THE VOLUME OF DUALS AND SECTIONS OF POLYTOPES

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Abstract. An explicit formula is given for the volume of the polar dual of a polytope. Using this formula, we prove a geometric criterion for critical (w.r.t. volume) sections of a regular simplex.

Introduction. The aim of this paper is to develop new methods for the study of isoperimetric problems for polytopes. Our major new result is a geometric characterization of critical sections of certain polytopes. In the final section, we shall briefly discuss how this method can be used to investigate other isoperimetric problems. These ideas originally arose in the theory of general hypergeometric functions recently introduced by Gel'fand *et al.* (see [**10**], [**11**] and [**12**]).

Let Q be a polytope in $\mathbf{R}^n$ with $0 \in \operatorname{int} Q$, and let $V(Q \cap L)$ be the volume of the intersection of Q by a d-dimensional subspace L. We shall focus on two problems about $V(Q \cap L)$.

1. Find a formula for $V(Q \cap L)$ in terms of the Plücker coordinates of L.
2. Find a geometric criterion for "critical sections", *i.e.*, a section $Q \cap L_0$ such that the function $L \mapsto V(Q \cap L)$ is differentiable and has a critical point at the d-flat L_0.

We shall see that the function $V(Q \cap L)$ is piecewise rational in the Plücker coordinates of L, and so is not differentiable everywhere. A description of where $V(Q \cap L)$ is differentiable appears in the final section.

In Section 1, we prove the following theorem which can be used to solve problem 1 (see also [**16**]).

THEOREM 1. *The volume of the polar P^* of a polytope P in $\mathbf{R}^d$ can be written as an alternating sum of the volumes of simplices "dual" to those in a triangulation of P. These volumes are expressed as quotients of $d \times d$ minors in the coordinates of the vertices of P.*

Given a formula for the volume of the polar dual, it is easy to get a formula for sections. Let $P = \Pi(Q : L)$ be the orthogonal projection of Q into L. Then the usual duality between sections and projections implies

$$P^* = Q^* \cap L.$$

The reason why Theorem 1 is so useful for sections is that it is much easier to find coordinates for the vertices of P from a basis of L than those of the section P^*.

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A complete solution to problem 2 when Q is a regular simplex is given in Section 2.

THEOREM 8. *If P^* is a critical section of a regular simplex through its centroid, then each facet F^* of P^* is normal to the line connecting the centroid of F^* to the centroid of P^*.*

Conditions similar to those in Theorem 8 have been discovered for related extremum problems (see **[4]** and **[15]**).

§1. *The volume of the dual.* The goal of this section is to establish the formula for the volume of the polar dual of a convex polytope. The formula will be written in a bracket notation which is described below.

Let $P = \text{conv}\{u_1, \dots, u_n\}$, $u_i \in \mathbf{R}^d$, be a convex polytope with $0 \in \text{int}\, P$. Associated to P is an $n \times (d+1)$ matrix

$$\mathbf{X} = \begin{bmatrix} u_{11} & \dots & u_{1d} & 1 \\ u_{21} & \dots & u_{2d} & 1 \\ \vdots & & \vdots & \vdots \\ u_{n1} & \dots & u_{nd} & 1 \end{bmatrix}, \tag{1.1}$$

whose i-th row, x_i, contains the standard coordinates of u_i, together with a final column of ones. Given

$$\lambda \in \Lambda(n, d+1) = \{(\lambda_1, \dots, \lambda_{d+1}) \mid 1 \leq \lambda_1 < \dots < \lambda_{d+1} \leq n\},$$

we denote the corresponding $(d+1)\times(d+1)$ minor of $\mathbf{X}$ by

$$[\lambda] = \det[x_{\lambda_1}, \dots, x_{\lambda_{d+1}}] = \det \begin{bmatrix} u_{\lambda_1 1} & \dots & u_{\lambda_1 d} & 1 \\ \vdots & & \vdots & \vdots \\ u_{\lambda_{d+1} 1} & \dots & u_{\lambda_{d+1} d} & 1 \end{bmatrix}. \tag{1.2}$$

The cofactors of $[\lambda]$ from its last column can also be written as $(d+1)\times(d+1)$ determinants using the vector $\theta = (0, \dots, 0, 1)$;

$$[\lambda \backslash \lambda_i] = \det[x_{\lambda_1}, \dots, x_{\lambda_{i-1}}, \theta, x_{\lambda_{i+1}}, \dots, x_{\lambda_{d+1}}]. \tag{1.3}$$

If $j = \lambda_i$, we will often write $[\lambda \backslash j]$ for $[\lambda \backslash \lambda_i]$.

THEOREM 1. *Let Δ be a triangulation of P with vertices in $\{u_1, \dots, u_n\}$, such that 0 does not belong to the union of the affine hyperplanes spanned by vertices of Δ. Then*

$$V(P^*) = \frac{(-1)^d}{d!} \sum_{\lambda \in \Delta} \text{sign}\, \Delta_\lambda \frac{[\lambda]^d}{\prod_{i=1}^{d+1} [\lambda \backslash \lambda_i]}, \tag{1.4}$$

where $\text{sign}\, \Delta_\lambda$ *is the orientation of the labeled simplex* $\Delta_\lambda = \text{conv}\{u_{\lambda_1}, \dots, u_{\lambda_{d+1}}\}$.

A triangulation of P satisfying the requirements of the theorem can always be produced by triangulating the boundary of P and joining the boundary to one generic interior point. For such triangulations, (1.4) reduces to Lawrence's formula in [**16**].

