WITTEN-MORSE THEORY FOR CELL COMPLEXES

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In [Fo2] we introduced a combinatorial Morse theory for CW complexes (see [Fo1] for an informal summary). We showed that, in addition to the Morse inequalities, many of the basic notions of differential topology, such as the gradient vector field and the corresponding gradient flow, can be realized in this combinatorial setting. In [Fo3] we studied the notion of a general combinatorial vector field and found that one can find combinatorial analogues of much of the theory of vector fields, including the study of zeta functions and their special values. In this paper, we develop combinatorial analogues of some aspects of global analysis and present a proof of the combinatorial Morse inequalities of [Fo2] along the lines of Witten's Hodge-theoretic proof of the standard (smooth) Morse inequalities as presented in [Wi]. This combinatorial version retains much of the structure of the smooth theory, and provides combinatorial analogues of the main ingredients of the proof in [Wi]. On the other hand, some difficulties in the smooth theory, arising from the infinite dimensional nature of the analysis, or questions of transversality, do not appear. Moreover, the combinatorial theory can be applied to very general cell complexes, not just those arising from cell decompositions of manifolds.

This work was motivated by the analysis in [Wi], in which Witten gave a beautiful new proof of the Morse inequalities based on techniques from Hodge theory. Moreover, he showed, using ideas from quantum physics, how one could analytically derive the entire Morse complex, a differential complex built out of the critical points of a Morse function which has the same homology as the underlying manifold (see [Mi] and [Kl] for earlier topological treatments of the Morse complex, [C-F-K-S] and especially [H-S] for a mathematical treatment of Witten's ideas, and [Bo] for a wonderful overview of the subject).

In [Fo2] we presented a combinatorial Morse theory for CW complexes. In this paper we present a (finite-dimensional) linear algebra proof of the Morse inequalities for CW complexes along the lines of Witten's proof. We also derive the Morse complex in this setting. In particular, we exhibit a combinatorial version of the quantum phenomenon of tunneling. The reader may use this paper as an introduction to the rather daunting analysis of [H-S] (see also [B-Z] chapter VIII) which is required to carry through this program in the smooth category. We find it fascinating that there are finite dimensional analogues of most of the key ingredients of the proof in [H-S]. Moreover, the finite dimensional nature

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of our analysis allows us to avoid some of the difficulties which arise when attempting to make rigorous the arguments of [Wi]. Thus, in a couple of places our proof more closely follows the presentation of [Wi] than the proof in [H-S]. In particular, in §4 and, more explicitly, in §5 our proof makes use of (well-defined) discrete path integrals.

We now give a very brief look at the main ideas of the paper. Let M be a simplicial complex (although the results in this paper hold for more general cell complexes), K the set of simplices of M, and K_p the simplices of dimension p. A *(discrete) Witten-Morse function* is a function f on K, i.e., an assignment of a single real number to each simplex, satisfying some conditions (which we will not describe until later, see Definition 0.6). We will also define later (see Definition 0.3) the notion of a *critical simplex* for f. Whether a simplex σ of dimension p is critical or not depends only on comparisons between $f(\sigma)$ and $f(\tau)$ where τ runs over all neighboring simplices of dimension $p+1$ and $p-1$.

Consider the real simplicial chain complex of M

$$0 \longrightarrow C_n(M,\mathbf{R}) \xrightarrow{\partial} C_{n-1}(M,\mathbf{R}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(M,\mathbf{R}) \longrightarrow 0.$$

Our first goal is to study the homology of this complex, i.e.,

$$H_p(M,\mathbf{R}) \equiv \frac{\operatorname{Ker} \partial : C_p(M,\mathbf{R}) \longrightarrow C_{p-1}(M,\mathbf{R})}{\operatorname{Im} \partial : C_{p+1}(M,\mathbf{R}) \longrightarrow C_p(M,\mathbf{R})}.$$

Following Witten [Wi], we deform the boundary operator ∂, replacing it with

$$\partial_t = e^{tf} \partial e^{-tf}$$

and consider the induced Laplace operators

$$\Delta_p(t) = \partial_t \partial_t^* + \partial_t^* \partial_t : C_p(M,\mathbf{R}) \longrightarrow C_p(M,\mathbf{R})$$

where ∂_t^* is the adjoint of ∂_t with respect to the inner product on $C_*(M,\mathbf{R})$ such that the simplices are orthonormal. For each $t \in \mathbf{R}$

$$\text{Kernel } (\Delta_p(t)) \cong H_p(M,\mathbf{R}).$$

Letting $t \to \infty$, and expressing $\Delta_p(t)$ in matrix form with respect to the basis consisting of simplices, we will find that if f is a Witten-Morse function then

$$\Delta_p(\infty) = \begin{array}{cc} \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} & \tag{0.1} \\ \begin{array}{cc} \text{critical} & \text{non-critical} \\ p\text{-simplices} & p\text{-simplices} \end{array} & \end{array}$$

where D is a diagonal matrix with entries in $\mathbf{Z}_{>0} \cup \{+\infty\}$. So that

$$\dim \text{Kernel } (\Delta_p(\infty)) \;=\; \#\text{ of critical } p\text{-simplices}.$$

Standard arguments imply that for each p

$$\dim \text{Kernel } (\Delta_p(\infty)) \geq \dim \text{Kernel } (\Delta_p(t)) \qquad t \in \mathbf{R}$$

and

$$\sum_{p=0}^{n}(-1)^p \dim \text{Kernel } (\Delta_p(\infty)) = \sum_{p=0}^{n}(-1)^p \dim \ker(\Delta_p(t)) \qquad t \in \mathbf{R}.$$

It follows immediately that for each p

(0.2) $$\# \text{ of critical } p \text{ simplices} \geq \dim H_p(M, \mathbf{R})$$

and

$$\sum_{p=0}^{n}(-1)^p\{\# \text{ of critical } p \text{ simplices}\} = \sum_{p=0}^{n}(-1)^p\{\dim H_p(M,\mathbf{R})\}.$$

These are the *Weak Morse Inequalities.*

In fact, we show in §2 that this is sufficient to prove the *Strong Morse Inequalities*

$$m_k - m_{k-1} + \cdots \geq b_k - b_{k-1} + \cdots \quad \forall \ k = 0, 1, 2, \ldots$$

where

$$m_k = \# \text{ of critical simplices of dimension } k$$
$$b_k = \dim H_k(M, \mathbf{R})$$

(see also [Wi] and [Bo] for the corresponding argument in the smooth category).

In general, (0.2) will not be an equality, since we may have

$$\dim \ker \Delta_p(\infty) > \dim \ker \Delta_p(t) \qquad t \in \mathbf{R}.$$

We investigate the nature of these extra zero eigenvalues that occur at $t = \infty$. We learn that these small eigenvalues go to 0 exponentially fast as $t \to \infty$, and the corresponding eigenfunctions are concentrated, with exponentially small error, at the critical simplices. A closer look leads us to the Morse complex

$$\mathcal{M} : 0 \longrightarrow \mathcal{M}_n \xrightarrow{\tilde{\partial}} \mathcal{M}_{n-1} \xrightarrow{\tilde{\partial}} \cdots \xrightarrow{\tilde{\partial}} \mathcal{M}_0 \longrightarrow 0$$

where $\mathcal{M}_p \subseteq C_p(M, \mathbf{R})$ is the span of the critical p-simplices, and $\tilde{\partial}$ is a differential constructed from the "gradient paths" of f (this is explained in §3). We then prove that this complex has the same homology as the underlying manifold. That is,

$$H_*(\mathcal{M}) \cong H_*(M, \mathbf{R}).$$

Lastly, we single out a special class of Witten-Morse functions which we call *flat* (see Definition 0.7). We show that every Witten-Morse function is equivalent (in a precise sense) to a flat Witten-Morse function, and if f is flat, then

$$\Delta_p(\infty) = \begin{array}{cc} \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \\ \begin{array}{cc} \text{critical} & \text{non-critical} \\ p\text{-simplices} & p\text{-simplices} \end{array} \end{array} \tag{0.3}$$

where I is the diagonal matrix with all diagonal entries equal to 1. Thus, any quantity which can be expressed as a function of $\det(\Delta_p(t))$ becomes, as $t \to \infty$, a function of just the small eigenvalues, and hence, as we will see, depends solely on the information contained in the Morse complex $\mathcal{M}$. As an example, we consider (after choosing a representation Φ of $\pi_1(M)$ — this is all reviewed in §5) the Reidemeister Torsion $T(M, \Phi)$, which Ray and Singer showed in [R-S] could be expressed in terms of the determinants of Laplacians twisted by Φ. We show that the Morse complex $\mathcal{M}$ can be twisted by Φ to a complex $\mathcal{M}_\Phi$ and the torsion of this complex is equal to the Reidemeister Torsion of M. That is,

$$T(\mathcal{M}_\Phi) = T(M, \Phi).$$

We will now add some precision to our discussion. We begin with a brief review of the Morse theory developed in [Fo2]. We will work in the category of CW complexes (see [L-W] for definitions and basic properties), but the reader may prefer to think only of simplicial complexes. Let M be a finite CW complex, and let K denote the set of (open) cells of M, with K_p the cells of dimension p. For $\sigma \in K$ we will write $\sigma^{(p)}$ to indicate that $\dim \sigma = p$, and for $\sigma, \tau \in K$ we will write $\sigma < \tau$ or $\tau > \sigma$ to indicate that σ is contained in the boundary of τ, and we say that σ *is a face of* τ (we write $\sigma \leq \tau$ if $\sigma < \tau$ or $\sigma = \tau$). Suppose $\sigma^{(p)}$ is a face of $\tau^{(p+1)}$. Let B be a closed ball of dimension $p+1$, and

$$h : B \longrightarrow M$$

the characteristic map for τ, so that, in particular, h is a continuous map that maps interior(B) homeomorphically onto τ.

Definition 0.1. Say $\sigma^{(p)}$ is a *regular face* of $\tau^{(p+1)}$ if

(i) $h : h^{-1}(\sigma) \longrightarrow \sigma$ is a homeomorphism
(ii) $\overline{h^{-1}(\sigma)}$ is a closed p-ball.

Otherwise we say σ is an *irregular face* of τ.

We note that if M is a regular CW complex (and hence if M is a simplicial complex or a polyhedron) then all faces are regular. Of crucial importance is the following property. Suppose $\sigma^{(p)}$ is a regular face of $\tau^{(p+1)}$. Choose an orientation for each cell in M and consider σ and τ as elements in the cellular chain groups $C_p(M, \mathbf{Z})$ and $C_{p+1}(M, \mathbf{Z})$, respectively. Then

$$\langle \partial\tau, \sigma \rangle = \pm 1 \tag{0.4}$$

where $\langle \partial\tau, \sigma \rangle$ is the incidence number of τ and σ (for a proof see Corollary V.3.6 of [L-W]).

This brings us to the definition of a Morse function. Roughly speaking, for a function from K to $\mathbf{R}$ to be a Morse function, higher dimensional simplices must be assigned higher values, with at most one exception, locally, at each simplex (see [Fo1] and [Fo2] for further explanation and examples).

Definition 0.2. A function $f : K \to \mathbf{R}$ is a *discrete Morse function* if for every $\sigma^{(p)} \in K_p$

(i) If σ is an irregular face of $\tau^{(p+1)}$ then $f(\tau) > f(\sigma)$. Moreover,

$$\#\{\tau^{(p+1)} > \sigma \mid f(\tau) \leq f(\sigma)\} \leq 1.$$

(ii) If $\upsilon^{(p-1)}$ is an irregular face of σ then $f(\upsilon) < f(\sigma)$. Moreover,

$$\#\{\upsilon^{(p-1)} < \sigma \mid f(\upsilon) \geq f(\sigma)\} \leq 1.$$

We declare $\sigma^{(p)}$ to be a critical point (of index p) if there are no exceptions at σ, i.e.,

Definition 0.3. Let f be a discrete Morse function on M. Say $\sigma^{(p)}$ is a critical point (of index p) if

1) $\#\{\tau^{(p+1)} > \sigma \mid f(\tau) \leq f(\sigma)\} = 0$
2) $\#\{\upsilon^{(p-1)} < \sigma \mid f(\upsilon) \geq f(\sigma)\} = 0.$

Note that if a cell of dimension p is a critical point, it is necessarily a critical point of index p. See [Fo1] for a heuristic justification for this definition.

Let m_p denote the number of critical points of index p, and for any coefficient field $\mathbf{F}$ let $b_p(\mathbf{F})$ denote the p^{th} Betti number

$$b_p(\mathbf{F}) = \dim H_p(M, \mathbf{F}).$$

In [Fo2] we proved

Theorem 4. **(The Strong Morse Inequalities).** *For each $k = 0, 1, 2, \ldots$*

$$m_k - m_{k-1} + m_{k-2} - \cdots \pm m_0 \geq b_k(\mathbf{F}) - b_{k-1}(\mathbf{F}) + b_{k-2}(\mathbf{F}) - \cdots \pm b_0(\mathbf{F}).$$

As a corollary, we have the weak Morse Inequalities.

Corollary 5. **(The Weak Morse Inequalities).**

(i) *For each $k = 0, 1, 2, \ldots$*
$m_k \geq b_k(\mathbf{F})$.
(ii) $m_0 - m_1 + m_2 - \cdots = b_0(\mathbf{F}) - b_1(\mathbf{F}) + b_2(\mathbf{F}) - \cdots.$

See [Mi] for a discussion of the Morse inequalities. In this paper we will only discuss the Morse inequalities in the case that the coefficient field is the field of real numbers. With this in mind, let

$$b_p = b_p(\mathbf{R}) = \dim H_p(M, \mathbf{R}).$$

Before presenting our analytical proof of the Strong Morse Inequalities for CW complexes, we need to strengthen our definition of a Morse function. Note that Definition 0.2 is equivalent to

Definition 0.2′. A function $f : K \to \mathbf{R}$ is a *discrete Morse function* if, for every $\sigma^{(p)} \in K_p$,

1) whenever $\tau_1^{(p+1)} > \sigma$ and $\tau_2^{(p+1)} > \sigma$ satisfy $\tau_1 \neq \tau_2$, or $\tau_1 = \tau_2$ and σ is an irregular face of τ_1,
$$f(\sigma) < \max\{f(\tau_1), f(\tau_2)\}$$
2) whenever $v_1^{(p-1)} < \sigma$ and $v_2^{(p-1)} < \sigma$ satisfy $v_1 \neq v_2$, or $v_1 = v_2$ and v_1 is an irregular face of σ,
$$f(\sigma) > \min\{f(v_1), f(v_2)\}.$$

In the analytic approach we will be following, we need stronger inequalities. Compare Definition 0.2′ with the following definition.

