, \mathbf{p}_{T2})$ satisfy the equalities $\mathbf{p}_{T1} \cdot \mathbf{s} = \mathbf{p}_{T2} \cdot \mathbf{s} = 1$, and are short because $\varphi_T(\mathbf{s}) = 1$, which is the minimal non-zero value for integer vectors; the coordinate vectors $\mathbf{l}_1, \mathbf{l}_2$ satisfy the equalities $\mathbf{p}_{T1} \cdot \mathbf{l}_1 = \mathbf{p}_{T2} \cdot \mathbf{l}_2 = 2, \mathbf{p}_{T1} \cdot \mathbf{l}_2 = \mathbf{p}_{T2} \cdot \mathbf{l}_1 = 1$, and are long because $\varphi_T(\mathbf{l}_1) = \varphi_T(\mathbf{l}_2) = \frac{3}{2}$, which is the minimal possible value greater than one for integer vectors. The three triangles with verticesWe associate with T the two perfect vectors $\mathbf{p}_T^1 = [0, 0, 1, -1, 0, 0], \mathbf{p}_T^2 = [0, 0, 1, 0, -1, 0]$ in $\mathcal{P}_T$, with the properties that $\mathbf{p}_T^1 \cdot \mathbf{s} = 1$ for $\mathbf{s} \in \mathcal{S}_T$, $\mathbf{p}_T^1 \cdot \mathbf{l}_1 = 2, \mathbf{p}_T^1 \cdot \mathbf{l}_2 = 1$, and $\mathbf{p}_T^2 \cdot \mathbf{s} = 1$ for $\mathbf{s} \in \mathcal{S}_T$, $\mathbf{p}_T^{21} \cdot \mathbf{l}_1 = 1, \mathbf{p}_T^2 \cdot \mathbf{l}_2 = 2$. Because of these properties, the equation $f(\mathbf{x}) = -\frac{1}{2}(\mathbf{p}_T^1 + \mathbf{p}_T^2) \cdot \mathbf{x} + \varphi_T(\mathbf{x}) = 0$ determines an ellipsoid $\mathcal{E}_T$ which circumscribes T. It can also be shown that the only elements of $\mathbb{Z}^6$ contained in $\mathcal{E}_T$ are the nine vertices of T, and that $\mathcal{E}_T$ is empty. Therefore, T is Delaunay with respect to the form φ_T.

The tiles in the Delaunay star can also be generated using the reÆection group E_6. More speciøcally, there is a representation $\mathcal{G}_G$ of E_6 in $GL(6, \mathbb{Z})$

so that the Delaunay star is the $\mathcal{G}_G$—orbit of G. The dual action øxes the set of perfect vectors $\mathcal{P}_G$, and if $\mathbf{U} \in \mathcal{G}_G$, then the perfect vector $\mathbf{U}^\circ\mathbf{p}_G$ is associated with the Delaunay polytope $\mathbf{U}G$ ($\mathbf{U}^\circ = (\mathbf{U}^{-1})^T$ is the matrix dual to $\mathbf{U}$). That is, $\mathbf{p}_{\mathbf{U}G} = \mathbf{U}^\circ\mathbf{p}_G$. The empty ellipsoid $\mathcal{E}_{\mathbf{U}G}$, with equation $-(\mathbf{U}^\circ\mathbf{p}_G)\cdot\mathbf{x} + \varphi_G(\mathbf{x}) = 0$, circumscribes the polytope $\mathbf{U}G$.

Theorem 30 The action of the group $\mathcal{G}_S$ on the long triangle $\mathbf{l}_0+\mathbf{l}_1+\mathbf{l}_2 = 0$ generates 45 long triangles and $270 = 6 \times 45$ long vectors.

Proof. Since $(\mathcal{G}_S)^\circ$ acts transitively on $\mathcal{P}_S$, there are $5 \times 27/3 = 45$ perfect triangles in $\mathcal{P}_S$. Permutation of the last øve coordinates leaves $\mathcal{P}_S$ invariant, so these actions belong to $(\mathcal{G}_S)^\circ$. These permutations also act transitively on the triangles containing $\mathbf{p}_0$, so the 45 triangles in $\mathcal{P}_S$ lie in a single $(\mathcal{G}_S)^\circ$—orbit. It follows that the direct action of $\mathcal{G}_S$ generates 45 long triangles.

Since each long vector has a scalar product of 2 with only two perfect vectors, and these belong to a dual perfect triangle, distinct long triangles have no long vectors in common. Therefore the number of long vectors generated is $2 \times 3 \times 45 = 270$. (The factor 2 is added for the sign.) ■

Consider the long triangle ctors., a lattice triangle can be constructed at the origin by ørst following $\mathbf{p}_0$, then $-\mathbf{p}_1$ and then $-\mathbf{p}_2$. By permuting the vectors of the tripple $\{\mathbf{p}_0, -\mathbf{p}_1, -\mathbf{p}_2\}$ øve other triangles can be traced through the origin, which together with the original form the familiar hexagonal pattern of six triangles in the linear space spanned by these vectors. There are øve such linear dependencies formed from the columns of $\mathbf{P}_S(\mathbf{p}_0)$, the equalities $\mathbf{p}_0-\mathbf{p}_i-\mathbf{p}_{i+1}=0, i = 1, 3, ..., 9$. It is easy to see that there are no other triangles in $\mathcal{P}_S$ that include $\mathbf{p}_0$.

