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Maximum planar sets that determine k distances

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Abstract

Maximum planar sets that determine k distances are identified for $k \leqslant 5$. Evidence is presented for the conjecture that all maximum sets for $k \geqslant 7$ are subsets of the triangular lattice.

1. Introduction

Let $g(k)$ denote the maximum number of points in the Euclidean plane that determine exactly k different distances. Clearly, $g(1) = 3$, which is realized only by the vertices of an equilateral triangle. We determine $g(k)$ for each $k \leqslant 5$ and identify all $g(k)$-point planar sets that have exactly k interpoint distances for $k \leqslant 4$. We also present evidence for larger k that supports the following conjecture.

Conjecture 1. For every $k \geqslant 3$, some $g(k)$-point subset of the triangular lattice

$$L_{\triangle} = \{a(1,0) + b(1/2, \sqrt{3}/2)\colon a, b \in \mathbb{Z}\}$$

has exactly k interpoint distances. Moreover, if $k \geqslant 7$, every $g(k)$-point subset of the plane that determines k different distances is similar to a subset of $L_{\triangle}$.

Two configurations are *similar* if one can be mapped into the other by rotation about a point, reflection about a line, translation and uniform rescaling. We use $k \geqslant 3$ in the first part of the conjecture because $g(2)$ is realized only by the vertices of a regular pentagon. Avoidance of $k = 6$ in the latter part of Conjecture 1 is explained by

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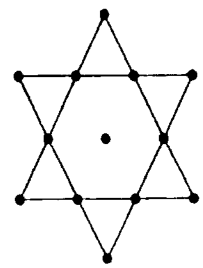


Conjecture 2. $g(6) = 13$, and this is realized only by the vertices of a regular 13-gon, or the center and the vertices of a regular 12-gon, or the preceding subset of $L_\triangle$.

This is the only configuration in $L_\triangle$ with more than 12 points and less than seven distances. We prove below that $g(5) = 12$. Hence, every 13-point subset of the plane determines at least six distances. Conjecture 2 asserts that every 14-point set has at least seven interpoint distances.

Erdős [3] considered the minimum number $f(n)$ of different distances determined by n points in the plane. By our definitions, $f(g(k)) \leqslant k$ with equality if $g(k-1) < g(k)$. The presently-best bounds on f are

$$n^{4/5}/(\log n)^c \leqslant f(n) \leqslant cn/(\log n)^{1/2}.$$

The lower bound is from Chung et al. [2]. The upper bound was shown by Erdős to follow from a square subset (side length $\sqrt{n}$) of the integer lattice

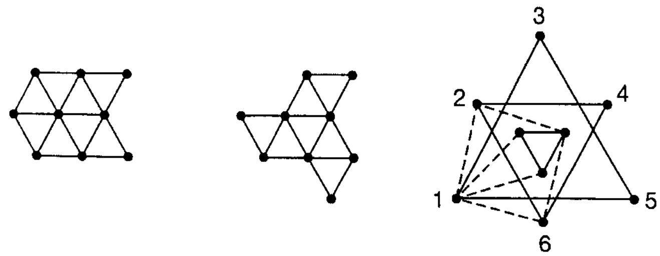
$$L_\square = \{a(1,0) + b(0,1)\colon a, b \in \mathbb{Z}\}.$$

The same upper bound, perhaps with a different constant c, can be proved with $L_\triangle$. Evidence presented below for Conjecture 1 suggests that $c_\triangle < c_\square$.

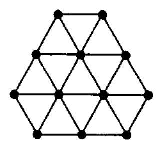
Let R_n denote the vertices of a regular n-gon, and let R_n^+ be R_n augmented by the center of the n-gon. We observe that R_6^+ is a seven-point set that is similar to a subset of $L_\triangle$. Fig. 1 identifies three other subsets of $L_\triangle$ involved in our main theorem along with a set not in $L_\triangle$. The last of these is composed of three equilateral triangles with the same center and a horizontal edge. The smallest distance applies to the sides of the inner triangle and from a vertex of it to the nearest vertex of the intermediate triangle. The next-larger distance is illustrated by the dashed lines, four of which form a square. The diagonal of the square, or a side of the intermediate triangle, has the next-to-largest distance. The largest distance applies to the sides of the big triangle and from a vertex of it to the farthest vertex of the intermediate triangle.