Definition 0.6. A function $f : K \to \mathbf{R}$ is a *discrete Witten-Morse function* if, for every $\sigma^{(p)} \in K_p$,

1) whenever $\tau_1^{(p+1)} > \sigma$ and $\tau_2^{(p+1)} > \sigma$ satisfy $\tau_1 \neq \tau_2$, or $\tau_1 = \tau_2$ and σ is an irregular face of τ_1,
$$f(\sigma) < \text{average } \{f(\tau_1), f(\tau_2)\}$$
2) whenever $v_1^{(p-1)} < \sigma$ and $v_2^{(p-1)} < \sigma$ satisfy $v_1 \neq v_2$, or $v_1 = v_2$ and v_1 is an irregular face of σ,
$$f(\sigma) > \text{average } \{f(v_1), f(v_2)\}.$$

Clearly, every Witten-Morse function is, in fact, a Morse function. In section 4 we will have to strengthen the definition even more.

Definition 0.7. Say a discrete Witten-Morse function f is *flat* if, for every $\sigma^{(p)} \in K_p$

1) whenever $\tau_1^{(p+1)} > \sigma$ and $\tau_2^{(p+1)} > \sigma$ satisfy $\tau_1 \neq \tau_2$
$$f(\sigma) \leq \min\{f(\tau_1), f(\tau_2)\}$$
2) whenever $v_1^{(p-1)} < \sigma$ and $v_2^{(p-1)} < \sigma$ satisfy $v_1 \neq v_2$
$$f(\sigma) \geq \max\{f(v_1), f(v_2)\}.$$

Although its role in the proof in §4 will be obvious, the ultimate meaning of this definition is not completely clear to the author. However, in this and in other work, flat Witten-Morse functions seem to have shown themselves to be the appropriate combinatorial analogue of smooth non-degenerate Morse functions.

The following figure exhibits 3 Morse functions on the solid triangle. The first is a Morse function, but not a Witten-Morse function. The second is a Witten-Morse function, but is not flat. The third is a flat Witten-Morse function.

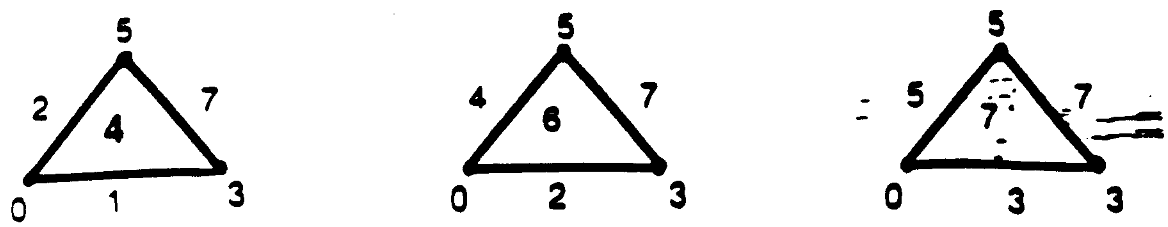

FIGURE 0.1

These 3 Morse functions are *equivalent* (see Definition 1.2).

In section 1 of this paper we show that, although the definition of a flat Morse function is more restrictive than that of a Morse function, for every Morse function there is an equivalent (in the sense of Definition 1.2) flat Morse function.

In section 2 we give a (discrete-) Hodge theoretic proof of the Strong Morse Inequalities (with coefficient field $\mathbf{F} = \mathbf{R}$) for CW complexes. Namely, we begin with the cellular chain complex

$$0 \longrightarrow C_n(M, \mathbf{R}) \xrightarrow{\partial} C_{n-1}(M, \mathbf{R}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(M, \mathbf{R}) \longrightarrow 0$$

and we investigate a combinatorial analogue of the perturbed differential studied. in the smooth category. in [Wi]. That is. we set

$$\partial_t = e^{tf} \partial e^{-tf}$$

where f is a Witten-Morse function. We then consider the Laplacian

$$\Delta_p(t) = \partial_t \partial_t^* + \partial_t^* \partial_t : C_p(M, \mathbf{R}) \longrightarrow C_p(M, \mathbf{R})$$

where ∂_t^* is the adjoint of ∂_t with respect to the inner product on the chain spaces defined by declaring the cells to be an orthonormal basis. We study the behavior of $\Delta_p(t)$ as t goes to $+\infty$. In particular. after presenting a more explicit description of $\Delta_p(t)$, it will follow immediately from Definitions (0.2) and (0.6) that for each p

$$\#\{\text{eigenvalues of } \Delta_p(t) \text{ which tend to } 0 \text{ as } t \to \infty\} \;=\; m_p.$$

We include a discussion of why this is sufficient to prove the Strong Morse Inequalities (see also [Bo]).

In section 3 we define the Morse complex

$$\mathcal{M} : 0 \longrightarrow \mathcal{M}_n \xrightarrow{\tilde{\partial}} \mathcal{M}_{n-1} \xrightarrow{\tilde{\partial}} \cdots \xrightarrow{\tilde{\partial}} \mathcal{M}_0 \longrightarrow 0$$

where $\mathcal{M}_p \subseteq C_p(M, \mathbf{R})$ is the subspace consisting of the linear combinations of the critical cells (so that $\dim \mathcal{M}_p = m_p$). Let $\tau^{(p+1)}$ and $\sigma^{(p)}$ be critical cells. We define the differential $\tilde{\partial}$ by setting $< \tilde{\partial}\tau, \sigma >$ to be the number of gradient paths (see Definition 3.1) of f from $\partial\tau$ to σ (counted according to an algebraic multiplicity).

In section 4 we present an analytic derivation of the Morse complex. We first introduce the one-parameter family of Witten complexes

$$\mathcal{W}(t) : 0 \longrightarrow W_n(t) \xrightarrow{\partial_t} W_{n-1}(t) \xrightarrow{\partial_t} \cdots \xrightarrow{\partial_t} W_0(t) \longrightarrow 0$$

where $W_p(t) \subseteq C_p(M, \mathbf{R})$ is the span of the eigenspaces of $\Delta_p(t)$ corresponding to the eigenvalues which tend to 0 as $t \to \infty$. It follows immediately that for all t

$$H_*(\mathcal{W}(t)) \cong H_*(M, \mathbf{R}). \tag{0.5}$$

We then prove that if f is a flat Witten-Morse function then, after a change of coordinates, as $t \to \infty$ the complex $\mathcal{W}(t)$ approaches the complex $\mathcal{M}$.

In section 5 we show that the equality (0.5) holds in the limit as $t \to \infty$. That is,

$$H_*(\mathcal{M}) \cong H_*(M, \mathbf{R}).$$

A topological proof of this fact is presented in [Fo2].

In section 6 we broaden our view and fix an orthogonal representation

$$\Phi : \pi_1(M) \longrightarrow O(k, \mathbf{R})$$

where $\pi_1(M)$ denotes the fundamental group of M and $O(k, \mathbf{R})$ denotes the group of real orthogonal $k \times k$ matrices. We recall the definition of the Reidemeister Torsion $T(M, \Phi)$, a combinatorial invariant of the complex M. We then show how the Morse complex $\mathcal{M}$ can be modified to a complex $\mathcal{M}(\Phi)$ which takes into account the representation Φ. The main result of the section is that $T(M, \Phi)$ can be calculated from the Morse complex $\mathcal{M}(\Phi)$. Namely, one can define a torsion of any differential complex and we show

$$T(M, \Phi) = \text{Torsion}(\mathcal{M}(\Phi)).$$

This identity, in the smooth category, is a key step in the recent proofs that Reidemeister torsion equals the analytic torsion of Ray and Singer [R-S] ([T], [B-Z], [B-F-K]).

§1. Flat Witten-Morse Functions.

In [Fo2] we proved the existence of many nontrivial Morse functions. We also examined their basic properties. Of importance to us is the following lemma (Lemma 2.5 of [Fo2]). Definition 0.3 list two conditions for a p-cell σ to be critical, which we denote 0.3(1) and 0.3(2).

Lemma 1.1. *For any p-cell σ, at least one of* 0.3(1) *and* 0.3(2) *must hold. That is,*

$$\#\{\tau^{(p+1)} > \sigma \mid f(\tau) \leq f(\sigma)\} \ + \ \#\{v^{(p-1)} < \sigma \mid f(v) \geq f(\sigma)\} \leq 1.$$

For a proof, see section 2 of [Fo2].

In this section we extend the existence to flat Witten-Morse functions. Namely, we prove that for every Morse function there is an equivalent flat Witten-Morse function.

Definition 1.2. If $f,\ g : K \to \mathbf{R}$ are 2 non-degenerate functions we say *f and g are equivalent* if, for every p and every $\sigma^{(p)} < \tau^{(p+1)}$

$$f(\sigma) < f(\tau) \longleftrightarrow g(\sigma) < g(\tau).$$

It follows directly from the definitions that if f and g are equivalent, then f is a Morse function if and only if g is a Morse function. If they are Morse functions, they have precisely the same critical points and (as we will see in section 3) they induce canonically isomorphic Morse complexes.

It will be convenient to require a bit more structure of our Witten-Morse functions.

Definition 1.3. If f is a combinatorial Morse function on M, say f is *self-indexing* if

1) Image$(f) \subseteq [0, \dim M]$
2) For every critical point σ of f

$$f(\sigma) = \dim \sigma.$$

Theorem 1.4. *Let M be a CW complex and f a discrete Morse function on M. Then there is a discrete self-indexing flat Witten-Morse function g on M which is equivalent to f.*

Proof. The proof will be by induction on the dimension of M.

If $\dim M = 0$ then the trivial function g, where $g(v) = 0$ for all v, satisfies the desired properties.

Suppose the theorem is true for all complexes of dimension $\leq k-1$ and $\dim M = k$. Let N be the $(k-1)$-skeleton of M. Restricting f to N yields a Morse function f_N on N. By induction there is an equivalent self-indexing flat Witten-Morse function $\tilde{g}$ on N (we will not label this function g_N since it is not quite the restriction of the desired function g to N). We begin our construction of g by setting

$$g(\sigma) = \tilde{g}(\sigma) \quad \text{if} \quad \dim \sigma \leq k-2.$$

We must modify $\tilde{g}$ slightly on the $(k-1)$-cells. To see this, suppose $\sigma_1^{(k-1)}$ and $\sigma_2^{(k-1)}$ are $(k-1)$-faces of $\tau^{(k)}$ and $f(\sigma_1) \geq f(\tau) > f(\sigma_2)$. Then we must have $g(\sigma_1) = g(\tau) > g(\sigma_2)$ so, in particular, $g(\sigma_1) > g(\sigma_2)$. However, this relationship need not hold for $\tilde{g}$. This difficulty is the motivation for the following definition. Given a $(k-1)$-cell σ, define

$$(1.4) \qquad d(\sigma) = \sup\{r \mid \exists \text{ a sequence}$$
$$\sigma = \sigma_0,\ \tau_0^{(k)},\ \sigma_1^{(k-1)},\ \tau_1^{(k)}, \ldots, \sigma_{r-1}^{(k-1)},\ \tau_{r-1}^{(k)},\ \sigma_r^{(k-1)}$$

such that for each $i = 0, 1, \ldots, r-1$

$$\tau_i > \sigma_i \neq \sigma_{i+1} < \tau_i \quad \text{and} \quad f(\sigma_i) \geq f(\tau_i) \geq f(\sigma_{i+1})\}$$

and let

$$D = \sup_{\sigma^{(k-1)}} d(\sigma).$$

Now set, for any $(k-1)$-cell σ

$$g(\sigma) = \tilde{g}(\sigma) + \frac{d(\sigma)}{2D}.$$

We make the following observations

(i) If $\exists v^{(k-2)} < \sigma^{(k-1)}$ with $f(v) \geq f(\sigma)$ then, by Lemma 1.1, $d(\sigma) = 0$ so $g(\sigma) = \tilde{g}(\sigma)$. Otherwise, $g(\sigma^{(k-1)}) \geq \tilde{g}(\sigma)$. It follows directly from the definitions that g is a flat Witten-Morse function on N that is equivalent to $\tilde{g}$, and hence to f_N.

(ii) If $\sigma^{(k-1)}$ is critical for f on M then $d(\sigma) = 0$ (since there is no $\tau^{(k)} > \sigma$ with $f(\tau) \leq f(\sigma)$). Moreover, σ is critical for f restricted to N. Thus

$$g(\sigma) = \tilde{g}(\sigma) = k - 1.$$

(iii) Since Image$(\tilde{g}) \leq [0, k-1]$ we have that restricted to N

$$\text{Image}(g) \subseteq \left[0, k - \frac{1}{2}\right].$$

(iv) If $d(\sigma^{(k-1)}) \neq 0$ then there is a $\tau^k > \sigma$ with $f(\tau) \leq f(\sigma)$. Using Lemma 1.1 we see that σ is critical for f_N so that $\tilde{g}(\sigma) = k - 1$.

We now define g on k-cells. If $\tau^{(k)}$ is critical for f, set

$$g(\tau) = k. \tag{1.5}$$

If $\tau^{(k)}$ is not critical, there must be a $\sigma^{(k-1)} < \tau$ with $f(\sigma) \geq f(\tau)$ (so that $d(\sigma) > 0$). Such a σ must be unique by condition 2 of Definition 0.1. In this case, set

$$g(\tau) = g(\sigma). \tag{1.6}$$

It remains to show that g satisfies the conclusions of the theorem. We must first show that g is equivalent to f. Since, restricted to N, g is equivalent to f_N (by observation (i)), it is sufficient to check that if $\sigma^{(k-1)} < \tau^{(k)}$ then

$$f(\sigma) < f(\tau) \longleftrightarrow g(\sigma) < g(\tau).$$

Suppose $f(\sigma) \geq f(\tau)$. Then $g(\sigma) = g(\tau)$ by (1.6). Suppose $f(\sigma) < f(\tau)$. If τ is critical for f then

$$g(\tau) = k > k - \frac{1}{2} \geq g(\sigma)$$

(by observation (iii)). If τ is not critical then there is a $\tilde{\sigma}^{(k-1)} < \tau$ with $f(\tilde{\sigma}) \geq f(\tau)$. It follows from (1.4) that $d(\tilde{\sigma}) \geq d(\sigma) + 1$ (since if $\sigma, \tau_0, \ldots, \sigma_r$ is any sequence as in (1.4) of length r beginning with σ, then $\tilde{\sigma}, \tau, \sigma, \tau_0, \ldots, \sigma_r$ is a sequence of length $r + 1$ beginning with $\tilde{\sigma}$). Thus, since $\tilde{g}(\tilde{\sigma}) = k - 1 \geq \tilde{g}(\sigma)$ (by observations (iii) and (iv))

$$g(\tau) = g(\tilde{\sigma}) = k - 1 + \frac{d(\tilde{\sigma})}{2D} > k - 1 + \frac{d(\sigma)}{2D} \geq g(\sigma)$$

as desired.

It follows immediately from the construction that

$$\text{Image}(g) \leq [0, k].$$

If $\dim \sigma \leq k - 2$ and σ is critical for f then σ is critical for f_N and

$$g(\sigma) = \tilde{g}(\sigma) = \dim \sigma.$$

It follows from observation (ii) and (1.5) that g is self-indexing on M.

