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Note

On the generalized Erdős–Szekeres Conjecture — a new upper bound

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Abstract

We prove the following result: For every two natural numbers n and q, $n \geqslant q + 2$, there is a natural number $E(n,q)$ satisfying the following:

(1) Let S be any set of points in the plane, no three on a line. If $|S| \geqslant E(n,q)$, then there exists a convex n-gon whose points belong to S, for which the number of points of S in its interior is $0 \pmod q$.

(2) For fixed q, $E(n,q) \leqslant 2^{c(q)\cdot n}$, $c(q)$ is a constant depends on q only.

Part (1) was proved by Bialostocki et al. [2] and our proof is aimed to simplify the original proof. The proof of Part (2) is completely new and reduces the huge upper bound of [2] (a super-exponential bound) to an exponential upper bound.

1. Introduction

In their classical paper [3] Erdős and Szekeres proved:

Theorem A. *For every integer $n \geqslant 3$ there is an integer $f(n)$ such that if S is any set of $f(n)$ points in the plane, no three on a line. Then there are n points of S forming a convex n-gon.*

The determination of $f(n)$, $n \geqslant 6$, as well as the determination of the order of magnitude of $f(n)$ are still open problems. The best known bounds for $f(n)$ are [5]

$$2^{n-2} + 1 \leqslant f(n) \leqslant \binom{2n-4}{n-2} + 1$$

and it is a famous conjecture of Erdős that the lower bound is in fact equal to $f(n)$. It was conjectured for a while (some 20 years ago) that for every $n \geqslant 3$ there exists a $g(n)$

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such that if S is any set of $g(n)$ points in the plane, no three on a line. Then there are n points of S forming a convex n-gon whose interior is empty. This was proved by Harborth [6] for $n = 3, 4, 5$, but was disproved by Horton [7] for $n \geqslant 7$. Motivated by the positive result of Erdős and Szekeres and the negative result of Horton [7], Bialostocki et al. [2] raised the following weaker conjecture (see e.g. [1, 2]).

Conjecture 1. For every two positive integers n and q, $n \geqslant 3$, there is a natural number $C(n, q)$ satisfying the following: Let S be any set of $C(n, q)$ points in the plane, no three on a line. Then there are n points of S forming a convex n-gon whose interior contains $0 \pmod q$ points of S.

They proved conjecture 1 whenever $n \geqslant q + 2$ showing that

$$C(n, q) \leqslant f(R_3(\underbrace{n, n, \ldots, n}_{q\text{-times}})),$$

where $f(n)$ is the Erdős–Szekeres function and

$$R_3((\underbrace{n, n, \ldots, n}_{q\text{-times}})),$$

is the Ramsey number for the complete 3-uniform hypergraph K_n^3 using q colors (a large number indeed). Let us consider the following more general setting, already anticipated in [1, 2]. Suppose S is a set of points in the plane, no three on a line, and suppose G is a finite abelian group. Assume further that each point x of S is labeled with an element of G say $w(x) = g \in G$. We say that a convex n-gon K has a zero-sum interior (mod G) if

$$\sum_{x \in \text{interior } K} w(x) = 0 \quad (\text{in } G).$$

Thus Conjecture 1 deals with the case $G = Z_q$, $w(x) \equiv 1$, $x \in S$. We shall prove the following.

Theorem 1. *For every two integers n and q, $n \geqslant q + 2$, there is an integer $E(n, q)$ satisfying the following:*

(1) *Let S be a set of points in the plane, no three on a line, and let G be an abelian group of order q. Assume $w: S \to G$ and $|S| \geqslant E(n, q)$. Then there are n points of S forming a convex n-gon having a zero-sum interior.*

(2) *For a fixed q, $E(n, q) \leqslant 2^{c(q) \cdot n}$, $c(q)$ depends only on q but not on n or on the structure of G.*