Theorem 1. $g(2) = 5$, $g(3) = 7$, $g(4) = 9$ *and* $g(5) = 12$. R_5 *is the only* 5-*point set with exactly two interpoint distances; the only* 7-*point sets that determine three distances are* R_7 *and* R_6^+; *a* 9-*point set with exactly four distances must be* R_9 *or one of the configurations at the top of Fig.* 1; *one* 12-*point set that determines five distances is the configuration in* $L_\triangle$ *at the bottom of Fig.* 1.

Can this say something about the oriented matroid?



Three 9-point configurations that determine 4 distances



The only known 12-point configuration for 5 distances

Fig. 1.

We suspect that our example for $g(5) = 12$ is unique. The complexity of proving this is discussed in Section 4.

The top right configuration on Fig. 1 is a curiosity in that it is the only verified or conjectured realizer of a $g(k)$ that is not an R_n or R_n^+ or subset of $L_\triangle$. As k gets larger, R_n and R_n^+ drop out of contention since we can always do better with a subset of $L_\triangle$. We say more about this in Section 5, where we also compare $L_\square$ to $L_\triangle$. The next three sections present our proof of Theorem 1, and Section 6 concludes the paper with a brief discussion.

2. Proof approach

The examples of Theorem 1 give $g(2) \geqslant 5$, $g(3) \geqslant 7$, $g(4) \geqslant 9$, and $g(5) \geqslant 12$. We assume these inequalities henceforth.

Let $d(x, y)$ denote the distance between $x, y \in \mathbb{R}^2$, and let $D = D(S)$ be the diameter of finite $S \subseteq \mathbb{R}^2$. Also let

$$S_D = \{x \in S\colon d(x, y) = D \text{ for some } y \in S\}.$$

We organize our proof for each k around possibilities for S_D when S has specified cardinality. We recall that two length-D segments in S must cross if they do not share an end point, and that there are at most $|S|$ such segments. Two further facts will be used extensively.

Lemma 1. *Let D be the diameter of an n-point planar set S with $n \geqslant 3$, and let $m = |S_D|$, so $2 \leqslant m \leqslant n$. Then*

(a) *if* $m \geqslant 3$, *the points in* S_D *are the vertices of a convex* m*-gon*;

(b) D *can be eliminated as an interpoint distance by removing at most* $\lceil m/2 \rceil$ *points from* S.

Proof. (a) Suppose $m \geqslant 3$, $d(x, y) = D$ for $x, y \in S$, and x is not a vertex of the convex hull of S_D. Let $[p, q]$ be a side of the convex hull such that either $x \in [p, q]$ or the extension of $[x, y]$ in the direction y to x intersects $[p, q]$. Then either $d(p, y) > D$ or $d(q, y) > D$, a contradiction.

(b) The result is obvious if $m \leqslant 3$. Given $m \geqslant 4$, let A and B be sets of $\lceil m/2 \rceil$ consecutive vertices of the m-gon of (a) such that $A \cup B = S_D$. If each of $A \backslash B$ and $B \backslash A$ has a length-D segment, we obtain the contradiction that two length-D segments with different end points do not cross. □

We will use Lemma 1(a) when m is large relative to a value n proposed for $g(k)$, and proceed with the convex m-gon. Smaller m use Lemma 1(b) to reduce the number of interpoint distances from k to at most $k - 1$ by removals from S. The next lemma is applied to case (a). We let $R_n - r$ for $0 \leqslant r \leqslant n - 3$ denote a set of $n - r$ vertices of R_n. When $r \geqslant 2$, dissimilar versions of $R_n - r$ obtain when different combinations of r vertices are removed from R_n.

