CONVEX BODIES AND ALGEBRAIC GEOMETRY

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During the last decade a new area of research has developed relating two subjects which until now had very little in common: convexity and algebraic geometry. An initial success was Stanley's solution of the so-called upper bound conjecture for combinatorial spheres ([16]; for polytopes solved by McMullen, published in [14]): Among all combinatorial spheres with v vertices (faces are convex polytopes), the convex hulls of v points on a moment curve $\{(t, t^2, \ldots, t^n) | t \in \mathbb{R}\}$ (called cyclic polytopes) possess maximal numbers of faces in all dimensions. This result has been pursued further by Kind and Kleinschmidt [10]. Another result is the solution of McMullen's conjecture about characterizing those vectors $f(P) = (f_0(P), \ldots, f_{d-1}(P))$ for which $f_j(P)$ is the number of j-faces of a polytope P. The necessity of McMullen's condition has been shown by Stanley [17] and the sufficiency by Billera and Lee [2].

A foundation for Stanley's, Billera's, and Lee's work was laid by Hochster [8]. This paper by Hochster was also one of the starting points for a development which is to be outlined in what follows. We emphasize two main achievements: First, a characterization of Milnor's number of critical points of complex algebraic functions by numbers assigned to convex polytopes (Kouchnirenko [11]); second, characterizations of mixed volumes of convex bodies as the intersection index of certain varieties, and an alternative proof of the Alexandrov–Fenchel inequality (Bernstein [1], Burago and Zalgaller [4], Teissier [19]). In order to make the exposition comprehensible to nonspecialists, we restrict the discussion to elementary explanations.

1

We first recall some basic facts from the theory of convex bodies (see [3, 6, 7, 12, 14]). A compact convex subset K of $\mathbb{R}^n$ will be called a *convex body*. If $K = \text{conv}\,(M)$ is the convex hull of a finite set of points M (or equivalently, provided it is bounded, the intersection of finitely many closed half spaces) it is called a *convex polytope*. The faces of a convex polytope are again convex polytopes; they form a cell complex of dimension $n - 1$. The cells of dimension 0 are called *vertices* of K. If, in addition, all vertices are lattice points, K is said to be a *lattice polytope*.

Every convex body K is the limit, with respect to the Hausdorff metric, of a sequence $\{P_j\}$ of convex polytopes P_j. The surface measure (area) and volume of the P_j then lead to the surface measure and volume of K, respectively. Linear combinations $\lambda_1 K_1 + \cdots + \lambda_r K_r$ of convex bodies $K_1, \ldots, K_r$ are defined by

$$\lambda_1 K_1 + \cdots + \lambda_r K_r := \{\lambda_1 x_1 + \cdots + \lambda_r x_r | x_1 \in K_1, \ldots, x_r \in K_r\},$$

resulting again in convex bodies. We restrict our attention to nonnegative λ_i. For the volume of a linear combination, one shows:

$$V(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum \lambda_{i_1}, \ldots, \lambda_{i_n} V(K_{i_1}, \ldots, K_{i_n}),$$

where the sum is taken over all nonnegative $i_1, \ldots, i_n$. The coefficient $V(K_{i_1}, \ldots, K_{i_n})$ is called the *mixed volume* of $K_{i_1}, \ldots, K_{i_n}$. In particular, $V(K, \ldots, K) = V(K)$ is the volume of K. The surface measure of K equals $nV(K, \ldots, K, B)$, where B is a unit ball in $\mathbb{R}^n$. Most mixed volumes, however, do not have an obvious geometrical meaning.