We shall derive Theorem 1 as a corollary of a lemma about characteristic functions of cells in a hyperplane arrangement in $\mathbf{R}^d$. The proof of this lemma will be simplified by embedding P and P^* in $\mathbf{R}^{d+1}$ so we can use the terminology of dual cones. We begin with a few basic properties of dual cones which can be found in [**13**].

If $x \in \mathbf{R}^{d+1}$, let

$$H_x = \{y \in \mathbf{R}^{d+1} \mid \langle x, y\rangle = 0\},$$
$$H_x^- = \{y \in \mathbf{R}^{d+1} \mid \langle x, y\rangle \leq 0\}, \qquad \text{and}$$
$$H_x^+ = \{y \in \mathbf{R}^{d+1} \mid \langle x, y\rangle \geq 0\}.$$

If C is a closed, convex cone in $\mathbf{R}^{d+1}$ with vertex at the origin, then the usual polar dual of C in $\mathbf{R}^{d+1}$ can be written as

$$C^* = \bigcap_{x \in C} H_x^- \tag{1.5}$$

(see [**13**, p. 49]). The set C^* is also a closed, convex cone with vertex at the origin. Some further properties of C^* are

$$z \in C^* \Longleftrightarrow C \subseteq H_z^-, \tag{1.6}$$
$$z \in \operatorname{int} C^* \Longleftrightarrow C \subseteq \operatorname{int} H_z^-, \tag{1.7}$$
$$C^{**} = \operatorname{pos} C. \tag{1.8}$$

We may embed the d-dimensional polytope P in $\mathbf{R}^{d+1}$ as a subset of $\theta + H_\theta$ using the points

$$x_i = (u_{i1}, \ldots, u_{id}, 1),$$

i.e., the rows of $\mathbf{X}$ in (1.1). The cone over P will be denoted by $CP = \operatorname{pos}\{x_1, \ldots, x_n\}$. The usual polar polytope P^* of P in $\mathbf{R}^d$ can be identified with

$$P^* = CP^* \cap \{-\theta + H_\theta\}. \tag{1.9}$$

Given a triangulation Δ of P as in Theorem 1, we shall associate with each $\lambda \in \Delta$ a closed simplex σ_λ in aff $P^* = -\theta + H_\theta$. By assumption, the set of points $\{x_0 = \theta, x_{\lambda_1}, \ldots, x_{\lambda_{d+1}}\}$ has a unique Radon partition $\{0, \lambda\} = \{0, \lambda^-\} \cup \{\lambda^+\}$ so that

$$\operatorname{conv}\{x_i \mid i \in \{0, \lambda^-\}\} \cap \operatorname{conv}\{x_i \mid i \in \lambda^+\} \neq \varnothing.$$

The elements of these two sets are given opposite signs in the corresponding circuit of the rank d oriented matroid on $\{\theta, x_1, \ldots, x_n\}$. If we set the sign of θ to be -1 in this circuit, then by Cramer's rule the sign of $i \in \lambda$ is

$$\varepsilon_\lambda^i = \operatorname{sign}\left\{\frac{[\lambda \backslash i]}{[\lambda]}\right\} = \operatorname{sign}\langle x_\lambda^i, \theta\rangle. \tag{1.10}$$

Here $\{x_\lambda^i\}$ is the dual basis to $\{x_i \mid i \in \lambda\}$ in $\mathbf{R}^{d+1}$, *i.e.*,

$$\langle x_\lambda^i, x_j\rangle = \delta_j^i, \qquad \forall i, j \in \lambda. \tag{1.11}$$

Now, consider the cone

$$C_\lambda = \operatorname{pos}\{\varepsilon_\lambda^i x_i \mid i \in \lambda\}. \tag{1.12}$$

From (1.10), the Radon partition for $\{\varepsilon_\lambda^i x_i \mid i \in \{0, \lambda\}\}$ is $\{0, \lambda\} = \{0\} \cup \{\lambda\}$, which implies $\theta \in \operatorname{int} C_\lambda$. Hence by (1.7) and (1.8), $C_\lambda^* \subset \operatorname{int} H_\theta^-$. The intersection

$$\sigma_\lambda = C_\lambda^* \cap \operatorname{aff} P^* \tag{1.13}$$

is therefore bounded and thus a simplex in aff P^*.

In order to keep our notation from becoming unwieldy in Lemma 2, we shall use the same symbol to represent a subset of $\mathbf{R}^{d+1}$ and its characteristic function.

LEMMA 2. *Let $z \in \operatorname{aff} P^*$ with $z \notin H_{x_i}$, $\forall i \in \{1, \ldots, n\}$, and let*

$$\varepsilon_\lambda = \prod_{i \in \lambda} \varepsilon_\lambda^i. \tag{1.14}$$

Then

$$P^*(z) = \sum_{\lambda \in \Delta} \varepsilon_\lambda \sigma_\lambda(z). \tag{1.15}$$

Similar results are proved in **[12]** and **[17]**.