Lemma 2. *Suppose* S *is the vertex set of a convex* n*-gon*, $n \geqslant 3$, *that determines exactly* t *different distances. Then* $t \geqslant \lfloor n/2 \rfloor$. *Moreover*:

(i) *if* n *is odd and* $t = (n-1)/2$, S *is* R_n;

(ii) *if* n *is even*, $t = n/2$, *and* $n \geqslant 8$, S *is* R_n *or* $R_{n+1} - 1$;

(iii) *if* $(n, t) = (4, 2)$, S *is one of* R_4, $R_5 - 1$, *the vertices of two equilateral triangles that share a side, and a set similar to* $\{1, 3, 4, 5\}$ *on Fig.* 1;

(iv) *if* $(n, t) = (6, 3)$, S *is one of* R_6, $R_7 - 1$, *and a set similar to* $\{1, 2, 3, 4, 5, 6\}$ *on Fig.* 1;

(v) *if* $(n, t) = (7, 4)$, S *is* $R_8 - 1$ *or an* $R_9 - 2$;

(vi) *if* $(n, t) = (9, 5)$, S *is* $R_{10} - 1$ *or an* $R_{11} - 2$.

Inequality $t \geqslant \lfloor n/2 \rfloor$, conjectured in [3], is proved in [1] along with Lemma 2(i). Parts (ii)–(v) are proved in [5], and (vi) is proved in [4].

To illustrate our approach, consider $k = 2$ with $|S| = 5$. Let $m = |S_D|$. If $m = 5$, Lemmas 1(a) and 2(i) give $S = R_5$. Suppose $m \leqslant 4$. Using Lemma 1(b), eliminate D by removing two points. This leaves R_3 by the result for $k = 1$. Let 1 be its side length. A point added to R_3 that restores $D > 1$ must be on a perpendicular bisector of a side at distance 1 from the side's vertices. However, the addition of two such points forces a third distance $> D$. Thus $m \leqslant 4$ cannot occur, and R_5 is the only 5-set with exactly two distances. It is impossible to add a point to R_5 without creating a third distance, so $g(2) < 6$ and the proof for $k = 2$ is complete.

We conclude this section with the proof of Theorem 1 for $k = 3$. Assume $k = 3$, $|S| = 7$, and let $m = |S_D|$. If $m = 7$, Lemmas 1(a) and 2(i) give $S = R_7$. If $m = 6$, Lemmas 1(a) and 2(iv) imply that S_D is R_6, $R_7 - 1$, or a configuration like

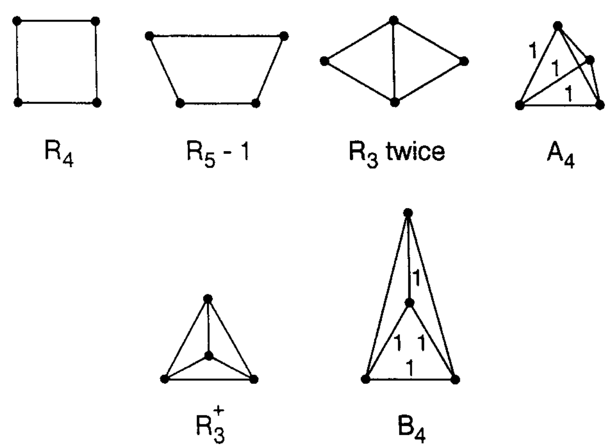


Fig. 2.

{1, 2, ..., 6} on Fig. 1. It is easily checked that the only way to add another point to one of these 6-sets so that the new point creates no new distance and does not have distance D to another point is to add the center to R_6. Thus, if $m \geqslant 6$, S is R_7 or R_6^+.

Suppose $m \leqslant 4$. By Lemma 1(b) and Theorem 1 for $k = 2$, D can be eliminated by removing two points, leaving R_5. But it is not possible to add two points to R_5 in any way without forcing at least two new distances. Hence, $m \leqslant 4$ cannot occur when $k = 3$ and $|S| = 7$.

This leaves $m = 5$. By Lemma 1(b), we remove three points to eliminate D and yield a 4-set that determines two distances. Fig. 2 shows the possibilities. Its four quadrilaterals are specified in Lemma 2(iii). The others require a fourth point in an isosceles triangle and can only be as shown at the bottom of the figure.