A number of inequalities for mixed volumes are known; for example, the *Minkowski inequalities*

$$V(K, \ldots, K, K')^n \geq V(K)^{n-1} V(K'),$$

$$V(K, \ldots, K, K')^2 \geq V(K) V(K, \ldots, K, K', K'),$$

which hold for arbitrary convex bodies K, K'. These inequalities play a decisive role in the solution of the isoperimetric problem. A generalization of the second inequality is the *Alexandrov–Fenchel inequality*:

$$V(K_1, \ldots, K_{n-1}, K_n)^2 \geq V(K_1, \ldots, K_{n-1}, K_{n-1}) V(K_1, \ldots, K_{n-2}, K_n, K_n).$$

2

We consider a ring R of functions $f: \mathbb{C}^n \to \mathbb{C}$ of the form

$$f(z) = \sum_{\rho \in I} c_\rho z^\rho,$$

where $z := (z_1, \ldots, z_n)$, $\rho := (\rho_1, \ldots, \rho_n)$, $z^\rho := z_1^{\rho_1} \ldots z_n^{\rho_n}$, $c_\rho \in \mathbb{C}$. ($\mathbb{C}$ can be replaced by any algebraically closed field.) If $I \subset \mathbb{Z}_+^n$ and if I is finite, R is the ring of polynomials $\mathbb{C}[z_1, \ldots, z_n]$ in n variables. If $I \subset \mathbb{Z}^n$ and I is finite, we obtain the ring $\mathbb{C}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]$ of Laurent polynomials. If only $I \subset \mathbb{Z}_+^n$ is required, $R = \mathbb{C}[[z_1, \ldots, z_n]]$ is the ring of formal power series.

The *support* supp f of f is given by $\{\rho | c_\rho \neq 0\}$. If I is finite, we call the lattice polytope

$$\mathcal{N}\omega(f) := \text{conv (supp } f)$$

the *Newton polytope* of f. One key to understanding what follows is the fact that many properties of the functions f or varieties defined by them depend only on $\mathcal{N}\omega(f)$ and not on supp f.

Let Δ be a lattice polytope such that $0 \notin \text{aff}\,\Delta$ (affine hull). We consider the cone $\sigma_\Delta := \{tx | t \in \mathbb{R}_+ \setminus \{0\}, | x \in \Delta\}$ and set

$$P_\Delta := \sigma_\Delta \cap \mathbb{Z}^n.$$

Then P_Δ is a semigroup of lattice vectors (FIGURE 1). It represents the subring R_Δ of $\mathbb{C}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]$ consisting of all Laurent polynomials whose support lies in P_Δ. The use of semigroups P_Δ is a second key to understanding what follows: transformation of algebraic or algebraic-geometric facts into combinatorial-geometric ones.

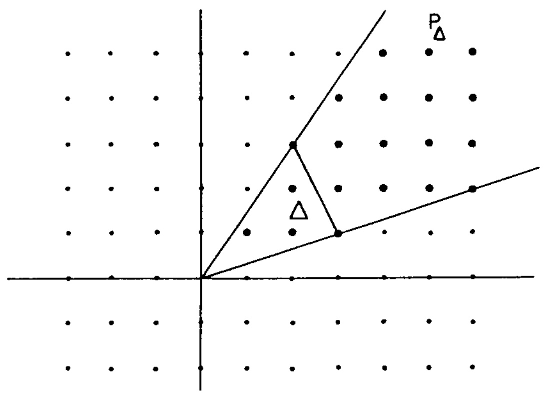

FIGURE 1

In a sense, the correspondence $R_\Delta \to P_\Delta$ is a logarithmic transformation. How it can be applied has been thoroughly studied by Hochster [8], Kempf *et al.* [9], Danilov [5], and others. Any time algebraic-geometric properties about singularities or blowups can be defined in local analytic coordinates by monomials, one may have a chance to apply the combinatorial-geometric technique of exponent semigroups. In [9], for example, this is done in a proof of a theorem about the existence of semistable reductions. The nucleus of the proof is a (55 page) proof of variants of the following combinatorial theorem about lattice points:

Let P be a lattice polytope of dimension n in $\mathbb{R}^n$. Then there exist a $v \in N$ and a subdivision of P into finitely many simplices T_j, such that for all j:

(a) *Every vertex of T_j lies in $(1/v)\mathbb{Z}^n$.*
(b) *The volume of T_j equals $1/v^n n!$.*