Lastly, it follows from the construction that g is flat, and hence is a Witten-Morse function. □

§2. The Morse Inequalities.

Theorem 2.1 (The Strong Morse Inequalities). *Let M be a finite CW complex and f a discrete Morse function on M with Morse numbers $\{m_i\}_{i=0,1,\dots,n}$, where $n = \dim M$. Then for every $k = 0, 1, 2, \dots$ we have the inequality*

$$m_k - m_{k-1} + \cdots + (-1)^{k-1} m_0 \geq b_k - b_{k-1} + \cdots + (-1)^{k-1} b_0$$

where $b_i = \dim H_i(M, \mathbf{R})$.

Proof. The Strong Morse Inequalities are equivalent to the following statement: There is a differential complex of finite dimensional real vector spaces

$$V : 0 \longrightarrow V^n \xrightarrow{d} V^{n-1} \xrightarrow{d} \cdots \xrightarrow{d} V^0 \longrightarrow 0$$

(so that $d^2 = 0$), satisfying for each $i = 0, 1, \dots, n$, $\dim V^i = m_i$

$$\dim H^i(V) \left[\equiv \dim \frac{\ker d : V^i \to V^{i-1}}{\operatorname{Im} d : V^{i-1} \to V^i} \right] = b_i.$$

To see how such differential complexes arise, consider the cellular chain complex of M

$$C : 0 \longrightarrow C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0 \longrightarrow 0$$

where $C_i = C_i(M, \mathbf{R})$ is the vector space of real cellular i-chains on M, and ∂ is the usual boundary operator. Endow each C_i with the inner product $<,>$ in which the cells of M form an orthonormal basis. Let ∂^* be the adjoint of ∂ with respect to these inner products. For each i we can consider the Laplacian

$$\Delta_i \equiv \partial \partial^* + \partial^* \partial : C_i \longrightarrow C_i.$$

It follows from standard linear algebra that

$$b_i = \dim \ker \Delta_i.$$

The operator Δ_i is symmetric, and hence diagonalizable. For each $\lambda \in \mathbf{R}$, let

$$E_i(\lambda) = \{c \in C_i \mid \Delta_i c = \lambda c\}$$

denote the λ eigenspace of Δ_i. Since $\partial \Delta = \Delta \partial$, ∂ preserves the eigenspaces. That is, for each $\lambda \in \mathbf{R}$ we have a differential complex

$$E(\lambda) : 0 \longrightarrow E_n(\lambda) \xrightarrow{\partial} E_{n-1}(\lambda) \xrightarrow{\partial} \cdots \xrightarrow{\partial} E_0(\lambda) \longrightarrow 0$$

where ∂ is simply the restriction of the boundary operator to the λ eigenspace. It is easy to check that

$$\dim H_i(E(\lambda)) = \begin{cases} b_i & \text{if } \lambda = 0 \\ 0 & \text{if } \lambda \neq 0. \end{cases} \tag{2.2}$$

For $\Lambda \in \mathbf{R}$, define

$$W_i(\Lambda) = \bigoplus_{\lambda \leq \Lambda} E_i(\lambda)$$

and consider the Witten complex

$$\mathcal{W}(\Lambda) = \bigoplus_{\lambda \leq \Lambda} E(\lambda) : 0 \longrightarrow W_n(\Lambda) \xrightarrow{\partial} W_{n-1}(\Lambda) \xrightarrow{\partial} \cdots \xrightarrow{\partial} W_0(\Lambda) \longrightarrow 0.$$

It follows from (2.2) that

$$\dim H_i(\mathcal{W}(\Lambda)) = \begin{cases} 0 & \text{if } \Lambda < 0 \\ b_i & \text{if } \Lambda \geq 0. \end{cases}$$

Therefore, to prove the Strong Morse Inequalities, it is sufficient to find a $\Lambda \geq 0$ so that for each i

$$\dim W_i(\Lambda) = m_i.$$

We note that

$$\dim W_i(\Lambda) = \#\{\text{eigenvalues of } \Delta_i \leq \Lambda\}$$

where the eigenvalues are counted according to their multiplicity.

This is not quite how we will prove the Morse Inequalities, since, in particular, we have not yet used the Morse function. Let f be a discrete Witten-Morse function. For each i, define the automorphism

$$e^{tf} : C_i \longrightarrow C_i$$

by setting

$$e^{tf}(\sigma) = e^{tf(\sigma)}\sigma$$

for each oriented i-cell σ of M, and extending linearly to all of C_i. We now consider the Witten cellular chain complex

$$(\mathcal{C}, \partial_t) : 0 \longrightarrow C_n \xrightarrow{\partial_t} C_{n-1} \xrightarrow{\partial_t} \cdots \xrightarrow{\partial_t} C_0 \longrightarrow 0$$

where

$$\partial_t = e^{tf} \partial e^{-tf}.$$

For each $t \in \mathbf{R}$ this complex has the same homology as $\mathcal{C} = (\mathcal{C}, \partial_0)$ so, preceeding exactly as before, to prove the Strong Morse Inequalities it is sufficient to find a $t \in \mathbf{R}$ and a $\Lambda \geq 0$ such that for all i

$$\#\{\text{eigenvalues of } \Delta_i(t) < \Lambda\} = m_i$$

where

$$\Delta_i(t) = \partial_t \partial_t^* + \partial_t^* \partial_t : C_i \longrightarrow C_i$$

and ∂_t^* is the adjoint of ∂_t with respect to the inner product $<,>$.

It is now simply a matter of finding a more explicit representation for $\Delta_i(t)$. Let σ be an oriented i-cell of M. Then

$$\partial\sigma = \sum_{v^{(i-1)}<\sigma} \langle \partial\sigma, v\rangle\, v$$

and

$$\partial_t\sigma = e^{tf}\partial e^{-tf}\sigma = \sum_{v^{(i-1)}<\sigma} \langle \partial\sigma, v\rangle\, e^{t(f(v)-f(\sigma))}v. \tag{2.3}$$

Similarly,

$$\partial^*\sigma = \sum_{\tau^{(i+1)}>\sigma} \langle \partial\tau, \sigma\rangle\, \tau$$

and

$$\partial_t^*\sigma = e^{-tf}\partial^* e^{tf}\sigma = \sum_{\tau^{(i+1)}>\sigma} \langle \partial\tau, \sigma\rangle\, e^{t(f(\sigma)-f(\tau))}\tau. \tag{2.4}$$

Combining these we learn

$$\begin{aligned}
\Delta_i(t)\sigma &= \partial_t\partial_t^* + \partial_t^*\partial_t\sigma \\
&= \sum_{v^{(i-1)}<\sigma}\ \sum_{\tilde\sigma^{(i)}>v} \langle\partial\sigma, v\rangle\,\langle\partial\tilde\sigma, v\rangle\, e^{t(2f(v)-f(\sigma)-f(\tilde\sigma))}\tilde\sigma \\
&\quad + \sum_{\tau^{(i+1)}>\sigma}\ \sum_{\tilde\sigma^{(i)}<\tau} \langle\partial\tau, \sigma\rangle\,\langle\partial\tau, \tilde\sigma\rangle\, e^{t(f(\sigma)+f(\tilde\sigma)-2f(\tau))}\tilde\sigma \\
&= \sum_{\tilde\sigma^{(i)}}\Bigg[\sum_{\substack{v^{(i-1)}\text{s.t.}\\ v<\sigma, v<\tilde\sigma}} \langle\partial\sigma, v\rangle\,\langle\partial\tilde\sigma, v\rangle\, e^{t(2f(v)-f(\sigma)-f(\tilde\sigma))} \\
&\quad + \sum_{\substack{\tau^{(i+1)}\text{s.t.}\\ \tau>\sigma, \tau>\tilde\sigma}} \langle\partial\tau, \sigma\rangle\,\langle\partial\tau, \tilde\sigma\rangle\, e^{t(f(\sigma)+f(\tilde\sigma)-2f(\tau))}\Bigg]\tilde\sigma.
\end{aligned}$$

If $\sigma^{(i)} \neq \tilde\sigma^{(i)}$ then for every $v^{(i-1)}$ which is a face of both σ and $\tilde\sigma$

$$2f(v) - f(\sigma) - f(\tilde\sigma) < 0$$

by condition 1 of Definition 0.7. If $\tau^{(i+1)}$ has both σ and $\tilde\sigma$ as faces then

$$f(\sigma) + f(\tilde\sigma) - 2f(\tau) < 0$$

by condition 2 of Definition 0.7. Therefore

$$\begin{aligned}
\Delta_i(t)\sigma &= \langle\Delta(t)\sigma, \sigma\rangle\,\sigma + O(e^{-tc}) \\
&= \left[\sum_{v^{(i-1)}<\sigma} \langle\partial\sigma, v\rangle^2 e^{2t(f(v)-f(\sigma))} + \sum_{\tau^{(i+1)}>\sigma} \langle\partial\tau, \sigma\rangle^2 e^{2t(f(\sigma)-f(\tau))}\right]\sigma + O(e^{-tc})
\end{aligned}$$

for some $c > 0$. Therefore, as $t \to \infty$, $\Delta_i(t)$ becomes diagonal with respect to the basis of C_i consisting of the i-cells of M. In particular, up to exponentially small errors as $t \to \infty$, the eigenvalues of $\Delta_i(t)$ are the diagonal entries

$$
\begin{aligned}
\langle \Delta_i(t)\sigma, \sigma \rangle = & \sum_{v^{(i-1)} < \sigma} \langle \partial\sigma, v \rangle^2 \, e^{2t(f(v) - f(\sigma))} \\
& + \sum_{\tau^{(i+1)} > \sigma} \langle \partial\tau, \sigma \rangle^2 \, e^{2t(f(\sigma) - f(\tau))}
\end{aligned}
\tag{2.5}
$$

We now observe that it follows directly from Definition 0.3 that all of the exponents in (2.5) are negative (and go to $-\infty$ as $t \to \infty$) if and only if σ is critical. Otherwise at least one exponent is nonnegative. For such an exponent, say $f(v^{(p-1)}) - f(\sigma^{(p)})$, v must be a regular face of σ. Applying (0.2) we learn $\langle \Delta_i(t)\sigma, \sigma \rangle \geq 1$. Therefore, for all i and any t large enough

$$\#\{\text{eigenvalues of } \ \Delta_i(t) \leq \frac{1}{2}\} = m_i$$

which completes the proof. $\square$

§3. The Morse Complex.

In this section we briefly describe the Morse complex associated to a discrete Morse function. A more complete presentation can be found in [Fo2] (for a discussion of the Morse complex in the smooth category see [Kl], [Wi] and [H-S]). Fix a discrete Morse function f. For each p let $\mathcal{M}_p(f)$ (or simply $\mathcal{M}_p$ if f is understood) denote the subspace of $C_p(M, \mathbf{R})$ consisting of linear combinations of critical p-cells (so that $\dim \mathcal{M}_p = m_p$). We will build a differential complex

$$\mathcal{M} : 0 \longrightarrow \mathcal{M}_n \xrightarrow{\tilde{\partial}} \mathcal{M}_{n-1} \xrightarrow{\tilde{\partial}} \cdots \xrightarrow{\tilde{\partial}} \mathcal{M}_0 \longrightarrow 0$$

satisfying, for each p,

$$\dim H_p(\mathcal{M}) = \dim H_p(M, \mathbf{R}).$$

To define the differential $\tilde{\partial}$ we must introduce the notion of a gradient path.

Definition 3.1. *A gradient path of dimension p from $\sigma^{(p)}_{\text{initial}}$ to $\sigma^{(p)}_{\text{final}}$* is a sequence

$$\gamma : \sigma^{(p)}_{\text{initial}} = \sigma_0^{(p)},\ \tau_0^{(p+1)},\ \sigma_1^{(p)},\ \tau_1^{(p+1)},\ \sigma_2^{(p)}, \ldots, \tau_{k-1}^{(p+1)},\ \sigma_k^{(p)} = \sigma^{(p)}_{\text{final}} \tag{3.1}$$

such that for every $i = 0, 1, \ldots, k-1$

1) $\sigma_i < \tau_i$ and $\sigma_{i+1} < \tau_i$
2) $\sigma_i \neq \sigma_{i+1}$
3) $f(\sigma_i) \geq f(\tau_i) > f(\sigma_{i+1})$.

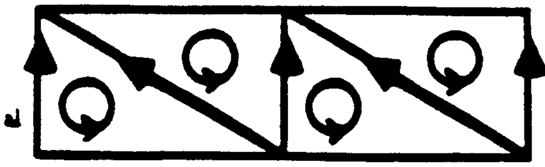

FIGURE 3.1

Our next goal is to define the algebraic multiplicity of a gradient path. Suppose, for the moment, that for each i, σ_i and σ_{i+1} are regular faces of τ_i. Then each σ_i, when endowed with an orientation, induces an orientation on τ_i (so that with this orientation $\langle \partial\tau_i, \sigma_i\rangle = -1$) which in turn induces an orientation on σ_{i+1} (so that with this orientation $\langle \partial\tau_i, \sigma_{i+1}\rangle = 1$). In this way an orientation on σ_{initial} induces an orientation on σ_{final} (see figure 3.1).

Suppose σ_{initial} and σ_{final} have been endowed with an orientation. Set the multiplicity of γ, denoted by $m(\gamma)$, to be $+1$ if the orientation on σ_{final} agrees with the orientation induced by the orientation on σ_{initial}, and -1 otherwise. Equivalently, suppose each cell of M is given an orientation, then

$$m(\gamma) = \prod_{i=0}^{k-1} - \langle \partial\tau_i, \sigma_i\rangle \langle \partial\tau_i, \sigma_{i+1}\rangle . \tag{3.3}$$

Definition 3.2. Suppose each cell of M is given an orientation and γ is a gradient path as in (3.1) (we no longer require σ_i and σ_{i+1} to be regular faces of τ_i). We define the algebraic multiplicity of γ, $m(\gamma)$, by the formula (3.3),

We can now define the differential $\tilde{\partial}$. Endow each cell of M with an orientation.

Definition 3.3. For any (oriented) critical cells $\tau^{(p+1)}$ and $\sigma^{(p)}$ set

$$\left\langle \tilde{\partial}\tau, \sigma \right\rangle = \sum_{\sigma_1^{(p)} < \tau} \langle \partial\tau, \sigma_1\rangle \sum_{\gamma\in\Gamma(\sigma_1,\sigma)} m(\gamma) \tag{3.4}$$

where $\Gamma(\sigma_1, \sigma)$ is the set of all gradient paths of dimension p from σ_1 to σ. Since the critical p-cells form a basis of $\mathcal{M}_p$, the formula (3.4) defines a unique linear map

$$\tilde{\partial} : \mathcal{M}_{p+1} \longrightarrow \mathcal{M}_p.$$

It is not at all obvious that the Morse complex has the same homology as M, or even that $\tilde{\partial}^2 = 0$. These will be proved in the next 2 sections.