2. The proof of Theorem 1

Proof of Part (1). The proof combines ideas from [2] and some simple but useful observations. Suppose first $n > q$, $n \equiv 2 \pmod q$. Let $E(n, q) = f(R(K_{n-1}, q) + 1)$

where f is the Erdős–Szekeres function and $R(K_{n-1},q)$ is the Ramsey number for K_{n-1} using q colors. Recall that in [1] the authors used $R(K_n^3,q)$ which is much larger. By Theorem A there are $R(K_{n-1},q)+1$ points of S forming a convex polygon. Pick an arbitrary point of this polygon and mark it by z. Number all the other points from 1 to $R(K_{n-1},q)$. Define the following coloring on the pairs (i,j), $1 \leqslant i < j \leqslant R(K_{n-1},q)$, $w(i,j) = \sum w(x)$: x is inside the triangle (i,j,z). Since $|G| = q$ this is a coloring of pairs using q colors and by the definition of the Ramsey numbers there are $n-1$ points say $x_1, x_2, \ldots, x_{n-1}$ such that $w(x_i,x_j) = g \in G$ for every choice of $1 \leqslant i < j \leqslant n-1$.

In particular we may assume that $x_1, x_2, \ldots, x_{n-1}$ appear in this order on the boundary of the polygon and hence $z, x_1, x_2, \ldots, x_{n-1}$ is a convex n-gon K (in that order).

Claim. *All the triangles in K are monochromatic of the same color (have the same element of G as a sum of the points in their interior).*

Indeed we may consider only two types of triangles.

(1) a triangle of type (x_i,x_j,z), but all of them have (because of Ramsey) the color $g \in G$.

(2) a triangle of type (x_i,x_j,x_k), $i<j<k$. But then consider the quadruple $A = (x_i,x_j,x_k,z)$. Clearly

$$w(A) = \sum_{x \in \text{interior } A} w(x) = w(x_i,x_j,z) + w(x_j,x_k,z)$$
$$= w(x_i,x_k,z) + w(x_i,x_j,x_k).$$

Clearly this implies $g+g = g + w(x_i,x_j,x_k)$ hence $w(x_i,x_j,x_k) = g$ as claimed. Now as $n \equiv 2 \pmod q$, the convex n-gon K is covered by exactly $0 \pmod q$ triangles, namely (z,x_i,x_{i+1}), $i = 1,2,\ldots,n-2$. Hence

$$\bar{w}(K) = \sum_{x \in \text{interior } K} w(x) = \sum_{i=1}^{n-2} w(z,x_i,x_{i+1}) = \underbrace{g+g+\cdots+g}_{n\text{-}2 \text{ times}} = 0 \quad (\text{in } G),$$

because $n-2 \equiv 0 \pmod q$, $|G| = q$ and recall Lagrange's theorem. So K has a zero-sum interior and $|K| \equiv 2 \pmod q$. Suppose now $n \geqslant q+3$, and write $n = n_0 + t$ where $n_0 \equiv 2 \pmod q$ and $1 \leqslant t \leqslant q-1$. Once again we will show $E(n,q) \leqslant f(R(K_{n-1},q)+1)$. Indeed, as above, there exists a convex n-gon K all of whose triangles are monochromatic (have the same interior sum in G).

Let K be listed clockwise, $x_1, x_2, \ldots, x_n$, and let the fixed sum of the interior points of a triangle in K be $g \in G$. If $g = 0$ then clearly $w(K) = 0$ and we are done. Hence assume $g \neq 0$ which means that each triangle contains a point from S in its interior (since by convention $w(\phi) = 0$). Consider the convex n_0-gon, $n_0 = n - t$ given by the points $B = \{x_1, x_3, x_5, \ldots, x_{2t-1}, x_{2t+1}, x_{2t+2}, x_{2t+3}, \ldots, x_{t+n_0-1}, x_{t+n_0} = x_n\}$. Since

$n_0 \equiv 2 \pmod q$), $w(B) = 0$. Consider the following t points $y_1, y_2, \ldots, y_t$ where y_i is inside the triangle $x_{2i-1}, x_{2i}, x_{2i+1}$ and has the property that the triangle x_{2i-1}, y_i, x_{2i+1} contains no point of S. The point y_i may be selected as the point of S in the triangle $x_{2i-1}, x_{2i}, x_{2i+1}$ closest to the line segment x_{2i-1}, x_{2i+1}. Clearly $w(x_{2i-1}, y_i, x_{2i+1}) = w(\phi) = 0$, hence the convex n-gon given by $\{x_1, y_1, x_3, y_2, x_5, y_3, \ldots, x_{2t-1}, y_t, x_{2t+1}, x_{2t+2}, \ldots, x_n\}$ has a zero-sum interior as needed. This completes the proof of Part (1).