3

Let us discuss in more detail a geometric characterization of Milnor's number of an isolated critical point of an algebraic set defined by a polynomial equation $f(z) = 0$ (Kouchnirenko [11]). What is Milnor's number? In order to understand its geometric meaning let us first look at an example.

$$f(z_1, z_2) = z_1^p + z_2^q = 0, \qquad p, q \in N\backslash\{0, 1\}, \tag{1}$$

defines in $\mathbb{C}^2$ or $\mathbb{R}^4$ an algebraic set V. Both partial derivatives vanish at 0; f has an isolated critical point there. In order to investigate the critical point further, we consider the 3-sphere S_ε about 0 with radius $\varepsilon > 0$. The intersection $C := S_\varepsilon \cap V$ is a knotted curve or a link of several curves. $S_\varepsilon\backslash V$ can be considered a fiber space by setting

$$\phi: S_\varepsilon\backslash V \to \mathbb{C}; \qquad \phi(z) := \frac{f(z)}{|f(z)|}. \tag{2}$$

A fiber $F_\theta := \phi^{-1}(e^{i\theta})$ is a surface homotopically equivalent to a bouquet $S^1 \vee \cdots \vee S^1$ of $\mu(f)$ circles. We now replace **(1)** by any polynomial equation

$$f(z) := f(z_1, \ldots, z_n) = 0, \qquad n \geq 2, \tag{3}$$

such that 0 is again an isolated critical point of the algebraic set V defined by **(3)**. Let S_ε be a $(2n-1)$-sphere about 0 whose radius $\varepsilon > 0$ is so small that inside S_ε there is no critical point $\neq 0$ and such that $f^{-1}(0)$ is a transversal to $S_{\varepsilon'}$ for all $0 < \varepsilon' < \varepsilon$. The map **(2)** provides a fibration of $S_\varepsilon\backslash V$, the fiber being homotopic to a bouquet $S^{2n-3} \vee \cdots \vee S^{2n-3}$ of $\mu(f)$ spheres. $\mu(f)$ is called *Milnor's number* of the isolated critical point 0 of f (see [13]).

It is possible to calculate $\mu(f)$ directly from f (Palamodov [15]):

$$\mu(f) = \dim_{\mathbb{C}} \mathbb{C}[[z_1, \ldots, z_n]] \Big/ \left(\frac{\partial f}{\partial z}\right),$$

where $(\partial f/\partial z)$ denotes the ideal generated by all partial derivatives of f. In example **(1)**, $\mu(f) = (p-1)(q-1)$ is easily obtained.

Following the work of Kouchnirenko [11], let us look at the Newton polytope $\mathcal{N}\omega(f)$ of f. We call f *permissible* if $\mathcal{N}\omega(f)$ touches all coordinate axes. The union of all faces Δ of $\mathcal{N}\omega(f)$ which are "visible" from 0, that is, for which $\mathcal{N}\omega(f) \cap \operatorname{conv}(\{0\} \cup \Delta) = \Delta$, 0 not in the affine hull of Δ, is called *Newton's boundary* $\Gamma(f)$ of f (FIGURE 2). f is called *nondegenerate* on $\Gamma(f)$ if $(z_1(\partial f/\partial z_1), \ldots, z_n(\partial f/\partial z_n))$ restricted to $\Gamma(f)$ is $\neq 0$ in $(\mathbb{C}\backslash\{0\})^n$.

We set $\Gamma_-(f) := \bigcup \operatorname{conv}(\{0\} \cup \Delta)$ for all $\Delta \subset \Gamma(f)$. Let V_n be the n-dimensional volume of $\Gamma_-(f)$, and let

