From Ewald:

2.1 Linear transforms, Gale transforms, and applications

We turn now to a method which, in a way, dualizes problems of fans and polytopes. It is useful in the solution of classification problems and in finding the structure of the combinatorial Picard group. Two further applications will be presented here.

Let $X = (x_1, ..., x_v)$ be a sequence of (not necessarily different) vectors in an n-dimensional vector space U, and let $x_1, ..., x_v$ span U. We consider a v-dimensional vector space V and a basis $b_1, ..., b_v$ of V. Then there is a well-defined linear map

$$L_1 : V \to U$$

for which $L_1(b_i) = x_i, i = 1, ..., v$. We introduce a third vector space W and a linear map L_2 such that the sequence

$$0 \to W \overset{L_2}{\to} V \overset{L_1}{\to} U \to 0$$

is exact. Then the dual sequence

$$0 \leftarrow W^* \overset{L_2^*}{\leftarrow} V^* \overset{L_1^*}{\leftarrow} U^* \leftarrow 0$$

is also exact.

Let $b_1^*, ..., b_v^*$ be the basis of V^* dual to $b_1, ..., b_v$.

2.1.1 Definition We set $\bar{x}_i := L_2^*(b_i^*), i = 1, , v$, and call the finite sequence $\bar{X} := (\bar{x}_1, ..., \bar{x}_v)$ a *linear transform* of X.

By dualizing twice we find:

2.1.2 Lemma *If $\bar{X}$ is a linear transform of X, then X is a linear transform of $\bar{X}$.*

From the definitions we readily deduce:

2.1.3 Lemma *a) If $L_U : U \to U$ is a bijective linear map, then $\bar{X}$ is also a linear transform of $L_U(X)$*

b) If $L_{W^*} : W^* \to W^*$ is a bijective linear map, then $L_{W^*}(\bar{X})$ is a linear transform of X as well.

2.1.4 Definition We set $\mathcal{L}(X) := ker L_1 \subset V = span\{b_1, ..., b_v\}$ and call $\mathcal{L}(X)$ the space of linear dependencies of X. It is convenient to write $\alpha \in \mathcal{L}(X)$ as a column vector $\alpha = (\alpha_1, ..., \alpha_v)^t$ (with respect to the basis $b_1, ..., b_v$). α is called an *affine dependency* (of X) if $\alpha_1 + ... + \alpha_v = 0$.

2.1.5 Lemma $(\alpha_1, \ldots, \alpha_v)' \in \mathcal{L}(X)$ *if and only if there exists a vector* $a \in W$ *such that*

$$\alpha_i = \bar{x}_i(a) \quad \textit{for } i = 1, \ldots, v.$$

Proof Ew II.4.8.

As a further analysis shows (see, for example, Ew II, sections 4 and 5), linear transformss may be calculated as follows. If the sequence $X = (x_1, \ldots, x_v)$ is given and spans the vector space U, find a basis $\alpha^{(1)}, \ldots, \alpha^{(v-n)}$ of the space $\mathcal{L}(\mathcal{X})$ of linear dependencies of X. Then the columns of the matrix

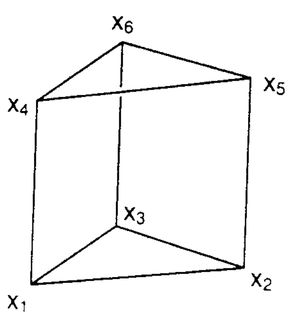
$$A = \begin{pmatrix} \alpha^{(1)} \\ \vdots \\ \alpha^{(v-n)} \end{pmatrix}$$

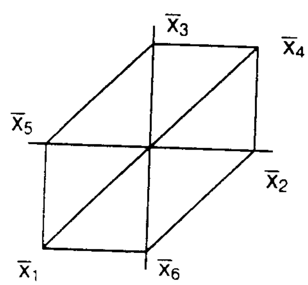
provide a linear transform of X.

Example Let $x_1, \ldots, x_6$ be the vertices of a prism in real affine 3-space H and consider H as a hyperplane in $\mathbf{R}^4$ such that 0 is not in H (Figure 1). Then $(-1,1,0,1,-1,0)$ and $(-1,0,1,1,,0,-1)$ are linear dependencies which provide a basis of $\mathcal{L}(\mathcal{X})$, and the colums of

$$A = \begin{pmatrix} -1 & 1 & 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 1 & 0 & -1 \end{pmatrix} = (\ \bar{x}_1, \ldots, \bar{x}_6\ )$$

are the elements of a linear transform of X (Figure 2).