The question for Fig. 2 is whether three points can be added to one of its 4-sets so that the three create only one new distance D with D greater than the other two, and such that exactly two of the original four points have a D distance to an added point with $m = 5$ overall. The answer, obtained by examining potential placements on perpendicular bisectors of segments of the 4-sets, is no. The only real contenders are 'R_3 twice' and R_3^+, where we are forced to add three that complete R_6^+. But R_6^+ has $m = 6$, not $m = 5$.

We conclude that S is R_7 or R_6^+ when $k = 3$ and $|S| = 7$. It is impossible to add a point to R_7 or R_6^+ without creating a new distance, so $g(3) < 8$ and the proof for $k = 3$ is complete.

3. Proof for $k = 4$

This section identifies all 9-sets that determine 4 distances. It is easily seen that any other point added to a determined set forces a new distance, so $g(4) = 9$.

Assume that $k = 4$ and $|S| = 9$. Let $m = |S_D|$. If $m = 9$, Lemmas 1(a) and 2(i) imply $S = R_9$. If $m = 8$, Lemmas 1(a) and 2(ii) give R_8 or $R_9 - 1$ for S_D. A new point (e.g. center) added to R_8 forces a new distance, and a new point added to $R_9 - 1$ that preserves $m = 8$ forces a new distance. Hence, $m = 8$ is impossible when $k = 4$. Suppose $m = 7$. Lemmas 1(a) and 2(i) and (v) imply that S_D is R_7, $R_8 - 1$, or an $R_9 - 2$. The only point that can be added to R_7 without producing at least two new distances is the center. If the center is added to $R_8 - 1$ or $R_9 - 2$, a fifth distance appears, and other additions that preserve $m = 7$ force new distances. Hence $S = R_9$ if $m \geqslant 7$.

Suppose $m \leqslant 6$, so D can be eliminated by removing one, two or three points. The remove-one case is impossible since $g(3) < 8$. Suppose D is eliminated by removing two points. Then, by Theorem 1 for $k = 3$, the remaining 7-set is R_7 or R_6^+. Only R_6^+ needs further consideration. Let R_6^+ be the seven inner points in the star of Conjecture 2. Then the only feasible D-inducing additions are the six outer points. We can use only two adjacent outer points, else a fifth distance occurs. The resulting 9-set is shown on the upper left of Fig. 1.

Finally, suppose three points must be removed to eliminate D, so $m \in \{5, 6\}$ by Lemma 1(b). This leaves six points that determine three distances. If the six form a convex hexagon, we have R_6, $R_7 - 1$ or a set similar to $\{1, 2, \ldots, 6\}$ of Fig. 1: see Lemma 2(iv). If R_6 obtains, we can make only two additions (see R_6^+ in the preceding paragraph) since the center cannot be part of S_D, thus falling one short of the desired nine points, and neither $R_7 - 1$ nor $\{1, 2, \ldots, 6\}$ allows the desired additions. For $\{1, 2, \ldots, 6\}$ of Fig. 1, every addition on a perpendicular bisector of a segment of $\{1, 2, \ldots, 6\}$ that duplicates old distances and qualifies for a new greater distance introduces at least two new distances. Hence, no new configurations for $g(4) = 9$ arise in the remove-three case when what remains forms a convex hexagon.

Suppose henceforth for the remove-three case that the six remaining points do not form a convex hexagon. Let T be the set of the six remaining points with diameter $E < D$, and let T_E be the subset of T involved with distance E. By Lemma 1(a), $|T_E| \leqslant 5$. We consider subcases for elimination of E.

Subcase 1: E can be eliminated by removing one point from T. Then the remaining five points must be R_5. However, any point added to R_5 on a perpendicular bisector of a side or chord of R_5 that duplicates the shorter R_5 distance while giving a new longest distance E must in fact yield two new distances.