Before we prove Lemma 2, we shall show that it implies Theorem 1. The vertices of σ_λ are given by

$$y_\lambda^i = -\frac{[\lambda]}{[\lambda \backslash i]} x_\lambda^i, \qquad i \in \lambda \tag{1.16}$$

(see (1.13)). It follows immediately that the volume of σ_λ, denoted by $|\sigma_\lambda|$, is

$$\operatorname{sign} \sigma_\lambda |\sigma_\lambda| = \frac{(-1)^d [\lambda]^d}{d! \prod_{i \in \lambda} [\lambda \backslash i]}. \tag{1.17}$$

Note that one power of $[\lambda]$ has cancelled since the determinant of the dual basis is

$$\det [x_\lambda^{\lambda_1}, \ldots, x_\lambda^{\lambda_{d+1}}] = \frac{1}{[\lambda]}.$$

From (1.10),

$$\begin{aligned} \varepsilon_\lambda &= \operatorname{sign}\left\{\prod_{i \in \lambda} \frac{[\lambda]}{[\lambda \backslash i]}\right\} = \operatorname{sign}[\lambda] \operatorname{sign}\left\{\frac{[\lambda]^d}{\prod_{i \in \lambda} [\lambda \backslash i]}\right\} \\ &= (-1)^d \operatorname{sign} \Delta_\lambda \operatorname{sign} \sigma_\lambda. \end{aligned} \tag{1.18}$$

Substituting (1.17) and (1.18) in (1.4), we may restate Theorem 1 as

$$V(P^*) = \sum_{\lambda \in \Delta} \varepsilon_\lambda |\sigma_\lambda|. \tag{1.19}$$

However,

$$\sum_{\lambda \in \Delta} \varepsilon_\lambda |\sigma_\lambda| = \sum_{\lambda \in \Delta} \varepsilon_\lambda \int \sigma_\lambda(z) dz = \int \sum_{\lambda \in \Delta} \varepsilon_\lambda \sigma_\lambda(z) dz = \int P^*(z) dz = V(P^*),$$

by Lemma 2. This proves that Theorem 1 is a consequence of Lemma 2.

Proof of Lemma 2. We first assume $z \notin P^*$. For every $\lambda \in \Delta$, define a new cone

$$D_\lambda = \text{pos}\,\{\varepsilon_i x_i \mid i \in \lambda\}, \qquad \text{where} \tag{1.20}$$

$$\varepsilon_i = \text{sign}\,\langle x_i, -z\rangle. \tag{1.21}$$

According to (1.6), $z \in C_\lambda^*$, if, and only if, $C_\lambda \subseteq H_z^-$. Comparing (1.20) with (1.12), it follows that

$$z \in \sigma_\lambda \iff C_\lambda = D_\lambda \iff \theta \in D_\lambda. \tag{1.22}$$

The hypothesis of Lemma 2 implies that $D_\lambda \cap H_z = \{0\}$, $\forall \lambda \in \Delta$. Moreover, $z \notin P^*$ implies $H_z \cap \text{int}\, P \neq \varnothing$. Since the number of cones D_λ, $\lambda \in \Delta$, is finite, there exists a vector $\theta' \in \mathbf{R}^{d+1}$ such that

$$\theta' \in \text{int}\, P \cap \text{int}\, H_z^- \qquad \text{and} \tag{1.23}$$

$$\theta' \notin D_\lambda, \qquad \forall \lambda \in \Delta. \tag{1.24}$$

Now consider the sum

$$s = \sum_{\lambda \in \Delta} \varepsilon_\lambda D_\lambda(\theta'), \tag{1.25}$$

where

$$\varepsilon_\lambda = \prod_{i \in \lambda} \varepsilon_i. \tag{1.26}$$

Comparing (1.24) and (1.25), we see that $s = 0$. The key idea in the proof is to move θ' to θ by a path in $\text{int}\, P \cap \text{int}\, H_z^-$, and show that s remains 0 after θ' crosses the interior of any facet of $\{D_\lambda, \lambda \in \Delta\}$. When θ' reaches θ, the sums in (1.15) and (1.25) are identical by (1.22).

Let $\mu \in \Lambda(n, d)$ and let

$$D_\mu = \text{pos}\,\{\varepsilon_i x_i \mid i \in \mu\} \tag{1.27}$$

be a facet of some D_λ, $\lambda = (a, \mu) \in \Delta$. If θ' crosses D_μ, then there must exist an adjacent cone $D_{\lambda'}$, $\lambda' = (b, \mu) \in \Delta$, having D_μ as a facet. Since x_a and x_b lie on opposite sides of lin D_μ,

$$\varepsilon_\lambda = \varepsilon_{\lambda'} \iff \text{sign}\,\langle x_a, z\rangle = \text{sign}\,\langle x_b, z\rangle \iff \text{int}\, D_\lambda \cap \text{int}\, D_{\lambda'} = \varnothing \qquad \text{and}$$

$$\varepsilon_\lambda = -\varepsilon_{\lambda'} \iff \text{sign}\,\langle x_a, z\rangle \neq \text{sign}\,\langle x_b, z\rangle \iff \text{int}\, D_\lambda \cap \text{int}\, D_{\lambda'} \neq \varnothing.$$

This shows s remains 0 after θ' crosses D_μ, and that the sum in (1.15) equals zero when $z \notin P^*$.

In case $z \in \text{int}\, P^*$, then by (1.7) $P \subseteq \text{int}\, H_z^-$ and $\varepsilon_i = 1$, $\forall i \in \{1, \ldots, n\}$. Hence by (1.22), $z \in \sigma_\lambda$, if, and only if, $\theta \in \text{pos}\,\{x_i \mid i \in \lambda\}$. Since Δ is a triangulation of P, there is exactly one simplex Δ_{λ^0} for which the latter holds. The theorem is completed by noting that $\varepsilon_{\lambda^0} = 1$.

In the next section, we will need a generalization of Theorem 1 to the faces of P^*. This shall follow from a couple of simple corollaries of Lemma 2.