2.1.6 Definition Let $X = \{x_1, \ldots, x_k\}$ be a set of non-zero vectors such that $\sigma = pos X$ is a cone with apex 0, and let $\bar{X}$ be a linear transform of X. For $Y \subset X$ call $C(Y) := \{\bar{x}_i | x_i \notin Y\}$ the *coface* of Y.

2.1.7 Theorem *Let* $\sigma = pos\{x_1, \ldots, x_k\}$ *be a cone with apex* 0, *and let* Y *be a subset of* $X = \{x_1, \ldots, x_k\}$ *Then* $pos Y$ *is a face of* σ *if and only if*

$$0 \in relint conv C(Y).$$

Proof Ew II.4.14.
Note that $pos\emptyset = \{0\}$, so that the theorem is also true for the improper face σ ($Y = X$).

Example In the above example, x_1, x_2, x_4, x_5 span a face, and, in fact, 0 is inside the line segment $[\bar{x}_3, \bar{x}_4]$. However, x_1, x_5 do not span a face, and 0 is not in the interior of the quadrangle with vertices $\bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_6$.

It is often useful to have an affine version of the linear transform. It is readily obtained by considering the following fact.

2.1.8 Lemma *For a linear transform $\bar{X}$ of X the following three conditions are equivalent:*
(a) The vectors of X represent points in an affine hyperplane H of U for which $0 \notin H$.
(b) $\bar{x}_1 + ... + \bar{x}_v = 0$.
(c) If $\bar{X}$ is constructed as above by means of a basis of $\mathcal{L}(\mathcal{X})$, the linear dependencies $\alpha^{(1)}, ..., \alpha^{(v)}$ are affine dependencies.

Proof Ew II.4.15.

2.1.9 Definition If $X = (x_1, ..., x_v)$ is a finite sequence of affine points in an affine space H considered a hyperplane not containing 0 in a vector space U then we call a linear transform of X with respect to U also a *Gale transform* or an *affine transform* of X with respect to H.
If the Gale transform is constructed by means of the matrix A we need not speak about U but proceed as in the case of linear transforms. We consider $v - n - 1$ affine dependencies if n is the dimension of H. The above example can then be considered as a Gale transform without any change.
We use the same definition of "coface" as in 2.1.6. Theorem 2.1.7 then carries over to the following theorem:

2.1.10 Theorem *Let $P = conv\{x_1, ..., x_k\}$ be a polytope, and let Y be a subset of $X = \{x_1, ..., x_k\}$ (chosen as vertices of P). Then $convY$ is a face of P if and only if*

$$0 \in relintconvC(Y)$$

We present now an application of Gale transforms to toric divisors. If X_Σ is an n-dimensional toric variety given by the fan Σ in $\mathbf{R}^n$, we obtain an (n-1)-dimensional toric subvariety D_ρ by considering the star of a 1-cone ρ in Σ and projecting it perpendicularly onto the hyperplane normal to ρ. *Toric Weil divisors* may then be introduced as formal linear combinations

$$D = n_1 D_{\rho_1} + ... + n_k D_{\rho_k}$$

where $n_1, ..., n_k$ are integers, and $\rho_1, ..., \rho_k$ are the 1-cones of Σ. (For a general definition of divisors see section 2.3 below). Let a_i be a generator of $\rho_i, i = 1, ..., k.D$ is *Cartier* if for every $\sigma \in \Sigma$ the generators $\{x_{i_1}, ..., x_{i_r}\} =: X_\sigma$ can be chosen such that $P_\sigma := convX_\sigma$ is a polytope of dimension $dim\sigma - 1$. A toric Cartier divisor is ample if Σ is complete, if $n_1 = ... = n_k$, and if the P_σ can be chosen to be the faces of an n-polytope P. A theorem which Shephard proved in 1971 can be used to characterize ampleness as follows.

2.1.11 Theorem *(Shephard's ampleness criterion) Let X_Σ be a complete toric variety, and let $\rho_1 = pos\{a_1\}, ..., \rho_k = pos\{a_k\}$ be the 1-cones of Σ, so that for each $\sigma \in \Sigma$ there exists a subset $X_\sigma \subset \{a_1, ..., a_k\} =: X$ for which*

$$\sigma = posX_\sigma.$$

A toric Weil divisor $D = D_{\rho_1} + ... + D_{\rho_k}$ is ample if and only if the following condition is true:

(1) For a Gale transform $\bar{X}$ of X we have

$$\bigcap_{\sigma \in \Sigma} relintconvC(X_\sigma) \neq \emptyset.$$

Proof Ew II.4.8.