§4. The Limit of the Witten Complex.

An immediate corollary of our proof in section 2 of the Strong Morse Inequalities is the observation that the differential complex (which we will call the Witten complex)

$$\mathcal{W}(t) : 0 \longrightarrow W_n(t) \xrightarrow{\partial_t} W_{n-1}(t) \xrightarrow{\partial_t} \cdots \xrightarrow{\partial_t} W_0(t) \longrightarrow 0$$

satisfies for all t

$$\dim H_i(\mathcal{W}(t)) = \dim H_i(M, \mathbf{R})$$

where $\partial_t = e^{tf}\partial e^{-tf}$ and $W_p(t)$ is the span of the eigenfunctions of $\Delta_p(t)$ corresponding to eigenvalues which tend to 0 as $t \to \infty$. It also follows from the proof of Theorem 2 that for each p

$$\lim_{t\to\infty} W_p(t) = \mathcal{M}_p.$$

This can be seen very clearly from the form of $\Delta_p(\infty)$ as shown in (0.1). In fact, the main theorem of this section is that, as $t \to \infty$, the Witten complex, slightly modified, converges to the Morse complex.

Before embarking on the rigorous proofs of the main results of this section, we will provide a brief overview. The key step is to define an isomorphism $\pi(t)$ from $\mathcal{M}_p$ to $W_p(t)$ (which satisfies $\lim_{t\to\infty} \pi(t)$ is the identity). The proof then boils down to showing that (after a simple rescaling)

$$\lim_{t\to\infty} \pi^{-1}(t)\partial_t\pi(t) = \tilde{\partial}.$$

The main ingredient is the observation that (as is hinted at in the expression for $\Delta_p(\infty)$ in (0.3)) if f is a flat Witten-Morse function, then $\pi(t)$ is very close to $1 - \Delta(t)$. In fact, up to negligible error, $\pi(t)$ can be replaced by $(1-(\Delta(t))^N)^N$ for sufficiently large N. To proceed further, we express the operator $\Delta(t)$ as a matrix with respect the standard basis consisting of cells. We note that the non-zero entries in $\Delta(t)$ correspond to adjacent cells, so that the entries of $(\Delta(t))^N$ are determined by paths of length N. Thus, $\pi(t)$ is expressed as a sum over paths, which we then analyze in greater detail.

The interested reader may wish to compare this analysis with Witten's argument appearing on pages 671 and 672 of [Wi]. Witten, in essence, constructs $\pi(t)$ from an examination of an appropriate Lagrangian on the path space of the manifold. From an operator point of view, he is studying the heat operator $e^{-\Delta(t)}$. It is easy to see why this operator can be used in the smooth case and $(1-(\Delta(t))^N)^N$ in the combinatorial setting. The goal is to construct $\pi(t)$ which is uniquely characterized by the property of being the identity operator or the eigenspaces corresponding to the eigenvalues which go to 0, i.e., $W(t)$, and 0 on the orthogonal complement. From (0.3) we see that as $t \to \infty$, $\Delta(t)$ approaches 0 on $W(t)$ and 1 on the complement. Hence $1 - \Delta(t)$ is close to $\pi(t)$. Simple estimates show that as $N \to \infty$ $(1-(\Delta(t))^N)^N$ approaches $\pi(t)$ exponentially fast. In Witten's smooth setting as $t \to \infty$ the eigenvalues of $\Delta(t)$ go to 0 on $W(t)$, while all other eigenvalues go to ∞. Hence $e^{-\Delta(t)}$ is an operator which goes to 1 on $W(t)$ and 0 on its orthogonal complement, as desired.

We now work more precisely. Let f be a flat Witten-Morse function and consider

$$\Delta(t) = \partial_t\partial_t^* + \partial_t^*\partial_t.$$

Following the proof of Theorem 2 we see that the off-diagonal terms of $\Delta(t)$, as well as the diagonal terms corresponding to critical cells, are $O(e^{-ct})$ for some $c > 0$. The diagonal terms corresponding to non-critical cells are all $1 + O(e^{-ct})$.

To study the behavior of the complex $W(t)$ as $t \to \infty$ we will choose a convenient basis for the W_p. Let $\pi_p(t)$ denote the orthogonal projection from $C_p(M, \mathbf{R})$ to $W_p(t)$. For each critical p-cell σ let

$$g_\sigma(t) = \pi(t)\sigma (= \sigma + O(e^{-tc}))$$

For sufficiently large t the g_σ's form a basis of $W_p(t)$, but not an orthonormal basis. Let G denote the square matrix with rows and columns indexed by the critical p-cells, and where (suppressing the t's)

$$G_{\sigma_1\sigma_2} = \langle g_{\sigma_1}, g_{\sigma_2}\rangle \, (= \delta_{\sigma_1\sigma_2} + O(e^{-tc})).$$

Let

$$h_\sigma = G^{-1/2} g_\sigma (= \sigma + O(e^{-tc})).$$

The h_σ form an orthonormal basis of $W_p(t)$. The main theorem of this section is

Theorem 4.1. *For any critical cells $\sigma^{(p)}$ and $\tau^{(p+1)}$*

$$\langle \partial_t h_\tau, h_\sigma\rangle = e^{t(f(\sigma)-f(\tau))}\left[\left\langle \tilde{\partial}\tau, \sigma\right\rangle + O(e^{-tc})\right]$$

for some $c > 0$, where $\tilde{\partial}$ is as in (3.4).

Before proving the theorem, we consider some implications. Let $H(t)$ denote the linear map on $W(t)$ which sends h_σ to $f(\sigma)h_\sigma$, and

$$\tilde{\partial}_t = e^{-tH}\partial_t e^{-tH}.$$

Then for all t the complex $(W(t), \tilde{\partial}_t)$ has the same homology as $(W(t), \partial_t)$ and hence the same homology as the underlying manifold M. Moreover, for all critical τ and σ

$$h_\tau \overset{t\to\infty}{\longrightarrow} \tau, \; h_\sigma \overset{t\to\infty}{\longrightarrow} \sigma$$

and from Theorem 4.1

$$\left\langle \tilde{\partial}_t h_\tau, h_\sigma\right\rangle = \left\langle \tilde{\partial}\tau, \sigma\right\rangle + O(e^{-tc}) \overset{t\to\infty}{\longrightarrow} \left\langle \tilde{\partial}\tau, \sigma\right\rangle.$$

Thus, we have

Corollary 4.2. *As $t \to \infty$, the complex $(W(t), \tilde{\partial}_t)$ converges to the Morse complex $\mathcal{M}$ defined in section 3. Since $\tilde{\partial}_t^2 = 0$ for all t, we learn that $\tilde{\partial}^2 = 0$. In addition, it immediately follows that for each p*

$$\dim H_p(\mathcal{M}) \geq \dim H_p(M, \mathbf{R}) \tag{4.2}$$

and

$$\Sigma(-1)^p \dim H_p(\mathcal{M}) = \Sigma(-1)^p \dim H_p(M, \mathbf{R}).$$

These are the Weak Morse Inequalities with the Morse numbers $\{m_p\}$ replaced by $\{\dim H_p(\mathcal{M})\}$. In fact, it also follows easily that the Strong Morse Inequalities hold. We will not prove this, since in section 5 we will prove that in (4.2) we actually have equality.

We now begin our preparations for the proof of Theorem 4.1.

Definition 4.3. Define a *p-step* on M to be a triple

$$\beta : \sigma_0^{(p)},\ \tau,\ \sigma_1^{(p)}$$

where either

1) $\dim \tau = p+1$ and $\sigma_0 < \tau > \sigma_1$

or

2) $\dim \tau = p-1$ and $\sigma_0 > \tau < \sigma_1$

($\sigma_0 = \sigma_1$ is permitted). We define the algebraic multiplicity of β, $m(\beta)$, by

$$m(\beta) = \begin{cases} -\langle \partial\tau, \sigma_0\rangle \langle \partial\tau, \sigma_1\rangle & \text{if } \dim\tau = p+1 \\ -\langle \partial\sigma_0, \tau\rangle \langle \partial\sigma_1, \tau\rangle & \text{if } \dim\tau = p-1 \end{cases}$$

and the action of β, $s(\beta)$, by

$$s(\beta) = \mid f(\sigma_0) + f(\sigma_1) - 2f(\tau) \mid .$$

We observe that

$$s(\beta) \geq \mid f(\sigma_0) - f(\sigma_1) \mid \tag{4.3}$$

with equality if and only if either

$$f(\sigma_0) \geq f(\tau) \geq f(\sigma_1)$$

or

$$f(\sigma_1) \geq f(\tau) \geq f(\sigma_0).$$

In either case, since f is flat, we must have either $f(\sigma_0) = f(\tau)$ or $f(\sigma_1) = f(\tau)$.

Definition 4.4. Define a *p-path of length r from* $\sigma^{(p)}_{\text{initial}}$ *to* $\sigma^{(p)}_{\text{final}}$ to be a sequence of r p-steps $\beta_0, \beta_1, \ldots, \beta_{r-1}$ where each step begins where the previous step ends. That is

$$\gamma : \left(\sigma^{(p)}_{\text{initial}} = \sigma_0^{(p)},\ \tau_0,\ \sigma_1^{(p)}\right),\ \left(\sigma_1^{(p)},\ \tau_1, \sigma_2^{(p)}\right), \ldots, \left(\sigma_{r-1}^{(p)},\ \tau_{r-1},\ \sigma_r^{(p)} = \sigma^{(p)}_{\text{final}}\right)$$

where

$$\beta_i = \sigma_i^{(p)},\ \tau_i,\ \sigma_{i+1}^{(p)}.$$

We then set

$$m(\gamma) = \prod_{i=0}^{r-1} m(\beta_i)$$

$$s(\gamma) = \sum_{i=0}^{r-1} s(\beta_i).$$

It follows from (4.3) that

$$s(\gamma) \geq | f(\sigma_{\text{initial}}) - f(\sigma_{\text{final}}) | \tag{4.4}$$

with equality if and only if either

$$f(\sigma_0) \geq f(\tau_0) \geq f(\sigma_1) \geq \cdots \geq f(\tau_{r-1}) \geq f(\sigma_r)$$

or

$$f(\sigma_r) \geq f(\tau_{r-1}) \geq f(\sigma_{r-1}) \geq \cdots \geq f(\tau_0) \geq f(\sigma_0).$$

In either case, by (4.1), for $i = 0, 1, \ldots, r-1$ we must have

$$f(\sigma_{i+1}) = f(\tau_i) \quad \text{or} \quad f(\sigma_{i+1}) = f(\tau_i). \tag{4.5}$$

We now use the notion of a p-path to define a distance function on the set of p-cells.

Definition 4.5. For any p-cells $\sigma_0^{(p)}$ and $\sigma_1^{(p)}$, let

$$D(\sigma_0, \sigma_1) = \min_{\gamma} s(\gamma)$$

where γ runs over all p-paths from σ_0 to σ_1.

It follows from (4.4) that

$$D(\sigma_0, \sigma_1) \geq |f(\sigma_0) - f(\sigma_1)|.$$

Moreover, we have an obvious triangle inequality. For all $\sigma_0^{(p)}$, $\sigma_1^{(p)}$ and $\sigma_2^{(p)}$

$$D(\sigma_0, \sigma_1) + D(\sigma_1, \sigma_2) \geq D(\sigma_0, \sigma_2). \tag{4.6}$$

Lemma 4.6. *If $\sigma_0^{(p)}$ and $\sigma_1^{(p)}$ are critical p-cells and $\sigma_0 \neq \sigma_1$ then*

$$D(\sigma_0, \sigma_1) \; > | \; f(\sigma_0) - f(\sigma_1) \; | \, .$$

Proof. Let γ be a p-path from σ_0 to σ_1. We must see that

$$s(\gamma) \; > | \; f(\sigma_0) - f(\sigma_1) \; | \, .$$

Suppose, on the contrary, that

$$s(\gamma) \; = | \; f(\sigma_0) - f(\sigma_1) \; |$$

and write

$$\gamma : (\sigma_0^{(p)}, \; \tau_0, v_1^{(p)}), \; (v_1^{(p)}, \; \tau_1, \; v_2^{(p)}), \ldots, (v_{r-1}^{(p)}, \tau_{r-1}, \; \sigma_1^{(p)}).$$

We may assume that

$$\sigma_0 \neq v_1 \neq v_2 \neq \cdots \neq \sigma_1$$

since, if $v_i = v_{i+1}$ we can remove the step (v_i, τ_i, v_{i+1}) and the action $s(\gamma)$ will not increase. By relabeling if necessary, we may assume that

$$f(\sigma_0) \geq f(\tau_0) \geq f(v_1) \geq f(\tau_1) \geq \cdots \geq f(\tau_{r-1}) \geq f(\sigma_1).$$

In particular, we must have

$$f(\sigma_0) > f(v_1) > f(v_2) > \cdots > f(\sigma_1)$$

which implies

$$\tau_0 \neq \tau_1 \neq \tau_2 \neq \cdots \neq \tau_{r-1}. \tag{4.7}$$

Since σ_0 is critical

$$f(\sigma_0) \neq f(\tau_0).$$

By (4.5) this implies

$$f(\tau_0) = f(v_1).$$

From condition 1 of Definition 0.2 and (4.7) we learn

$$f(v_1) \neq f(\tau_1).$$

Again applying (4.5),

$$f(\tau_1) = f(v_2).$$

Continuing in this fashion we must have

$$f(\tau_{r-1}) = f(\sigma_1).$$

This contradicts the hypothesis that σ_1 is critical. $\square$

Proof of Theorem 4.1. Our first goal is to reduce the theorem to statements about the simpler functions g_σ rather than h_σ. We will prove that for all critical $\sigma_0^{(p)}$, $\sigma_1^{(p)}$ and $\tau^{(p+1)}$

$$G_{\sigma_0\sigma_1} = \langle g_{\sigma_0}, g_{\sigma_1}\rangle = \begin{cases} O(e^{-tD(\sigma_0\sigma_1)}) & \text{if } \ \sigma_0 \neq \sigma_1 \\ 1 + O(e^{-tc}), c > 0 & \text{if } \ \sigma_0 = \sigma_1 \end{cases} \tag{4.8}$$

$$\langle \partial_t g_\tau, g_\sigma\rangle = e^{t(f(\sigma)-f(\tau))}(\left\langle \tilde{\partial}\tau, \sigma\right\rangle + O(e^{-tc})) \quad \text{for some } \ c > 0. \tag{4.9}$$

This is sufficient to prove the theorem since, assuming (4.8), and using (4.6) we see

$$(G^{-1/2})_{\sigma_0\sigma_1} = \begin{cases} O(e^{-tD(\sigma_0,\sigma_1)}) & \text{if } \ \sigma_0 \neq \sigma_1 \\ 1 + O(e^{-tc}), c > 0 & \text{if } \ \sigma_0 = \sigma_1. \end{cases}$$

For all critical $\sigma^{(p)}$ and $\tau^{(p+1)}$

$$\langle \partial_t h_\tau, h_\sigma \rangle = \sum_{\substack{\sigma_1^{(p)}, \tau_1^{(p+1)} \\ \text{critical}}} (G^{-1/2})_{\tau\tau_1} \langle \partial_t g_{\tau_1}, g_{\sigma_1} \rangle (G^{-1/2})_{\sigma_1\sigma}$$

$$= (G^{-1/2})_{\tau\tau} \langle \partial_t g_\tau, g_\sigma \rangle (G^{-1/2})_{\sigma\sigma} + \sum_{\substack{\sigma_1, \tau_1 \text{ critical} \\ (\sigma_1,\tau_1) \neq (\sigma,\tau)}} (G^{-1/2})_{\tau\tau_1} \langle \partial_t g_{\tau_1}, g_{\sigma_1} \rangle (G^{-1/2})_{\sigma_1\sigma}.$$

Assuming (4.9), the first term is equal to

$$e^{t(f(\sigma)-f(\tau))}(\left\langle \tilde{\partial}\tau, \sigma \right\rangle + O(e^{-tc}))$$

for some $c > 0$. The second term is

$$\sum_{\substack{\sigma_1, \tau_1 \text{ critical} \\ (\sigma_1,\tau_1) \neq (\sigma,\tau)}} O\left(e^{-tD(\tau,\tau_1)}\right) e^{t(f(\sigma_1)-f(\tau_1))} O\left(e^{-tD(\sigma_1,\sigma)}\right).$$

From Lemma 4.5 we see that each of these terms is

$$O\left(e^{t[(f(\sigma)-f(\tau))-c]}\right)$$

for some $c > 0$, and thus does not contribute to the leading term.