Any sequence $\lambda \in \Delta$ has a unique partition $\lambda = \lambda^+ \cup \lambda^-$ (the Radon partition of $\{\theta, x_{\lambda_1}, \ldots, x_{\lambda_{d+1}}\}$), where

$$\lambda^+ = \{i \in \lambda \mid \varepsilon^i_\lambda > 0\} \qquad \text{and} \qquad \lambda^- = \{i \in \lambda \mid \varepsilon^i_\lambda < 0\}.$$

This partition can be used to change σ_λ (defined in (1.13)) into a half-open simplex

$$\hat{\sigma}_\lambda = \left(\bigcap_{i \in \lambda^+} H_i^-\right) \cap \left(\bigcap_{i \in \lambda^-} \operatorname{int} H_i^+\right) \cap \operatorname{aff} P^*, \tag{1.28}$$

where $H_i = H_{x_i}$. That is, we obtain $\hat{\sigma}_\lambda$ by removing the facets of σ_λ corresponding to the indices in λ^-. These modified simplices can be used to give an expansion of the entire characteristic function of P^*.

COROLLARY 3.

$$P^* = \sum_{\lambda \in \Delta} \varepsilon_\lambda \hat{\sigma}_\lambda. \tag{1.29}$$

Proof. Let $\mathscr{H} = \{H_i \cap \operatorname{aff} P^* \mid i = 1, \ldots, n\}$ be the hyperplane arrangement which has P^* as a cell, and let $H = \bigcup_{i=1}^n H_i$. Lemma 2 implies that both sides of (1.29) match on $\{\operatorname{aff} P^*\} \backslash H$. It suffices therefore to prove (1.29) on any face F^* of $\mathscr{H}$.

If

$$\nu = \{i \mid F^* \subseteq H_i\}, \tag{1.30}$$

then every point $z \in \operatorname{rel\,int} F^*$ satisfies $\langle x_i, z\rangle = 0 \Leftrightarrow i \in \nu$. Thus the ray

$$R = \{\alpha z - (1-\alpha)\theta \mid \alpha < 1\}$$

lies in

$$R \subseteq \bigcap_{i \in \nu} \operatorname{int} H_i^-. \tag{1.31}$$

Moreover, we can find a point $r \in R$ near enough to z so that

$$\operatorname{conv}\{z, r\} \cap H = \{z\}.$$

The relationship of r to z allows us to compare $\hat{\sigma}(z)$ and $\sigma(r)$. For example, if $\lambda \in \Delta$ and $\lambda \cap \nu = \varnothing$, then

$$z \in \hat{\sigma}_\lambda \iff \operatorname{conv}\{z, r\} \subseteq \operatorname{int} \hat{\sigma}_\lambda \iff r \in \sigma_\lambda \tag{1.32}$$

since $\operatorname{int} \hat{\sigma}_\lambda = \operatorname{int} \sigma_\lambda$. On the other hand, if $\lambda \cap \nu \neq \varnothing$, then (1.28) and (1.31) imply

$$z \in \hat{\sigma}_\lambda \iff z \in \sigma_\lambda \qquad \text{and} \qquad \lambda \cap \nu \subseteq \lambda^+ \iff r \in \sigma_\lambda. \tag{1.33}$$

Combining (1.32) and (1.33) gives

$$\sum_{\lambda \in \Delta} \varepsilon_\lambda \hat{\sigma}_\lambda(z) = \sum_{\lambda \in \Delta} \varepsilon_\lambda \sigma_\lambda(r). \tag{1.34}$$

However, it follows from Lemma 2 that the second sum is 0 if $F^* \not\subseteq P^*$, and 1 if $F^* \subseteq P^*$. This completes the proof of (1.29).

The next step is to generalize Corollary 3 to the faces of P^*. In Section 2, P shall always be in general position. Thus we may greatly reduce our work by assuming P is simplicial, or equivalently that the hyperplane arrangement $\mathscr{H}$ is "simple", *i.e.*,

$$|\{i \mid F^* \subseteq H_i\}| = d - \dim F^* \tag{1.35}$$

for any face F^* of $\mathscr{H}$. Suppose

$$F_\nu^* = P^* \cap \left(\bigcap_{i \in \nu} H_i \right) \tag{1.36}$$

is a face of P^* with $\dim F_\nu^* = d - |\nu|$. If $\lambda \in \Delta$ and $\nu \subset \lambda$, we define

$$\hat{\sigma}_{\lambda|\nu} = \left(\bigcap_{i \in \lambda^+ \setminus \nu} H_i^- \right) \cap \left(\bigcap_{i \in \lambda^- \setminus \nu} \operatorname{int} H_i^+ \right) \cap \left(\bigcap_{i \in \nu} H_i \right) \cap \operatorname{aff} P^*. \tag{1.37}$$

LEMMA 4. *If P is simplicial, then*

$$F_\nu^* = \sum_{\lambda \in \Delta, \lambda \supset \nu} \left(\prod_{i \in \lambda \setminus \nu} \varepsilon_\lambda^i \right) \hat{\sigma}_{\lambda|\nu}. \tag{1.38}$$

Proof. The proof will be by induction on $|\nu|$. Let $\nu = \{i\}$ and let $F^* \subset H_i$ be a facet of $\mathscr{H}$. We define three sets

$$A = \{\lambda \in \Delta \mid i \in \lambda^+\}, \qquad B = \{\lambda \in \Delta \mid i \in \lambda^-\} \qquad \text{and} \qquad C = \{\lambda \in \Delta \mid i \notin \lambda\}.$$

