CHAPTER ONE

Grassmann algebra

This chapter presents a systematic account of Grassmann (exterior) algebra, with emphasis on aspects useful for geometric measure theory, and with strict adherence to the principles of naturality. The reader is assumed to be familiar with the category of vector spaces and linear maps, but no knowledge of multilinear algebra (or determinants) is presupposed. The field of scalars will be the field **R** of real numbers, except where another field is explicitly specified. Of course much of the theory is applicable more generally, even to modules.

The development of exterior algebra was begun over a century ago by H. Grassmann, and received its most significant impetus from the works of É. Cartan. A sketch of the classical history may be found in [BO, Livre II, Chapitre III]. Among the more recent improvements adopted here is the treatment of the exterior algebra as a Hopf algebra; the diagonal map was introduced in [CC] under the name "analyzing mapping". The concepts of mass and comass originated in [WH 4], the proof of Wirtinger's inequality is taken from [F 20], and the content of 1.4.5 from [F 15].

The two concluding sections of this chapter treat symmetric algebra by methods analogous to those used for exterior algebra in the earlier part. The definition of polynomial function was taken from [GG].

1.1. Tensor products

1.1.1. A function f which maps a cartesian product

$$V_1 \times V_2 \times \cdots \times V_n$$

of n vectorspaces $V_1, V_2, \ldots, V_n$ into some other vectorspace W is called n **linear** if and only if, for any i and any $v_j \in V_j$ corresponding to all $j \neq i$, the function on V_i carrying x into

$$f(v_1, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_n)$$

is a linear map of V_i into W.

By the **tensor product** of $V_1, \ldots, V_n$ one means a vectorspace

$$V_1 \otimes V_2 \otimes \cdots \otimes V_n$$

together with a particular n linear map

$$\mu\colon V_1 \times V_2 \times \cdots \times V_n \to V_1 \otimes V_2 \otimes \cdots \otimes V_n$$

which are jointly characterized as follows:

For each n linear map f of $V_1 \times \cdots \times V_n$ into any vectorspace W there exists a unique linear map g of $V_1 \otimes \cdots \otimes V_n$ into W such that $f = g \circ \mu$.

It is customary to write

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n$$

in place of $\mu(v_1, v_2, \ldots, v_n)$, whenever $v_j \in V_j$ for $j = 1, \ldots, n$.

The uniqueness (up to a linear isomorphism) of the tensor product follows immediately from the above characterization. To prove its existence one considers the one to one map

$$\phi\colon V_1 \times V_2 \times \cdots \times V_n \to F,$$

where F is the vectorspace consisting of those real valued functions on $V_1 \times \cdots \times V_n$ which vanish outside some (varying) finite set, and $\phi(v_1, \ldots, v_n)$ is the function with value 1 at $(v_1, \ldots, v_n)$ and value 0 elsewhere in $V_1 \times \cdots \times V_n$. Letting G be the vectorsubspace of F generated by all elements of the two types

$$\phi(v_1, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_n) + \phi(v_1, \ldots, v_{i-1}, y, v_{i+1}, \ldots, v_n)$$
$$- \phi(v_1, \ldots, v_{i-1}, x+y, v_{i+1}, \ldots, v_n)$$

and

$$\phi(v_1, \ldots, v_{i-1}, c\, v_i, v_{i+1}, \ldots, v_n) - c\, \phi(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_n)$$

with $c \in \mathbf{R}$, one defines $V_1 \otimes \cdots \otimes V_n = F/G$ and takes μ to be the composition of ϕ and the canonical map of F onto F/G.

The construction of tensor products is natural in the sense that *for any linear maps*

$$f_1\colon V_1 \to V_1', \ldots, f_n\colon V_n \to V_n'$$

there exists a unique linear map

$$f_1 \otimes \cdots \otimes f_n\colon V_1 \otimes \cdots \otimes V_n \to V_1' \otimes \cdots \otimes V_n'$$

such that

$$(f_1 \otimes \cdots \otimes f_n)(v_1 \otimes \cdots \otimes v_n) = f_1(v_1) \otimes \cdots \otimes f_n(v_n)$$

whenever $v_j \in V_j$ for $j = 1, \ldots, n$.

1.1.2. One frequently uses the following linear isomorphisms:

For each permutation λ of $\{1,\dots,n\}$,

$$V_1\otimes\cdots\otimes V_n\simeq V_{\lambda(1)}\otimes\cdots\otimes V_{\lambda(n)},$$

where $v_1\otimes\cdots\otimes v_n$ is mapped onto $v_{\lambda(1)}\otimes\cdots\otimes v_{\lambda(n)}$.

For $m<n$,

$$(V_1\otimes\cdots\otimes V_m)\otimes(V_{m+1}\otimes\cdots\otimes V_n)\simeq V_1\otimes\cdots\otimes V_n,$$

where $(v_1\otimes\cdots\otimes v_m)\otimes(v_{m+1}\otimes\cdots\otimes v_n)$ is mapped onto $v_1\otimes\cdots\otimes v_n$.

For each vectorspace V the scalar multiplication is a bilinear map of $\mathbf{R}\times V$ into V, inducing the isomorphism $\mathbf{R}\otimes V\simeq V$, where $c\otimes x$ is mapped onto cx.

If $V\simeq P\oplus Q$ (direct sum), then

$$V\otimes W\simeq(P\otimes W)\oplus(Q\otimes W).$$

In fact, if $f: V\to P$, $\phi: P\to V$, $g: V\to Q$, $\psi: Q\to V$ are linear maps for which

$$f\circ\phi=\mathbf{1}_P,\qquad g\circ\psi=\mathbf{1}_Q,\qquad \phi\circ f+\psi\circ g=\mathbf{1}_V,$$

then $f\otimes\mathbf{1}_W$, $\phi\otimes\mathbf{1}_W$, $g\otimes\mathbf{1}_W$, $\psi\otimes\mathbf{1}_W$ are linear maps for which

$$(f\otimes\mathbf{1}_W)\circ(\phi\otimes\mathbf{1}_W)=\mathbf{1}_{P\otimes W},\qquad (g\otimes\mathbf{1}_W)\circ(\psi\otimes\mathbf{1}_W)=\mathbf{1}_{Q\otimes W},$$

$$(\phi\otimes\mathbf{1}_W)\circ(f\otimes\mathbf{1}_W)+(\psi\otimes\mathbf{1}_W)\circ(g\otimes\mathbf{1}_W)=\mathbf{1}_{V\otimes W}.$$

It follows that *if B_j is a basis of V_j for each j, then the elements $b_1\otimes\cdots\otimes b_n$, with $b_j\in B_j$, form a basis of $V_1\otimes\cdots\otimes V_n$.* Consequently

$$\dim(V_1\otimes\cdots\otimes V_n)=\prod_{j=1}^{n}\dim V_j.$$

1.1.3. As an *illustrative example* we consider the special case where each V_j is the set of all real valued functions on some set S_j, W is the set of all real valued functions on the cartesian product $S_1\times\cdots\times S_n$, and

$$f(v_1,\dots,v_n)(s_1,\dots,s_n)=v_1(s_1)\cdot v_2(s_2)\dots v_n(s_n)$$

whenever $v_j\in V_j$ and $s_j\in S_j$ for $j=1,\dots,n$. Here the corresponding linear map g is a monomorphism, as seen by induction with respect to n; in case $n=2$ one readily verifies that

$$\sum_{k=1}^{m} v_{1,k}\otimes v_{2,k}\notin\ker g$$

whenever $v_{j,1}, v_{j,2},\dots,v_{j,m}$ are linearly independent elements of V_j, for $j=1,2$. We also observe that g is an epimorphism if and only if at least $n-1$ of the n sets S_j are finite.

1.1.4. We will use the natural linear transformation

$$\phi:\ \mathrm{Hom}(V,\mathbf{R})\otimes W\to\mathrm{Hom}(V,W);$$

for each linear function $\alpha: V\to\mathbf{R}$ and each $y\in W$, the linear function $\phi(\alpha\otimes y): V\to W$ maps $x\in V$ onto $\alpha(x)\cdot y\in W$. Making use of a basis of W, one readily sees that ϕ is always a monomorphism, and that ϕ is an epimorphism unless $\dim V=\infty=\dim W$.

We also recall that, if $n=\dim V<\infty$ and $e_1,\dots,e_n$ form a basis of V, then the **dual basis** of $\mathrm{Hom}(V,R)$ consists of the real valued linear functions $\omega_1,\dots,\omega_n$ characterized by the conditions

$$\omega_i(e_i)=1,\qquad \omega_i(e_j)=0\ \text{ for } i\neq j.$$

1.2. Graded algebras

1.2.1. For the purpose of this book a **graded algebra** will be a vectorspace A with a specified direct sum decomposition

$$A=\bigoplus_{m=0}^{\infty}A_m$$

and a bilinear function (multiplication) $\mu: A\times A\to A$ such that

$$\mu(A_m\times A_n)\subset A_{m+n}$$

whenever m and n are nonnegative integers; we ordinarily use a product notation, like $x\cdot y$, in place of $\mu(x,y)$.

Usually the multiplication will be **associative**, and there will be a linear isomorphism $\mathbf{R}\simeq A_0$ mapping 1 onto a unit element of the ring A.

Several, but not all, of the algebras considered will satisfy the **anti-commutative law:**

$$\eta\cdot\xi=(-1)^{mn}\xi\cdot\eta\ \text{ for }\ \xi\in A_m,\ \eta\in A_n.$$

1.2.2. If A and B are graded algebras, then the **graded tensor product**

$$A\otimes B=\bigoplus_{m=0}^{\infty}\ \bigoplus_{p+q=m}A_p\otimes B_q$$

can be made a graded algebra with either of the following two standard definitions of multiplication:

(1) To obtain the **commutative product** $A\otimes B$, let

$$(a\otimes b)\cdot(c\otimes d)=(a\cdot c)\otimes(b\cdot d)$$

whenever $a\in A$, $b\in B$, $c\in A$, $d\in B$.

(2) To obtain the **anticommutative product** $A \otimes B$, let

$$(a \otimes b) \cdot (c \otimes d) = (-1)^{qr} (a \cdot c) \otimes (b \cdot d)$$

whenever $a \in A_p$, $b \in B_q$, $c \in A_r$, $d \in B_s$.

These two definitions are motivated by the simple fact that the commutative product of two commutative algebras is a commutative algebra, while the anticommutative product of two anticommutative algebras is an anticommutative algebra. (Nevertheless it is sometimes also useful to consider the commutative product of two anticommutative algebras!)

The anticommutative products $A \otimes B$ and $B \otimes A$ are isomorphic; the standard isomorphism maps $a \otimes b$ onto $(-1)^{pq} b \otimes a$, whenever $a \in A_p$ and $b \in B_q$. (The analogous isomorphism of commutative products omits the factor $(-1)^{pq}$.)

For every graded algebra A there is a unique linear map

$$\Phi\colon\ A \otimes A \to A$$

such that $\Phi(x \otimes y) = x \cdot y$ whenever $x, y \in A$. In case A is an associative commutative (anticommutative) algebra, then Φ is a graded algebra homomorphism of the commutative (anticommutative) product $A \otimes A$ into A.

1.2.3. We shall now construct, for each vectorspace V, a particular graded algebra

$$\otimes_* V = \bigoplus_{n=0}^{\infty} \otimes_n V,$$

called the **tensor algebra** of V. We let

$$\otimes_0 V = \mathbf{R}, \qquad \otimes_1 V = V, \qquad \otimes_2 V = V \otimes V, \qquad \ldots;$$

in general $\otimes_m V$ is the m fold tensor product with all m factors equal to V. We define multiplication in $\otimes_* V$ so that its restriction to $\otimes_m V \times \otimes_n V$ is simply the (bilinear) composition

$$\otimes_m V \times \otimes_n V \to \otimes_m V \otimes \otimes_n V \simeq \otimes_{m+n} V.$$

One readily verifies the associative law and the fact that the element 1 of $\otimes_0 V$ is a unit element of the ring $\otimes_* V$.

Among all graded associative algebras with a unit, whose direct summand of index 1 is linearly isomorphic to V, the tensor algebra $\otimes_* V$ is characterized (up to isomorphism) by the following universal mapping property:

For every graded associative algebra A with a unit element, each linear map of V into A_1 can be uniquely extended to a unit preserving algebra homomorphism of $\otimes_* V$ into A, carrying $\otimes_m V$ into A_m for each m.

Finally we take note of the naturality of the construction $\otimes_*$: *Each linear map $f\colon V \to V'$ can be uniquely extended to a unit preserving algebra homomorphism*

$$\otimes_* f\colon\ \otimes_* V \to \otimes_* V'.$$

Moreover f is the direct sum of the linear maps

$$\otimes_m f\colon\ \otimes_m V \to \otimes_m V'.$$

1.3. The exterior algebra of a vectorspace

1.3.1. In the associative tensor algebra $\otimes_* V$ we consider the **two sided ideal** $\mathfrak{A} V$ generated by all the elements $x \otimes x$ in $\otimes_2 V$ corresponding to $x \in V$. The quotient algebra

$$\wedge_* V = \otimes_* V / \mathfrak{A} V$$

is called the **exterior algebra** of the vectorspace V. Clearly $\mathfrak{A} V$ is a homogeneous ideal, in fact

$$\mathfrak{A} V = \bigoplus_{m=2}^{\infty} (\otimes_m V \cap \mathfrak{A} V)$$

and therefore

$$\wedge_* V = \bigoplus_{m=0}^{\infty} \wedge_m V$$

where

$$\wedge_m V = \otimes_m V / (\otimes_m V \cap \mathfrak{A} V);$$

in particular $\wedge_0 V = \mathbf{R}$ and $\wedge_1 V = V$. The elements of $\wedge_m V$ are called **m-vectors of** V. The multiplication in $\wedge_* V$ is called **exterior multiplication** and denoted by the wedge symbol $\wedge$. It follows that if $v_1, \ldots, v_m \in V$, then the canonical homomorphism maps the product $v_1 \otimes \cdots \otimes v_m \in \otimes_m V$ onto the product

$$v_1 \wedge \cdots \wedge v_m \in \wedge_m V.$$

Clearly $\wedge_m V$ is the vectorspace generated by all such products.

If x and y belong to V, then

$$x \otimes y + y \otimes x = (x+y) \otimes (x+y) - x \otimes x - y \otimes y \in \mathfrak{A} V,$$

hence $y \wedge x = -x \wedge y$. Therefore

$$\begin{aligned}(v_{p+1} \wedge \cdots \wedge v_{p+q}) \wedge (v_1 \wedge \cdots \wedge v_p)\\ = (-1)^{pq} (v_1 \wedge \cdots \wedge v_p) \wedge (v_{p+1} \wedge \cdots \wedge v_{p+q})\end{aligned}$$

whenever $v_1, \ldots, v_{p+q} \in V$, which implies that *the anticommutative law holds for exterior multiplication.*

Among all anticommutative associative algebras with a unit, whose direct summand of index 1 is linearly isomorphic to V, the exterior algebra $\wedge_* V$ is characterized (up to isomorphism) by the following property:

For every anticommutative associative algebra A with a unit element, each linear map of V into A_1 can be uniquely extended to a unit preserving algebra homomorphism of $\wedge_* V$ into A, carrying $\wedge_m V$ into A_m for each m.

Such an extension is unique because the algebra $\wedge_* V$ is generated by $V \cup \{1\}$. To prove its existence, we recall that each linear map of V into A_1 can be extended to an algebra homomorphism h of $\otimes_* V$ into A; since A is anticommutative and $\mathbf{R}$ has characteristic different from 2, $a^2 = 0$ whenever $a \in A_1$, hence $\mathfrak{A} V \subset \ker h$, and h is divisible by the canonical homomorphism of $\otimes_* V$ onto $\wedge_* V$.