To prove the estimates on the g_σ we need a convenient representation for $\pi(t)$, the orthogonal projection onto $W(t)$. The operator $\pi(t)$ is uniquely characterized by the property

$$\pi(t) = \begin{cases} 1 & \text{on } W(t) \\ 0 & \text{on } W^\perp(t) \end{cases}$$

where $W^\perp(t)$ is the orthogonal complement to $W(t)$. Equivalently, $W^\perp(t)$ is the span of the eigenfunctions of $\Delta(t)$ corresponding to eigenvalues which do not go to 0 as $t \to \infty$. Recall that there is a $c > 0$ such that

$$\Delta(t) = \begin{cases} O(e^{-ct}) & \text{on } W(t) \\ 1 + O(e^{-ct}) & \text{on } W^\perp(t). \end{cases}$$

For any $N > 0$

$$\Delta^N(t) = \begin{cases} O(e^{-cNt}) & \text{on } W(t) \\ 1 + O(e^{-ct}) & \text{on } W^\perp(t). \end{cases}$$

so that

$$1 - \Delta^N(t) = \begin{cases} 1 + O(e^{-cNt}) & \text{on } W(t) \\ O(e^{-ct}) & \text{on } W^\perp(t) \end{cases}$$

and

$$(1 - \Delta^N(t))^N = \begin{cases} 1 + O(e^{-cNt}) & \text{on } W(t) \\ O(e^{-cNt}) & \text{on } W^\perp(t) \end{cases} = \pi(t) + O(e^{-cNt}).$$

Therefore, for N large enough, it is sufficient to prove (4.8) and (4.9) with $g_\sigma = \pi\sigma$ replaced by $(1-\Delta^N)^N\sigma$, i.e., to prove that for all critical $\sigma^{(p)}$, $\sigma_1^{(p)}$ and $\tau^{(p+1)}$

$$\left\langle (1-\Delta^N(t))^N\sigma,\ (1-\Delta^N(t))^N\sigma_1 \right\rangle = \begin{cases} O(e^{-tD(\sigma,\sigma_1)}) & \text{if } \sigma \neq \sigma_1 \\ 1+O(e^{-ct}) & \text{if } \sigma = \sigma_1 \end{cases} \tag{4.10}$$

$$\left\langle \partial_t(1-\Delta^N)^N\tau,\ (1-\Delta^N)^N\sigma \right\rangle = e^{t(f(\sigma)-f(t))}\left(\left\langle \tilde{\partial}\tau,\sigma\right\rangle + O(e^{-ct})\right). \tag{4.11}$$

In particular, (4.10) and (4.11) imply (4.8) and (4.9) as long as N is chosen large enough so that for all σ, σ_1 and τ

$$cN > D(\sigma,\sigma_1)$$
$$cN > f(\tau) - f(\sigma).$$

Proof of (4.10). For p-cells σ_0 and σ_1 let $\mathcal{P}_r(\sigma_0,\sigma_1)$ denote the set of p-paths of length r from σ_0 to σ_1. Then

$$\langle \Delta\sigma_0, \sigma_1\rangle = \sum_{\gamma\in\mathcal{P}_1(\sigma_0,\sigma_1)} -m(\gamma)e^{-ts(\gamma)}$$

and

$$\langle \Delta^N\sigma_0,\sigma_1\rangle = \sum_{\gamma\in\mathcal{P}_N(\sigma_0,\sigma_1)} (-1)^N m(\gamma)e^{-ts(\gamma)} = O\left(e^{-tD(\sigma_0,\sigma_1)}\right).$$

Thus

$$\langle (1-\Delta^N)\sigma_0,\sigma_1\rangle = O\left(e^{-tD(\sigma_0,\sigma_1)}\right)$$

and for any $K \geq 1$

$$\begin{aligned} &\langle (1-\Delta^N)^K\sigma_0,\sigma_1\rangle \\ &= \sum_{v_1^{(p)},v_2^{(p)},\ldots,v_{K-1}^{(p)}} \langle (1-\Delta^N)\sigma_0, v_1\rangle \langle (1-\Delta^N)v_1,v_2\rangle \cdots \langle (1-\Delta^N)v_{K-1},\sigma_1\rangle \\ &= O\left(e^{-tD(\sigma_0,v_1)}\right) O\left(e^{-tD(v_1,v_2)}\right)\cdots O\left(e^{-tD(v_{K-1},\sigma_1)}\right) \\ &= O\left(e^{-tD(\sigma_0,\sigma_1)}\right) \end{aligned}$$

by the triangle inequality. It follows immediately that for any σ_0, σ_1

$$\langle (1-\Delta^N)^N\sigma_0,\ (1-\Delta^N)^N\sigma_1\rangle = \langle (1-\Delta^N)^{2N}\sigma_0,\sigma_1\rangle = O(e^{-tD(\sigma_0,\sigma_1)}).$$

If σ is critical, then for any $K \geq 1$

$$\Delta^K\sigma = O(e^{-tc}).$$

Hence

$$\langle (1-\Delta^N)^N\sigma,\ (1-\Delta^N)^N\sigma\rangle = \langle\sigma,\sigma\rangle + O(e^{-tc}) = 1 + O(e^{-tc}).$$

This completes the proof of (4.10).

Proof of (4.11). We observe that ∂_t commutes with $\Delta(t) = \partial_t\partial_t^* + \partial_t^*\partial_t$. In fact

$$\partial_t\Delta(t) = \Delta(t)\partial_t = \partial_t(\partial_t^*\partial_t) = (\partial_t\partial_t^*)\partial_t.$$

Therefore, for critical $\sigma^{(p)}$ and $\tau^{(p+1)}$

$$\begin{aligned}(4.12)\qquad &\langle \partial_t(1-\Delta^N)^N\tau,\ (1-\Delta^N)^N\sigma\rangle = \langle \partial_t\tau,\ (1-(\partial_t\partial_t^*)^N)^{2N}\sigma\rangle \\ &= \sum_{\sigma_1<\tau} \langle \partial_t\tau, \sigma_1\rangle \langle \sigma_1,\ (1-(\partial_t\partial_t^*)^N)^{2N}\sigma\rangle \\ &= \sum_{\sigma_1<\tau} \langle \partial\tau, \sigma_1\rangle e^{t(f(\sigma_1)-f(\tau))} \langle \sigma_1,\ (1-(\partial_t\partial_t^*)^N)^{2N}\sigma\rangle .\end{aligned}$$

We now focus our attention on the expression

$$\langle \sigma_1,\ (1-(\partial_t\partial_t^*)^N)^{2N}\sigma\rangle$$

keeping track only of terms which are on the order of $e^{-t|f(\sigma_1)-f(\sigma)|}$. The first step is to show that we can replace the operator $(1-(\partial_t\partial_t^*)^N)^{2N}$ by the simpler $(1-(\partial_t\partial_t^*))^{2N}$. This is the content of Lemma 4.7.

Lemma 4.7. *If $\sigma^{(p)}$ is critical then for N large enough there is a $c>0$ such that for all $\sigma_1^{(p)}$*

$$\langle \sigma_1,\ (1-(\partial_t\partial_t^*)^N)^{2N}\sigma\rangle - \langle \sigma_1,\ (1-(\partial_t\partial_t^*))^{2N}\sigma\rangle = O(e^{-t(|f(\sigma_1)-f(\sigma)|+c)}).$$

We will postpone the proof of Lemma 4.7 until later, and continue with the proof of Theorem 4.1.

To study the operator $\partial_t\partial_t^*$ we introduce some new definitions (which will also be used in the proof of Lemma 4.7).

Define an *upper p-step* to be a p-step (see Definition 4.3)

$$\sigma_0^{(p)},\ \tau,\ \sigma_1^{(p)}$$

where $\dim\tau = p+1$. We define an *upper p-path* to be a sequence of upper p-steps, each step beginning where the previous step ends, and we let $\mathcal{P}_r^+(\sigma_0,\sigma_1)$ denote the set of upper p-steps of length r from σ_0 to σ_1. Note that

$$\mathcal{P}_r^+(\sigma_0,\sigma_1) \subseteq \mathcal{P}_r(\sigma_0,\sigma_1).$$

It can easily be seen from (2.3) and (2.4) that

$$\langle(\partial_t\partial_t^*)\sigma_0, \sigma_1\rangle = \sum_{\gamma\in\mathcal{P}_1^+(\sigma_0,\sigma_1)} -m(\gamma)e^{-ts(\gamma)}$$

so that

$$\langle \sigma_0,\ (1-(\partial_t\partial_t^*))\sigma_1\rangle = \delta_{\sigma_0,\sigma_1} + \sum_{\gamma\in\mathcal{P}_1^+(\sigma_0,\sigma_1)} m(\gamma)e^{-ts(\gamma)}.$$

We will now simplify the expression on the right hand side. Let

$$\Sigma_p = \{\sigma^{(p)} \mid \exists \tau^{(p+1)} > \sigma \quad \text{with} \quad f(\tau) = f(\sigma)\}.$$

For each $\sigma^{(p)} \in \Sigma_p$, there is a zero energy upper p-step from σ to itself, which we will refer to as z_τ. Namely,

$$z_\sigma = \sigma,\ \tau^{(p+1)},\ \sigma$$

where $\tau > \sigma$ satisfies $f(\sigma) = f(\tau)$, so that

$$m(z_\sigma) = -1, \quad s(z_\sigma) = 0.$$

For $\sigma \notin \Sigma_p$ we now define a similar object. That is, for $\sigma \notin \Sigma_p$, let 1_σ denote a trivial stationary upper p-step from σ to itself satisfying

$$m(1_\sigma) = +1, \quad s(1_\sigma) = 0.$$

That is, 1_σ is not an actual p-step in that there is no τ such that $1_\sigma = \sigma, \tau, \sigma$. However, it will often be useful to allow paths to "rest" at a $\sigma \notin \Sigma_p$, and we do this by adding the 1_σ's to the set of allowable steps. Let $\widetilde{\mathcal{P}}_1^+(\sigma_0, \sigma_1) = \mathcal{P}_1^+(\sigma_0, \sigma_1) - \{z_\sigma\}_{\sigma \in \Sigma_p} \cup \{1_\sigma\}_{\sigma \notin \Sigma_p}$. Then

$$\langle \sigma_0,\ (1 - (\partial_t \partial_t^*))\sigma_1 \rangle = \sum_{\gamma \in \widetilde{\mathcal{P}}_1^+(\sigma_0, \sigma_1)} m(\gamma) e^{-ts(\gamma)}.$$

Let $\widetilde{P}_{2N}^-(\sigma_0, \sigma_1)$ denote the upper p-steps of length $2N$ from σ_0 to σ_1 with the modification that the upper step z_σ is not permitted if $\sigma \in \Sigma$, but if $\sigma \notin \Sigma$ the trivial step 1_σ may be used. Then

$$\langle \sigma_0,\ (1 - (\partial_t \partial_t^*))^{2N} \sigma \rangle = \sum_{\gamma \in \widetilde{\mathcal{P}}_{2N}^+(\sigma_0, \sigma_1)} m(\gamma) e^{-ts(\gamma)}. \tag{4.13}$$

Let

$$D^+(\sigma_0, \sigma) = \min_{\gamma \in \widetilde{\mathcal{P}}_{2N}^+(\sigma_0, \sigma)} s(\gamma).$$

Then

$$D^+(\sigma_0, \sigma) \geq D(\sigma_0, \sigma).$$

It is clear that (4.13) is $O(e^{-tD^+(\sigma_0, \sigma)})$. Following the proof of Lemma 4.6 in this context we see that if $\sigma_0 \notin \Sigma_p$, $\sigma \notin \Sigma_p$, $\sigma_0 \neq \sigma$,

$$D^+(\sigma_0, \sigma) > |f(\sigma_0) - f(\sigma)|. \tag{4.14}$$

Therefore, since we are interested only in terms which are on the order of $e^{-t|f(\sigma_0) - f(\sigma)|}$ and σ is critical (and hence $\notin \Sigma$) we may ignore $\sigma_0 \neq \sigma$ satisfying $\sigma_0 \notin \Sigma_p$.