Suppose first that $F^* \not\subset P^*$. Let $z \in \operatorname{rel\,int} F^*$ and choose points $z^\pm \in H_i^\mp$ near z on either side of H_i. The simplicity of $\mathscr{H}$ implies that for $\lambda \in C$, $z^+ \in \sigma_\lambda$, if, and only if, $z^- \in \sigma_\lambda$. In this case, Lemma 2 gives

$$0 = \sum_{A \cup C} \varepsilon_\lambda \sigma_\lambda(z^+) = \sum_{B \cup C} \varepsilon_\lambda \sigma_\lambda(z^-),$$

and

$$0 = \sum_A \varepsilon_\lambda \sigma_\lambda(z^+) - \sum_B \varepsilon_\lambda \sigma_\lambda(z^-). \tag{1.39}$$

Since $\operatorname{rel\,int} \hat{\sigma}_{\lambda|i} = \operatorname{rel\,int} \sigma_\lambda \cap H_i$,

$$z \in \hat{\sigma}_{\lambda|i} \iff \lambda \in A \text{ and } z^+ \in \sigma_\lambda, \text{ or } \lambda \in B \text{ and } z^- \in \sigma_\lambda. \tag{1.40}$$

Substituting (1.40) in (1.39) yields

$$0 = \sum_{\lambda \in \Delta, \lambda \ni i} \varepsilon_\lambda^i \varepsilon_\lambda \hat{\sigma}_{\lambda|i}(z). \tag{1.41}$$

In the case when F^* is a face of P^*, a similar argument shows that the sum in (1.41) is 1.

From (1.41) we can conclude that if $\nu = \{i\}$, (1.38) holds on $H_i \setminus H$. The proof that (1.38) holds on all of H_i is identical to the proof of Corollary 3 with $\mathscr{H}$ replaced by its restriction to H_i.

The argument above uses Corollary 3, which is (1.38) with $\nu = \varnothing$, to derive (1.38) with $|\nu| = 1$. The general inductive step from $|\nu| = r$ to $|\nu| = r + 1$ follows the same pattern.

In [**14**], the following determinantal formula is given for the volume of $\sigma_{\lambda|\nu}$

$$\frac{(-1)^d \operatorname{sign} \Delta_\lambda [\lambda]^{d-r}}{(d-r)! \prod_{i\in\lambda\setminus\nu} [\lambda\setminus i]} = \left(\prod_{i\in\lambda\setminus\nu} \varepsilon_\lambda^i\right) \frac{|\sigma_{\lambda|\nu}|}{|u_\nu|}, \tag{1.42}$$

where $|u_\nu|$ is the volume of the parallelotope spanned by $\{u_{\nu_1}, \ldots, u_{\nu_{r+1}}\}$. Combining Lemma 4 with (1.42), we obtain the generalization of Theorem 1.

THEOREM 5. *Suppose Δ is a triangulation of a simplicial polytope P satisfying the conditions of Theorem* 1. *Then for any $(r-1)$-dimensional face F_ν of P, the dual face in P^* has volume*

$$\frac{V(F_\nu^*)}{|u_\nu|} = \frac{(-1)^d}{(d-r)!} \sum_{\lambda\in\Delta, \lambda\supset\nu} \operatorname{sign} \Delta_\lambda \frac{[\lambda]^{d-r}}{\prod_{i\in\lambda\setminus\nu} [\lambda\setminus i]}. \tag{1.43}$$

§2. *Sections of the regular simplex.* The formula for the volume of the dual in Theorem 1 allows us to transfer most of the theory of critical projections of polytopes in [7] to critical sections. Here we shall deal only with sections of a regular simplex through its centroid, and postpone remarks on sections of other polytopes until Section 3.

Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbf{R}^n$. The set

$$T^{n-1} = \operatorname{conv}\{e_1, \ldots, e_n\} - \{e_1 + \ldots + e_n\}/n$$

is a regular $(n-1)$-dimensional simplex with centroid at the origin. In this section, it is more convenient to work with P and P^* as subsets of $\mathbf{R}^d$. Thus we let $P = \operatorname{conv}\{u_1, \ldots, u_n\}$ and let L be the d-dimensional subspace of $\mathbf{R}^n$ spanned by the first d columns of $\mathbf{X}$ in (1.1). If $\Pi(T^{n-1} : L)$ is the orthogonal projection of T^{n-1} into L, then [7, Theorem 4] implies

$$P = \Pi(T^{n-1} : L) \iff X'X = \begin{bmatrix} I & 0 \\ 0 & n \end{bmatrix}. \tag{2.1}$$

In the case when (2.1) holds,

$$P^* = (T^{n-1})^* \cap L = (-nT^{n-1}) \cap L \tag{2.2}$$

is a d-dimensional section of $-nT^{n-1}$ through its centroid.

An immediate corollary of Theorem 1 is

THEOREM 6. *The volume of $T^{n-1} \cap L$ is a piecewise rational function in the $d \times d$ minors of* $\mathbf{X}$ *in* (1.1), *homogeneous of degree* -1.

These minors are the Plücker coordinates of L in the usual embedding of the Grassmannian $G(d, n)$ of d-dimensional subspaces of $\mathbf{R}^n$ into the exterior algebra $\bigwedge_d \mathbf{R}^n$. The pieces described in Theorem 6 correspond to the open regions in the decomposition of $G(d, n+1)$ induced by the oriented matroid of $\{0, u_1, \ldots, u_n\}$ (see [**2**], Chapter 4).