The construction $\wedge_*$ is natural: *Each linear map $f\colon V \to V'$ can be uniquely extended to a unit preserving algebra homomorphism*

$$\wedge_* f\colon \wedge_* V \to \wedge_* V'.$$

Moreover $\wedge_* f$ is the direct sum of the linear maps

$$\wedge_m f\colon \wedge_m V \to \wedge_m V'.$$

1.3.2. *The function $\wedge_*$ converts direct sums of vectorspaces into anticommutative products of algebras: If $V \simeq P \oplus Q$, then $\wedge_* V \simeq \wedge_* P \otimes \wedge_* Q$.*

In fact, if $f\colon V \to P$, $\phi\colon P \to V$, $g\colon V \to Q$, $\psi\colon Q \to V$ are linear maps for which

$$f \circ \phi = \mathbf{1}_P, \quad g \circ \psi = \mathbf{1}_Q, \quad \phi \circ f + \psi \circ g = \mathbf{1}_V,$$

then there is a unique unit preserving algebra homomorphism

$$\alpha\colon \wedge_* V \to \wedge_* P \otimes \wedge_* Q \text{ (anticommutative product)}$$

such that $\alpha(v) = f(v) \otimes 1 + 1 \otimes g(v)$ whenever $v \in V$; moreover the composition β of algebra homomorphisms

$$\wedge_* P \otimes \wedge_* Q \xrightarrow{\wedge_* \phi \otimes \wedge_* \psi} \wedge_* V \otimes \wedge_* V \xrightarrow{\Phi} \wedge_* V$$

(where Φ is induced by the multiplication of $\wedge_* V$) is inverse to α, because $\beta \circ \alpha$ and $\alpha \circ \beta$ induce identity maps on the direct summands of degree 1.

Since $\wedge_* \mathbf{R} = \mathbf{R} \oplus \mathbf{R}$, it follows that, *if $e_1, e_2, e_3, \ldots$ form a basis of V, then the products*

$$e_\lambda = e_{\lambda(1)} \wedge e_{\lambda(2)} \wedge \cdots \wedge e_{\lambda(m)}$$

corresponding to all increasing m termed sequences λ form a basis of $\wedge_m V$. In case V has finite dimension n, this implies that

$$\dim \wedge_m V = \binom{n}{m} \text{ for } m \leq n, \quad \wedge_m V = \{0\} \text{ for } m > n.$$

In fact $\wedge_m V$ has a basis equipotent with the set

$$\Lambda(n, m)$$

of all increasing maps of $\{1, \ldots, m\}$ into $\{1, \ldots, n\}$.

1.3.3. The **diagonal map** of $\wedge_* V$ is the unit preserving algebra homomorphism

$$\Psi\colon \wedge_* V \to \wedge_* V \otimes \wedge_* V \text{ (anticommutative product)}$$

such that $\Psi(v) = v \otimes 1 + 1 \otimes v$ whenever $v \in V$.

For $v_1, \ldots, v_m \in V$ we compute the product

$$\Psi(v_1 \wedge \cdots \wedge v_m) = \prod_{i=1}^{m} (v_i \otimes 1 + 1 \otimes v_i)$$

using the rules

$$(v_i \otimes 1) \cdot (1 \otimes v_j) = (v_i \otimes v_j) = -(1 \otimes v_j) \cdot (v_i \otimes 1).$$

The result can be conveniently expressed in terms of the notion of **shuffle of type** (p, q), meaning a permutation of $\{1, 2, \ldots, p+q\}$ which is increasing on each of the two sets $\{1, \ldots, p\}$ and $\{p+1, \ldots, p+q\}$. Letting $\mathrm{Sh}(p, q)$ be the set of all shuffles of type (p, q), we find that the product equals

$$\sum_{p=0}^{m} \sum_{\sigma \in \mathrm{Sh}(p, m-p)} \mathrm{index}(\sigma) \cdot (v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(p)}) \otimes (v_{\sigma(p+1)} \wedge \cdots \wedge v_{\sigma(m)}).$$

The **index** of any permutation σ equals $(-1)^N$, where N is the number of pairs (i, j) such that $i < j$ and $\sigma(i) > \sigma(j)$.

The diagonal map Ψ of $\wedge_* V$ is associative, which means that the following diagram is commutative:

$$\begin{array}{ccc} & \wedge_* V & \\ \swarrow^{\Psi} & & \searrow^{\Psi} \\ \wedge_* V \otimes \wedge_* V & & \wedge_* V \otimes \wedge_* V \\ \downarrow^{\Psi \otimes 1} & & \downarrow^{1 \otimes \Psi} \\ (\wedge_* V \otimes \wedge_* V) \otimes \wedge_* V & \simeq & \wedge_* V \otimes (\wedge_* V \otimes \wedge_* V) \end{array}$$

In fact the two vertical compositions of algebra homomorphisms agree on V, mapping v onto $v \otimes 1 \otimes 1 + 1 \otimes v \otimes 1 + 1 \otimes 1 \otimes v$.

The *diagonal map* Ψ *of* $\wedge_* V$ *is anticommutative*, which means that $\alpha \circ \Psi = \Psi$, where α is the automorphism of the algebra $\wedge_* V \otimes \wedge_* V$ which maps $x \otimes y$ onto $(-1)^{pq} y \otimes x$ whenever $x \in \wedge_p V$ and $y \in \wedge_q V$. This is true because $\alpha \circ \Psi$ and Ψ agree on V.

The diagonal map is a natural transformation: If f is a linear map of V into a vectorspace V', with diagonal map Ψ', then

$$\Psi' \circ \wedge_* f = (\wedge_* f \otimes \wedge_* f) \circ \Psi.$$

1.3.4. We conclude this section by defining and computing the **determinant** of a linear map $f: V \to V$, where $\infty > \dim V = n$. Since $\dim \wedge_n V = 1$, there exists a unique real number $\det(f)$ such that

$$(\wedge_n f)\, \xi = \det(f) \cdot \xi \quad \text{whenever} \quad \xi \in \wedge_n V.$$

Relative to any choice of base vectors $e_1, \ldots, e_n$ of V, the endomorphism f can be described by the matrix a consisting of real coefficients $a_{i,j}$ such that

$$f(e_i) = \sum_{j=1}^{n} a_{i,j}\, e_j \quad \text{for } j = 1, \ldots, n.$$

Then we find that

$$(\wedge_n f)(e_1 \wedge \cdots \wedge e_n) = f(e_1) \wedge \cdots \wedge f(e_n) = \sum_{\lambda} \left(\prod_{i=1}^{n} a_{i,\lambda(i)} \right) e_\lambda,$$

where the summation is over the set of all permutations λ of $\{1, \ldots, n\}$, and since $e_\lambda = \operatorname{index}(\lambda) \cdot e_1 \wedge \cdots \wedge e_n$ we obtain

$$\det(f) = \sum_{\lambda} \operatorname{index}(\lambda) \prod_{i=1}^{n} a_{i,\lambda(i)}.$$

If g is another endomorphism of V, then

$$\wedge_n(g \circ f) = (\wedge_n g) \circ (\wedge_n f), \quad \text{hence} \quad \det(g \circ f) = \det(g) \cdot \det(f).$$

Again using base vectors $e_1, \ldots, e_n$ of V we associate with each permutation λ of $\{1, \ldots, n\}$ the endomorphism $\phi(\lambda)$ of V which maps e_i onto $e_{\lambda(i)}$. Since ϕ and det are multiplicative homomorphisms, so is $\operatorname{index} = \det \circ \phi$.

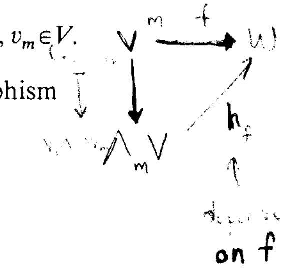
1.4. Alternating forms and duality

1.4.1. An m linear function f which maps the m fold cartesian product V^m of a vector space V into some other vectorspace W, is called **alternating** if and only if $f(v_1, \ldots, v_m) = 0$ whenever $v_1, \ldots, v_m \in V$ and $v_i = v_j$ for some $i \neq j$. We let

$$\wedge^m(V, W)$$

be the vectorspace of all m linear alternating functions (forms) mapping V^m into W. If $f \in \wedge^m(V, W)$ and $g: \otimes_m V \to W$ is the corresponding linear function, then $\mathfrak{A}\, V \cap \otimes_m V \subset \ker g$, hence there exists a unique linear function $h: \wedge_m V \to W$ such that

$$f(v_1, \ldots, v_m) = h(v_1 \wedge \cdots \wedge v_m) \quad \text{whenever} \quad v_1, \ldots, v_m \in V.$$



Thus associating h with f, we obtain the linear isomorphism

$$\wedge^m(V, W) \simeq \operatorname{Hom}(\wedge_m V, W).$$

Moreover there is an obvious linear isomorphism

$$\operatorname{Hom}(\wedge_m V, W) \simeq \operatorname{Hom}^m(\wedge_* V, W),$$

where the right side means the set of those linear maps of $\wedge_* V$ into W which vanish on $\wedge_n V$ whenever $n \neq m$. The above isomorphisms remain true for $m = 0$ with the convention $\wedge^0(V, W) = W$. We define

$$\wedge^*(V, W) = \bigoplus_{m=0}^{\infty} \wedge^m(V, W).$$

Most frequently we shall deal with the case when $W = \mathbf{R}$; we therefore abbreviate

$$\wedge^m(V, \mathbf{R}) = \wedge^m V, \; \wedge^*(V, \mathbf{R}) = \wedge^* V.$$

The elements of $\wedge^m V$ are called m-**covectors** of V.

In an extension of the usual notation

$$\langle \xi, h \rangle = h(\xi) \quad \text{for} \quad \xi \in \wedge_m V, \; h \in \operatorname{Hom}(\wedge_m V, W),$$

we shall also write $\langle \xi, f \rangle = \langle \xi, h \rangle = \langle \xi, k \rangle$ whenever $f \in \wedge^m(V, W)$ and $k \in \operatorname{Hom}^m(\wedge_* V, W)$ correspond to h under the above isomorphisms.

Each linear map $f: V \to V'$ induces a dual linear map

$$\wedge^*(f, W): \wedge^*(V', W) \to \wedge^*(V, W)$$

which is the direct sum of the linear maps

$$\wedge^m(f, W): \wedge^m(V', W) \to \wedge^m(V, W)$$

characterized by the equations

$$\langle \xi, \wedge^m(f, W)\, \phi \rangle = \langle (\wedge_m f)\, \xi, \phi \rangle$$

for $\xi \in \wedge_m V$ and $\phi \in \wedge^m(V', W)$.

We abbreviate $\wedge^*(f, \mathbf{R}) = \wedge^* f$.

For example, in case $V' = V$ with $\infty > \dim V = n$, then

$$(\wedge^n f)\,\phi = \det(f)\cdot\phi \quad\text{whenever}\quad \phi\in\wedge^n(V,W),$$

because, for each $\xi\in\wedge_n V$,

$$\langle \xi, \wedge^n(f,W)\,\phi\rangle = \langle(\wedge_n f)\,\xi,\phi\rangle = \langle\det(f)\cdot\xi,\phi\rangle = \langle\xi,\det(f)\cdot\phi\rangle.$$

1.4.2. Whenever W is an (ungraded) algebra over $\mathbf{R}$, we shall use the diagonal map Ψ of $\wedge_* V$ to turn the graded vectorspace $\wedge^*(V,W)$ into a graded algebra, called the **alternating algebra of V with coefficients in W.** Recalling that

$$\wedge^*(V,W) \simeq \bigoplus_{m=0}^{\infty} \operatorname{Hom}^m(\wedge_* V, W)$$

we define, for $\alpha\in\operatorname{Hom}^p(\wedge_* V,W)$ and $\beta\in\operatorname{Hom}^q(\wedge_* V,W)$, the product $\alpha\wedge\beta\in\operatorname{Hom}^{p+q}(\wedge_* V,W)$ to be the composition

$$\wedge_* V \xrightarrow{\Psi} \wedge_* V\otimes\wedge_* V \xrightarrow{\alpha\otimes\beta} W\otimes W \xrightarrow{\nu} W,$$

where $\nu(s\otimes t) = s\cdot t$ for $s,t\in W$. Taking account of the associativity, anticommutativity and naturality of Ψ, one easily verifies:

If the multiplication of W is associative, then $\wedge$ is associative.

If the multiplication of W is commutative, then $\wedge$ is anticommutative.

A multiplicative unit element of W acts also as a unit element for $\wedge$.

For each linear map $f\colon V\to V'$, the linear map $\wedge^*(f,W)$ is a multiplicative homomorphism.

The alternating product $\alpha\wedge\beta\in\wedge^{p+q}(V,W)$ of $\alpha\in\wedge^p(V,W)$ and $\beta\in\wedge^q(V,W)$ is defined through the isomorphisms $\wedge^m(V,W)\simeq\operatorname{Hom}^m(\wedge_* V,W)$. However, the shuffle formula for $\Psi(v_1\wedge\cdots\wedge v_m)$ leads immediately to the following explicit formula for $\alpha\wedge\beta$:

$$(\alpha\wedge\beta)(v_1,\dots,v_{p+q}) = \sum_{\sigma\in\operatorname{Sh}(p,q)} \operatorname{index}(\sigma)\cdot\alpha(v_{\sigma(1)},\dots,v_{\sigma(p)})\cdot\beta(v_{\sigma(p+1)},\dots,v_{\sigma(p+q)})$$

whenever $v_1,\dots,v_{p+q}\in V$. Assuming the multiplication of W to be associative, one readily obtains by induction a similar formula for the product of m alternating forms

$$\alpha_i\in\wedge^{p(i)}(V,W)$$

corresponding to $i=1,\dots,m$. Abbreviating

$$s(i) = \sum_{j\le i} p(j),$$

one defines a **shuffle of type** $[p(1),\dots,p(m)]$ as a permutation of the set $\{1,\dots,s(m)\}$ which is increasing on each of the m sets $\{s(i-1)+1,\dots,s(i)\}$, and one finds that

$$(\alpha_1\wedge\cdots\wedge\alpha_m)(v_1,\dots,v_{s(m)}) = \sum_{\sigma\in\operatorname{Sh}[p(1),\dots,p(m)]} \operatorname{index}(\sigma)\prod_{i=1}^{m}\alpha_i(v_{\sigma[s(i-1)+1]},\dots,v_{\sigma[s(i)]})$$

whenever $v_1,\dots,v_{s(m)}\in V$.

Therefore, in case $p(i)=1$ for $i=1,\dots,m$, we have

$$(\alpha_1\wedge\cdots\wedge\alpha_m)(v_1,\dots,v_m) = \sum_\sigma \operatorname{index}(\sigma)\prod_{i=1}^m \alpha_i(v_{\sigma(i)}),$$

where the summation extends over all permutations σ of $\{1,\dots,m\}$. In particular, if $\alpha_i(v_j)=0$ whenever $j<i$, we obtain $\prod_{i=1}^m \alpha_i(v_i)$.