$$V_k := \sum \operatorname{vol}_k(\Gamma_-(f) \cap U^k),$$

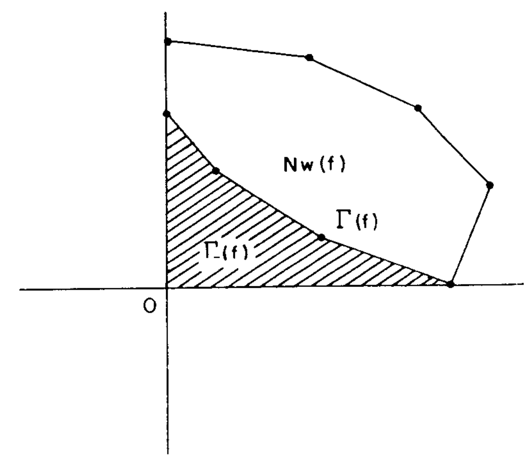

FIGURE 2

where the sum is taken over all k-dimensional coordinate subspaces U^k, $k < n$. We define

$$\nu(f) := n!\,V_n - (n-1)!\,V_{n-1} \pm \cdots + (-1)^{n-1}1!\,V_1 + (-1)^n$$

to be *Newton's number.*

THEOREM 1 (Kouchnirenko [11]). *Let f be a permissible polynomial, and let* (3) *have 0 as an isolated critical point. Then*
(a) $\mu(f) \geq \nu(f)$,
(b) $\mu(f) = \nu(f)$ *iff f is nondegenerate on* $\Gamma(f)$.

It should be noted that (a) and (b) depend only on $\mathcal{N}\omega(f)$, not on supp f.

A related result is obtained as follows. Let $\tilde{\mu}(f)$ denote the sum of all Milnor numbers of f. Then

$$\tilde{\mu}(f) = \dim_{\mathbb{C}} \mathbb{C}[z_1, \ldots, z_n] \Big/ \left(\frac{\partial f}{\partial z}\right).$$

We set $\tilde{\Gamma}_-(f) := \operatorname{conv}(\{0\} \cup \mathcal{N}\omega(f))$ and denote by $\tilde{\Gamma}(f)$ the union of all faces of $\tilde{\Gamma}_-(f)$ that do not contain 0. If, in the definition of $\nu(f)$, we replace $\Gamma_-(f)$, $\Gamma(f)$ by $\tilde{\Gamma}_-(f)$, $\tilde{\Gamma}(f)$, respectively, we obtain a number $\tilde{\nu}(f)$.

THEOREM 2 (Kouchnirenko [11]). *Under the assumptions of* THEOREM 1 *we have*
(a) $\tilde{\mu}(f) \leq \tilde{\nu}(f)$,
(b) $\tilde{\mu}(f) = \tilde{\nu}(f)$ *iff f is nondegenerate on* $\tilde{\Gamma}(f)$.

In the above example (1), the Newton polytope $\mathcal{N}\omega(f)$ is the line segment joining $(p, 0)$ and $(0, q)$. We find $V_2 = \frac{1}{2}pq$, $V_1 = p + q$; hence, $\nu(f) = pq - p - q + 1 = (p-1)(q-1)$. Since f is nondegenerate on $\Gamma(f) = \tilde{\Gamma}(f)$, we obtain $\mu(f) = (p-1)(q-1) = \tilde{\mu}(f)$. ($f$ only has 0 as a critical point.)

Kouchnirenko has extended THEOREM 2 to Laurent polynomials. The proof of all three theorems makes use of the theory of graded rings, the Cohen–Macaulayness of the rings R_A (proved by Hochster [8]), and work with Koszul complexes.

4

We consider a system of Laurent polynomial equations in n variables

$$\begin{aligned} f_1(z) &= 0 \\ &\vdots \\ f_n(z) &= 0. \end{aligned} \tag{4}$$

By the *number of typical solutions* $\ell(f_1, \ldots, f_n)$ we mean the total number of solutions for almost all systems (4) in the following sense. Let n be the total number of lattice points that occur in at least one f_i. We consider solutions that occur in an open, everywhere dense set of the coefficient space $\mathbb{C}^n$; hereby $\mathcal{N}\omega(f_1), \ldots, \mathcal{N}\omega(f_n)$ is the same set of polytopes. Furthermore, we choose only solutions in $(\mathbb{C}\setminus\{0\})^n$.