Since the group $\mathcal{G}_S$ acts transitively on $\mathcal{P}_S$ each perfect vector is included in øve triangles, and there are a total of $(27 \times 5)/3 = 45$ triangles in $\mathcal{P}_S$.

Theorem 31 The group $\mathcal{G}_S$ act transitively on the triangles in $\mathcal{P}_S$.

Proof. The stability group $\mathcal{G}_S(\mathbf{p}_0)$ of $\mathbf{p}_0$ is the set of elements $\mathbf{U} \in \mathcal{G}_S$ with dual action satisfying the equality $\mathbf{U}^\circ\mathbf{p}_0 = \mathbf{p}_0$. Since an arbitrary permutation of the last øve coordinates øxes $\mathbf{p}_0$, the symmetric group $\mathcal{S}_5 \subset \mathcal{G}_S(\mathbf{p}_0)$. In addition, the matrix

$$\mathbf{U}^{\circ} = \begin{bmatrix} -1 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & 0 \\ -1 & \cdot 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}^{\circ} = \begin{bmatrix} -1 & 0 & 0 & -1 & -1 & -1 \\ 2 & 0 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

øxes $\mathbf{p}_0$ and $\mathcal{P}_S$ so $\mathbf{U} \in \mathcal{G}_S(\mathbf{p}_0)$. leaves permutes the ørst and second, and third and fourth columns of $\mathbf{P}_S(\mathbf{p}_0)$, øxing the rest, and $\mathbf{p}_0$, and leaves rthe It is clear that this action must permute the columns of $\mathbf{P}_S(\mathbf{p}_0)$. ■

the set $triangle$ sin $theset$the sethat i-the ,fectors tripple. formed, each corresponding to a distinct ordering of the vectors of the tripple . These form the familiar hexagonal pattern in the linear space spanned by the tripple., sin ceai, The columns are the ten ordered elements of $\mathcal{P}_S$ that are sides of lattice triangles that have $\mathbf{p}_0$ as a side. For example, starting at the origin and following , traces out a lattice triangle in $\mathbb{Z}^6$ since ; the vectors $\mathbf{p}_1, \mathbf{p}_2$ are the ørst two columns of $\mathbf{P}_S(\mathbf{p}_0)$. , in ter the three vectorsthe This tripple accounts for 6 lattice triangles at the origin, the number of distinct closed pathse aere are ten triangles at the origin of this type, two for each of the equalities $\mathbf{p}_0 - \mathbf{p}_i - \mathbf{p}_{i+5} = 0, i = 1, 5$;The columns $\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_{10}$ are grouped so that $\mathbf{p}_i + \mathbf{p}_{i+1} - \mathbf{p}_0 = 0, i = 1, 3, ..., 9$.In total there are øve such equalities that can be formed, the equalities $\mathbf{p}_0 - \mathbf{p}_1 - \mathbf{p}_2 = 0$

$$\mathbf{L}_S(\mathbf{p}_0) = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

there is a doubling because the return path depends on the order in which the two vectors $\mathbf{p}_i, \mathbf{p}_{i+5}$ are followed.

The ten columns $\mathbf{l}_1, \mathbf{l}_2, ..., \mathbf{l}_{10} \in \mathbf{L}_S(\mathbf{p}_0)$ each satisfy the equality $\mathbf{l}_i \cdot \mathbf{p}_0 = 2$. We will refer to these vectors as long.

Theorem 32 Deøne the stability group $\mathcal{G}(\mathbf{p}_0)$ of $\mathbf{p}_0$ to be set of elements $\mathbf{U} \in \mathcal{G}_S$ with dual action satisfying the equality $\mathbf{U}^{\circ}\mathbf{p}_0 = \mathbf{p}_0$. Then the direct action $\mathbf{l}_i \to \mathbf{U}\mathbf{l}_i, \mathbf{U} \in \mathcal{G}(\mathbf{p}_0)$ øxes the set $\mathcal{L}_S(\mathbf{p}_0) = \{\mathbf{l}_1, \mathbf{l}_2, ..., \mathbf{l}_{10}\}$, and acts transitively on this set.

.is a consequence of the two ways to return to the originresults in the two ways the vectors $\mathbf{p}_i, \mathbf{p}_{i+5}$ can be ordered.six . triangle inP There are in total 10 trianglesAlso consider the matrix of scalar products $\mathbf{M}_S^{+1} = (\mathbf{P}_S^{+1})^T \mathbf{L}_S^{+1}$.p

$$\mathbf{M}_S^{+1} = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

. of forms and each of these forms is uniquely determined by the inhomogeneous systems of equations $\vartheta_S(\mathbf{p}) = 1, \mathbf{p} \in \mathcal{P}_S, \vartheta_T(\mathbf{p}) = 1, \mathbf{p} \in \mathcal{P}_T$. The matrices for these two forms are given by In addition, any form ϑ satisfying the homogeneous system $\vartheta(p) = 0, p \in \mathcal{P}_S \cap \mathcal{P}_T = C$ is necessarily a scalar multiple of the form $\vartheta_S - \vartheta_T$. This being the case, form... These sets are adjacent because the intersection $\mathcal{P}_S \cap \mathcal{P}_T = C$ vectors common to the two sets

14.1 To be continued

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