1.4.3. Next we take $W=\mathbf{R}$ and observe that, *if $\omega_1,\dots,\omega_n$ are linearly independent in $\wedge^1 V$, and $m\le n$, then the products*

$$\omega_\lambda = \omega_{\lambda(1)}\wedge\cdots\wedge\omega_{\lambda(m)}$$

corresponding to all $\lambda\in\Lambda(n,m)$ are linearly independent in $\wedge^m V$. In fact, choosing $e_j\in V$ so that $\langle e_j,\omega_j\rangle=1$ and $\langle e_j,\omega_i\rangle=0$ whenever $i\neq j$, we find that $\langle e_\lambda,\omega_\lambda\rangle=1$ and $\langle e_\mu,\omega_\lambda\rangle=0$ whenever $\lambda\neq\mu\in\Lambda(n,m)$. *In case $\infty>\dim V=n$, then the products ω_λ form a basis of $\wedge^m V$,* because in this case

$$\dim\wedge^m V = \dim\wedge_m V = \binom{n}{m};$$

we also note the equations

$$\phi = \sum_{\lambda\in\Lambda(n,m)}\langle e_\lambda,\phi\rangle\,\omega_\lambda \quad\text{for}\quad \phi\in\wedge^m V,$$

$$\xi = \sum_{\lambda\in\Lambda(n,m)}\langle\xi,\omega_\lambda\rangle\,e_\lambda \quad\text{for}\quad \xi\in\wedge_m V.$$

The coefficients $\langle e_\lambda,\phi\rangle$ and $\langle\omega_\lambda,\xi\rangle$ are called **Grassmann coordinates** of ϕ and ξ, and will usually be denoted ϕ_λ and ξ_λ.

1.4.4. The identity $\operatorname{Hom}(V,\mathbf{R})=\wedge^1 V$ leads to a unique unit preserving algebra homomorphism

$$\Omega\colon \wedge_*\operatorname{Hom}(V,\mathbf{R})\to\wedge^* V$$

such that $\Omega(\alpha)=\alpha$ whenever $\alpha\in\operatorname{Hom}(V,\mathbf{R})$. We see that *$\Omega$ is a monomorphism*, because for each choice of base elements of $\operatorname{Hom}(V,\mathbf{R})$, Ω maps their m-fold exterior products, which form a base of $\wedge_m\operatorname{Hom}(V,\mathbf{R})$, onto their alternating products in $\wedge^m V$, which are linearly independent.

Moreover Ω *is an epimorphism in case* $\infty > \dim V = n$, because then $\wedge_m \operatorname{Hom}(V, \mathbf{R})$ and $\wedge^m V$ both have dimension $\binom{n}{m}$. If $\infty = \dim V$, then im Ω does not contain $\wedge^m V$ for any $m \geq 2$.

(While the identity $\operatorname{Hom}(V, W) = \wedge^1(V, W)$ allows one to define a similar algebra homomorphism of $\wedge_* \operatorname{Hom}(V, W)$ into $\wedge^*(V, W)$, for any associative algebra W with a unit element, this homomorphism is not injective unless $\dim W = 1$; it is surjective whenever $\dim V < \infty$. On the other hand, in case W is also commutative, one obtains a W linear homomorphism

$$\wedge_*^W \operatorname{Hom}(V, W) \to \wedge^*(V, W),$$

which is an isomorphism is case $\dim V < \infty$. Here $\wedge_*^W$ means the exterior algebra constructed with W replacing $\mathbf{R}$ as coefficient ring.)

1.4.5. For any two vectorspaces V and W the *commutative product* $\wedge^* V \otimes \wedge_* W$ is an associative algebra, which is neither commutative nor anticommutative (unless $V \simeq \mathbf{R}$ or $W \simeq \mathbf{R}$). However *the subalgebra*

$$A = \bigoplus_{m=0}^{\infty} \wedge^m V \otimes \wedge_m W$$

is commutative.

Assuming $\dim V = n < \infty$ we recall (1.1.4) the natural isomorphism

$$\wedge^1 V \otimes W \simeq \operatorname{Hom}(V, W),$$

whose inverse Γ can be computed as follows: If $e_1, \dots, e_n$ and $\omega_1, \dots, \omega_n$ are dual basic sequences of V and $\wedge^1 V$, then to each $f \in \operatorname{Hom}(V, W)$ corresponds

$$\Gamma(f) = \sum_{i=1}^{n} \omega_i \otimes f(e_i) \in \wedge^1 V \otimes W.$$

Using the multiplication in A we compute the divided m-th power

$$\Gamma(f)^m/m! = \sum_{\lambda \in \Lambda(n, m)} \omega_\lambda \otimes (\wedge_m f)\, e_\lambda = \Gamma(\wedge_m f),$$

where the symbol Γ on the right designates the inverse of the natural isomorphisms

$$\wedge^m V \otimes \wedge_m W \simeq \wedge^1(\wedge_m V) \otimes \wedge_m W \simeq \operatorname{Hom}(\wedge_m V, \wedge_m W).$$

In case $V = W$ we define trace $\in \operatorname{Hom}(A, \mathbf{R})$ so that

$$\operatorname{trace}(\phi \otimes \xi) = \langle \xi, \phi \rangle \quad \text{for } \phi \in \wedge^m V,\ \xi \in \wedge_m V,$$

and observe that

$$\operatorname{trace}[\zeta \cdot \Gamma(\mathbf{1}_{\wedge_j V})] = \binom{n-m}{j} \operatorname{trace}(\zeta)$$

for $\zeta \in \wedge^m V \otimes \wedge_m V$ and $j = 0, 1, \dots, n-m$. Moreover

$$\operatorname{trace}[\Gamma(f)^n/n!] = \operatorname{trace}[\Gamma(\wedge_n f)] = \det(f)$$

for $f \in \operatorname{Hom}(V, V)$. Using the binomial theorem we find that, whenever $t \in \mathbf{R}$,

$$\Gamma(t\,\mathbf{1}_V - f)^n/n! = [t\,\Gamma(\mathbf{1}_V) - \Gamma(f)]^n/n! = \sum_{m=0}^{n} t^{n-m}\,\Gamma(\mathbf{1}_{\wedge_{n-m} V})(-1)^m\,\Gamma(\wedge_m f),$$

hence the value of the **characteristic polynomial** of f at t equals

$$\det(t\,\mathbf{1}_V - f) = \sum_{m=0}^{n} t^{n-m}(-1)^m \operatorname{trace}[\Gamma(\wedge_m f)].$$

Similarly one obtains the formula

$$\det(\mathbf{1}_V + t\,f) = \sum_{m=0}^{n} t^m \operatorname{trace}[\Gamma(\wedge_m f)].$$

Hereafter we abbreviate $\operatorname{trace}[\Gamma(f)] = \operatorname{trace}(f)$.

If $f, g \in \operatorname{Hom}(V, V)$, then

$$\operatorname{trace}(f \circ g) = \operatorname{trace}(g \circ f);$$

in fact this equation is bilinear, and in the special case when $\Gamma(f) = \alpha \otimes v$, $\Gamma(g) = \beta \otimes w$ it holds because

$$\Gamma(f \circ g) - \Gamma(g \circ f) = \alpha(w)\,\beta \otimes v - \beta(v)\,\alpha \otimes w.$$

If $f \in \operatorname{Hom}(V, V)$, then $\operatorname{trace}(\wedge^1 f) = \operatorname{trace}(f)$.

1.5. Interior multiplications

1.5.1. These operations are bilinear maps

$$\lrcorner : \wedge_p V \times \wedge^q(V, W) \to \wedge^{q-p}(V, W)$$

$$\llcorner : \wedge_q V \times \wedge^p V \to \wedge_{q-p} V$$

defined for $p \leq q$, and characterized by the conditions:

$$\langle \xi, \eta \lrcorner \phi \rangle = \langle \xi \wedge \eta, \phi \rangle \quad \text{whenever } \xi \in \wedge_{q-p} V,\ \eta \in \wedge_p V,\ \phi \in \wedge^q(V, W);$$

$$\langle \zeta \llcorner \alpha, \beta \rangle = \langle \zeta, \alpha \wedge \beta \rangle \quad \text{whenever } \zeta \in \wedge_q V,\ \alpha \in \wedge^p V,\ \beta \in \wedge^{q-p} V.$$

The interior multiplications $\lrcorner$ and $\llcorner$ may be constructed by essentially dual procedures as follows:

Right exterior multiplication by η maps $\wedge_{q-p} V$ into $\wedge_q V$; the induced map

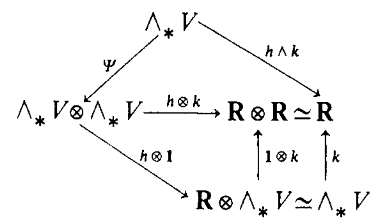
$$\operatorname{Hom}(\wedge_q V, W) \simeq \wedge^q(V, W) \to \operatorname{Hom}(\wedge_{q-p} V, W) \simeq \wedge^{q-p}(V, W)$$

carries ϕ onto $\eta \lrcorner \phi$.

The diagonal map Ψ of $\wedge_* V$ and the map $h \in \operatorname{Hom}^p(\wedge_* V, \mathbf{R})$ corresponding to α lead to the composition

$$\wedge_* V \xrightarrow{\Psi} \wedge_* V \otimes \wedge_* V \xrightarrow{h \otimes 1} \mathbf{R} \otimes \wedge_* V \simeq \wedge_* V$$

which carries ζ onto $\zeta \llcorner \alpha$. Letting $k \in \operatorname{Hom}^{q-p}(\wedge_* V, \mathbf{R})$ correspond to β, we derive the characteristic condition from the commutativity of the diagram:



We note that, whenever $r+s \leq t$,

$$(\xi \wedge \eta) \lrcorner \phi = \xi \lrcorner (\eta \lrcorner \phi) \text{ for } \xi \in \wedge_r V, \ \eta \in \wedge_s V, \ \phi \in \wedge^t V;$$

$$\zeta \llcorner (\alpha \wedge \beta) = (\zeta \llcorner \alpha) \llcorner \beta \text{ for } \xi \in \wedge_t V, \ \alpha \in \wedge^r V, \ \beta \in \wedge^s V.$$

1.5.2. If $n = \dim V < \infty$, $e_1, \ldots, e_n$ and $\omega_1, \ldots, \omega_n$ are dual bases of V and $\Lambda^1 V$, $\lambda \in \Lambda(n,p)$ and $\mu \in \Lambda(n,q)$, the characteristic conditions immediately yield the following values of the interior products $e_\lambda \lrcorner \omega_\mu$ and $e_\mu \llcorner \omega_\lambda$:

In case $\operatorname{im} \lambda \not\subset \operatorname{im} \mu$, then

$$e_\lambda \lrcorner \omega_\mu = 0 \quad \text{and} \quad e_\mu \llcorner \omega_\lambda = 0.$$

In case $\operatorname{im} \lambda \subset \operatorname{im} \mu$, then

$$e_\lambda \lrcorner \omega_\mu = (-1)^M \omega_\nu \quad \text{and} \quad e_\mu \llcorner \omega_\lambda = (-1)^N e_\nu,$$

where $\nu \in \Lambda(n, q-p)$, $\operatorname{im} \lambda \cup \operatorname{im} \nu = \operatorname{im} \mu$, and

$$M = \text{the number of pairs } (i,j) \in \operatorname{im} \lambda \times \operatorname{im} \nu \text{ with } i<j,$$

$$N = \text{the number of pairs } (i,j) \in \operatorname{im} \lambda \times \operatorname{im} \nu \text{ with } i>j.$$

Note that $M+N = p(q-p)$.

In particular, in case μ is the identity map of $\{1, \ldots, n\}$, one finds that right interior multiplication by ω_μ, and left interior multiplication by e_μ, give linear isomorphisms

$$\mathbf{D}_p \colon \wedge_p V \simeq \wedge^{n-p} V \quad \text{and} \quad \mathbf{D}^p \colon \wedge^p V \simeq \wedge_{n-p} V.$$

Moreover $\mathbf{D}_p$ and $\mathbf{D}^{n-p}$ are inverse to each other. Note that these isomorphisms depend only on the dual base vectors e_μ and ω_μ of $\wedge_n V$ and $\wedge^n V$.

The final equations of 1.5.1 imply, whenever $r+s \leq n$, that

$$\mathbf{D}_{r+s}(\xi \wedge \eta) = \xi \lrcorner \mathbf{D}_s \eta \text{ for } \xi \in \wedge_r V, \ \eta \in \wedge_s V;$$

$$\mathbf{D}^{r+s}(\alpha \wedge \beta) = (\mathbf{D}^r \alpha) \llcorner \beta \text{ for } \alpha \in \wedge^r V, \ \beta \in \wedge^s V.$$

1.5.3. *If $v \in V$ and $\alpha \in \wedge^1 V$ with $\langle v, \alpha \rangle = 1$, then*

$$\phi = v \lrcorner (\phi \wedge \alpha) + (v \lrcorner \phi) \wedge \alpha$$

whenever $\phi \in \wedge^j V$ with $j \geq 1$. One readily verifies this equation after expressing $\phi = \beta \wedge \alpha + \psi$ with $\beta \in \wedge^{j-1} V$, $\psi \in \wedge^j V$, $v \lrcorner \beta = 0$, $v \lrcorner \psi = 0$. The equation implies that

$$\phi \wedge \alpha = 0 \textit{ if and only if } \phi = \beta \wedge \alpha \textit{ for some } \beta \in \wedge^{j-1} V,$$

$$v \lrcorner \phi = 0 \textit{ if and only if } \phi = v \lrcorner \gamma \textit{ for some } \gamma \in \wedge^{j+1} V,$$

and yields the direct sum decomposition

$$\wedge^j V = \{\beta \wedge \alpha \colon \beta \in \wedge^{j-1} V\} \oplus \{v \lrcorner \gamma \colon \gamma \in \wedge^{j+1} V\}.$$

We also note the dual equation:

$$\xi = (v \wedge \xi) \llcorner \alpha + v \wedge (\xi \llcorner \alpha) \text{ for } \xi \in \wedge_j V.$$

1.6. Simple m-vectors

1.6.1. An element of $\wedge_m V$ is called **simple** (or **decomposable**) if and only if it equals the exterior product of m elements of V. **We shall see that there is a close connection between simple m-vectors and m dimensional vectorsubspaces of V.**

With each $\xi \in \wedge_m V$ we associate the vectorsubspace

$$T = V \cap \{v \colon \xi \wedge v = 0\}.$$

Assuming $\xi \neq 0$, we claim that $k = \dim T \leq m$ *and for any base vectors* $e_1, \ldots, e_k$ *of T there exists a $\xi' \in \wedge_{m-k} V$ such that*

$$\xi = e_1 \wedge \cdots \wedge e_k \wedge \xi'.$$

We observe that it suffices to verify this assertion in case $n = \dim V < \infty$, and choose $e_{k+1}, \ldots, e_n \in V$ so that $e_1, \ldots, e_n$ form a basis of V. Expanding

$$\xi = \sum_{\lambda \in \Lambda(n,m)} \xi_\lambda e_\lambda$$

and multiplying by e_i, where $i \leq k$, we find that $\xi_\lambda = 0$ unless $i \in \operatorname{im} \lambda$, because the products $e_\lambda \wedge e_i$ with $i \notin \operatorname{im} \lambda$ are linearly independent.

We deduce the following four corollaries:

A nonzero m-vector ξ is simple if and only if its associated subspace T has dimension m; in this case ξ equals the exterior product of m suitable base vectors of T.

The associated subspaces of two nonzero simple m-vectors ξ and η are equal if and only if $\xi = c\,\eta$ with $0 \neq c \in \mathbf{R}$.

If ξ is a nonzero simple m-vector and η is a nonzero simple n-vector, then $\xi \wedge \eta \neq 0$ if and only if the subspace associated with $\xi \wedge \eta$ is the direct sum of the two subspaces associated with ξ and η.

The subspace associated with a nonzero simple m-vector ξ is contained in the subspace associated with a nonzero simple n-vector η if and only if $\eta = \xi \wedge \zeta$ for some $\zeta \in \wedge_{n-m} V$.