Suppose $\sigma_0 \in \Sigma_p$. Let $\gamma \in \widetilde{\mathcal{P}}_{2N}^+(\sigma_0, \sigma)$ be a path from σ_0 to σ satisfying

$$s(\gamma) = |f(\sigma_0) - f(\sigma)|.$$

If $\gamma = \gamma_1 \circ \gamma_0$ where γ_0 is a path from σ_0 to σ_1 and γ_1 is a path from σ_1 to σ, then we must have

$$s(\gamma_0) = |f(\sigma_0) - f(\sigma_1)|$$
$$s(\gamma_1) = |f(\sigma_1) - f(\sigma)|.$$

Thus, we must have $\sigma_1 \in \Sigma_p$ or $\sigma_1 = \sigma$. That is, every p-cell in γ except σ must belong to Σ_p, and hence γ must look like

(4.15)
$$\gamma : \left(\sigma_0^{(p)}, \tau_0^{(p+1)}, \sigma_1^{(p)}\right)\left(\sigma_1^{(p)}, \tau_1^{(p+1)}, \sigma_2^{(p)}\right)\cdots \left(\sigma_{r-1}^{(p)}, \tau_{r-1}^{(p+1)}, \sigma_r^{(p)} = \sigma^{(p)}\right)(1_\sigma)^{2N-r}$$

where for each $i = 0, 1, \ldots, r-1$

(4.16)
$$\sigma_i < \tau_i \quad \sigma_{i+1} < \tau$$

and

(4.17)
$$f(\sigma_i) = f(\tau_i) > f(\sigma_{i+1}).$$

Conversely, every path γ of the form (4.15) satisfying (4.16) and (4.17) also satisfies

$$s(\gamma) = |f(\sigma_0) - f(\sigma)| = f(\sigma_0) - f(\sigma).$$

Moreover (4.16) and (4.17) are precisely the properties which characterize the gradient paths of f from σ_0 to σ (see Definition 3.1). That is, $\gamma \in \widetilde{\mathcal{P}}_{2N}^{+}(\sigma_0, \sigma)$ satisfies

$$s(\gamma) = |f(\sigma_0) - f(\sigma)|$$

if and only if γ is of the form

(4.18)
$$\gamma = \gamma^* \circ (1_\sigma)^{2N-r}$$

for some r where $\gamma^* \in \Gamma(\sigma_0, \sigma)$ has length r. In addition,

$$m(\gamma) = m(\gamma^*), \; s(\gamma) = s(\gamma^*) = f(\sigma_0) - f(\sigma).$$

Thus, if $\sigma_0 \in \Sigma_p$

(4.19)
$$\begin{aligned}\langle \sigma_0, (1 - (\partial_t \partial_t^*))^{2N} \sigma \rangle &= \sum_{\gamma \in \Gamma(\sigma_0, \sigma)} m(\gamma) e^{-t(f(\sigma_0) - f(\sigma))} + O(e^{-t(f(\sigma_0) - f(\sigma) + c)}) \\ &= e^{-t(f(\sigma_0) - f(\sigma))} \left(\sum_{\gamma \in \Gamma(\sigma_0, \sigma)} m(\gamma) + O(e^{-tc}) \right).\end{aligned}$$

We now observe that the formula (4.18) holds even if $\sigma_0 \notin \Sigma_p$. Namely, if $\sigma_0 \notin \Sigma_p$, $\sigma_0 \neq \sigma$ then there are no gradient paths from σ_0 to σ so both sides are $O(e^{-t(|f(\sigma_0)-f(\sigma)|+c)})$. If $\sigma_0 = \sigma$ then there is only the trivial gradient path from σ_0 to σ (of length 0) so both sides are $1 + O(e^{-tc})$.

Combining (4.12) and (4.19) with Lemma 4.7 yields

$$\begin{aligned}\langle \partial_t (1-\Delta^N)^N \tau,\ (1-\Delta^N)^N \sigma\rangle &= \sum_{\sigma_1 < \tau} \langle \partial\tau, \sigma_1\rangle\, e^{t(f(\sigma_1)-f(\tau))} \\ &\quad \left[e^{-t(f(\sigma_1)-f(\sigma))} \left(\sum_{\gamma\in\Gamma(\sigma_1,\sigma)} m(\gamma) + O(e^{-tc}) \right) \right] \\ &= \sum_{\sigma_1<\tau} \langle \partial\tau, \sigma_1\rangle\, e^{t(f(\sigma)-f(\tau))} \left(\sum_{\gamma\in\Gamma(\sigma_1,\sigma)} m(\gamma) + O(e^{-tc}) \right)\end{aligned}$$

which is precisely (4.11).

This completes the proof of Theorem 4.1. □

Proof of Lemma 4.7. (The reader will note that we will make use of definitions, notation and observations introduced in the proof of Theorem 4.1 after the statement of Lemma 4.7.)

Since

$$(1-(\partial_t\partial_t^*)^N)^{2N} = (1-(\partial_t\partial_t^*))^{2N}(1+(\partial_t\partial_t^*)+(\partial_t\partial_t^*)^2+\cdots+(\partial_t\partial_t^*)^{N-1})^{2N}$$

to prove Lemma 4.7 it is sufficient to prove that for any critical σ and any $K \geq 1$

$$\langle \sigma_1,\ (1-\partial_t\partial_t^*)^N (\partial_t\partial_t^*)^K \sigma\rangle = O(e^{-t(|f(\sigma_1)-f(\sigma)|+c)}).$$

We write

$$\begin{aligned}\langle \sigma_1,\ (1-\partial_t\partial_t^*)^N (\partial_t\partial_t^*)^K \sigma\rangle &= \langle (1-\partial_t\partial_t^*)^N \sigma_1,\ (\partial_t\partial_t^*)^K\sigma\rangle \\ &= \sum_{\sigma_2} \langle (1-\partial_t\partial_t^*)^N\sigma_1,\ \sigma_2\rangle \langle \sigma_2,\ (\partial_t\partial_t^*)^K\sigma\rangle\end{aligned}$$

and observe that

$$\begin{aligned}&\langle (1-\partial_t\partial_t^*)^N\sigma_1, \sigma_2\rangle = O(e^{-tD^+(\sigma_1,\sigma_2)}) = O(e^{-t|f(\sigma_1)-f(\sigma_2)|}) \\ (4.20)\qquad &\langle \sigma_2,\ (\partial_t\partial_t^*)^K\sigma\rangle = O(e^{-tD^+(\sigma_2,\sigma)}) = O(e^{-t|f(\sigma_2)-f(\sigma)|}).\end{aligned}$$

It is sufficient to prove that for any σ_2 either

$$\langle (1-\partial_t\partial_t^*)^N\sigma_1, \sigma_2\rangle = O(e^{-t(|f(\sigma_1)-f(\sigma_2)|+c)})$$

or

$$\langle \sigma_2,\ (\partial_t\partial_t^*)^K\sigma\rangle = O(e^{-t(|f(\sigma_2)-f(\sigma)|+c}).$$

First, if $\sigma_2 = \sigma$ then, since σ is critical, for any $K \geq 1$

$$\langle \sigma_2, \ (\partial_t \partial_t^*)^K \sigma \rangle = O(e^{-tc}) = O(e^{-t(|f(\sigma_2)-f(\sigma)|+c)}).$$

Second, if $\sigma_2 \neq \sigma$ and $\sigma_2 \notin \Sigma_p$, then, as observed in (4.14)

$$D^+(\sigma_2, \sigma) > |f(\sigma_2) - f(\sigma)|$$

so the result follows from (4.20).

Last, suppose $\sigma_2 \in \Sigma_p$ and consider

$$\langle (1 - (\partial_t \partial_t^*)^N)\sigma_1, \sigma_2 \rangle = \sum_{\gamma \in \mathcal{P}_N^+(\sigma_1, \sigma_2)} m(\gamma) e^{-ts(\gamma)}.$$

Let $\gamma \in \widetilde{\mathcal{P}}_N^+(\sigma_1, \sigma_2)$ be a path with $s(\gamma) = |f(\sigma_1) - f(\sigma_2)|$. Clearly γ contains no closed loops, so if N is large enough γ must contain a trivial step (i.e., a step from some σ_3 to itself satisfying $s(\beta) = 0$). Since $\widetilde{\mathcal{P}}_N^+(\sigma_1, \sigma_2)$ does not include any zero energy steps from a p-cell $\sigma_3 \in \Sigma_p$ to itself, γ must contain a p-cell $\sigma_3 \notin \Sigma_p$. Thus we can write $\gamma = \gamma_1 \circ \gamma_2$ with γ_1 a path from σ_1 to σ_3 satisfying

$$s(\gamma_1) = |f(\sigma_1) - f(\sigma_3)|$$

and γ_2 a path from σ_3 to σ_2 satisfying

$$s(\gamma_2) = |f(\sigma_3) - f(\sigma_2)|.$$

As we saw in (4.18), γ_1 must be of the form

$$\gamma_1 = \gamma_1^* \circ (1_{\sigma_3})^r$$

for some r, where γ_1^* is a gradient path from σ_1 to σ_3. This implies, in particular, that $f(\sigma_3) < f(\sigma_1)$ (or else there are no gradient path from σ_1 to σ_3). Similarly $f(\sigma_3) < f(\sigma_2)$. Therefore

$$s(\gamma) = s(\gamma_1) + s(\gamma_2) = |f(\sigma_3) - f(\sigma_1)| + |f(\sigma_3) - f(\sigma_2)| > |f(\sigma_1) - f(\sigma_2)|$$

which is a contradiction. $\square$

§5. $\dim \mathbf{H_p(M, R)} = \dim \mathbf{H_p}(\mathcal{M})$.

It follows from the work in section 4 that

$$\dim H_p(M, \mathbf{R}) \leq \dim H_p(\mathcal{M})$$

so in order to prove the equality stated in the title of this section we prove

$$\dim H_p(M, \mathbf{R}) \geq \dim H_p(\mathcal{M}).$$

The method will be to find a $1-1$ linear map from $H_p(\mathcal{M})$ to $H_p(M, \mathbf{R})$. In fact, we will construct the map on the chain level. This map will be constructed via a discrete path integral. In (5.1)–(5.5) we present the 5 fundamental properties that our path integral needs to satisfy. In fact, any operator satisfying these properties is sufficient to complete the proof. It seems possible that such an axiomatic approach to the path integral may also help clarify the situation in the smooth case.

For each p, let

$$\rho : C_p(M, \mathbf{R}) \longrightarrow \mathcal{M}_p \subseteq C_p(M, \mathbf{R})$$

denote the canonical projection. That is, if

$$c = \sum_{\sigma \in K_p} c_\sigma \sigma, \quad c_\sigma \in \mathbf{R}$$

is a p-chain, then

$$\rho(c) = \sum_{\sigma^{(p)} \text{critical}} c_\sigma \sigma.$$

We will define an operator

$$L : C_p(M, \mathbf{R}) \longrightarrow C_p(M, \mathbf{R})$$

with the following properties

$$L^2 = L \tag{5.1}$$

$$L \circ \rho \circ L = L \tag{5.2}$$

$$\partial \circ L = L \circ \partial \tag{5.3}$$

and, restricted to $\mathcal{M}$

$$\rho \circ L = 1 \tag{5.4}$$

$$\tilde{\partial} = \rho \circ L \circ \partial. \tag{5.5}$$

Before defining such an operator, we will see that the existence of L implies the desired result.

Theorem 5.1. *Suppose there is an operator L satisfying* (5.1)–(5.5), *then for each p*

$$\dim H_p(\mathcal{M}) \leq \dim H_p(M, \mathbf{R}).$$

Proof. Consider the diagram of differential complexes

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{M}_n \arrow[r, "\partial"] \arrow[d, shift right, "L"'] & \mathcal{M}_{n-1} \arrow[r, "\partial"] \arrow[d, shift right, "L"'] & \cdots \arrow[r, "\partial"] & \mathcal{M}_0 \arrow[r] \arrow[d, shift right, "L"'] & 0 \\
0 \arrow[r] & C_n(M, \mathbf{R}) \arrow[r, "\partial"] \arrow[u, shift right, "\rho \circ L"'] & C_{n-1}(M, \mathbf{R}) \arrow[r, "\partial"] \arrow[u, shift right, "\rho \circ L"'] & \cdots \arrow[r, "\partial"] & C_0(M, \mathbf{R}) \arrow[r] \arrow[u, shift right, "\rho \circ L"'] & 0.
\end{tikzcd}$$

It follows from the properties of L that

$$L \circ \tilde{\partial} = L \circ \rho \circ L \circ \partial = L \circ \partial = \partial \circ L.$$

Thus, L maps Kernel $(\tilde{\partial})$ to Kernel (∂) and Image $(\tilde{\partial})$ to Image (∂) and hence induces a map

$$L_* : H_p(\mathcal{M}) \longrightarrow H_p(M, \mathbf{R}).$$

Namely, if $\alpha \in \mathcal{M}_p$ satisfies $\tilde{\partial}\alpha = 0$ then α represents an element $[\alpha] \in H_p(\mathcal{M})$ and

$$L_*[\alpha] = [L\alpha] \in H_p(M, \mathbf{R}).$$

Similarly

$$\begin{aligned} \tilde{\partial} \circ (\rho \circ L) &= \rho \circ L \circ \partial \circ \rho \circ L = \rho \circ \partial \circ L \circ \rho \circ L \\ &= \rho \circ \partial \circ L = (\rho \circ L) \circ \partial \end{aligned}$$

so that $(\rho \circ L)$ induces a map

$$(\rho \circ L)_* : H_p(M, \mathbf{R}) \longrightarrow H_p(\mathcal{M})$$

where, if $\alpha \in C_p(M, \mathbf{R})$ satisfies $\partial\alpha = 0$ then α represents $[\alpha] \in H_p(M, \mathbf{R})$ and

$$(\rho \circ L)_*[\alpha] = [(\rho \circ L)(\alpha)] \in H_p(\mathcal{M}).$$

Consider the map

$$(\rho \circ L)_* \circ L_* : H_p(\mathcal{M}) \longrightarrow H_p(\mathcal{M}).$$

If $\alpha \in \mathcal{M}_p$ satisfies $\tilde{\partial}\alpha = 0$ then

$$((\rho \circ L)_* \circ L_*)[\alpha] = [((\rho \circ L) \circ L)(\alpha)].$$

From (5.1) and (5.4) it follows that for $\alpha \in \mathcal{M}$

$$(\rho \circ L \circ L)(\alpha) = (\rho \circ L)(\alpha) = \alpha.$$

Therefore, $(\rho \circ L)_* \circ L_*$ is the identity on $H_p(\mathcal{M})$. In particular, L_* must be 1–1. This implies the conclusion of the theorem. □

We note that it follows from the work of section 4 that

$$\dim H_p(\mathcal{M}) \geq \dim H_p(M, \mathbf{R})$$

and hence L_* must be an isomorphism.

The operator L will be defined as a sum over paths.

Definition 5.2. For $\sigma_0^{(p)}, \sigma_1^{(p)} \in K_p$ let $P(\sigma_0, \sigma_1)$ denote the set of p-paths

$$\gamma : \left(\sigma_0^{(p)} = v_0^{(p)}, \tau_0, v_1^{(p)}\right)\left(v_1^{(p)}, \tau_1, v_2^{(p)}\right)\cdots\left(v_{r-1}^{(p)}, \tau_{r-1}, v_r^{(p)} = \sigma_1^{(p)}\right) \tag{5.6}$$

such that at least one of the p-cells $\sigma_0 = v_0, v_1, \ldots, v_r = \sigma_1$ is critical, and such that for each $i = 0, 1, 2, \ldots, r-1$

1) $v_i \neq v_{i+1}$
2) Either
   $\dim \tau_i = p + 1$ and $v_i < \tau_i,\ v_{i+1} < \tau_i$
   or
   $\dim \tau_i = p - 1$ and $v_i > \tau_i,\ v_{i+1} > \tau_i$
3) $f(v_i) \geq f(\tau_i) \geq f(v_{i+1})$.