THEOREM 3 (Bernstein [1]). *The typical number of solutions of* (4) *equals $n!$ times the mixed volume of the Newton polytopes of the f_i:*

$$\ell(f_1, \ldots, f_n) = n!\,V(\mathcal{N}\omega(f_1), \ldots, \mathcal{N}\omega(f_n)). \tag{5}$$

Again, there is a dependence only on $\mathcal{N}\omega(f_j)$, not on supp f_j, $j = 1, \ldots, n$.

EXAMPLE. Let two quadratic equations be given:

$$\begin{aligned} f_1(z_1, z_2) &= a_1 z_1^2 + a_2 z_2^2 + a_3 = 0, \\ f_2(z_1, z_2) &= b_1 z_1^2 + b_2 z_2^2 + b_3 = 0 \end{aligned}$$

all coefficients being $\neq 0$. The typical number of solutions is obviously 4 (FIGURE 3).

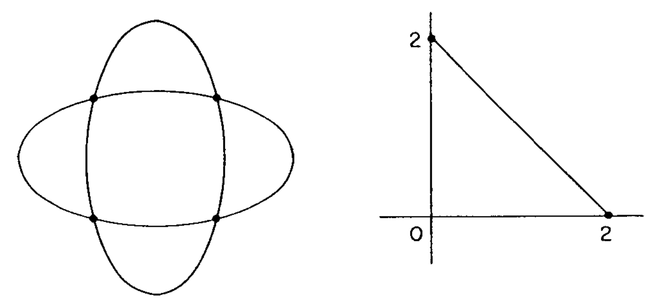

FIGURE 3

On the other hand, $\mathcal{N}\omega(f_1) = \mathcal{N}\omega(f_2)$ is the triangle with vertices (0, 0), (2, 0), and (0, 2). We have $V(\mathcal{N}\omega(f_1), \mathcal{N}\omega(f_2)) = V(\mathcal{N}\omega(f_1)) = 2$ (area of the triangle). Hence, $2!\,V(\mathcal{N}\omega(f_1), \mathcal{N}\omega(f_2)) = 4$.

Bernstein's proof of THEOREM 3 makes use of Puiseux series. Meanwhile, there is an alternative proof. Burago and Zalgaller mention in their book "Geometric Inequalities" [4] the possibility of proving (5) by characterizing the functionals on both sides of (5) uniquely by certain properties. Such properties have been found independently by W. J. Firey and by P. McMullen (unpublished).

When (5) *is established it is possible, by elementary arguments, to prove the Alexandrov–Fenchel inequality (see Section 2) for lattice polytopes and, by an approximation argument, extend the proof to arbitrary convex bodies.*

[**Note added in proof:** Meanwhile, the elementary proof by Phedotow published in [4] turned out to be false.]

5

A further characterization of mixed volumes and a proof of the Alexandrov–Fenchel inequality use Hodge Theory. Both have been achieved by Teissier [18, 19].

Let $K_1, \ldots, K_r$ be lattice polytopes in $\mathbb{R}^n$, $n \geq 2$. We consider the support function H of $K := K_1 + \cdots + K_r$ (Minkowski sum):

$$H(u) := \max_{x \in K} u \cdot x.$$

H is piecewise linear and convex. The same is true for the support functions H_i of K_i, $i = 1, \ldots, r$.

Let Δ be a face of K, and let p be a point in the relative interior of Δ. We denote the cone of "outer normals" in p by σ_Δ, that is,

$$\sigma_\Delta := \{u - p \mid \|u - p\| \leq \|u - q\| \text{ for all } q \in K\}.$$

σ_Δ only depends on Δ, not on p. It is readily seen that $\{\sigma_\Delta\}$, for $\Delta \neq \phi$, being a face of K, is a decomposition of $\mathbb{R}^n$ such that no two σ_Δ have a relative interior point in common. We denote the convex dual cone of σ_Δ by $\check{\sigma}_\Delta$:

$$\check{\sigma}_\Delta := \{x \in \mathbb{R}^n \mid u \cdot x \geq 0 \text{ for all } u \in \sigma_\Delta\}.$$

$\check{\sigma}_\Delta$ has dimension n.