Subcase 2: E can be eliminated only by removing three of T's points. By Lemma 1(b), this implies $|T_E| = 5$. By Lemma 1(a), the points in T_E are the vertices of a convex pentagon. If T_E determines only two distances, it is R_5, and T can only be R_5^+. But then it is impossible to add three more points to produce only one more distance $D > E$ for the 9-set. Suppose T_E determines three distances. There are 15 convex pentagons with this property. They are shown in Fig. 2 in [6]. Of these 15, seven (denoted by $P_3, P_4, P_6, P_8, P_{11}, P_{14}, P_{15}$) have all five vertices at ends of E segments. But none of those seven accommodates an internal point that has only the two shorter distances to the vertices of the pentagon. Hence, subcase 2 does not yield a 9-set with four distances.

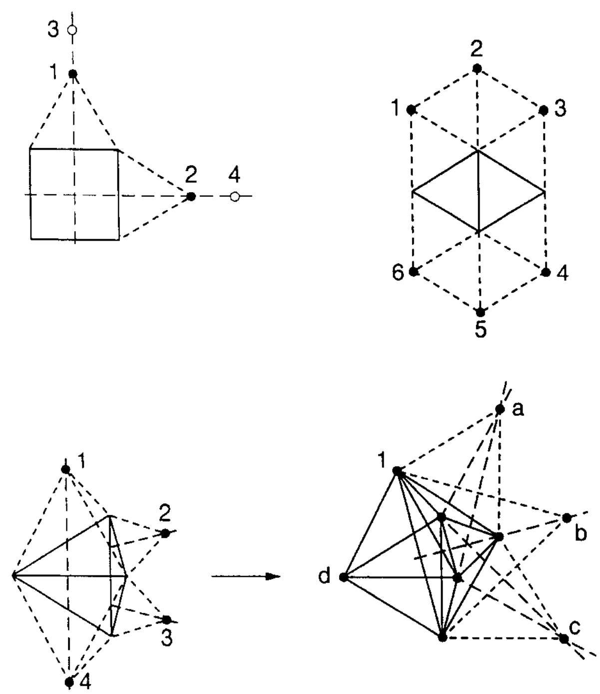


Fig. 3.

Subcase 3: E can be eliminated by removing two vertices but not one vertex from T. Then the remaining 4-set has two distances, so it is one of the sets in Fig. 2. In each case, we try to add two points on perpendicular bisectors of the six line segments of the 4-set so that the additions determine only the original two distances ($d_1 > d_2$) and one new greater distance $E > d_1$ that arises for each addition independently. We avoid convex hexagons here since they were considered above. After the two additions for E, we consider three more additions that determine $D > E$ and no other new distance. This second step avoids R_6^+, which was analyzed previously. We consider each 4-set in turn.

(3.1): R_4. Potential additions for E are shown on the top left of Fig. 3. Points 1 and 2 are d_2 from the nearest corner of the square, and 3 and 4 are d_1 from the nearest corner. Similar potential additions occur to the left of and below the square, but two additions off opposite sides are infeasible since they create a fourth distance. The only feasible pair of additions is $\{1,2\}$ because $\{1,4\}$ and $\{3,4\}$ force a fourth distance. Given $\{1,2\}$ to complete our six-point set with R_4, only two more additions are possible for D, namely the similar points to 1 and 2 to the left of and below the square. Hence, R_4 does not produce a 9-set with $k = 4$ under the present restrictions.

(3.2): $R_5 - 1$. There is no feasible pair for E.

(3.3): R_3 twice, a part of $L_\triangle$. Potential additions for E are shown on the top right of Fig. 3. There are two dissimilar pairs of additions for E, $\{1,2\}$ and $\{1,6\}$. Each resulting 6-set has further lattice points as potential additions for D. There is only one 3-point addition for D that avoids R_6^+. It is pictured as the middle diagram on the top of Fig. 1.

(3.4): A_4. The four potential additions for E are shown on the bottom left of Fig. 3. They form equilateral triangles with points in A_4, and 1, 2, and the bottom left points in A_4 form a square. Up to similarity, $\{1,2\}$ is the only feasible addition pair for E, which is the length of the diagonal of the square. Feasible additions for D to $A_4 \cup \{1,2\}$ are shown on the bottom right of Fig. 3. Collectively, $\{a,b,c\}$ adds only one new distance, which is $D = ad = bd = cd = ac$. The result is shown on the upper right of Fig. 1.