1.6.2. The above association maps the set of all nonzero simple m-vectors of V onto the **Grassmann manifold**

$$\mathbf{G}(V, m)$$

of all m **dimensional subspaces of** V. With respect to this map, ξ and η are equivalent if and only if $\xi = c\,\eta$ for some nonzero real number c.

One also considers the following somewhat finer equivalence relation on the set of all nonzero simple m-vectors of V: ξ and η are equivalent if and only if $\xi = c\,\eta$ for some *positive* number c. The equivalence classes now obtained are called **oriented** m **dimensional subspaces of** V, and the identification space will be denoted

$$\mathbf{G}_0(V, m).$$

We shall abbreviate

$$\mathbf{G}(\mathbf{R}^n, m) = \mathbf{G}(n, m), \qquad \mathbf{G}_0(\mathbf{R}^n, m) = \mathbf{G}_0(n, m).$$

1.6.3. An element of $\wedge^m V$ is called **simple** (or **decomposable**) if and only if it equals the alternating product of m elements of $\wedge^1 V$. In case $n = \dim V < \infty$, the isomorphism (1.4.4)

$$\Omega\colon \wedge_* \wedge^1 V \simeq \wedge^* V$$

shows that simple m-covectors behave just like simple m-vectors; in particular, for $0 \neq \phi \in \wedge^m V$, the associated subspace

$$\wedge^1 V \cap \{\alpha\colon \phi \wedge \alpha = 0\}$$

has dimension $\leq m$, with equality holding if and only if ϕ is simple. Moreover the isomorphisms $\mathbf{D}_p$ and $\mathbf{D}^p$ defined in 1.5.2 preserve simplicity. *If T is the subspace associated with a nonzero simple p-vector ξ,*

then the subspace associated with the simple $n-p$ covector $\mathbf{D}_p(\xi)$ equals

$$\wedge^1 V \cap \{\alpha\colon \langle v, \alpha\rangle = 0 \text{ for all } v \in T\},$$

the annihilator of T. This is obvious because

$$\mathbf{D}_p(e_1 \wedge \cdots \wedge e_p) = (-1)^{p(n-p)}\, \omega_{p+1} \wedge \cdots \wedge \omega_n.$$

Since the annihilator of the intersection of two subspaces equals the vector sum of their annihilators, we obtain the following corollary:

If T and U are the subspaces associated with simple nonzero p- and r-vectors ξ and η, then

$$\dim(T \cap U) = p + r - n \text{ if and only if } \mathbf{D}_p(\xi) \wedge \mathbf{D}_r(\eta) \neq 0;$$

when these conditions hold, then

$$T \cap U \text{ is associated with } \mathbf{D}^{2n-p-r}[\mathbf{D}_p(\xi) \wedge \mathbf{D}_r(\eta)].$$

1.6.4. Regarding the dual pairing of simple m-vectors and simple m-covectors we shall prove:

If $e_1, \ldots, e_m \in V$, $\xi = e_1 \wedge \cdots \wedge e_m \neq 0$ and $\alpha_1, \ldots, \alpha_m \in \wedge^1 V$, then

$$\langle \xi, \alpha_1 \wedge \cdots \wedge \alpha_m \rangle = \det(f),$$

where f is the endomorphism of the subspace associated with ξ such that

$$f(e_i) = \sum_{j=1}^{m} \langle e_i, \alpha_j \rangle\, e_j \text{ for } i = 1, \ldots, m.$$

By the naturality of $\langle\ ,\ \rangle$ we may assume that $e_1, \ldots, e_m$ form a base of V, and choose $\omega_j \in \wedge^1 V$ so that $\langle e_j, \omega_j \rangle = 1$ and $\langle e_i, \omega_j \rangle = 0$ whenever $j \neq i$. Then $(\wedge^* f)\,\omega_j = \alpha_j$, hence

$$\alpha_1 \wedge \cdots \wedge \alpha_m = (\wedge^* f)(\omega_1 \wedge \cdots \wedge \omega_m) = \det(f) \cdot (\omega_1 \wedge \cdots \wedge \omega_m),$$
$$\langle \xi, \alpha_1 \wedge \cdots \wedge \alpha_m \rangle = \det(f) \cdot \langle \xi, \omega_1 \wedge \cdots \wedge \omega_m \rangle = \det(f).$$

1.6.5. Here we recall 1.5.2 and use the linear maps

$$P\colon V \to V \times V,\ Q\colon V \to V \times V,\ g\colon V \to V \times V,\ f\colon V \times V \to V,$$
$$P(x) = (x, 0),\ Q(x) = (0, x),\ g(x) = (x, x),\ f(x, y) = x - y \text{ for } x, y \in V.$$

If $\xi \in \wedge_k V$ and $\eta \in \wedge_l V$ with $k + l \geq n$, then

$$(\langle \xi, \wedge_k P \rangle \wedge \langle \eta, \wedge_l Q \rangle) \llcorner \langle \omega_1 \wedge \cdots \wedge \omega_n, \wedge^n f \rangle$$
$$= (-1)^{(n-k)l} \langle \mathbf{D}^{2n-k-l}(\mathbf{D}_k \xi \wedge \mathbf{D}_l \eta), \wedge_{k+l-n} g \rangle.$$

We observe that a change of dual bases multiplies both members of this equation by the determinant of the corresponding automorphism of V.

To verify the equation in case ξ and η are simple with $\mathbf{D}_k\xi\wedge\mathbf{D}_l\eta\neq 0$ we choose $e_1,\ldots,e_n$ and $\omega_1,\ldots,\omega_n$ so that

$$\xi=e_1\wedge\cdots\wedge e_k,\qquad \eta=e_{n-l+1}\wedge\cdots\wedge e_n,$$

and we compute

$$\mathbf{D}^{n-k+n-l}(\mathbf{D}_k\xi\wedge\mathbf{D}_l\eta)=\xi\llcorner\mathbf{D}_l\eta$$
$$=(e_1\wedge\cdots\wedge e_k)\llcorner\omega_1\wedge\cdots\wedge\omega_{n-l}=e_{n-l+1}\wedge\cdots\wedge e_k.$$

We also let $a_j=2^{-1}(e_j,-e_j)$ and $b_j=(e_j,e_j)$, note that

$$P(e_j)\wedge Q(e_j)=a_j\wedge b_j \text{ for } j\in\{1,\ldots,n\},$$

and infer that, for all $\phi\in\wedge^{k+l-n}(V\times V)$,

$$\langle P(e_1)\wedge\cdots\wedge P(e_k)\wedge Q(e_{n-l+1})\wedge\cdots\wedge Q(e_n),(\omega_1\circ f)\wedge\cdots\wedge(\omega_n\circ f)\wedge\phi\rangle$$
$$=\langle P(e_1)\wedge\cdots\wedge P(e_{n-l})\wedge a_{n-l+1}\wedge\cdots\wedge a_k\wedge b_{n-l+1}\wedge\cdots\wedge b_k$$
$$\wedge Q(e_{k+1})\wedge\cdots\wedge Q(e_n),(\omega_1\circ f)\wedge\cdots\wedge(\omega_n\circ f)\wedge\phi\rangle$$
$$=(-1)^{(n-k)(k+l-n)+n-k}\langle b_{n-l+1}\wedge\cdots\wedge b_k,\phi\rangle.$$

1.6.6. Assuming that V and W are vectorspaces over the field $\mathbf{C}$ of complex numbers we define

$$\wedge^m_{\mathbf{C}}(V,W)$$

as the subset of $\wedge^m(V,W)$ consisting of those forms ϕ for which

$$\phi(v_1,\ldots,v_{j-1},c\,v_j,v_{j+1},\ldots,v_m)=c\,\phi(v_1,\ldots,v_m)$$

whenever $j\in\{1,\ldots,m\}$, $c\in\mathbf{C}$ and $v_1,\ldots,v_m\in V$. We also let

$$\wedge^*_{\mathbf{C}}(V,W)=\bigoplus_{m=0}^{\infty}\wedge^m_{\mathbf{C}}(V,W).$$

Clearly $\wedge^*_{\mathbf{C}}(V,\mathbf{C})$ is a $\mathbf{C}$ subalgebra of $\wedge^*(V,\mathbf{C})$.

Complex conjugation is an automorphism of $\wedge^*(V,\mathbf{C})$. To each $\alpha\in\wedge^m(V,\mathbf{C})$ correspond $\sigma,\tau\in\wedge^m(V,\mathbf{R})$ such that $\alpha=\sigma+\mathbf{i}\,\tau$, $\bar\alpha=\sigma-\mathbf{i}\,\tau$, hence

$$\alpha\wedge\bar\alpha=\sigma\wedge\sigma+\tau\wedge\tau \text{ for even } m,\quad \alpha\wedge\bar\alpha=-2i\,\sigma\wedge\tau \text{ for odd } m.$$

If $\varepsilon_1,\ldots,\varepsilon_n$ and $\alpha_1,\ldots,\alpha_m$ are dual $\mathbf{C}$ bases of V and $\wedge^1_{\mathbf{C}}(V,\mathbf{C})$, and if $\alpha_j=\sigma_j+\mathbf{i}\,\tau_j$ with $\sigma_j,\tau_j\in\wedge^1(V,\mathbf{R})$, then

$$\varepsilon_1,\mathbf{i}\,\varepsilon_1,\ldots,\varepsilon_n,\mathbf{i}\,\varepsilon_n \quad\text{and}\quad \sigma_1,\tau_1,\ldots,\sigma_n,\tau_n$$

are dual $\mathbf{R}$ bases for V and $\wedge^1(V,\mathbf{R})$. Moreover the products

$$\alpha_{\lambda(1)}\wedge\cdots\wedge\alpha_{\lambda(p)}\wedge\bar\alpha_{\mu(1)}\wedge\cdots\wedge\bar\alpha_{\mu(q)}$$

corresponding to all $\lambda\in\Lambda(n,p)$, $\mu\in\Lambda(n,q)$ with $p+q=m$ form a $\mathbf{C}$ base of $\wedge^m(V,\mathbf{C})$, and those products which correspond to $p=m$, $q=0$ form a $\mathbf{C}$ base of $\wedge^m_{\mathbf{C}}(V,\mathbf{C})$. We also note that

$$\sigma_1\wedge\tau_1\wedge\cdots\wedge\sigma_n\wedge\tau_n=(\mathbf{i}/2)^n\alpha_1\wedge\bar\alpha_1\wedge\cdots\wedge\alpha_n\wedge\bar\alpha_n$$
$$=(\mathbf{i}/2)^n(-1)^{n(n-1)/2}\alpha_1\wedge\cdots\wedge\alpha_n\wedge\bar\alpha_1\wedge\cdots\wedge\bar\alpha_n.$$

If f is any $\mathbf{C}$ linear endomorphism of V, then $\det(f)\geq 0$. To prove this we choose bases as above and observe that

$$(\alpha_1\circ f)\wedge\cdots\wedge(\alpha_n\circ f)=d\alpha_1\wedge\cdots\wedge\alpha_n$$

for some complex number d (the $\mathbf{C}$ determinant of f), hence

$$(\bar\alpha_1\circ f)\wedge\cdots\wedge(\bar\alpha_n\circ f)=\bar d\bar\alpha_1\wedge\cdots\wedge\bar\alpha_n,$$
$$(\wedge^{2n}f)(\sigma_1\wedge\tau_1\wedge\cdots\wedge\sigma_n\wedge\tau_n)=d\,\bar d\sigma_1\wedge\tau_1\wedge\cdots\wedge\sigma_n\wedge\tau_n$$

and $\det(f)=d\,\bar d=|d|^2\geq 0$.

A nonzero simple m vector $\xi\in\wedge_m V$ is called **complex** if and only if the $\mathbf{R}$ vectorsubspace of V associated with ξ is a $\mathbf{C}$ vectorsubspace of V. It follows that ξ is complex if and only if m is even, say $m=2p$, and

$$\xi=r\,v_1\wedge\mathbf{i}\,v_1\wedge\cdots\wedge v_p\wedge\mathbf{i}\,v_p$$

for some $r\in\mathbf{R}$ and $v_1,\ldots,v_p\in V$. Moreover $\operatorname{sign}(r)$ is uniquely determined by ξ, because for any two $\mathbf{C}$ bases $v_1,\ldots,v_p$ and $w_1,\ldots,w_p$ of the vectorspace associated with ξ there exists a $\mathbf{C}$ linear automorphism g of this vectorspace which maps v_j onto w_j, hence

$$w_1\wedge\mathbf{i}\,w_1\wedge\cdots\wedge w_p\wedge\mathbf{i}\,w_p=\det(g)\,v_1\wedge\mathbf{i}\,v_1\wedge\cdots\wedge v_p\wedge\mathbf{i}\,v_p,$$

with $\det(g)>0$ according to the preceding paragraph. We term ξ **positive** in case $r>0$.

1.7. Inner products

1.7.1. We recall that the bilinear functions

$$B\colon V\times V\to\mathbf{R}$$

are in one to one correspondence with the linear maps

$$\beta\colon V\to\wedge^1 V$$

through the connecting formula

$$B(x,y)=\langle x,\beta(y)\rangle \text{ for } x,y\in V.$$

One calls B **symmetric** if and only if $B(x,y)=B(y,x)$ whenever $x,y\in V$; in this case we call the corresponding linear map β a **polarity**.

An **inner product** is a symmetric bilinear function B satisfying the condition

$$B(x,x)>0 \text{ if and only if } 0\neq x\in V.$$

When discussing a vectorspace V with a particular fixed inner product B, we generally use the dot product notation $x\bullet y$ in place of $B(x,y)$, and also define the **norm**

$$|x|=(x\bullet x)^{\frac{1}{2}}.$$

For example we usually write

$$x\bullet y=\sum_{i=1}^{n} x_i y_i \text{ for } x=(x_1,\dots,x_n) \text{ and } y=(y_1,\dots,y_n)\in\mathbf{R}^n.$$

Assuming henceforth that $\bullet$ is an inner product for V, we use the fact that

$$t^2(x\bullet x)+2t(x\bullet y)+(y\bullet y)=(t\,x+y)\bullet(t\,x+y)\geq 0$$

whenever $t\in\mathbf{R}$ to obtain the inequalities

$$x\bullet y\leq|x|\cdot|y|, \quad\text{hence}\quad |x+y|\leq|x|+|y|$$

for $x,y\in V$; both inequalities are strict in case x and y are linearly independent.

A sequence $v_1,\dots,v_p$ satisfying the conditions $v_i\bullet v_i=1$, and $v_i\bullet v_j=0$ for $i\neq j$, is called **orthonormal**. For every linearly independent sequence $u_1,\dots,u_p\in V$ there exists an orthonormal sequence $v_1,\dots,v_p$ such that, for $k=1,\dots,p$ the sets $\{u_1,\dots,u_k\}$ and $\{v_1,\dots,v_k\}$ generate the same vector subspace of V. Therefore, in case $\dim V<\infty$, V has an orthonormal base.

We metrize V so that the distance between x and y equals $|x-y|$. It follows that V is boundedly compact if and only if $\dim V<\infty$.

1.7.2. A linear map $f\colon V\to V'$, where V' is another vectorspace with an inner product (also denoted by $\bullet$) is called an **orthogonal injection** if and only if $f(x)\bullet f(y)=x\bullet y$ whenever $x,y\in V$. The set of all orthogonal injections of $\mathbf{R}^m$ into $\mathbf{R}^n$ will be denoted

$$\mathbf{O}(n,m).$$

Moreover $\mathbf{O}(n)=\mathbf{O}(n,n)$ is the orthogonal group of $\mathbf{R}^n$.