$$R_\Delta := \mathbb{C}[\check{\sigma}_\Delta \cap \mathbb{Z}^n]$$

defines a subalgebra of Laurent polynomials (see Section 2). We consider the sub-R_Δ-module $L_{i\Delta}$ of $\mathbb{C}(\mathbb{Z}^n)$ (FIGURE 4) generated by

$$L'_{i\Delta} := \{x \in \mathbb{Z}^n \mid u \cdot x \geq H_i(u) \text{ for all } u \in \sigma_\Delta\}.$$

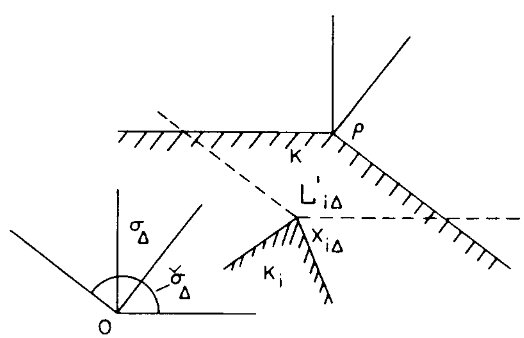

FIGURE 4

We can describe $L'_{i\Delta}$ geometrically as follows. An elementary property of convex polytopes is that each face Δ of K is the Minkowski sum of faces Δ_i of K_i,

$$\Delta = \Delta_1 + \cdots + \Delta_r.$$

We choose a vertex $m_{i\Delta}$ of Δ_i. Then $L'_{i\Delta}$ is obtained from $\check{\sigma}_\Delta \cap \mathbb{Z}^n$ by a translation: $L'_{i\Delta} = (\check{\sigma}_\Delta \cap \mathbb{Z}^n) + m_{i\Delta}$. Adding any vector of $\check{\sigma}_\Delta \cap \mathbb{Z}^n$ to $L'_{i\Delta}$ maps $L'_{i\Delta}$ into itself; therefore, $L_{i\Delta}$ is a sub-R_Δ-module of $\mathbb{C}(\mathbb{Z}^n)$. Furthermore, any vector of $L'_{i\Delta}$ is representable as $y = x + m_{i\Delta}$, where $x \in \check{\sigma}_\Delta \cap \mathbb{Z}^n$; so $z^{m_{i\Delta}}$ generates $L_{i\Delta}$.

Using sheaf-theoretic means one can glue up the affine varieties Spec $\mathbb{C}[\check{\sigma}_\Delta \cap \mathbb{Z}^n]$ (points, irreducible curves, irreducible surfaces, etc., represented by prime ideals) to a compact, normal, integral, and rational algebraic variety X. Hereby one takes advantage of the complex structure of $\{\sigma_\Delta\}$. Also the $L_{i\Delta}$ with any fixed i glue up together into an invertible sheaf of fractional ideals L_i. Let $L_j^{\nu_j}$ denote the ν_j-fold tensor product of L_j. As is known from sheaf theory, there is a polynomial expression for the coherent Euler-characteristic of $L_1^{\nu_1} \otimes \cdots \otimes L_r^{\nu_r}$:

$$\chi(X, L_1^{\nu_1} \otimes \cdots \otimes L_r^{\nu_r}) = \sum_{\substack{\alpha \in N^r \\ |\alpha| = n}} \frac{1}{\alpha_1! \cdots \alpha_r!} S_\alpha \nu_1^{\alpha_1} \cdots \nu_r^{\alpha_r}$$

$$+ \text{ polynomial of degree } \leq n - 1.$$

It is now readily shown that

$$S_\alpha = n!\, V(\underbrace{K_1, \ldots, K_1}_{\alpha_1}, \ldots, \underbrace{K_r, \ldots, K_r}_{\alpha_r}).$$

This is another characterization of mixed volumes. *The Alexandrov–Fenchel inequality now follows from the Hodge Index Theorem* (Teissier [17]), according to which the (topological) index of a Kähler manifold M coincides with an alternating sum of complex cohomology group dimensions of M.

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