(3.5): R_3^+, a part of $L_\triangle$. We obtain the result of (3.3).

(3.6): B_4. An upside down version of B_4 appears in the lower middle of the final diagram on Fig. 3. There are two feasible pairs for E, $\{1,d\}$ and $\{1,b\}$. The only feasible additions for D complete the diagram.

This completes our analysis when three points must be removed to eliminate D, so the proof of Theorem 1 for $k = 4$ is complete.

4. Proof for $k = 5$

We are to prove that $g(5) < 13$. Comments on the difficulty of determining all 12-sets that determine five distances appear at the end of the section.

We suppose that some S with $|S| = 13$ determines only five distances and obtain a contradiction. Let $m = |S_D|$. By Lemmas 1(a) and 2, $m \geqslant 12$ is impossible. If $m \in \{9, 10, 11\}$ then Lemmas 1(a) and 2(i), (ii) and (vi) imply that S_D is R_9 with four distances or one of $R_{10}, R_{10} - 1$, $R_{11}, R_{11} - 1$, and $R_{11} - 2$ with five distances. Additions that bring the total number of points to 12 or more force a sixth distance, so a contradiction obtains when $m \geqslant 9$.

Suppose $m \leqslant 8$. By Lemma 1(b), removal of four points eliminates D. The resulting 9-set has four distances, so it is either R_9 or one of the sets on the top of Fig. 1. Two or more additions to R_9 force at least two more distances. If the 9-set is one of the top two subsets of $L_\triangle$ on Fig. 1, feasible additions for D as the fifth distance lie at adjacent lattice points. In either case, the only way to add *three* points and not force a sixth distance is shown on the bottom of Fig. 1. If another point is added to bring the total to 13, we contradict $k = 5$. Finally, every plausible D addition to the 9-set on the upper right of Fig. 1 forces at least two new distances. Hence, $m \leqslant 8$ also allows no 13-point realization for $k = 5$, so $g(5) < 13$.

The preceding analysis applied to $|S| = 12$ shows that the 12-set of Fig. 1 is the only 12-set that determines five distances when $m \geqslant 9$ or $m \leqslant 6$. Difficulties arise when $m \in \{7, 8\}$ and four points must be removed to eliminate D. Of the (a) and (b) approaches with Lemma 1, (b) seems more tractable. That route leaves an 8-set with 4 distances. The family of all 8-sets that determine four distances includes R_7^+, the convex octagons of Lemma 2(ii), and every 8-point subset of the three 9-sets on Fig. 1.

But it may include other realizations, so we presently have no guarantee that the 12-set of Fig. 1 is the only realizer of $g(5) = 12$.

5. Lattices

Fig. 4 shows maximum or near-maximum subsets of $L_\triangle$ that determine k distances for $k \in \{7, 8, 9, 10, 11, 13\}$. We omit $k = 12$ because we have no example that exceeds the 27 points at $k = 11$. It might be true that $g(12) = g(11)$. The counts on the figure and straightforward extensions show that an R_n or R_n^+ never does as well as a subset of $L_\triangle$ when $k \geqslant 7$.

Table 1 compares $L_\triangle$ and $L_\square$. We use regular hexagonal arrays of $L_\triangle$ with s points on a side, $n = 6\binom{s}{2} + 1$ total points, and $k \leqslant s^2 - 1$ distinct distances. If $\langle i, j \rangle$ represents the distance obtained from moving i units in one direction followed by j units in a direction 60° from the first in the direction of travel, then $\langle 7, 0 \rangle = \langle 5, 3 \rangle$, $\langle 9, 1 \rangle = \langle 6, 5 \rangle$, and so forth. We use square arrays of $L_\square$ with s points on a side, $n = s^2$, and $k \leqslant (s + 2)(s - 1)/2$ distinct distances. If $[i, j]$ is the distance