In case $\dim V<\infty$, the polarity corresponding to the inner product of V is a linear isomorphism of V onto $\wedge^1 V$, and one endows $\wedge^1 V$ with the inner product which makes this polarity orthogonal. The resulting norms on V and $\wedge^1 V$ are dual, in the sense that

$$|\alpha|=\sup\{\langle x,\alpha\rangle\colon x\in V, |x|\leq 1\}$$

whenever $\alpha\in\wedge^1 V$. Moreover the polarity maps each orthonormal basis of V onto the dual basis of $\wedge^1 V$.

1.7.3. *For each symmetric bilinear function* $S\colon V\times V\to\mathbf{R}$, *with* $\dim V<\infty$, *there exists an orthonormal base* $e_1,\dots,e_n$ *of* V *such that*

$$S(e_i,e_i)\geq S(e_j,e_j) \quad\textit{and}\quad S(e_i,e_j)=0 \textit{ for } i<j.$$

Proceeding inductively, one may choose e_i in the compact set

$$C_i=\{x\colon |x|=1,\ x\bullet e_k=0 \text{ whenever } k<i\}$$

so that $S(e_i,e_i)\geq S(x,x)$ for $x\in C_i$; for $i<j$ the fact that

$$|e_i+t\,e_j|^{-1}(e_i+t\,e_j)\in C_i$$

whenever $t\in\mathbf{R}$ implies $S(e_i,e_j)=0$.

An endomorphism f of V is called

symmetric in case $f(x)\bullet y=x\bullet f(y)$ for $x,y\in V$,

skewsymmetric in case $f(x)\bullet y=-x\bullet f(y)$ for $x,y\in V$.

Every symmetric or skewsymmetric endomorphism f is *orthogonally reducible* in the following sense: *If W is a subspace of V, with the orthogonal complement*

$$W'=V\cap\{x\colon x\bullet y=0 \text{ for all } y\in W\},$$

then $f(W)\subset W$ implies $f(W')\subset W'$. In case $\dim V<\infty$ it follows that V is the direct sum of finitely many mutually orthogonal subspaces W which are *minimal* with respect to the property that $f(W)\subset W$.

In case f is symmetric, these minimal subspaces have dimension 1. One associates with f the symmetric bilinear function S such that $S(x,y)=f(x)\bullet y$ for $x,y\in V$, and obtains an orthonormal base $e_1,\dots,e_n$ of V such that $f(e_i)\bullet e_j=0$ for $i\neq j$, hence $f(e_i)=\lambda_i e_i$ with $\lambda_i\in R$.

In case f is skewsymmetric, the minimal subspaces have dimension ≤ 2. This is true because $f^2=f\circ f$ is symmetric, and $f^2(e_i)=\lambda_i e_i$ implies that f maps the subspace generated by e_i and $f(e_i)$ into itself. As a corollary one obtains the following representation of alternating 2-forms:

If $\phi\in\wedge^2 V$, with $\dim V<\infty$, then there exists an orthonormal sequence $\omega_1,\omega_2,\dots,\omega_{2m-1},\omega_{2m}\in\wedge^1 V$ and a sequence of nonnegative numbers $\lambda_1,\dots,\lambda_m$ such that

$$\phi=\sum_{j=1}^{m}\lambda_j\,\omega_{2j-1}\wedge\omega_{2j}.$$

This is trivial if $\dim V\leq 2$. To obtain the general case we consider the skewsymmetric endomorphism f of V such that $f(x)\bullet y=\phi(x,y)$ for

$x, y \in V$, we decompose V into the direct sum of mutually orthogonal subspaces $W_1, \dots, W_s$ with $f(W_j) \subset W_j$ and $\dim W_j \leq 2$, and observe that $\phi(x,y) = 0$ whenever $x \in W_j$, $y \in W_k$, $j \neq k$.

If f: $V \to V'$ is a linear map of inner product spaces, with $\dim V < \infty$, *then V has an orthonormal base $e_1, \dots, e_n$ such that $f(e_i) \bullet f(e_j) = 0$ for $i \neq j$. In case* $\dim V \leq \dim V'$, *there exists a symmetric endomorphism g of V and an orthogonal injection h: $V \to V'$ such that $h \circ g = f$.* Choosing e_i adapted to the symmetric bilinear function S such that $S(x,y) = f(x) \bullet f(y)$ for $x, y \in V$, one may define

$$g(e_i) = |f(e_i)| \cdot e_i \text{ for all } i,$$

and choose h so that

$$h(e_i) = |f(e_i)|^{-1} \cdot f(e_i) \text{ whenever } f(e_i) \neq 0,$$

while h maps $\ker f$ orthogonally into the orthogonal complement of $\operatorname{im} f$ in V'. Similarly, *in case* $\dim V \geq \dim V'$ *there exists a symmetric endomorphism k of V' and* (*see* 1.7.4) *an orthogonal projection p: $V \to V'$ such that $k \circ p = f$*; this assertion may be proved by applying the preceding proposition to f^*.

1.7.4. Now suppose V and V' are finite dimensional vector spaces with inner products, and with the corresponding polarities β and β'. With each linear map f: $V \to V'$ one associates the **adjoint** linear map f^*: $V' \to V$ by means of the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\beta} & \wedge^1 V \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle \wedge^1 f} \\ V' & \xrightarrow{\beta'} & \wedge^1 V' \end{array}$$

or equivalently by the condition

$$x \bullet f^*(y) = f(x) \bullet y \text{ for } x \in V,\ y \in V'.$$

If g: $V' \to V''$ is also linear, then $(g \circ f)^* = f^* \circ g^*$.

In case $V = V'$, f is symmetric if and only if $f^* = f$; f is skewsymmetric if and only if $f^* = -f$.

Always $f^{**} = f$. The endomorphisms $f^* \circ f$, $f \circ f^*$ of V, V' are symmetric.

We observe that f is an orthogonal injection if and only if $f^* \circ f = \mathbf{1}_V$. In case f is an orthogonal injection, we call f^* an **orthogonal projection.** Hence a linear map g: $V' \to V$ is an orthogonal projection if and only if $g \circ g^* = \mathbf{1}_V$. We shall frequently consider the set

$$\mathbf{O}^*(n,m) = \{f^*\colon f \in \mathbf{O}(n,m)\}$$

of all orthogonal projections of $\mathbf{R}^n$ onto $\mathbf{R}^m$. In particular, to each $\lambda \in \Lambda(n,m)$ corresponds the map $\mathbf{p}_\lambda \in \mathbf{O}^*(n,m)$ such that

$$\mathbf{p}_\lambda(x) = (x_{\lambda(1)}, \dots, x_{\lambda(m)}) \text{ for } x = (x_1, \dots, x_n) \in \mathbf{R}^n.$$

If W is any m dimensional vectorsubspace of $\mathbf{R}^n$, with the inclusion map h: $W \to \mathbf{R}^n$, then $\wedge^m h$ is an epimorphism, hence $(\wedge^m h)\,\omega_\lambda \neq 0$ for some $\lambda \in \Lambda(n,m)$, where $\omega_1, \dots, \omega_n$ form the standard base of $\wedge^1 \mathbf{R}^n$; since ω_λ generates $\operatorname{im} \wedge^m \mathbf{p}_\lambda$, it follows that $\wedge^m(\mathbf{p}_\lambda \circ h) \neq 0$, $\mathbf{p}_\lambda \circ h$ is an isomorphism, hence

$$W \cap \ker \mathbf{p}_\lambda = \{0\}.$$

If f: $V \to W_1$, g: $V \to W_2$ are orthogonal projections and W_3 is an inner product space such that

$$\dim W_1 + \dim W_2 - \dim V \geq \dim W_3,$$

then there exist orthogonal projections p: $W_1 \to W_3$, q: $W_2 \to W_3$ with $p \circ f = q \circ g$. In fact, since $\dim(\operatorname{im} f^* \cap \operatorname{im} g^*) \geq \dim W_3$, there exist orthogonal injections u: $W_3 \to W_1$, v: $W_3 \to W_2$ with $f^* \circ u = g^* \circ v$, and we can take $p = u^*$, $q = v^*$.

1.7.5. Next we discuss the manner in which *inner products for the spaces $\wedge_m V$ are induced by the given inner product for V*: The polarity β: $V \to \wedge^1 V$ can be uniquely extended to a unit preserving algebra homomorphism γ: $\wedge_* V \to \wedge^* V$, which is the direct sum of linear maps

$$\gamma_m\colon \wedge_m V \to \wedge^m V.$$

Composing γ_m with $\wedge^m V \simeq \wedge^1(\wedge_m V)$, we obtain linear maps

$$\beta_m\colon \wedge_m V \to \wedge^1(\wedge_m V)$$

which satisfy the condition

$$\langle \xi, \beta_m(\eta) \rangle = \langle \eta, \beta_m(\xi) \rangle \text{ for } \xi, \eta \in \wedge_m V.$$

It suffices to verify that this holds true if ξ and η are simple, say

$$\xi = v_1 \wedge \cdots \wedge v_m \text{ and } \eta = w_1 \wedge \cdots \wedge w_m,$$

in which case the permutation formula for the alternating product of 1-forms (1.4.2) gives

$$\begin{aligned} \langle \xi, \beta_m(\eta) \rangle &= \langle v_1 \wedge \cdots \wedge v_m, \beta(w_1) \wedge \cdots \wedge \beta(w_m) \rangle \\ &= \sum_\sigma \operatorname{index}(\sigma) \prod_{i=1}^m v_{\sigma(i)} \bullet w_i = \sum_\sigma \operatorname{index}(\sigma^{-1}) \prod_{j=1}^m v_j \bullet w_{\sigma^{-1}(j)} \\ &= \langle w_1 \wedge \cdots \wedge w_m, \beta(v_1) \wedge \cdots \wedge \beta(v_m) \rangle = \langle \eta, \beta_m(\xi) \rangle. \end{aligned}$$

Thus β_m is a polarity, and we define a symmetric bilinear function $\bullet$ on $\wedge_m V \times \wedge_m V$ by the formula

$$\xi \bullet \eta = \langle \xi, \beta_m(\eta) \rangle \text{ for } \xi, \eta \in \wedge_m V.$$

We shall soon see that $\xi \bullet \xi > 0$ in case $\xi \neq 0$, so that $\bullet$ is in fact an inner product for $\wedge_m V$.

The above permutation formula shows that if some w_i is orthogonal to all v_j, then $(v_1 \wedge \cdots \wedge v_m) \bullet (w_1 \wedge \cdots \wedge w_m) = 0$.

For any $v_1, \ldots, v_m \in V$ we can express $v_i = u_i + w_i$, where u_i belongs to the subspace generated by $\{v_k : k < i\}$ and w_i is orthogonal to this subspace; then $v_1 \wedge \cdots \wedge v_m = w_1 \wedge \cdots \wedge w_m$ and the permutation formula implies

$$(v_1 \wedge \cdots \wedge v_m) \bullet (v_1 \wedge \cdots \wedge v_m) = \prod_{i=1}^{m} v_i \bullet w_i \leq \prod_{i=1}^{m} v_i \bullet v_i,$$

equality holding if and only if $v_1, \ldots, v_m$ are mutually orthogonal.

Therefore, if $e_1, \ldots, e_n$ form an orthonormal base for V, then the base vectors e_λ of $\wedge_m V$, corresponding to $\lambda \in \Lambda(n,m)$, are likewise orthonormal. For any m-vectors ξ and η the representations

$$\xi = \sum_{\lambda \in \Lambda(n,m)} \xi_\lambda e_\lambda, \qquad \eta = \sum_{\lambda \in \Lambda(n,m)} \eta_\lambda e_\lambda$$

and the bilinearity of $\bullet$ lead to the formula

$$\xi \bullet \eta = \sum_{\lambda \in \Lambda(n,m)} \xi_\lambda \eta_\lambda.$$

In case $\xi = \eta \neq 0$, we obtain

$$\xi \bullet \xi = \sum_\lambda (\xi_\lambda)^2 > 0.$$

One now readily estimates the norm of the exterior product of a p-vector ξ and a q-vector η:

In case ξ or η is simple, then $|\xi \wedge \eta| \leq |\xi| \cdot |\eta|$.

In case both ξ and η are simple and nonzero, equality holds if and only if the subspaces associated with ξ and η are orthogonal.

Always

$$|\xi \wedge \eta| \leq \binom{p+q}{p}^{\frac{1}{2}} |\xi| \cdot |\eta|.$$

To prove the last inequality we represent

$$\xi = \sum_{\lambda \in \Lambda(n,p)} \xi_\lambda e_\lambda, \qquad \eta = \sum_{\mu \in \Lambda(n,q)} \eta_\mu e_\mu$$

with $\xi_\lambda, \eta_\mu \in \mathbf{R}$, we define

$$S(v) = \{(\lambda, \mu) : \lambda \in \Lambda(n,p), \mu \in \Lambda(n,q), e_\lambda \wedge e_\mu = \pm e_v\}$$

whenever $v \in \Lambda(n, p+q)$, and observe that

$$\begin{aligned} |\xi \wedge \eta|^2 &\leq \sum_{v \in \Lambda(n,p+q)} \Big(\sum_{(\lambda,\mu) \in S(v)} |\xi_\lambda \eta_\mu| \Big)^2 \\ &\leq \sum_{v \in \Lambda(n,p+q)} \operatorname{card} S(v) \sum_{(\lambda,\mu) \in S(v)} (\xi_\lambda \eta_\mu)^2 \\ &\leq \binom{p+q}{p} \sum_{\lambda \in \Lambda(n,p)} (\xi_\lambda)^2 \sum_{\mu \in \Lambda(n,q)} (\eta_\mu)^2. \end{aligned}$$

The maps β, γ_m, β_m occurring in the preceding construction are related by the commutative diagram:

$$\begin{array}{ccccc} & & \wedge_m V & & \\ & \swarrow^{\wedge_m \beta} & \downarrow^{\gamma_m} & \searrow^{\beta_m} & \\ \wedge_m \wedge^1 V & \xrightarrow{\Omega} & \wedge^m V & \simeq & \wedge^1 \wedge_m V \end{array}$$

In case $\dim V < \infty$, the maps β_m and γ_m are linear isomorphisms, and one endows $\wedge^1 \wedge_m V$ and $\wedge^m V$ with inner product so that β_m and γ_m become orthogonal. If $e_1, \ldots, e_n$ and $\omega_1, \ldots, \omega_n$ are dual orthogonal bases of V and $\wedge^1 V$, then $\gamma_m(e_\lambda) = \omega_\lambda$ for $\lambda \in \Lambda(n,m)$. Also

$$|\langle \xi, \phi \rangle| \leq |\xi| \cdot |\phi| \text{ whenever } \xi \in \wedge_m V, \ \phi \in \wedge^m V;$$

equality holds if and only if $\gamma_m(\xi)$ and ϕ are linearly dependent.

1.7.6. Suppose $f: V \to V'$ is a linear map, where V and V' are finite dimensional inner product spaces. From the commutative diagram

$$\begin{array}{ccccccc} \wedge_m V & \xrightarrow{\wedge_m \beta} & \wedge_m \wedge^1 V & \simeq & \wedge^m V & \simeq & \wedge^1 \wedge_m V \\ \uparrow{\scriptstyle \wedge_m f^*} & & \uparrow{\scriptstyle \wedge_m \wedge^1 f} & & \uparrow{\scriptstyle \wedge^m f} & & \uparrow{\scriptstyle \wedge^1 \wedge_m f} \\ \wedge_m V' & \xrightarrow{\wedge_m \beta'} & \wedge_m \wedge^1 V' & \simeq & \wedge^m V' & \simeq & \wedge^1 \wedge_m V' \end{array}$$

we see that $\wedge_m f^* = (\wedge_m f)^*$.