We now define, for $\sigma \in K_p$

$$L(\sigma) = \sum_{\sigma_1 \in K_p} \sum_{\gamma \in P(\sigma, \sigma_1)} m(\gamma)\sigma_1 \tag{5.7}$$

and extend L linearly to all of $C_p(M, \mathbf{R})$. At the end of section 4 we proved that for all N large enough

$$\lim_{t \to \infty} e^{-tf}(1 - (\partial_t \partial_t^*))^N e^{tf} = \widetilde{L}$$

where, for $\sigma \in K_p$

$$\widetilde{L}(c) = \sum_{\sigma_1 \in K_p} \sum_{\gamma \in \Gamma(\sigma, \sigma_1)} m(\gamma)\sigma_1.$$

The same argument, applied with the operator $(\partial_t \partial_t^*)$ replaced by $\Delta(t)$ yields that for all N large enough

$$\lim_{t \to \infty} e^{-tf}(1 - \Delta(t))^N e^{tf} = L. \tag{5.8}$$

The rest of this section is devoted to proving that L satisfies (5.1)–(5.5). Although it is not difficult to prove this by working directly from the definition (5.7), the formula (5.8) leads to some simplification, particularly in the proofs of (5.1), (5.3) and (5.5).

Proof of (5.1). From (5.8)

$$\begin{aligned} L^2 &= \lim_{t \to \infty} e^{-tf}(1 - \Delta(t))^N e^{tf} \cdot \lim_{t \to \infty} e^{-tf}(1 - \Delta(t))^N e^{tf} \\ &= \lim_{t \to \infty} e^{-tf}(1 - \Delta(t))^{2N} e^{tf} = L. \end{aligned}$$

Proof of (5.3). Since

$$\partial = e^{-tf}\partial_t e^{tf}$$

and

$$\partial_t \Delta(t) = \Delta(t)\partial_t$$

we have

$$\begin{aligned}
\partial \circ L &= e^{-tf}\partial_t e^{tf} \lim_{t\to\infty} e^{-tf}(1-\Delta(t))^N e^{tf} \\
&= \lim_{t\to\infty} e^{-tf}\partial_t(1-\Delta(t))^N e^{tf} \\
&= \lim_{t\to\infty} e^{-tf}(1-\Delta(t))^N \partial_t e^{tf} \\
&= \left(\lim_{t\to\infty} e^{-tf}(1-\Delta(t))^N e^{tf}\right)\left(e^{-tf}\partial e^{tf}\right) \\
&= L \circ \partial.
\end{aligned}$$

Proof of (5.5). It follows from the work of section 4 that for any critical cells $\sigma^{(p)}$, $v^{(p-1)}$

$$\begin{aligned}
\left\langle \tilde{\partial}\sigma, v\right\rangle &= \lim_{t\to\infty} \langle \partial_t(1-\Delta(t))^N e^{tf}\sigma,\ (1-\Delta(t))^N e^{-tf} v\rangle \\
&= \left\langle \left(\lim_{t\to\infty} e^{-tf}\partial_t(1-\Delta(t))^{2N} e^{tf}\right)\sigma, v\right\rangle
\end{aligned}$$

so that

$$\begin{aligned}
\tilde{\partial} &= \rho \circ \lim_{t\to\infty} e^{-tf}\partial_t(1-\Delta(t))^{2N} e^{tf} \\
&= \rho \circ \lim_{t\to\infty} \partial e^{-tf}(1-\Delta(t))^{2N} e^{tf} \\
&= \rho \circ \partial \circ L.
\end{aligned} \tag{5.9}$$

Since ∂_t commutes with $\Delta(t)$, we can reverse the order of the terms ∂_t and $(1-\Delta(t))^{2N}$ in (5.9) to learn

$$\tilde{\partial} = \rho \circ L \circ \partial.$$

The proofs of (5.2) and (5.4) will require the following lemma.

Lemma 5.3. *For any $\sigma_0^{(p)}$, $\sigma_1^{(p)} \in K_p$ and any $\gamma \in \mathcal{P}(\sigma_0, \sigma_1)$ of the form (5.6) exactly one of the p-cells $\sigma_0 = v_0, v_1, \ldots, v_r = \sigma_1$ is critical. In particular there is a unique critical σ_2 such that $\gamma = \gamma_0 \circ \gamma_1$ with*

$$\gamma_0 \in P(\sigma_0, \sigma_2), \quad \gamma_1 \in P(\sigma_2, \sigma_1).$$

Proof. By definition, at least one of the p-cells is critical. Suppose v_i is critical. Since $f(v_i) \geq f(\tau_i)$ we must have

$$\dim \tau_i = p-1, \quad f(v_i) > f(\tau_i).$$

Then, since $f(\tau_i^{(p-1)}) \geq f(v_{i+1}^{(p)})$ we must have (by the flatness of f)

$$f(\tau_i) = f(v_{i+1}).$$

Since $\tau_i \neq \tau_{i+1}$ (or else $v_{i+1} = v_{i+2}$) and $f(v_{i+1}) \geq f(\tau_{i+1})$ we must have

$$\dim \tau_{i+1} = p-1, \quad f(v_{i+1}) > f(\tau_{i+1})$$

and thus

$$f(\tau_{i+1}) = f(v_{i+2}).$$

Continuing in this fashion, for each $j \geq 0$

$$f(\tau_{i+j}) = f(v_{i+j+1})$$

so that v_{i+j+1} is not critical.

Working in the reverse order, since v_i is critical and $f(\tau_{i-1}) \geq f(v_i)$ we must have

$$\dim \tau_{i-1} = p+1, \quad f(\tau_{i-1}) > f(v_i).$$

Then, since $f(v_{i-1}^{(p)}) \geq f(\tau_i^{(p+1)})$ we learn (by the flatness of f)

$$f(v_{i-1}) = f(\tau_i).$$

Continuing in this direction, for every $j > 0$

$$f(v_{i-j}^{(p)}) = f(\tau_{i-j}^{(p)})$$

so that v_{i-j} is not critical. □

Proof of (5.2). For any $\sigma_0^{(p)}, \sigma_1^{(p)} \in K_p$

$$\langle (L \circ \rho \circ L)\sigma_0, \sigma_1 \rangle = \sum_{\sigma_2^{(p)} \text{critical}} \langle L\sigma_0, \sigma_2 \rangle \langle L\sigma_2, \sigma_1 \rangle$$

$$= \sum_{\sigma_2^{(p)} \text{critical}} \sum_{\gamma_0 \in P(\sigma_0, \sigma_2)} \sum_{\gamma_1 \in P(\sigma_2, \sigma_1)} m(\gamma_0) m(\gamma_1)$$

$$= \sum_{\sigma_2^{(p)} \text{critical}} \sum_{\gamma_0 \in P(\sigma_0, \sigma_2)} \sum_{\gamma_1 \in P(\sigma_2, \sigma_1)} m(\gamma_0 \circ \gamma_1). \tag{5.10}$$

For any critical $\sigma_2^{(p)}$, $\gamma_0 \in P(\sigma_0, \sigma_2)$ and $\gamma_1 \in P(\sigma_2, \sigma_1)$ we have

$$\gamma_0 \circ \gamma_1 \in P(\sigma_0, \sigma_1).$$

Conversely, by Lemma 5.3 for any $\gamma \in P(\sigma_0, \sigma_1)$ there is a unique critical $\sigma_2^{(p)}$ such that γ can be written as

$$\gamma = \gamma_0 \circ \gamma_1$$

with

$$\gamma_0 \in P(\sigma_0, \sigma_2), \quad \gamma_1 \in P(\sigma_2, \sigma_1).$$

Hence (5.10) can be simplified to

$$\sum_{\gamma \in P(\sigma_0, \sigma_1)} m(\gamma) = \langle L\sigma_0, \sigma_1 \rangle.$$

Proof of 5.4. Suppose $\sigma_0^{(p)}$ and $\sigma_1^{(p)}$ are critical. Then

$$\langle L\sigma_0, \sigma_1\rangle = \sum_{\gamma \in P(\sigma_0, \sigma_1)} m(\gamma).$$

By Lemma 5.3, any $\gamma \in P(\sigma_0, \sigma_1)$ contains exactly 1 critical p-cell. Thus, since γ contains both σ_0 and σ_1, if $\sigma_0 \neq \sigma_1$, we must have

$$P(\sigma_0, \sigma_1) = \emptyset$$

so that

$$\langle L\sigma_0, \sigma_1\rangle = 0 \qquad \text{if } \sigma_0 \neq \sigma_1. \tag{5.10}$$

If $\sigma_0 = \sigma_1$ then

$$P(\sigma_0, \sigma_0) = \sigma_0$$

where the σ_0 on the right hand side indicates the trivial path of length 0 with $m(\sigma_0) = 1$. Thus,

$$\langle L\sigma_0, \sigma_0\rangle = 1. \tag{5.11}$$

Combining (5.9) and (5.10), for critical $\sigma_0^{(p)}$

$$(\rho \circ L)(\sigma_0) = \sigma_0$$

so that, restricted to $\mathcal{M}_p$

$$\rho \circ L = 1.$$

§6. The Morse Complex and Reidemeister Torsion.

In [Mi], Milnor showed that the Reidemeister Torsion of a smooth manifold (the definition will be reviewed shortly) is equal to the torsion of the Morse complex associated to a generic smooth Morse function. In this section we show that the analogous statement in the context of combinatorial Morse functions follows immediately from the work of section 4 of this paper.

We begin with a review of the notion of torsion. Suppose

$$V : 0 \longrightarrow V_n \xrightarrow{\delta} V_{n-1} \xrightarrow{\delta} V_{n-2} \xrightarrow{\delta} \cdots \xrightarrow{\delta} V_0 \longrightarrow 0 \tag{6.1}$$

is an *exact* differential complex and each V_p is endowed with an inner product (this is more data than is necessary but this definition will be sufficient for our purposes). We can then define the adjoint δ^* of δ with respect to these inner products, and the induced Laplacians

$$\Delta_p = \delta\delta^* + \delta^*\delta : V_p \longrightarrow V_p.$$

We define the torsion of the complex V, which we denote by $T(V)$, by

$$T(V) = \prod_{p=0}^{n} (\det \Delta_p)^{(-1)^{p+1} p/2}. \tag{6.2}$$

A couple of remarks are in order. We note first that since the complex V is exact, each Laplacian Δ_p is an isomorphism with all positive eigenvalues so that

$$\det \Delta_p > 0.$$

Second, the exponents in (6.2) may seem strange. A more suggestive formula (although less convenient for computations) is

$$T(V) = \prod_{p=0}^{n} (\det{}' \partial^* \partial : V_p \longrightarrow V_p)^{(-1)^p 1/2}$$

where $\det'$ denotes the product of the non-zero eigenvalues. The formula (6.2) first appeared in [R-S].

In order to apply this notion to a CW complex M, we need to associate to M an exact sequence. To do this, we "twist" the chain complex of M by a representation of $\pi_1(M)$. Let

$$\Phi : \pi_1(M) \longrightarrow O(k, \mathbf{R})$$

denote a homomorphism from $\pi_1(M)$, the fundamental group of M, to the group of $k \times k$ orthogonal real matrices. Let $\widetilde{M}$ denote the universal cover of M. Then $\pi_1(M)$ acts freely on $\widetilde{M}$. Moreover, $\widetilde{M}$ has a natural cell structure induced from that of M, and preserved by the $\pi_1(M)$ action, so that for all p

$$C_p(\widetilde{M}, \mathbf{R})/\pi_1(M) \cong C_p(M, \mathbf{R}).$$

Consider

$$(C_p(\widetilde{M}, \mathbf{R}))^k = \{(c_1, c_2, \ldots, c_k) \mid c_i \in C_p(\widetilde{M}, \mathbf{R})\}.$$

The key observation is that, given the representation Φ, $\pi_1(M)$ acts naturally on $(C_p(\widetilde{M}, \mathbf{R}))^k$. Namely, suppose $g \in \pi_1(M)$ and $c = (c_1, \ldots, c_k) \in (C_p(\widetilde{M}, \mathbf{R}))^k$. For any p-cell σ of $\widetilde{M}$ we set

$$\langle c, \sigma \rangle = (\langle c_1, \sigma \rangle, \ \langle c_2, \sigma \rangle, \ldots, \langle c_k, \sigma \rangle) \in \mathbf{R}^k.$$

We then define $g_*(c)$ to be the unique element of $(C_p(\widetilde{M}, \mathbf{R}))^k$ such that for all cells σ of $\widetilde{M}$

$$\langle g_*(c), \sigma \rangle = \Phi(g) \langle c, g^{-1}(\sigma) \rangle.$$

Let $C_p(M, \Phi)$ denote the elements of $(C_p(\widetilde{M}, \mathbf{R}))^k$ fixed by this action. Then $C_p(M, \Phi)$ is preserved by ∂ (since ∂ commutes with the action of $\pi_1(M)$) so we have a differential complex

$$C(M, \Phi) : 0 \longrightarrow C_n(M, \Phi) \xrightarrow{\partial} C_{n-1}(M, \Phi) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(M, \Phi) \longrightarrow 0.$$

We now assume that the complex $C(M, \Phi)$ is exact. That is, for each p

$$H_p(C(M, \Phi)) = 0.$$

Note that a necessary condition for this to hold is that $\chi(M) = 0$.

We now observe that there is a standard L^2 inner product on $C_p(M, \Phi)$. Namely, choose a lift $\tilde{\sigma}$ of every p-cell $\sigma \in K_p(M)$. Then, for every α, $\beta \in C_p(M, \Phi)$, set

$$\langle \alpha, \beta \rangle = \sum_{\sigma \in K_p(M)} \langle \alpha(\tilde{\sigma}), \beta(\tilde{\sigma}) \rangle .$$

The inner product is independent of the chosen lifts since Φ is an orthogonal representation.

With these definitions in hand, we are now ready to define the Reidemeister Torsion of M with respect to the representation Φ, $\operatorname{Tor}(M, \Phi)$, by

$$\operatorname{Tor}(M, \Phi) = T(C(M, \Phi)).$$

It is a theorem of Franz [Fr] that $\operatorname{Tor}(M, \Phi)$ is invariant under subdivision of the cell structure.