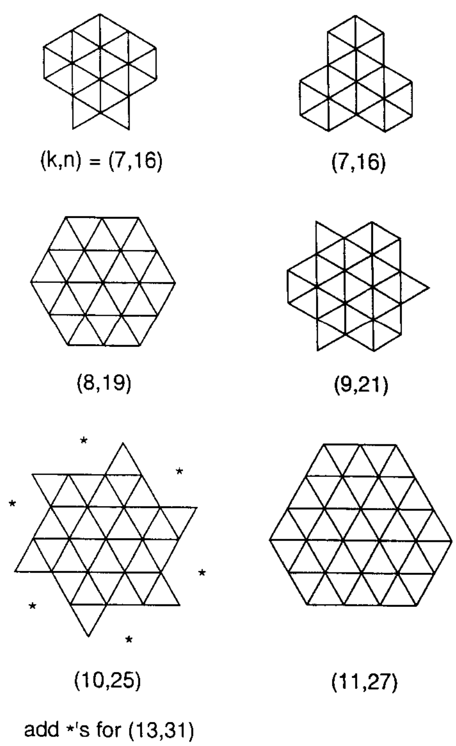


Fig. 4.

Table 1
Numbers of distinct distances k and points n determined by regular hexagonal subsets of $L_\triangle$ with s points on a side, and by square subsets of $L_\square$ with s points on a side.

$L_\triangle$			$L_\square$					
n	k	s	n	k	s	n	k	s
7	3	2	4	2	2	361	160	19
19	8	3	9	5	3	400	177	20
37	15	4	16	9	4	441	194	21
61	23	5	25	13	5	484	212	22
91	34	6	36	19	6	529	228	23
127	46	7	49	25	7	576	248	24
169	59	8	64	32	8	625	268	25
217	74	9	81	40	9	676	288	26
271	90	10	100	49	10	729	309	27
331	109	11	121	58	11	784	331	28
397	129	12	144	69	12	841	352	29
469	150	13	169	80	13	900	377	30
547	173	14	196	91	14	961	400	31
631	197	15	225	104	15	1024	425	32
721	223	16	256	118	16	1089	451	33
817	250	17	289	130	17	1156	474	34
919	280	18	324	146	18	1225	501	35
1027	312	19						
1141	345	20						
1261	382	21						

obtained from moving i units in one direction followed by j units in the perpendicular direction, then $[5,0] = [4,3]$, $[7,1] = [5,5]$, and so forth. The k values in the table account for all such duplications.

We see that $L_\triangle$ is substantially better than $L_\square$ in the n/k ratios. For approximately equal n, k for $L_\triangle$ is about 26% smaller than k for $L_\square$, and this figure is quite robust over values of $n \geqslant 100$ in the table. We do not claim that our choices of arrays are optimal, but it seems unlikely that other near-optimal choices would change matters by much.

6. Discussion

We have identified subsets of the plane for small k that determine k distances and have as many points as possible. Our results in conjunction with limited information about larger k values suggest that the maximum sets that determine k distances for $k \geqslant 7$ must be similar to subsets of the triangular lattice.

Several local problems in addition to Conjecture 2 have arisen. One is whether there is a unique 12-set that determines exactly five distances. Another is whether any subset of $L_\triangle$ has more than 27 points and no more than 12 distances.

The latter problem raises the question of whether $g(k) = g(k + 1)$ for some k. If so, is $g(k) = g(k + 1)$ for an infinite number of k? Let $\Delta g(k) = g(k + 1) - g(k)$. The average $\Delta g(k)$ is 2.25 for $k \leqslant 4$ and appears from $L_\triangle$ on Table 1 to be about 3.3 for larger k's shown there. Could it be true that $\Delta g(k) \to \infty$?

For small n, $f(n + 1) \leqslant f(n) + 1$. Is this true for all n? Perhaps there is a nice proof.

A refinement of Conjecture 1 asks whether the regular hexagonal subset of $L_\triangle$ with s points on each side is a maximum set for the k distances thus determined and, if so, is it the only maximum set for that k when $s \geqslant 3$.

Finally, we note that all verified maximum sets for k have the property that some point has all k distances to the others. Is this generally true? A similar result does not hold for f because there is an 8-point set for $f(8) = 4$ in which every point has only three distances to the others.

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