A second application of Gale transforms which we discuss now makes use of Shephard's criterion. It deals with cell decompositions, in particular triangulations, of lattice polytopes. We characterize a property of such decompositions ("coherent") which occurs in the theory of polytopal semigroup rings (see, for example, Bruns [1997]), and that of secondary polytopes (Gelfand, Kapranov, Zelevinsky [1994], chapter 7).

2.1.12 Definition Let P be an n-dimensional lattice polytope in $\mathbf{R}^n$. By a *polyhedral (or cell) decomposition* of P we mean a finite collection $\mathcal{C} = \{P_i | i \in I\}$ of lattice polytopes such that

(a) $P = \bigcup_{i \in I} P_i$

(b) If $P_i \in \mathcal{C}$ and F is a face of P_i, then $F \in \mathcal{C}$

(c) If $P_i, P_j \in \mathcal{C}$ then $P_i \cap P_j$ is a common face of P_i and P_j.

If all P_i are simplices, we call $\mathcal{C}$ a *triangulation* of P. Furthermore, $\mathcal{C}$ is called *coherent* if there exists a continuous function $f : P \to \mathbf{R}$, which is linear on each P_i, and which is concave, that is, satisfies

$$f(\alpha x + (1 - \alpha)y) \geq \alpha f(x) + (1 - \alpha)f(y)$$

for $0 \leq \alpha \leq 1$ and $x, y \in P$.

2.1.13 Theorem *Let $\mathcal{C} = \{P_i | i \in I\}$ be a polyhedral decomposition of a lattice polytope P in $\mathbf{R}^n$, and let $\bar{X}$ be a Gale transform of the set X of all lattice points which are a vertex of at least one $P_i, i \in I$. $\mathcal{C}$ is coherent if and only if the following condition is satisfied:*

(2) $$\bigcap_{i \in I} relintposC(P_i) \neq \emptyset.$$

For triangulations the theorem is proved in Gelfand et al. [1994], p. 225-226. We show that the general theorem is a consequence of Shephard's criterion (Theorem 2.1.12).

We extend $\mathbf{R}^n$ to $\mathbf{R}^{n+1}$ in such a way that the graph of a real-valued function $f : P \to \mathbf{R}$ consists of points $(x_1, ..., x_n, f(x_1, ..., x_n))$, where $(x_1, ..., x_n, 0) \in P$. Up to adding a constant function we may assume $f(x) > 0$ for all $x \in P$.

Let $a = (0, -1)$ and let b be a lattice point such that

$$a \in intconv(P \cup \{b\}).$$

Clearly, $conv(P \cap \{b\})$ is a pyramid over P. By a translation, we shift 0 to a and consider the complete fan Σ consisting of all $posP_i$ for $P_i \in \mathcal{C}$, all faces $pos(F \cup \{b\})$ of the pyramid (F a proper face of P), $\{0\}$, and $pos\{b\}$.

By Shephard's criterion (Theorem 2,1,11), Σ can be obtained by projecting the faces of an $(n+1)$-polytope R if and only if $\bigcap relintconvC(Q_j) \neq 0$ where Q_j runs over all $P_i \in \mathcal{C}$ and the pyramids $conv(F \cup \{b\})$, F a proper face of P, and, finally, $\{b\}$.

Let X be the set consisting of b and all 0-cells of $\mathcal{C}$. Since b is the apex of a pyramid, we have $\bar{b} = 0$. If Q_j is a P_i, then $b \in X \backslash vertP_i$ so that $0 \in C(Q_j)$. If $Q_j = conv(F \cup \{b\})$, F a proper face of P, then Q_j is a face of the pyramid $conv(P \cup \{b\})$ and hence, by 2.1.10, $0 \in relintconvC(Q_j)$ which is equivalent to $0 \in relintposC(Q_j)$. So we obtain that