In case $V = V'$, $\det(f^*) = \det(f)$ and $\operatorname{trace}(f^*) = \operatorname{trace}(f)$. If f is symmetric, so is $\wedge_m f$. If f is skewsymmetric, then $(\wedge_m f)^* = (-1)^m \wedge_m f$. If f is orthogonal, so is $\wedge_m f$, hence $\det(f)^2 = 1$.

In general, if f is an orthogonal injection, so is $\wedge_m f$. If f is an orthogonal projection so is $\wedge_m f$.

The **norm of** f is defined by the formula

$$\|f\| = \sup\{|f(x)| : x \in V \text{ and } |x| \leq 1\}.$$

It follows that $\|f\| = \|f^*\| = \|\wedge^1 f\|$ and

$$\|\wedge_m f\| = \|\wedge_m f^*\| = \|\wedge^m f\| \leq \|f\|^m \text{ for all } m.$$

To prove this we choose an orthonormal base $e_1, \dots, e_n$ of V such that $f(e_i) \bullet f(e_j) = 0$ for $i \neq j$; then the m-vectors $(\wedge_m f)\, e_\lambda$ corresponding to $\lambda \in \Lambda(n,m)$ are mutually orthogonal, hence $\xi \in \wedge_m V$ implies

$$|(\wedge_m f)\,\xi|^2 = \Big|\sum_\lambda \xi_\lambda (\wedge_m f)\, e_\lambda\Big|^2 = \sum_\lambda (\xi_\lambda)^2\, |(\wedge_m f)\, e_\lambda|^2 \le |\xi|^2\, \|f\|^{2m}.$$

In case $m = \dim V$, then $\|\wedge_m f\| = |(\wedge_m f)\,\xi|$ for every $\xi \in \wedge_m V$ with $|\xi| = 1$; hence $\|\wedge_m f\| > 0$ if and only if f is a monomorphism.

In case $m = \dim V'$, then $\|\wedge^m f\| = |(\wedge^m f)\,\phi|$ for every $\phi \in \wedge^m V'$, with $|\phi| = 1$; hence $\|\wedge_m f\| > 0$ if and only if f is an epimorphism.

In case $V = V'$ and $m = \dim V$, then $\|\wedge_m f\| = |\det(f)|$.

1.7.7. If $\xi \in \wedge_m \mathbf{R}^n$ and ξ is simple, then

$$|\xi| = \sup\{|(\wedge_m g)\,\xi| \colon g \in \mathbf{O}^*(n,m)\}.$$

In fact $g \in \mathbf{O}^*(n,m)$ implies $\|\wedge_m g\| = 1$, hence $|(\wedge_m g)\,\xi| \le |\xi|$. In case $\xi \neq 0$ we can choose $f \in \mathbf{O}(n,m)$ so that $\operatorname{im} f$ is the subspace T associated with ξ, hence $f \circ f^* | T = \mathbf{1}_T$ and $(\wedge_m f \circ \wedge_m f^*)\,\xi = \xi$; since $\|\wedge_m f\| = 1$, $|(\wedge_m f^*)\,\xi| \ge |\xi|$.

Similarly, if $\phi \in \wedge^m \mathbf{R}^n$ and ϕ is simple, then

$$|\phi| = \sup\{|(\wedge^m f)\,\phi| \colon f \in \mathbf{O}(n,m)\}.$$

1.7.8. Given $n = \dim V < \infty$ and $E \in \wedge_n V$ with $|E| = 1$ we define linear maps

$$*\colon \wedge_p V \to \wedge_{n-p} V, \qquad *\xi = E \llcorner \gamma_p(\xi) \text{ for } \xi \in \wedge_p V,$$
$$*\colon \wedge^p V \to \wedge^{n-p} V, \qquad *\phi = \gamma_{n-p}(E \llcorner \phi) \text{ for } \phi \in \wedge^p V.$$

These maps equal $\mathbf{D}^p \circ \gamma_p$ and $\gamma_{n-p} \circ \mathbf{D}^p$, where $\mathbf{D}^p$ is defined as in 1.5.2 with $e_\mu = E$, $\mu = (1, \dots, n)$; using the notation introduced there we find that

$$*e_\lambda = (-1)^N e_\nu \quad \text{and} \quad *\omega_\lambda = (-1)^N \omega_\nu,$$
$$*e_\nu = (-1)^M e_\lambda \quad \text{and} \quad *\omega_\nu = (-1)^M \omega_\lambda.$$

It follows that

$$**\xi = (-1)^{p(n-p)}\xi, \qquad \xi \wedge *\eta = (\xi \bullet \eta)\, e_{(1,\dots,n)},$$
$$**\phi = (-1)^{p(n-p)}\phi, \qquad \phi \wedge *\psi = (\phi \bullet \psi)\, \omega_{(1,\dots,n)},$$
$$\langle *\xi, *\phi\rangle = \langle \xi, \phi\rangle, \qquad \langle \xi, *\phi\rangle = (-1)^{p(n-p)}\langle *\xi, \phi\rangle$$

whenever $\xi, \eta \in \wedge_p V$ and $\phi, \psi \in \wedge^p V$.

1.7.9. Whenever V and W are inner product spaces, the space $V \otimes W$ can be given an inner product such that

$$(v \otimes w) \bullet (v' \otimes w) = (v \bullet v') \cdot (w \bullet w') \text{ for } v, v' \in V \text{ and } w, w' \in W.$$

Clearly this equation characterizes a unique symmetric bilinear function. Moreover, if $v_1, \dots, v_n \in V$ and $w_1, \dots, w_s \in W$ are orthonormal sequences, then the vectors $v_i \otimes w_j$ are orthonormal. Therefore we have indeed defined an inner product.

Assuming $\dim V = n < \infty$ and $f, g \in \operatorname{Hom}(V, W)$, hence $\Gamma(f), \Gamma(g) \in \wedge^1 V \otimes W$ according to 1.4.5, we will apply the above definition with V replaced by $\wedge^1 V$ to compute $\Gamma(f) \bullet \Gamma(g)$. Choosing dual orthonormal bases $e_1, \dots, e_n$ and $\omega_1, \dots, \omega_n$ of V and $\wedge^1 V$ we obtain

$$\Gamma(f) \bullet \Gamma(g) = \Big[\sum_{i=1}^n \omega_i \otimes f(e_i)\Big] \bullet \Big[\sum_{j=1}^n \omega_j \otimes g(e_j)\Big]$$
$$= \sum_{i=1}^n f(e_i) \bullet g(e_i) = \sum_{i=1}^n e_i \bullet (f^* \circ g)\, e_i$$
$$= \operatorname{trace}\Big[\sum_{i=1}^n \omega_i \otimes (f^* \circ g)\, e_i\Big] = \operatorname{trace}(f^* \circ g).$$

Replacing f, g by $\wedge_m f$, $\wedge_m g$ we find that

$$\Gamma(\wedge_m f) \bullet \Gamma(\wedge_m g) = \operatorname{trace}[\wedge_m (f^* \circ g)]$$

for every positive integer m; in particular

$$\Gamma(\wedge_n f) \bullet \Gamma(\wedge_n g) = \det(f^* \circ g),$$
$$|\Gamma(\wedge_n f) - \Gamma(\wedge_n g)|^2 = \det(f^* \circ f) + \det(g^* \circ g) - 2\det(f^* \circ g).$$

We note the inequalities

$$\|f\| \le |\Gamma(f)| \le n^{\frac{1}{2}}\, \|f\|.$$

Also, if s and t are orthogonal automorphisms of V and W, then $(\wedge^1 s) \otimes t$ is an orthogonal automorphism of $\wedge^1 V \otimes W$ mapping $\Gamma(f)$ onto $\Gamma(t \circ f \circ s)$, hence $|\Gamma(f)| = |\Gamma(t \circ f \circ s)|$.

Hereafter we abbreviate $\Gamma(f) \bullet \Gamma(g) = f \bullet g$, $|\Gamma(f)| = |f|$.

1.7.10. In case V is a finite dimensional inner product space we define the **discriminant** and the **trace** of a bilinear function $B\colon V \times V \to \mathbf{R}$ by letting

$$\operatorname{discr}(B) = \det(f), \qquad \operatorname{trace}(B) = \operatorname{trace}(f)$$

where f is the linear endomorphism of V such that

$$f(x) \bullet y = B(x, y) \text{ for } x, y \in V.$$

1.7.11. Assuming $n=\dim V<\infty$, $\sigma\in\wedge_k V$, $\tau\in\wedge_m V$, $\lambda=k+m-n\geq 0$, σ and τ are simple, $|\sigma|=1=|\tau|$, S and T are the vectorsubspaces of V associated with σ and τ, we consider here the linear map

$$f\colon S\times T\to V,\quad f(x,y)=x-y \text{ for } (x,y)\in S\times T.$$

We will prove that (see 1.7.8)

$$\|\wedge_n f\|=2^{\lambda/2}\,|(*\sigma)\wedge(*\tau)|.$$

For this purpose we select:

dual orthonormal bases $e_1,\dots,e_n$ and $\omega_1,\dots,\omega_n$ of V and $\wedge^1 V$ such that $e_i\in S$ for $i\leq k$;

dual orthonormal bases $e'_1,\dots,e'_n$ and $\omega'_1,\dots,\omega'_n$ of V and $\wedge^1 V$ such that $e'_i\in T$ for $i>n-m$.

Since $\dim(S\cap T)\geq\lambda$ we may also require that

$$e_i=e'_i,\text{ hence } \omega_i=\omega'_i,\text{ for } n-m<i\leq k.$$

Using the orthonormal basis of $S\times T$ consisting of the vectors

$$2^{-\frac12}(e_i,e_i)\text{ and } 2^{-\frac12}(e_i,-e_i)\text{ with } n-m<i\leq k,$$

$$(e_i,0)\text{ with } i\leq n-m,\qquad (0,e'_i)\text{ with } i>k,$$

and observing that $(e_i,e_i)\in\ker f$ for $n-m<i\leq k$, we compute

$$\begin{aligned}\|\wedge_n f\|&=\left|(\wedge_n f)\left[\bigwedge_{i=1}^{n-m}(e_i,0)\wedge\bigwedge_{i=n-m+1}^{k}2^{-\frac12}(e_i,-e_i)\wedge\bigwedge_{i=k+1}^{n}(0,e'_i)\right]\right|\\
&=2^{\lambda/2}\,|e_1\wedge\cdots\wedge e_k\wedge e'_{k+1}\wedge\cdots\wedge e'_n|\\
&=2^{\lambda/2}\,|\langle e_1\wedge\cdots\wedge e_k\wedge e'_{k+1}\wedge\cdots\wedge e'_n,\omega_1\wedge\cdots\wedge\omega_n\rangle|\\
&=2^{\lambda/2}\,|\langle e'_{k+1}\wedge\cdots\wedge e'_n,\omega_{k+1}\wedge\cdots\wedge\omega_n\rangle|\\
&=2^{\lambda/2}\,|\langle e'_1\wedge\cdots\wedge e'_n,\omega'_1\wedge\cdots\wedge\omega'_k\wedge\omega_{k+1}\wedge\cdots\wedge\omega_n\rangle|\\
&=2^{\lambda/2}\,|\omega'_1\wedge\cdots\wedge\omega'_k\wedge\omega_{k+1}\wedge\cdots\wedge\omega_n|\\
&=2^{\lambda/2}\,|\omega'_1\wedge\cdots\wedge\omega'_{n-m}\wedge\omega_{k+1}\wedge\cdots\wedge\omega_n|=2^{\lambda/2}\,|(*\tau)\wedge(*\sigma)|.\end{aligned}$$

1.7.12. If f is an endomorphism of a finite dimensional inner product space, then

$$2\operatorname{trace}(\wedge_2 f)=(\operatorname{trace}f)^2-\operatorname{trace}(f\circ f),$$

$$\operatorname{trace}[\wedge_2(f+f^*)]=2(\operatorname{trace}f)^2-\operatorname{trace}(f\circ f)-\operatorname{trace}(f^*\circ f).$$

To verify the first formula we expand both sides of the equation

$$\det(\mathbf{1}+t\,f)\det(\mathbf{1}-t\,f)=\det(\mathbf{1}-t^2 f\circ f)$$

in powers of t, using 1.4.5, and compare the coefficients of t^2. We obtain the second formula by applying the first to $f+f^*$.

1.7.13. Here we assume that V is a **Hilbert space,** which means that V has an inner product $\bullet$ and V is complete relative to the metric with value

$$|x-y|=[(x-y)\bullet(x-y)]^{\frac12}\text{ for } (x,y)\in V\times V.$$

If $a\in V$, C is a nonempty closed convex subset of V and

$$d=\operatorname{dist}(a,C)=\inf\{|a-x|\colon x\in C\},$$

then there exists a unique $c\in C$ with $d=|a-c|$. To prove this we consider for $0<\varepsilon\in\mathbf{R}$ the nonempty closed set

$$C_\varepsilon=C\cap\{x\colon |x-a|^2\leq d^2+\varepsilon^2\}$$

and observe that if $x,y\in C_\varepsilon$, then $(x+y)/2\in C$ and

$$\begin{aligned}d^2+\varepsilon^2&\geq(|x-a|^2+|y-a|^2)/2\\&=(|x+y-2a|^2+|x-y|^2)/4\geq d^2+(|x-y|/2)^2,\end{aligned}$$

hence $|x-y|\leq 2\varepsilon$; thus $\operatorname{diam}C_\varepsilon\leq 2\varepsilon$. Since $C_\delta\subset C_\varepsilon$ for $0<\delta<\varepsilon$ it follows that $\bigcap\{C_\varepsilon\colon \varepsilon>0\}$ consists of a single point $c\in C$. Moreover, *in case C is a vectorsubspace of V, then*

$$(a-c)\bullet x=0 \textit{ whenever } x\in C,$$

because $c+t\,x\in C$ for all $t\in\mathbf{R}$, and

$$|a-(c+t\,x)|^2=|a-c|^2-2t(a-c)\bullet x+t^2\,|x|^2$$

is smallest for $t=0$.

For every continuous linear map $f\colon V\to\mathbf{R}$ there exists a unique $u\in V$ such that $f(v)=u\bullet v$ for all $v\in V$. To prove this in case $f\neq 0$ we choose $a\in V$ with $f(a)=1$, apply the preceding proposition with $C=\ker f$, let

$$u=|a-c|^{-2}(a-c)$$

and infer that if $v\in V$, then $v-f(v)(a-c)\in C$,

$$(a-c)\bullet[v-f(v)(a-c)]=0,\qquad u\bullet v=f(v).$$

1.8. Mass and comass

1.8.1. Consider a finite dimensional inner product space V, with the induced dual inner products and norms (denoted • and | |) on the spaces of m-vectors and m-covectors (see 1.7.5). In addition to these Euclidean norms | | we shall use *another pair of dual norms* (denoted $\| \|$) *on* $\wedge_m V$ *and* $\wedge^m V$. These other norms are defined as follows:

For each $\phi \in \wedge^m V$, the **comass of** ϕ is

$$\|\phi\| = \sup\{\langle \xi, \phi\rangle : \xi \in \wedge_m V,\ \xi \text{ is simple},\ |\xi| \le 1\}.$$

Always

$$|\phi| \ge \|\phi\| \ge \binom{\dim V}{m}^{-\frac{1}{2}} |\phi|.$$

Moreover $|\phi| = \|\phi\|$ if and only if ϕ is simple.

For each $\xi \in \wedge_m V$, the **mass of** ξ is

$$\|\xi\| = \sup\{\langle \xi, \phi\rangle : \phi \in \wedge^m V,\ \|\phi\| \le 1\}.$$

Always

$$|\xi| \le \|\xi\| \le \binom{\dim V}{m}^{\frac{1}{2}} |\xi|.$$

Moreover $|\xi| = \|\xi\|$ if and only if ξ is simple.