The remainder of this section is devoted to studying the critical conditions for $V(T^{n-1} \cap L)$ on $G(d, n)$. The proof of Proposition 9 in [7] (with φ replaced by the gradient of $V(T^{n-1} \cap L)$) shows that these critical conditions are identical with the critical conditions for $V(P^*)$ under the restriction $\sum |u_i|^2 = d$.

Setting up the appropriate Lagrange multiplier problem, the critical conditions become

$$\alpha u_{il}=\sum_{\lambda\ni i}\sum_{\mu\subset\lambda,\mu\ni i}\frac{[\mu\backslash il]}{[\mu]}(-1)^d\operatorname{sign}\Delta_\lambda\left(\frac{[\lambda]^{d-1}}{(d-1)!\prod_{k\in\mu}[\lambda\backslash k]}-\frac{[\lambda]^d}{d!\prod_{k\in\lambda}[\lambda\backslash k]}\right),\tag{2.3}$$

where $|\mu|=d$ and $\alpha\in\mathbf{R}$. The brackets $[\lambda]$ and $[\lambda\backslash k]$ are defined in (1.2) and (1.3). The remaining brackets are

$$[\mu]=\det[u_{\mu_1},\ldots,u_{\mu_d}]\tag{2.4}$$

and $[\mu\backslash il]$, the cofactor of u_{il} in $[\mu]$.

Since we hope to obtain geometric conditions on the polytope P^* from (2.3), we need to rewrite these equations using geometrically meaningful quantities. First, note that

$$[\mu]=\sum_{l=1}^{d}u_{il}[\mu\backslash il].\tag{2.5}$$

This implies the vector in $\mathbf{R}^d$ whose l-th coordinate is $[\mu\backslash il]/[\mu]$ is u^i_μ, where

$$\langle u^i_\mu,u_k\rangle=\delta^i_k,\qquad \forall i,k\in\mu.\tag{2.6}$$

Substituting $u^i_{\bar{j}}$, $\bar{j}=\lambda\backslash j$, and (1.42) in (2.3) gives

$$\alpha u_i=\sum_{\lambda\ni i}\sum_{j\in\lambda,j\neq i}u^i_{\bar{j}}\left(\varepsilon^j_\lambda\varepsilon_\lambda\frac{|\sigma_{\lambda|j}|}{|u_j|}-\varepsilon_\lambda|\sigma_\lambda|\right).\tag{2.7}$$

Further simplification will require some equations relating the quantities in each term of (2.7). Throughout this discussion $\lambda\in\Delta$ will be fixed, so our notation may be condensed by letting

$$\sigma=\varepsilon_\lambda|\sigma_\lambda|\qquad\text{and}\tag{2.8}$$

$$\sigma_i=\varepsilon^i_\lambda\varepsilon_\lambda|\sigma_{\lambda|i}|.\tag{2.9}$$

Comparing the right and left sides of (1.42), we obtain

$$\frac{\sigma_i}{|u_i|[\lambda\backslash i]}=\frac{\sigma_j}{|u_j|[\lambda\backslash j]}=\frac{d\sigma}{[\lambda]}.\tag{2.10}$$

Recall that $u_i\in\mathbf{R}^d$ is obtained by dropping the final coordinate of x_i (see (1.1)). Similarly, we define v^i by dropping the last coordinate of y^i_λ, a vertex of σ_λ. An example of dual simplices in $\mathbf{R}^2$ appears in Fig. 1.

Equations (1.11) and (1.16) imply

$$\langle u_i,v^j\rangle=\frac{-[\lambda]}{[\lambda\backslash j]}\delta^i_j+1,\qquad\forall i,j\in\lambda.\tag{2.11}$$

Therefore, v^j is normal to the facet Δ_j of Δ_λ opposite u_j. Moreover, if $i\neq j$

$$\operatorname{sign}\langle v^j,u_j-u_i\rangle=\operatorname{sign}\left\{\frac{-[\lambda]}{[\lambda\backslash j]}\right\}=-\varepsilon^j_\lambda.\tag{2.12}$$

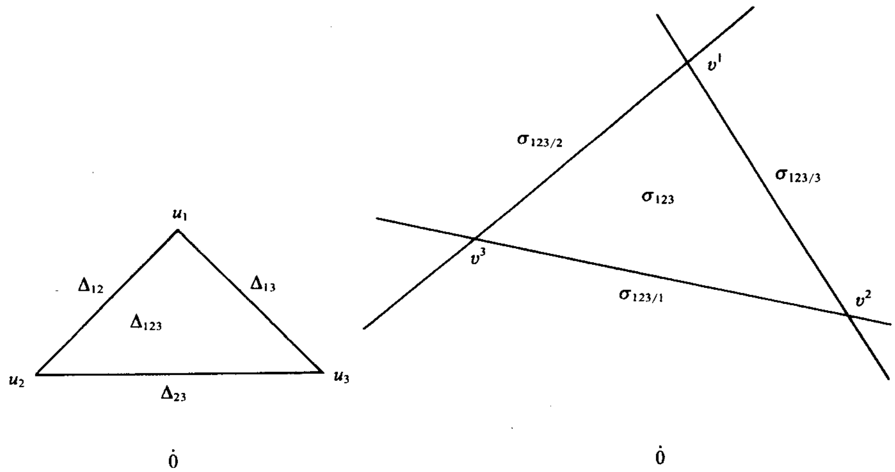

Figure 1

Thus $\varepsilon^j_\lambda v^j$ is an outer normal of Δ_λ. Since the distance from $\Delta_{\bar{j}}$ to the origin is $1/|v^j|$,

$$\frac{|\Delta_{\bar{j}}|}{d|v^j|}=\frac{1}{d!}|[\lambda\backslash j]|.$$

Applying Minkowski's theorem to the outer "area" normals of Δ_λ, we find that

$$0=\sum_{j\in\lambda}\frac{\varepsilon^j_\lambda v^j}{|v^j|}|\Delta_{\bar{j}}|,\qquad 0=\sum_{j\in\lambda}v^j[\lambda\backslash j],\qquad \text{and}\qquad 0=\sum_{j\in\lambda}v^j\frac{\sigma_j}{|u_j|},\tag{2.13}$$

by (2.10) and (2.12).