Proof of Part (2). Recall first the old theorem of Erdős–Ginzburg and Ziv [3]: Let G be a finite abelian group of order q, and let $g_1, g_2, \ldots, g_{2q-1}$ be a sequence of $2q-1$ elements of G. Then there exists $I \subset \{1, 2, \ldots, 2q-1\}$, $|I| = q$ such that $\sum_{i \in I} g_i = 0$. Recall also that by Part (1) we have proved $E(n,q) \leqslant f(R(K_{n-1}, q) + 1)$ and we shall use this bound for $q + 2 \leqslant n \leqslant 3q - 2$. Now suppose $n \geqslant 3q - 1$ and write $n = n_0 + tq$ where $2q - 1 \leqslant n_0 \leqslant 3q - 2$. We will show that for fixed q, $E(n,q) \leqslant f([R(K_{n_0-1}, q) + 1](t+1)q)$. Indeed by the Erdős–Szekeres theorem there exists a convex polygon of size $[R(K_{n_0-1}, q) + 1](t+1)q := z$. Label the points by $x_0, x_1, \ldots, x_{(t+1)q-1}, x_{(t+1)q}, \ldots, x_{2(t+1)q-1}, x_{2(t+1)q}, \ldots, x_{z-1}$. Observe that $\{x_0, x_{(t+1)q}, x_{2(t+1)q}, \ldots, x_{R(K_{n_0}-1,q)(t+1)q}\}$ form a convex polygon A of size $R(K_{n_0-1}, q) + 1$, hence by Part (1) must contain a convex n_0-gon B with a zero-sum interior. Say the points of this n_0-gon B are $y_1, y_2, \ldots, y_{n_0}$ and observe that for each i, y_i and y_{i+1} are separated by at least tq points of the original polygon, and further $n_0 \geqslant 2q - 1$. Pick for each $1 \leqslant i \leqslant n_0$ a point $x_i \in A \backslash B$, x_i is between y_i and y_{i+1} and define $w_i = w(y_i, x_i, y_{i+1}) = \sum w(x)$: $x \in$ interior (y_i, x_i, y_{i+1}). Since $n_0 \geqslant 2q - 1$ we have a sequence $w_1, w_2, \ldots, w_{n_0}$ of elements in G and by the Erdős–Ginzburg–Ziv theorem there is a subsequence of cardinality q which sum to 0. Say the points chosen are $x_1, \ldots, x_q$ then $B^* = y_1 x_1 y_2 x_2 \ldots y_q x_q y_{q+1} \ldots y_{n_0}$ is a convex $(n_0 + q)$-gon with zero-sum interior. Clearly by the construction this argument can be repeated at least t times giving lastly a convex n-gon with zero-sum interior as claimed. Thus $E(n,q) \leqslant f([R(K_{3q-3,q}) + 1](t+1)q) \leqslant f(c_1(q) \cdot n) \leqslant 2^{c(q) \cdot n}$ using the upper bound for the Erdős–Szekeres function

$$f(n) \leqslant \binom{2n-4}{n-2} + 1.$$

This completes the proof of Part (2). $\square$

3. Concluding remarks

(1) The accumulated experience with zero-sum problems strongly indicated that the upper bound $E(n,q) \leqslant f(c(q)n)$ is still too large and that we can expect an upper bound of the form $E(n,q) \leqslant f(c(q) + n)$. This would be the case once we can ensure in the proof of Part (2) that a convex polygon with $R(K_{n_0-1}, q) + 1 + (t+1)q$ points

suffices to guarantee a convex n_0-gon with zero-sum interior whose adjacent points are separated by tq points of the original polygon. Such an argument eluded me. On the other hand the results of Horton [7] imply that $E(n,q) \geqslant n+q$ for $n \geqslant 7$ and every q. So $E(n,q) \gg f(n)$ if q is sufficiently large and hence some dependence on q must occur.

(2) The assumption that G is a finite abelian group is not essential. Indeed one can easily modify the proof to the case when G is an arbitrary finite group, using a theorem of Olson [8] instead of the Erdős–Ginzburg–Ziv theorem.

(3) Certainly the most challenging problem now is to prove the existence of $E(n,q)$ in the case $3 \leqslant n \leqslant q+1$.

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