An alternate, more direct, characterization of the mass norm may be derived, using some elementary properties of convex sets (for which see [B F, pages 5, 9] or [B O, Livre V, Chapitre II, § 1] or [E G 2, pages 23, 35]). Since $\wedge^m V \simeq \wedge^1 \wedge_m V$, the set

$$C = \wedge_m V \cap \{\xi : \|\xi\| \le 1\}$$

is the convex hull of the compact connected set

$$S = \wedge_m V \cap \{\xi : \xi \text{ is simple and } |\xi| \le 1\},$$

hence C consists of all finite sums

$$\sum_{i=1}^{N} c_i \xi_i \ \text{ with } \ \xi_i \in S,\ c_i > 0, \quad \sum_{i=1}^{N} c_i = 1$$

and

$$N \le \dim \wedge_m V = \binom{\dim V}{m}.$$

It follows that *for each* $\xi \in \wedge_m V$ *there exist simple m-vectors* $\xi_1, \ldots, \xi_N$ *with*

$$\xi = \sum_{i=1}^{N} \xi_i, \qquad \|\xi\| = \sum_{i=1}^{N} |\xi_i|, \qquad N \le \binom{\dim V}{m}.$$

Consequently

$$\|\xi\| = \inf\left\{\sum_{i=1}^{N} |\xi_i| : \xi_i \text{ are simple and } \xi = \sum_{i=1}^{N} \xi_i\right\}.$$

If $\xi \in \wedge_p V$ and $\eta \in \wedge_q V$, then $\|\xi \wedge \eta\| \le \|\xi\| \cdot \|\eta\|$.

If $\phi \in \wedge^p V$ and $\psi \in \wedge^q V$, then

$$\|\phi \wedge \psi\| \le \binom{p+q}{p} \|\phi\| \cdot \|\psi\|;$$

in case ϕ or ψ is simple, then $\|\phi \wedge \psi\| \le \|\phi\| \cdot \|\psi\|$.

If $f: V \to V'$ is a linear map of finite dimensional inner product spaces, then

$$\|(\wedge^m f)\phi\| \le \|f\|^m \cdot \|\phi\| \ \text{ for } \ \phi \in \wedge^m V',$$
$$\|(\wedge_m f)\xi\| \le \|f\|^m \cdot \|\xi\| \ \text{ for } \ \xi \in \wedge_m V.$$

1.8.2. We assume here that V is a vectorspace over the field **C** of complex numbers, with a **Hermitian product** H. Thus

$$H: V \times V \to \mathbf{C}$$

is bilinear with respect to **R** and satisfies the conditions

$$H(v, \mathbf{i}\, w) = \mathbf{i}\, H(v, w), \quad H(w, v) = \overline{H(v, w)}, \quad H(v, v) > 0 \text{ in case } v \ne 0,$$

where $v, w \in V$, $\mathbf{i}^2 = -1$ and $\overline{}$ is complex conjugation. Expressing

$$H = B + \mathbf{i}\, A,$$

where B and A are real valued functions, we find that B *is an inner product* (hereafter denoted •) and A *is an alternating 2-form*. Moreover

$$|H(v, w)| \le |v| \cdot |w| \ \text{ for } \ v, w \in V;$$

equality holds if and only if v and w are linearly dependent over **C**. This follows from the inequality $|v \bullet (c\, w)| \le |v| \cdot |c\, w|$, when $c \in \mathbf{C}$ is chosen so that $|c| = 1$ and $H(v, c\, w) \in \mathbf{R}$.

For example the vectorspace $\mathbf{C}^\nu$ has the Hermitian product

$$H(v, w) = \sum_{j=1}^{\nu} \bar{v}_j w_j \ \text{ for } \ v, w \in \mathbf{C}^\nu.$$

The inner product B corresponds to the standard inner product of $\mathbf{R}^{2\nu}$ under the canonical isomorphism $\mathbf{C}^\nu \simeq \mathbf{R}^{2\nu}$. If $Z_1, \ldots, Z_\nu \in \wedge^1(\mathbf{C}^\nu, \mathbf{C})$ are the usual coordinate functions on $\mathbf{C}^\nu$, then

$$A = (\mathbf{i}/2) \sum_{j=1}^{\nu} Z_j \wedge \bar{Z}_j \in \wedge^2(\mathbf{C}^\nu, \mathbf{C}).$$

We shall now compute the comass of the μ'th exterior power $A^\mu \in \wedge^{2\mu} V$, where $\mu \le \dim_{\mathbf{C}} V$, by proving **Wirtinger's inequality:**

If $\xi \in \wedge_{2\mu} V$ and ξ is simple, then

$$\langle \xi, A^\mu \rangle \le \mu! \, |\xi|;$$

equality holds if and only if there exist $v_1, \ldots, v_\mu \in V$ such that

$$\xi = v_1 \wedge (\mathbf{i}\, v_1) \wedge \cdots \wedge v_\mu \wedge (\mathbf{i}\, v_\mu).$$

Consequently $\|A^\mu\| = \mu!$

We assume that $|\xi| = 1$.

In case $\mu = 1$, we let $\xi = v \wedge w$, where v and w are orthonormal. Then $H(v, w) = \mathbf{i}\, A(v, w)$, hence

$$\langle \xi, A \rangle = A(v, w) = H(\mathbf{i}\, v, w) = (\mathbf{i}\, v) \bullet w \le 1;$$

equality holds if and only if $\mathbf{i}\, v = w$.

In case $\mu > 1$, we consider the 2μ dimensional subspace T associated with ξ, the inclusion map $f\colon T \to V$, and the 2-form $(\wedge^2 f) A \in \wedge^2 T$. Then we choose dual orthonormal bases $e_1, \ldots, e_{2\mu}$ and $\omega_1, \ldots, \omega_{2\mu}$ of T and $\wedge^1 T$, and nonnegative numbers $\lambda_1, \ldots, \lambda_\mu$ such that

$$(\wedge^2 f) A = \sum_{j=1}^{\mu} \lambda_j (\omega_{2j-1} \wedge \omega_{2j}).$$

Noting that $\lambda_j = A(e_{2j-1}, e_{2j}) \le 1$ for each j, and that $\xi = \varepsilon\, e_1 \wedge \cdots \wedge e_{2\mu}$ with $\varepsilon = \pm 1$, we compute

$$(\wedge^{2\mu} f) A^\mu = \mu! \, \lambda_1 \ldots \lambda_\mu \, \omega_1 \wedge \cdots \wedge \omega_{2\mu}, \qquad \langle \xi, A^\mu \rangle = \varepsilon\, \mu! \, \lambda_1 \ldots \lambda_\mu \le \mu!;$$

equality holds if and only if $\varepsilon = 1$ and $\lambda_j = 1$, hence $e_{2j} = \mathbf{i}\, e_{2j-1}$, for each j.

1.8.3. Very little appears to be known about the structure of the convex sets

$$\wedge^m(R^n) \cap \{\phi\colon \|\phi\| \le 1\}.$$

What are their extreme points?

1.8.4. *Suppose S and T are mutually orthogonal subspaces of an inner product space V, $s\colon S \to V$ and $t\colon T \to V$ are the inclusion maps, $\xi \in \operatorname{im} \wedge_p s$ and $\eta \in \operatorname{im} \wedge_q t$. The equation*

$$\|\xi \wedge \eta\| = \|\xi\| \cdot \|\eta\|$$

holds if either ξ or η is simple. In case ξ is simple one can choose an orthonormal sequence $\omega_1, \ldots, \omega_p \in \wedge^1 V$ with

$$\langle \xi, \omega_1 \wedge \cdots \wedge \omega_p \rangle = \|\xi\| \quad \text{and} \quad T \subset \ker \omega_i \ \text{ for } i = 1, \ldots, p,$$

as well as select $\psi \in \wedge^q V$ with $\|\psi\| = 1$ and $\langle \eta, \psi \rangle = \|\eta\|$; it follows that

$$\langle \xi \wedge \eta, \omega_1 \wedge \cdots \wedge \omega_p \wedge \psi \rangle = \|\xi\| \cdot \|\eta\|$$

with $\|\omega_1 \wedge \cdots \wedge \omega_p \wedge \psi\| \le 1$, hence $\|\xi \wedge \eta\| \ge \|\xi\| \cdot \|\eta\|$. *I do not know whether the above equation holds always* (in case neither ξ nor η is simple).

1.9. The symmetric algebra of a vectorspace

1.9.1. Proceding similarly as in 1.3, we now consider in the tensor algebra $\otimes_* V$ of any vectorspace V the *two sided ideal* $\mathfrak{B}\, V$ generated by the elements

$$x \otimes y - y \otimes x \in \otimes_2 V$$

corresponding to all $x, y \in V$. The quotient algebra

$$\odot_* V = \otimes_* V / \mathfrak{B}\, V$$

is called the **symmetric algebra** of V. Clearly

$$\mathfrak{B}\, V = \bigoplus_{m=2}^{\infty} (\otimes_m V \cap \mathfrak{B}\, V)$$

is a homogeneous ideal, hence

$$\odot_* V = \bigoplus_{m=2}^{\infty} \odot_m V,$$

where

$$\odot_m V = \otimes_m V / (\otimes_m V \cap \mathfrak{B}\, V);$$

in particular $\odot_0 V = \mathbf{R}$ and $\odot_1 V = V$. The multiplication in $\odot_* V$ will be denoted by the symbol $\odot$. Therefore $\odot_m V$ is the vectorspace generated by all the products $v_1 \odot \cdots \odot v_m$ corresponding to $v_1, \ldots, v_m \in V$, and we see from the definition of $\mathfrak{B}\, V$ that *the symmetric multiplication $\odot$ is commutative.*

Among all commutative associative graded algebras with a unit, whose direct summand of index 1 is isomorphic with V, the symmetric algebra $\odot_* V$ is characterized (up to isomorphism) by the following property:

For every commutative associative graded algebra A with a unit element, every linear map of V into A_1 can be uniquely extended to a unit preserving algebra homomorphism of $\odot_* V$ into A, carrying $\odot_m V$ into A_m for each m.

It follows that *every linear map $f\colon V\to V'$ can be uniquely extended to a unit preserving algebra homomorphism*

$$\odot_* f\colon \odot_* V\to\odot_* V',$$

which is the direct sum of the linear maps

$$\odot_m f\colon \odot_m V\to\odot_m V'.$$

1.9.2. The functor $\odot_*$ *converts direct sums of vector spaces into commutative products of algebras:*

$$\odot_*(P\oplus Q)\simeq\odot_* P\otimes\odot_* Q.$$

If V has a basis consisting of a single element x, then $\odot_m V$ has a basis consisting of the m'th symmetric power

$$x^m = x\odot\cdots\odot x \quad (m \text{ factors}).$$

Therefore, *if V has a basis consisting of $e_1,\ldots,e_n$, then a basis of $\odot_m V$ is formed by the products*

$$e^\alpha=(e_1)^{\alpha_1}\odot\cdots\odot(e_n)^{\alpha_n}$$

corresponding to all n termed sequences α of nonnegative integers with

$$\Sigma\,\alpha=\sum_{i=1}^{n}\alpha_i=m;$$

designating the set of all such sequences by

$$\Xi(n,m)$$

we conclude that

$$\dim\odot_m V=\operatorname{card}\Xi(n,m)=\binom{m+n-1}{n-1}.$$

1.9.3. Suppose k and m are positive integers.

For $t=(t_1,\ldots,t_k)\in\mathbf{R}^k$ and $v=(v_1,\ldots,v_k)\in V^k$ the k-nomial theorem (which holds in every commutative ring) implies

$$(t_1 v_1+\cdots+t_k v_k)^m/m! = \sum_{\alpha\in\Xi(k,m)} t^\alpha v^\alpha/\alpha!$$

where $v^\alpha=(v_1)^{\alpha_1}\odot(v_2)^{\alpha_2}\odot\cdots\odot(v_k)^{\alpha_k}$ and

$$t^\alpha=\prod_{i=1}^{k}(t_i)^{\alpha_i},\qquad \alpha!=\prod_{i=1}^{k}(\alpha_i!).$$

Taking $k=m$ and letting T be the set of all functions mapping $\{1,\ldots,m\}$ into $\{1,-1\}$ we obtain the **polarization formula**

$$\sum_{t\in T}\prod_{i=1}^{m} t_i\Big(\sum_{j=1}^{m} t_j v_j\Big)^m\Big/m! = 2^m\, v_1\odot\cdots\odot v_m;$$

in fact the above sum equals

$$\sum_{\alpha\in\Xi(m,m)}\Big[\sum_{t\in T}\prod_{i=1}^{m}(t_i)^{1+\alpha_i}\Big]v^\alpha/\alpha!,$$

and the bracketed term equals 0 whenever $\alpha_i=0$ for some i, because summation over T is invariant under the permutation of T mapping t onto $(t_1,\ldots,t_{i-1},-t_i,t_{i+1},\ldots,t_m)$.

1.9.4. The **diagonal map** of $\odot_* V$ is the unit preserving algebra homomorphism

$$\Upsilon\colon \odot_* V\to\odot_* V\otimes\odot_* V \text{ (commutative product)}$$

such that $\Upsilon(v)=v\otimes 1+1\otimes v$ whenever $v\in V$.

For $v_1,\ldots,v_m\in V$ we compute the product

$$\Upsilon(v_1\odot\cdots\odot v_m)=(v_1\otimes 1+1\otimes v_1)\odot\cdots\odot(v_m\otimes 1+1\otimes v_m)$$
$$=\sum_{p=0}^{m}\sum_{\sigma\in\mathrm{Sh}(p,m-p)}(v_{\sigma(1)}\odot\cdots\odot v_{\sigma(p)})\otimes(v_{\sigma(p+1)}\odot\cdots\odot v_{\sigma(m)}).$$

Therefore, if $e_1,\ldots,e_n$ form a basis of V and $\alpha\in\Xi(n,m)$, then

$$\Upsilon(e^\alpha)=\sum_{p=0}^{m}\sum_{\alpha\ge\beta\in\Xi(n,p)}\binom{\alpha_1}{\beta_1}\cdot\ldots\cdot\binom{\alpha_m}{\beta_m}e^\beta\otimes e^{\alpha-\beta},$$

hence

$$\Upsilon(e^\alpha/\alpha!)=\sum_{p=0}^{m}\sum_{\alpha\ge\beta\in\Xi(n,p)}[e^\beta/\beta!]\otimes[e^{\alpha-\beta}/(\alpha-\beta)!].$$

We observe (in analogy with 1.3.3) that *the diagonal map Υ of $\odot_* V$ is associative and commutative*, and that it *is a natural transformation.*

1.10. Symmetric forms and polynomial functions

1.10.1. An m linear function f, which maps the m fold cartesian product V^m of a vectorspace V into some other vectorspace W, is called **symmetric** if and only if

$$f(v_{\sigma(1)},\ldots,v_{\sigma(m)})=f(v_1,\ldots,v_m)$$

whenever $v_1, \ldots, v_m \in V$ and σ is a permutation of $\{1, \ldots, m\}$. We let

$$\odot^m(V, W)$$

be the vectorspace of all m linear symmetric functions (forms) mapping V^m into W. We shall use the linear isomorphism

$$\odot^m(V, W) \simeq \mathrm{Hom}(\odot_m V, W),$$

where the corresponding $f \in \odot^m(V, W)$ and $h \in \mathrm{Hom}(\odot_m V, W)$ are related by the condition

$$f(v_1, \ldots, v_m) = h(v_1 \odot \cdots \odot v_m) \text{ whenever } v_1, \ldots, v_m \in V;$$

when this is the case we write

$$\langle \xi, f \rangle = \langle \xi, h \rangle = h(\xi) \text{ for } \xi \in \odot_m V.$$

Moreover there is an obvious linear isomorphism

$$\mathrm{Hom}(\odot_m V, W) \simeq \mathrm{Hom}^m(\odot_* V, W),$$

where the right member means the set of all those linear maps of $\odot_* V$ into W which vanish on $\odot_n V$ whenever $n \neq m$.