We can now prove the geometric criterion for critical sections.

THEOREM 7. *If P^* is a critical section of T^{n-1}, then its centroid must lie at the origin.*

Proof. Since the vertices of T^{n-1} project onto the set $\{u_1,\dots,u_n\}$, we have $\sum u_i=0$. Therefore summing over i in (2.7) gives

$$\begin{aligned}0&=\sum_{i=1}^{n}\sum_{\lambda\ni i}\sum_{j\in\lambda,j\neq i}\left(\frac{\sigma_j}{|u_j|}-\sigma\right)u^i_j=\sum_{\lambda\in\Delta}\sum_{i\in\lambda}\sum_{j\in\lambda,j\neq i}\left(\frac{\sigma_j}{|u_j|}-\sigma\right)u^i_j\\&=\sum_{\lambda\in\Delta}\sum_{j\in\lambda}\left(\frac{\sigma_j}{|u_j|}-\sigma\right)\sum_{i\in\lambda,i\neq j}u^i_j=\sum_{\lambda\in\Delta}\sum_{j\in\lambda}\left(\frac{\sigma_j}{|u_j|}-\sigma\right)v^j.\end{aligned}\tag{2.14}$$

The last equality follows from (2.6) and (2.11) since

$$1=\langle v^j,u_k\rangle=\left\langle\sum_{i\in\lambda,i\neq j}u^i_j,u_k\right\rangle,\qquad\forall j,k\in\lambda,\quad k\neq j,$$

implies

$$v^j=\sum_{i\in\lambda,i\neq j}u^i_j.\tag{2.15}$$

Substituting (2.13) in (2.14) then gives

$$0 = \sum_{\lambda \in \Delta} \sigma\left(\sum_{j \in \lambda} v^j\right). \tag{2.16}$$

However, the term in parenthesis is $(d+1)$ times the centroid of σ_λ. A similar argument to the one following (1.19) shows that the right-hand side of (2.16) is proportional to the centroid of P^*.

THEOREM 8. *If P^* is a critical section of T^{n-1}, then each facet of P^* is normal to the line from the origin to its centroid.*

Proof. Suppose u_i is a vertex of P. The critical equations (2.7) in our condensed notation are

$$\alpha u_i = \sum_{\lambda \ni i} \sum_{j \in \lambda, j \neq i} \left(\frac{\sigma_j}{|u_j|} - \sigma\right) u_j^i = \sum_{\lambda \ni i} \left(\sum_{j \in \lambda, j \neq i} \frac{\sigma_j}{|u_j|} u_j^i\right) - \left(\sigma \sum_{j \in \lambda, j \neq i} u_j^i\right). \tag{2.17}$$

In order to simplify these sums, note that

$$0 = \langle u_k, u_j^i \rangle = \langle u_k, v^i - v^j \rangle, \qquad \forall k \in \lambda \backslash \{i, j\},$$

by (2.6) and (2.11). Thus u_j^i must be parallel to $v^i - v^j$. The equations

$$\langle u_i, u_j^i \rangle = 1 \qquad \text{and} \qquad \langle u_i, v^i - v^j \rangle = -\frac{[\lambda]}{[\lambda \backslash i]},$$

therefore imply

$$u_j^i = -\frac{[\lambda \backslash i]}{[\lambda]}(v^i - v^j) \qquad \text{and} \tag{2.18}$$

$$u_i^j = -\frac{[\lambda \backslash j]}{[\lambda \backslash i]} u_j^i. \tag{2.19}$$

This enables us to rewrite the first sum in (2.17) as

$$\sum_{j \in \lambda, j \neq i} \frac{\sigma_j}{|u_j|} u_j^i = -\sum_{j \in \lambda, j \neq i} \frac{\sigma_j}{|u_j|} \frac{[\lambda \backslash j]}{[\lambda \backslash i]} u_i^j = -\frac{\sigma_i}{|u_i|} \sum_{j \in \lambda, j \neq i} u_i^j = -\frac{\sigma_i}{|u_i|} v^i \tag{2.20}$$

(see (2.10) and (2.15)).

The second sum in (2.17) is simplified using (2.18):

$$\sigma \sum_{j \in \lambda, j \neq i} u_j^i = -\frac{[\lambda \backslash i]}{[\lambda]} \sigma \sum_{j \in \lambda, j \neq i} (v^i - v^j) = -\frac{\sigma_i}{d|u_i|}(dv^i - dw^i), \tag{2.21}$$

where w^i is the centroid of $\sigma_{\lambda|i}$.

Substituting (2.20) and (2.21) in (2.17) gives

$$\alpha u_i = \sum_{\lambda \ni i} -\frac{\sigma_i}{|u_i|} v_i + \frac{\sigma_i}{d|u_i|}(dv^i - dw^i) = -\frac{1}{|u_i|} \sum_{\lambda \ni i} \sigma_i w^i. \tag{2.22}$$

Lemma 4 implies that this last sum is proportional to the centroid of F_i^*, which completes the theorem.