$$\bigcap relintconvC(Q_j) \neq \emptyset$$

if and only if

$$\bigcap relintconvC(P_i) \neq \emptyset.$$

Since $0 \in C(P_i)$ for each P_i, the latter condition is equivalent to

$$\bigcap relintposC(P_i) \neq \emptyset.$$

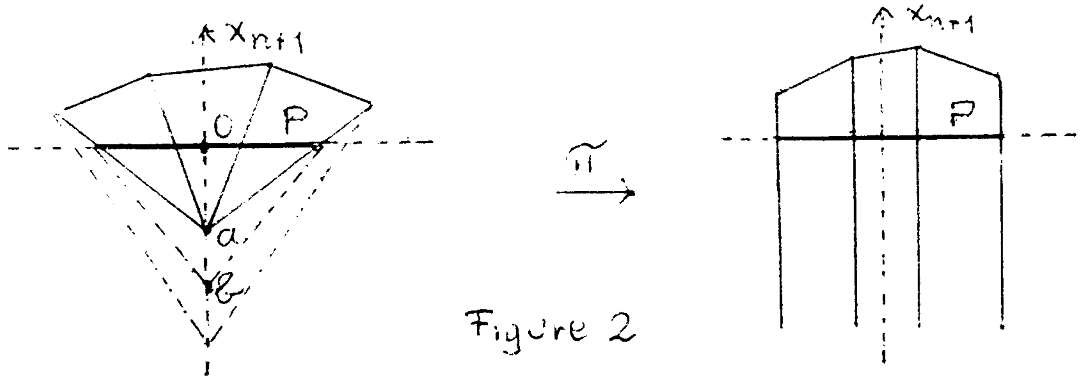


Figure 2

Suppose a polytope R is given whose faces span Σ. We extend $\mathbf{R}^{n+1}$ to a projective space and apply a central collineation π which leaves P pointwise fixed and transforms a to the the point at infinity of the x_{n+1}-axis ($x_1 = ... = x_n = 0$). If H_a is the hyperplane in $\mathbf{R}^{n+1}$ which contains a and is parallel to the affine hull of P, we choose the image of the projective extension of H_a under π as new "hyperplane at infinity". Then $\pi(Q)$ contains the graph of a concave function by which C is seen to be coherent. (Figure 2).

If, conversely, C is coherent, we may reverse the arguments and obtain a polytope R as above. This completes the proof of the theorem.

Standard examples of non-coherent triangulations of 2-polytopes are those presented in Figure 3. They may be sonsidered as special cases of a more general class of non-coherent cell decompositions:

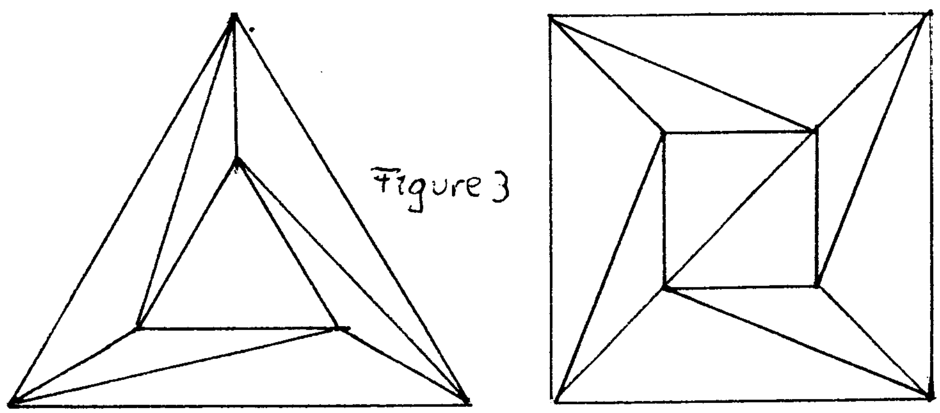


Figure 3

2.1.14 Definition Let C be a polyhedral decomposition of a lattice polytope $P \subset \mathbf{R}^n$. Suppose the line segments $s_i := [b_i, a_{i+1}], i = 1, ..., k; k+1 \equiv 1$, are cells of C such that $a_i \neq b_i$ and

$$(I) \quad b_{i+1} - b_i = \alpha_i(a_{i+1} - a_i), \ \alpha_i > 0, \ i = 1, ..., k-1,$$

furthermore,

$$(II) \quad \alpha_1 \geq ... \geq \alpha_{k-1}.$$

Then we call $\{s_1, ..., s_k\}$ a *whirl* (Figure 4).

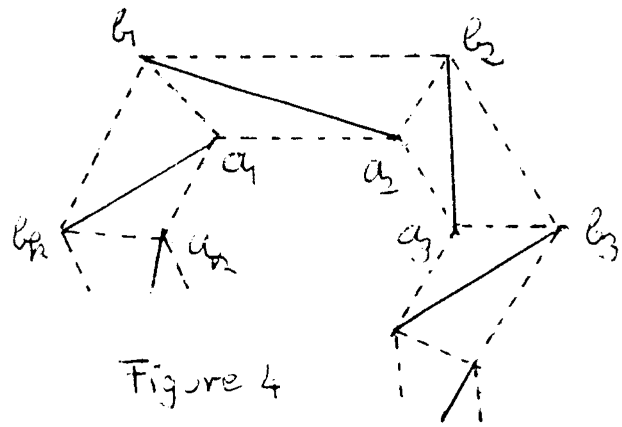


Figure 4

Figures 3 and 5 contain whirls. It should be noted that the line segments joining a_i and a_{i+1} or b_i and b_{i+1} need not be cells of $\mathcal{C}$ (Figure 5(b)). Figure 5(c) shows a whirl in a 3-dimensional $\mathcal{C}$. The numbers in Figure 5 indicate the α_i.