We define $\odot^0(V, W) = W$,

$$\odot^*(V, W) = \bigoplus_{m=0}^{\infty} \odot^m(V, W),$$

and abbreviate $\odot^m(V, \mathbf{R}) = \odot^m V$, $\odot^*(V, \mathbf{R}) = \odot^* V$.

Each linear map $f: V \to V'$ induces a dual linear map

$$\odot^*(f, W): \odot^*(V', W) \to \odot^*(V, W)$$

which is the direct sum of the linear maps

$$\odot^m(f, W): \odot^m(V', W) \to \odot^m(V, W)$$

characterized by the equations

$$\langle \xi, \odot^m(f, W)\phi \rangle = \langle (\odot_m f)\xi, \phi \rangle$$

for $\xi \in \odot_m V$ and $\phi \in \odot^m(V', W)$.

We abbreviate $\odot^*(f, \mathbf{R}) = \odot^* f$.

1.10.2. Whenever W is an (ungraded) algebra over $\mathbf{R}$, we shall use the diagonal map Υ of $\odot_* V$ (in analogy with 1.4.2) to turn the graded vectorspace $\odot^*(V, W)$ into the graded **algebra of symmetric forms of V with coefficients in W**. For

$$\phi \in \mathrm{Hom}^p(\odot_* V, W) \quad \text{and} \quad \psi \in \mathrm{Hom}^q(\odot_* V, W)$$

we define the symmetric product

$$\phi \odot \psi \in \mathrm{Hom}^{p+q}(\odot_* V, W)$$

to be the composition

$$\odot_* V \xrightarrow{\Upsilon} \odot_* V \otimes \odot_* V \xrightarrow{\phi \otimes \psi} W \otimes W \xrightarrow{\nu} W,$$

where ν corresponds to the multiplication of W.

If W is associative, or commutative, or has a unit element, then $\odot^*(V, W)$ has the same property. Each induced map $\odot^*(f, W)$ is a homomorphism of algebras.

From the shuffle formula for Υ we see that

$$(\phi \odot \psi)(v_1, \ldots, v_{p+q}) = \sum_{\sigma \in \mathrm{Sh}(p,q)} \phi(v_{\sigma(1)}, \ldots, v_{\sigma(p)}) \cdot \psi(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)})$$

whenever $\phi \in \odot^p(V, W)$, $\psi \in \odot^q(V, W)$ and $v_1, \ldots, v_{p+q} \in V$.

If $e_1, \ldots, e_n$ and $\omega_1, \ldots, \omega_n$ form dual bases of V and $\mathrm{Hom}(V, \mathbf{R}) = \odot^1 V$, *then the products*

$$\omega^\alpha = (\omega_1)^{\alpha_1} \odot \cdots \odot (\omega_n)^{\alpha_n} \in \odot^m V$$

corresponding to all $\alpha \in \Xi(n, m)$ form a basis of $\odot^m V$, which is dual to the base $\{e^\alpha/\alpha!: \alpha \in \Xi(n, m)\}$ of $\odot_m V$.

In the definition of the symmetric product of forms the multiplication of an algebra W may be replaced by any bilinear map $\mu: W_1 \times W_2 \to W_3$; thus one obtains

$$\phi \odot \psi \in \odot^{p+q}(V, W_3) \text{ for } \phi \in \odot^p(V, W_1),\ \psi \in \odot^q(V, W_2).$$

1.10.3. Proceeding just as in 1.5.1 we define for $p \leq q$ the **interior multiplications**

$$\lrcorner: \odot_p V \times \odot^q(V, W) \to \odot^{q-p}(V, W), \qquad \llcorner: \odot_q V \times \odot^p V \to \odot_{q-p} V.$$

If $e_1, \ldots, e_n$ and $\omega_1, \ldots, \omega_n$ are dual bases of V and $\odot^1 V$, and if $\alpha \in \Xi(n, p)$ and $\beta \in \Xi(n, q)$, then

$$e^\alpha \lrcorner (\omega^\beta/\beta!) = \omega^{\beta-\alpha}/(\beta-\alpha)!, \qquad (e^\beta/\beta!) \llcorner \omega^\alpha = e^{\beta-\alpha}/(\beta-\alpha)!$$

in case $\alpha \leq \beta$; otherwise these interior products equal 0.

1.10.4. A map $P: V \to W$ is called a **homogeneous polynomial function of degree** m if and only if there exists a form $\phi \in \odot^m(V, W)$ such that

$$P(x) = \langle x^m/m!, \phi \rangle \text{ for } x \in V.$$

The polarization formula (1.9.3) shows that $P=0$ if and only if $\phi=0$. Thus $\odot^m(V,W)$ *is linearly isomorphic with the vectorspace of all homogeneous polynomial functions of degree m mapping V into W.*

More generally, a map $P: V \to W$ is called a **polynomial function** if and only if there exists an integer $M \geq 0$ and forms $\phi_m \in \odot^m(V,W)$ corresponding to $m=0,\dots,M$ such that

$$P(x)=\sum_{m=0}^{M}\langle x^m/m!,\phi_m\rangle \text{ for } x\in V.$$

Since this formula implies

$$P(t\,x)=\sum_{m=0}^{M} t^m\langle x^m/m!,\phi_m\rangle \text{ for } x\in V \text{ and } t\in\mathbf{R},$$

we see that $P=0$ if and only if $\phi_m=0$ for $m=0,\dots,M$. Thus $\odot^*(V,W)$ *is linearly isomorphic with the vectorspace of all polynomial maps of V into W*, and we can define

$$\text{degree } P=\sup(\{m:\ \phi_m\neq 0\}\cup\{0\})$$

whenever P and $\phi_0,\dots,\phi_M$ are related as above. Moreover, *in case W is an algebra, the symmetric product $\odot$ corresponds under the preceding isomorphism to pointwise multiplication in the function space W^V*, hence *the algebra $\odot^*(V,W)$ is isomorphic with the algebra of polynomial maps of V into W; in fact whenever* $\phi\in\odot^p(V,W)$ and $\psi\in\odot^q(V,W)$ the shuffle formula shows that, for $x\in V$,

$$\begin{aligned}&\langle x^{p+q}/(p+q)!,\phi\odot\psi\rangle\\ &=\text{card Sh}(p,q)\cdot\langle x^p,\phi\rangle\cdot\langle x^q,\psi\rangle/(p+q)!=\langle x^p/p!,\phi\rangle\cdot\langle x^q/q!,\psi\rangle.\end{aligned}$$

Whenever P and $\phi_0,\dots,\phi_M$ are related as above we use the binomial theorem to obtain for $x,v\in V$ the **Taylor formula**

$$P(x+v)=\sum_{m=0}^{M}\sum_{i=0}^{m}\langle v^i/i!\odot x^{m-i}/(m-i)!,\phi_m\rangle=\sum_{i=0}^{M}\langle v^i/i!,S_i(x)\rangle,$$

where

$$S_i(x)=\sum_{m=i}^{M}x^{m-i}/(m-i)!\,\lrcorner\,\phi_m.$$

We observe that S_i is a polynomial function mapping V into $\odot^i(V,W)$; in 3.1.11 it will be identified with the i-th differential of P. Here we also note that

$$S_i(x+v)=\sum_{m=i}^{M}v^{m-i}/(m-i)!\,\lrcorner\,S_m(x),$$

because the right member of this equation equals

$$\begin{aligned}&\sum_{m=i}^{M}v^{m-i}/(m-i)!\,\lrcorner\sum_{j=m}^{M}x^{j-m}/(j-m)!\,\lrcorner\,\phi_j\\ &=\sum_{j=i}^{M}\sum_{m=i}^{j}(v^{m-i}/(m-i)!\odot x^{j-m}/(j-m)!)\,\lrcorner\,\phi_j\\ &=\sum_{j=i}^{M}(v+x)^{j-i}/(j-i)!\,\lrcorner\,\phi_j=S_i(v+x).\end{aligned}$$

To represent the polynomial function P in terms of a basis $e_1,\dots,e_n$ of V, we recall 1.9.3 and find that

$$P\left(\sum_{i=1}^{n}t_i\,e_i\right)=\sum_{m=0}^{M}\sum_{\alpha\in\Xi(n,m)}\langle e^\alpha/\alpha!,\phi_m\rangle\,t^\alpha$$

whenever $t\in\mathbf{R}^n$.

1.10.5. In analogy with 1.7.5, any given inner product of V can be used to construct inner products on the spaces $\odot_m V$, by extending the given polarity to an algebra homomorphism of $\odot_* V$ into $\odot^* V$. If $e_1,\dots,e_n$ are orthonormal in V, then the products $(\alpha!)^{-\frac{1}{2}}e^\alpha$ corresponding to all $\alpha\in\Xi(n,m)$ are orthonormal in $\odot_m V$. The norms corresponding to such inner products satisfy the inequality

$$|v_1\odot\cdots\odot v_m|\leq(m!)^{\frac{1}{2}}\,|v_1|\cdot\cdots\cdot|v_m| \text{ for } v_1,\dots,v_m\in V.$$

Other useful norms can be constructed by adapting the method of 1.8.1 as follows: Whenever V and W are normed vectorspaces and $\phi\in\odot^m(V,W)$ we define

$$\|\phi\|=\sup\{|\phi(v_1,\dots,v_m)|:\ v_i\in V \text{ and } |v_i|\leq 1 \text{ for } i=1,\dots,m\}.$$

Clearly $\|(a_1\odot\cdots\odot a_j)\,\lrcorner\,\phi\|\leq|a_1|\cdot\cdots\cdot|a_j|\cdot\|\phi\|$ in case $j\leq m$ and $a_1,\dots,a_j\in V$. The shuffle formula shows that

$$\|\phi\odot\psi\|\leq\binom{p+q}{p}\|\phi\|\cdot\|\psi\|$$

for $\phi\in\odot^p(V,W)$ and $\psi\in\odot^q(V,W)$, provided W is a normed algebra with $|w\cdot z|\leq|w|\cdot|z|$ for $w,z\in W$; moreover

$$\|\phi^k\|=k!\,\|\phi\|^k \text{ for } \phi\in\odot^1(V,W),\ k=1,2,3,\dots$$

in case W is associative and $|w^k|=|w|^k$ for $w\in W$. We also define

$$\|\xi\|=\sup\{\langle\xi,\phi\rangle:\ \phi\in\odot^m V \text{ and } \|\phi\|\leq 1\}$$

whenever $\xi \in \odot_m V$, and readily verify that

$$\|\xi \circ \eta\| \le \|\xi\| \cdot \|\eta\| \text{ for } \xi \in \odot_p V,\ \eta \in \odot_q V;$$
$$\|x^k\| = |x|^k \text{ for } x \in V,\ k = 1, 2, 3, \ldots .$$

If V and $\odot^1 V$ have dual basic sequences $e_1, \ldots, e_n$ and $\omega_1, \ldots, \omega_n$ with $|e_i| = \|\omega_i\| = 1$ for $i = 1, \ldots, n$, then

$$\alpha! \le \|\omega^\alpha\| \le (\Sigma\alpha)! \text{ and } \alpha!/(\Sigma\alpha)! \le \|e^\alpha\| \le 1 \text{ for } \alpha \in \Xi(n, m).$$

It may also be shown that $\|\xi\| > 0$ for $\xi \in \odot_m V \sim \{0\}$, and (compare 1.8.1) that $\|\xi\|$ equals the infimum of all finite sums

$$\sum_{j=1}^{N} |v_{1,j}| \cdot \cdots \cdot |v_{m,j}|$$

corresponding to $v_{i,j} \in V$ with

$$\xi = \sum_{j=1}^{N} (v_{1,j} \odot \cdots \odot v_{m,j});$$

if $\dim V < \infty$ one can take $N \le \dim \odot_m V$.

For every homogeneous polynomial function $P: V \to W$ of degree m we define

$$\|P\| = \sup\{|P(x)|: x \in V, |x| \le 1\}.$$

Choosing $\phi \in \odot^m(V, W)$ so that $P(x) = \langle x^m/m!, \phi\rangle$ whenever $x \in V$, we observe that

$$m!\, \|P\| \le \|\phi\| \le m^m\, \|P\|$$

as a consequence of the polarization formula; for the case when V is an inner product space it was shown in [H 1] that $m!\,\|P\| = \|\phi\|$. Recalling 1.9.3 we also obtain

$$\left|P\Big(\sum_{j=1}^{k} t_j v_j\Big)\right| \le \sum_{\alpha \in \Xi(k,m)} |t^\alpha/\alpha!| \cdot |\langle v^\alpha, \phi\rangle|$$
$$\le \|\phi\| \sum_{\alpha \in \Xi(k,m)} |t|^\alpha/\alpha! = \|\phi\| \Big(\sum_{j=1}^{k} |t_j|\Big)^m \Big/ m!$$

provided $|v_j| \le 1$ for $j = 1, \ldots, k$.

1.10.6. Assuming that V and W are inner product spaces with $\dim V = n < \infty$, we first endow $\odot^m V$ with the inner product such that the polarity described in the first paragraph of 1.10.5 is an orthogonal isomorphism mapping $\odot_m V$ onto $\odot^m V$, then use the method of 1.7.9 to define an inner product on

$$\odot^m(V, W) \simeq [\odot^m V] \otimes W.$$

Now suppose $e_1, \ldots, e_n$ and $\omega_1, \ldots, \omega_n$ are dual orthonormal bases of V and $\odot^1 V$, hence $\alpha!^{-\frac{1}{2}} e^\alpha$ and $\alpha!^{-\frac{1}{2}} \omega^\alpha$ corresponding to $\alpha \in \Xi(n, m)$ form dual orthonormal bases of $\odot_m V$ and $\odot^m V$. With each ϕ in $\odot^m(V, W)$ we associate

$$\sum_{\alpha \in \Xi(n,m)} \alpha!^{-\frac{1}{2}} \omega^\alpha \otimes \langle \alpha!^{-\frac{1}{2}} e^\alpha, \phi\rangle$$

in $[\odot^m V] \otimes W$ and compute

$$|\phi|^2 = \sum_{\alpha \in \Xi(n,m)} |\langle \alpha!^{-\frac{1}{2}} e^\alpha, \phi\rangle|^2.$$

Letting $\mathscr{S}(n, m)$ be the set of all functions mapping $\{1, \ldots, m\}$ into $\{1, \ldots, n\}$ and observing that for each $\alpha \in \Xi(n, m)$ there exist precisely $m!/\alpha!$ sequences $s \in \mathscr{S}(n, m)$ such that

$$\alpha(j) = \operatorname{card}\{i: s(i) = j\} \text{ for } j \in \{1, \ldots, m\},$$

we obtain the formula

$$m!\, |\phi|^2 = \sum_{s \in \mathscr{S}(n,m)} |\langle e_{s(1)} \odot \cdots \odot e_{s(m)}, \phi\rangle|^2.$$

It follows that

$$|\langle x^m/m!, \phi\rangle| \le m!^{-\frac{1}{2}} |x|^m |\phi| \text{ for } x \in V.$$

Recalling 1.10.5 we see that

$$\|\phi\| \le m^m\, m!^{-\frac{1}{2}} |\phi| \quad\text{and}\quad |\phi| \le m!^{-\frac{1}{2}} n^{m/2} \|\phi\|.$$