It is easy to show that α in (2.22) equals $-V(P^*)$, so this equation reduces to

$$V(P^*)u_i = \frac{V(F_i^*)c_i^*}{|u_i|}, \tag{2.23}$$

where c_i^* is the centroid of F_i^*. Equation (2.23) is our final simplification of the critical equation (2.7).

§3. *Comments.* As one might expect, the duality between sections and projections carries over to a duality of the corresponding extremum problems. The following is a summary of those properties of extremal projections found in [7] which have analogues for sections.

(a) *Combinatorial regions.* According to Theorem 6, the volume of a section $P^* = T^{n-1} \cap L$ is a rational function on a region of $G(d, n)$. The hypothesis of Theorem 1 seems to indicate that this region depends on the triangulation Δ of P. However, we can produce a triangulation that gives $V(P^*)$ for all sections P^* having the same combinatorial type. Just triangulate the boundary of P and join the boundary to a generic interior point of P. This shows $V(T^{n-1} \cap L)$ is a piecewise rational function on $G(d, n)$, and the pieces correspond to regions of $G(d, n)$ in which the sections have the same combinatorial type. The open regions correspond to sections which are simple (or projections which are simplicial [7, Theorem 1]).

(b) *Isoperimetric problem.* As mentioned in Section 2, the critical conditions for generic sections of T^{n-1} are identical with the critical conditions for the volume of a simple d-polytope P^* such that $\sum l_i^2 = d$, where $1/l_i$ is the distance from the origin to the i-th facet of P^*. This fact, together with Lindelöff's theorem (see [6]), implies that a minimal section circumscribed about the sphere also solves the isoperimetric problem, *i.e.*, it has the smallest surface area among polytopes with a fixed volume and the same combinatorial type as P^*.

(c) *Differentiability.* Here we describe the differentiability of the function $f(L) = V((T^{n-1})^* \cap L)$. Clearly f is differentiable if L is in the open regions of (a).

A plane L lies on the boundary of these open regions precisely when $P^* = (T^{n-1})^* \cap L$ is not simple. Equivalently, the projection $P = \Pi(T^{n-1} : L)$ is not simplicial. Suppose distinct vertices of T^{n-1} project to distinct vertices or interior points of P. Let Δ be a triangulation of P. Then in any nearby plane L^+, $\partial\Delta$ becomes a star-shaped set with respect to the origin. Thus, we may triangulate P^+ by adding simplices Δ_λ^+ in P^+ outside Δ. These simplices will give rise to new terms containing $|\sigma_\lambda^+|$ and $|\sigma_{\lambda|j}^+|$ in the derivative (2.7) of f. However, both $|\sigma_\lambda^+|$ and $|\sigma_{\lambda|j}^+|$ approach 0 as L^+ approaches L, which shows f is differentiable at L.

The only case in which f fails to be differentiable is when two or more vertices of T^{n-1} project to the same vertex of P. In this case the surface areas $|\sigma^+_{\lambda|j}|$ do not all approach 0 as L^+ approaches L.

If f is differentiable at L and P is not simplicial, then the only place where the proof of Theorem 8 must be modified is the step which uses Lemma 4. However when $|\nu| = 1$, a simple argument shows the conclusion of Lemma 4 holds unless P has doubled vertices. But this is not possible since f was assumed to be differentiable at L. Therefore, Theorem 8 holds whenever f is differentiable.

(d) *Minimal sections.* By analogy with [7], we expect that minimal sections occur in the open regions of (a), *i.e.*, when P^* is simple. Such sections would satisfy the critical conditions in Theorems 7 and 8.

(e) *Maximal sections.* Maximal sections should occur at planes L where $V((T^{n-1})^* \cap L)$ is not differentiable. This is true for the maximal sections of the cube discovered by Ball in [**1**].

Comparing results in [**8**] on minimal projections, and results in [**5**] and [**18**] on maximal sections, we conjecture that the maximum d-section and the minimum d-projection of T^{n-1} are always identical.

(f) *Symmetrical sections.* A trivial consequence of Theorem 8 is that any polytope whose symmetry group is irreducible when restricted to each facet must be a critical section of a regular simplex. This includes all the simple regular polytopes and many semi-regular polytopes as well (see [**3**]). It may be possible to show some of these polytopes are minimal sections (and thus solve the isoperimetric problem) using the representation theoretic approach found in [**9**].

(g) *Planar sections.* Using the critical condition in Theorem 8 and Jensen's inequality, it is easy to show that the minimum section of T^{n-1} by a plane is an equilateral triangle. The other regular polygons are critical points which we expect to be saddle points (compare [**9**, Section 3]).

(h) *Hyperplane sections.* Section 3 of [**7**] implies that the volume of a section P^* of T^{n-1} can be written entirely in terms of the coordinates of an appropriate Gale transform of its polar dual P. This implies results on minimal sections with small codimension should be fairly easy to obtain. Using this method, it can be shown that the smallest hyperplane section of T^{n-1} is a regular $(n-2)$-simplex parallel to a facet of T^{n-1}.

(i) *Other polytopes.* Theorem 1 can be used to give an explicit formula for a d-dimensional section of any polytope in $\mathbf{R}^n$ in terms of a basis of the d-plane containing the section. However, simple geometric interpretations of the critical conditions probably do not exist except for sections of the regular cube and cross-polytope.

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