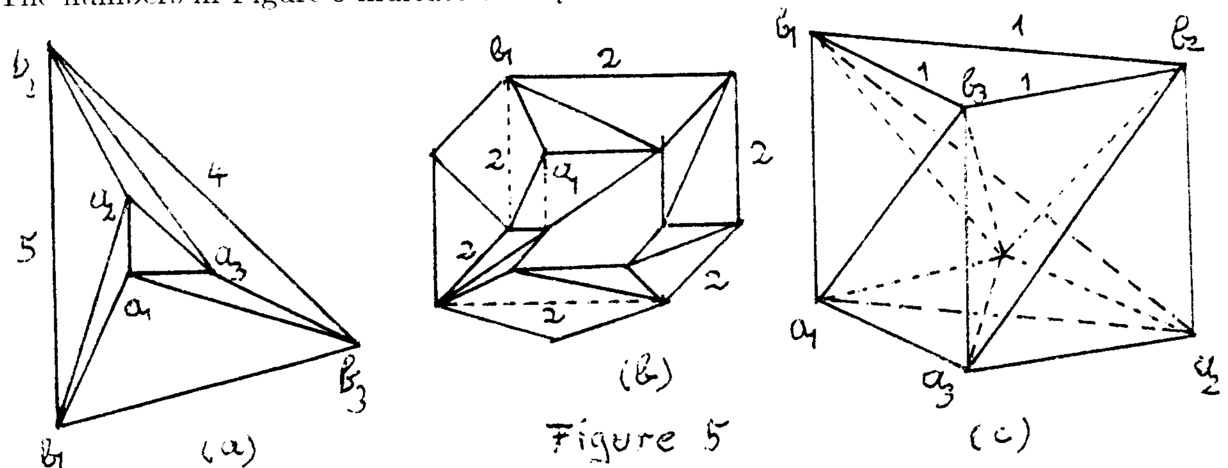


Figure 5

2.1.15 Theorem *If a polyhedral decomposition $\mathcal{C}$ of an n-polytope P contains a whirl then it is non-coherent.*

Proof Let $X = (a_1, a_2, ..., a_k, b_1, ..., b_k, ..., c)$ be the sequence of all 0-cells of $\mathcal{C}$, where the 0-cells following b_k are chosen in an arbitrary order. By (I), we obtain affine relations which obviously are linearly independent, and which can be extended to a basis of $\mathcal{L}(X)$ (rows of the matrix)

$$A = \left(\begin{array}{cccccccccc|c} \alpha_1 & -\alpha_1 & & & & -1 & 1 & & & & \\ & \alpha_2 & -\alpha_2 & & & & -1 & 1 & & & O \\ & & & \ddots & & & & & \ddots & & \\ & & & \alpha_{k-1} & -\alpha_{k-1} & & & & -1 & 1 & \\ \hline & & & & & * & & & & & \end{array}\right)$$

$$= (\bar{a}_1,\ \bar{a}_2,\ \bar{a}_3,\ ...\ \bar{a}_{k-1},\ \bar{a}_k,\ \bar{b}_1,\ \bar{b}_2,\ \bar{b}_3,\ ...,\ \bar{b}_{k-1},\ \bar{b}_k,\ ...)$$

We write the column vectors of $\mathbf{R}^{r-n-1}$ (r the number of 0-cells of $\mathcal{C}$) as

$$(x_1, ..., x_k, ..., x_{r-n-1}, ...)^t.$$

Then $C([b_i, a_{i+1}])$ is the convex hull of all columns of A except $\bar{b}_i$ and $\bar{a}_{i+1}$. Therefore, all points of $C([b_i, a_{i+1}])$ satisfy $x_i \geq 0$, $i = 1, ..., k$. Since $x_i > 0$ for at least one of the points, we find $x_i > 0$ for *relintpos*$([b_i, a_{i+1}])$. However, for the points of $C([b_k, a_1])$ we find from (II):

$$x_1 + ... + x_k \leq 0.$$

Therefore, condition (2) of Theorem 2.1.13 is violated. This completes the proof of the theorem.