The next step is to observe that to Φ and any discrete Morse function f we can associate a Morse complex

$$\mathcal{M}_f(\Phi) : 0 \longrightarrow \mathcal{M}_n(\Phi) \xrightarrow{\tilde{\partial}} \mathcal{M}_{n-1}(\Phi) \xrightarrow{\tilde{\partial}} \cdots \xrightarrow{\tilde{\partial}} \mathcal{M}_0(\Phi) \longrightarrow 0.$$

Namely, there is a canonical lift of f to a Morse function $\widetilde{f}$ on $\widetilde{M}$. The critical cells of $\widetilde{f}$ are precisely the lifts of the critical cells of M. Define

$$\mathcal{M}_p(\Phi) \subseteq C_p(M, \Phi)$$

to be the elements of $C_p(M, \Phi)$ which are supported on the critical cells of $\widetilde{f}$. For any $\alpha \in \mathcal{M}_p(\Phi)$ and any critical $(p-1)$-cell σ on $\widetilde{M}$, we define $\widetilde{\partial}$ by setting

$$\left\langle \widetilde{\partial}\alpha, \sigma \right\rangle = \sum_{\substack{\tau^{(p)} \in K_p(\widetilde{M}) \\ \tau \text{ critical}}} \sum_{\upsilon^{(p-1)} < \tau} \sum_{g \in \pi_1(M)} \sum_{\gamma \in \Gamma(\upsilon, g(\sigma))} \langle \partial\tau, \upsilon \rangle \, m(\gamma) \Phi(g^{-1}) \, \langle \alpha, \tau \rangle$$

where $\Gamma(\upsilon, g(\sigma))$ is the set of gradient paths of $\widetilde{f}$ from υ to $g(\sigma)$.

To see that $\widetilde{\partial}$ does map $\mathcal{M}_p(\Phi)$ to $\mathcal{M}_{p-1}(\Phi)$ we must show that for any $h \in \pi_1(M)$, $\alpha \in \mathcal{M}_p(\Phi)$ and $\sigma^{(p-1)}$ critical

$$\left\langle h_*(\widetilde{\partial}\alpha), \sigma \right\rangle = \left\langle \widetilde{\partial}\alpha, \sigma \right\rangle .$$

Using the fact that $\alpha \in C_p(M, \Phi)$

$$\begin{aligned}
\left\langle h_*(\tilde{\partial}\alpha), \sigma \right\rangle &\equiv \Phi(h) \left\langle \tilde{\partial}\alpha, h^{-1}(\sigma) \right\rangle \\
&= \sum_{\substack{\tau^{(p)} \in K_p(M) \\ \tau \text{ critical}}} \sum_{v^{(p-1)} < \tau} \sum_{g \in \pi_1(M)} \sum_{\gamma \in \Gamma(v, g(h^{-1}(\sigma)))} \langle \partial\tau, v \rangle \, m(\gamma) \Phi(h) \Phi(g^{-1}) \langle \alpha, \tau \rangle \\
&= \sum_{\tau^{(p)} \text{ critical}} \sum_{v^{(p-1)} < \tau} \sum_{(gh^{-1}) \in \pi_1(M)} \sum_{\gamma \in \Gamma(v, (gh^{-1})(\sigma))} \\
&\qquad \langle \partial\tau, v \rangle \, m(\gamma) \Phi((gh^{-1})^{-1}) \langle \alpha, \tau \rangle \\
&= \sum_{\tau^{(p)} \text{ critical}} \sum_{v^{(p-1)} < \tau} \sum_{g \in \pi_1(M)} \sum_{\gamma \in \Gamma(v, g(\sigma))} \langle \partial\tau, v \rangle \, m(\gamma) \Phi(g^{-1}) \langle \alpha, \tau \rangle \\
&= \left\langle \tilde{\partial}\alpha, \sigma \right\rangle .
\end{aligned}$$

We can now state the main theorem of this section.

Theorem 6.1. *For any discrete Morse function f on M, and any orthogonal representation Φ of $\pi_1(M)$ such that $H_*(M, \Phi) = 0$, the Reidemeister Torsion of M with respect to Φ is equal to the torsion of the Morse complex induced by f and Φ. That is*

$$\operatorname{Tor}(M, \Phi) = T(\mathcal{M}(\Phi)).$$

The only new ingredient is the following key lemma.

Lemma 6.2. *Let V denote a general exact sequence, as in (6.1), with each V_p endowed with an inner product. Suppose that for each p we have a one-parameter family of automorphisms*

$$\mathcal{U}_p(t) : V_p \longrightarrow V_p.$$

We can then consider the one-parameter family of exact sequences

$$V_t : 0 \longrightarrow V_n \xrightarrow{\partial_t} V_{n-1} \xrightarrow{\partial_t} \cdots \xrightarrow{\partial_t} V_0 \longrightarrow 0$$

where

$$\partial_t = \mathcal{U}(t) \, \partial \, \mathcal{U}^{-1}(t).$$

For all t

$$\frac{d}{dt} \log T(V_t) = \frac{1}{2} \sum_{p=0}^{n} (-1)^p \operatorname{Tr}(\theta_p(t) + \theta_p^*(t))$$

where

$$\theta_p(t) = \left(\frac{d}{dt} \mathcal{U}(t) \right) \mathcal{U}^{-1}(t) : V_p \longrightarrow V_p.$$

Proof. (This sort of lemma and the accompanying proof is standard in this subject. See for example [R-S] and [B-F-K].) By direct calculation

$$\frac{d}{dt}\log T(V_t) = \frac{1}{2}\sum_{p=0}^{n}(-1)^p \operatorname{Tr}\left(\left(\frac{d}{dt}\Delta_p\right)\Delta_p^{-1}\right)$$

and, letting $\cdot$ denote $\frac{d}{dt}$,

$$\begin{aligned}
\dot{\Delta}_p &= \dot{\partial}_t\partial_t^* + \partial_t\dot{\partial}_t^* + \dot{\partial}_t^*\partial_t + \partial_t^*\dot{\partial}_t \\
\dot{\partial}_t &= \dot{U}\partial U^{-1} - U\partial_t U^{-1}\dot{U}U^{-1} \\
&= \theta\partial_t - \partial_t\theta \\
\dot{\partial}_t^* &= \partial_t^*\theta^* - \theta^*\partial_t^*.
\end{aligned}$$

Using $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$ and

$$\partial_t\Delta_p^{-1}(t) = \Delta_{p-1}^{-1}(t)\partial_t,\ \ \partial_t^*\Delta_p^{-1}(t) = \Delta_{p+1}^{-1}(t)\partial_t^*$$

we find (supressing the t's)

$$\begin{aligned}
&\operatorname{Tr}(\dot{\Delta}_p\Delta_p^{-1}) \\
&= \operatorname{Tr}(\theta\partial\partial^*\Delta_p^{-1} - \partial\theta\partial^*\Delta_p^{-1} + \partial\partial^*\theta^*\Delta_p^{-1} - \partial\theta^*\partial^*\Delta_p^{-1} \\
&\qquad + \partial^*\theta^*\partial\Delta_p^{-1} - \theta\partial^*\partial\Delta_p^{-1} + \partial^*\theta\partial\Delta_p^{-1} - \partial^*\partial\theta\Delta_p^{-1}) \\
&= \operatorname{Tr}(\theta\partial\partial^*\Delta_p^{-1} - \theta\partial^*\partial\Delta_{p+1}^{-1} + \theta^*\partial\partial^*\Delta_p^{-1} - \theta^*\partial^*\partial\Delta_{p+1}^{-1} \\
&\qquad + \theta^*\partial\partial^*\Delta_{p-1}^{-1} - \theta^*\partial^*\partial\Delta_p^{-1} + \theta\partial\partial^*\Delta_{p-1}^{-1} - \theta\partial^*\partial\Delta_p^{-1}) \\
&= \operatorname{Tr}\left((\theta+\theta^*)\partial\partial^*\Delta_p^{-1} - (\theta+\theta^*)\partial^*\partial\Delta_p^{-1} - (\theta+\theta^*)\partial^*\partial\Delta_{p+1}^{-1} + (\theta+\theta^*)\partial\partial^*\Delta_{p-1}^{-1}\right).
\end{aligned}$$

Considering those terms involving $\partial\partial^*$

$$\begin{aligned}
&\sum_{p=0}^{n}(-1)^p p \operatorname{Tr}\left((\theta+\theta^*)\partial\partial^*\Delta_p^{-1} + (\theta+\theta^*)\partial\partial^*\Delta_{p-1}^{-1}\right) \\
&= \sum_{p=0}^{n}(-1)^p p \operatorname{Tr}\left((\theta+\theta^*)\partial\partial^*\Delta_p^{-1}\right) + \sum_{p=0}^{n}(-1)^{p+1}(p+1)\operatorname{Tr}\left((\theta+\theta^*)\partial\partial^*\Delta_p^{-1}\right) \\
&= \sum_{p=0}^{n}(-1)^{p+1}\operatorname{Tr}\left((\theta+\theta^*)\partial\partial^*\Delta_p^{-1}\right).
\end{aligned} \tag{6.3}$$

Similarly

$$\begin{aligned}
&\sum_{p=0}^{n}(-1)^p p \operatorname{Tr}\left(-(\theta+\theta^*)\partial^*\partial\Delta_p^{-1} - (\theta+\theta^*)\partial^*\partial\Delta_{p-1}^{-1}\right) \\
&= \sum_{p=0}^{n}(-1)^{p+1}\operatorname{Tr}\left((\theta+\theta^*)\partial^*\partial\Delta_p^{-1}\right).
\end{aligned} \tag{6.4}$$

Combining (6.3) and (6.4)

$$\sum_{p=0}^{n}(-1)^p p \operatorname{Tr}\left(\dot{\Delta}_p \Delta_p^{-1}\right) = \sum_{p=0}^{n}(-1)^{p+1} \operatorname{Tr}\left((\theta+\theta^*)(\partial\partial^* + \partial^*\partial)\Delta_p^{-1}\right)$$
$$= \sum_{p=0}^{n}(-1)^{p+1} \operatorname{Tr}(\theta+\theta^*). \quad \square$$

Proof of Theorem 6.1. Since equivalent Morse functions induce the same Morse complex, by Theorem 1.4 we may assume that f is a flat Witten-Morse function. Beginning with the complex $C(M,\Phi)$, define the automorphisms

$$\mathcal{U}_{1,p}(t) : C_p(M,\Phi) \longrightarrow C_p(M,\Phi)$$

by setting, for $\alpha \in C_p(M,\Phi)$ and any p-cell σ of $\widetilde{M}$

$$\langle \mathcal{U}_{1,p}(t)\alpha, \sigma\rangle = e^{tf(\sigma)} \langle \alpha, \sigma\rangle .$$

Setting

$$\partial_t = \mathcal{U}_1 \, \partial \, \mathcal{U}_1^{-1}$$

we get a family of complexes

$$C(M,\Phi,t) : 0 \longrightarrow C_n(M,\Phi) \xrightarrow{\partial_t} C_{n-1}(M,\Phi) \xrightarrow{\partial_t} \cdots \xrightarrow{\partial_t} C_0(M,\Phi) \longrightarrow 0.$$

Let

$$W_p(\Phi,t) \subseteq C_p(M,\Phi)$$

denote the span of the eigenfunctions of

$$\Delta_p(t) = \partial_t\partial_t^* + \partial_t^*\partial_t : C_p(M,\Phi) \longrightarrow C_p(M,\Phi)$$

corresponding to the eigenvalues which tend to 0 as $t \to \infty$. As in the untwisted case studied in section 4

$$\lim_{t\to\infty} W_p(\Phi,t) = \mathcal{M}_p(\Phi).$$

The complex $C(M,\Phi,t)$ splits into the direct sum of 2 complexes

$$W^\perp(\Phi,t) : 0 \longrightarrow W_n^\perp(\Phi,t) \xrightarrow{\partial_t} W_{n-1}^\perp(\Phi,t) \xrightarrow{\partial_t} \cdots \xrightarrow{\partial_t} W_0^\perp(\Phi,t) \longrightarrow 0$$
$$W(\Phi,t) : 0 \longrightarrow W_n(\Phi,t) \xrightarrow{\partial_t} W_{n-1}(\Phi,t) \xrightarrow{\partial_t} \cdots \xrightarrow{\partial_t} W_0(\Phi,t) \longrightarrow 0$$

where

$$W_p^\perp(\Phi,t) \subseteq C_p(M,\Phi,t)$$

is the orthogonal complement of $W_p(\Phi, t)$, or, equivalently, the span of the eigenfunctions of $\Delta_p(t)$ corresponding to the eigenvalues which do not tend to 0 as $t \to \infty$. It follows directly from the formula (6.2) that

$$T(C(M, \Phi, t)) = T(W(\Phi, t))T(W^{\perp}(\Phi, t)).$$

Let $e_1, \ldots, e_k$ denote the standard basis of $\mathbf{R}^k$. For each critical p-cell σ of f on M, choose a lift $\tilde{\sigma}$. Then there is a basis

$$\{\alpha_{\sigma,i}\}_{\sigma^{(p)} \in K_p(M) \text{ critical}, \ 1 \leq i \leq k}$$

of $\mathcal{M}_p(\Phi)$ characterized by the property that for all critical p-cells σ_0, σ_1, on M

$$\langle \alpha_{\sigma_0,i}, \sigma_1 \rangle = \delta_{\sigma_0,\sigma_1} e_i.$$

Let $g_{\sigma,i}(t)$ denote the orthogonal projection of $\alpha_{\sigma,i}$ onto $W_p(\Phi, t)$. For t large enough the $g_{\sigma,i}(t)$'s form a basis of $W_p(\Phi, t)$. Let $\{h_{\sigma,i}(t)\}$ denote the basis resulting from orthonormalizing the $g_{\sigma,i}$'s. For convenience we will assume that the $h_{\sigma,i}$'s are defined for all $t \geq 0$. It is easy to adjust the argument if this is not the case. Let

$$U_{2,p}(t) : W_p(\Phi, t) \longrightarrow W_p(\Phi, t)$$

denote the family of automorphisms defined by

$$U_{2,p}(t)(h_{\sigma,i}(t)) = e^{-tf(\sigma)} h_{\sigma,i}(t)$$

and let $W'(\Phi, t)$ denote the family of complexes

$$W'(\Phi, t) : 0 \longrightarrow W_n(\Phi, t) \xrightarrow{d_t} W_{n-1}(\Phi, t) \xrightarrow{d_t} \cdots \xrightarrow{d_t} W_0(\Phi, t) \longrightarrow 0$$

where

$$d_t = U_2(t) \partial_t U_2^{-1}(t).$$

Let $C'(M, \Phi, t)$ denote the sum of the complexes $W^{\perp}(\Phi, t)$ and $W'(\Phi, t)$, so that

$$T(C'(M, \Phi, t)) = T(W^{\perp}(\Phi, t))T(W'(\Phi, t)).$$

As in the untwisted case, all eigenvalues of $\Delta_p(t)$ which do not tend to 0 as $t \to \infty$ tend to 1. Thus

$$T(W^{\perp}(\Phi, t)) \xrightarrow{t \to \infty} 1.$$

In addition, as in the untwisted case,

$$W'(\Phi, t) \xrightarrow{t \to \infty} \mathcal{M}(\Phi).$$

Therefore

$$\lim_{t \to \infty} T(C'(M, \Phi, t)) = T(\mathcal{M}(\Phi)).$$

On the other hand

$$C'(M, \Phi, 0) = C(M, \Phi)$$

so

$$T(C'(M, \Phi, 0)) = \mathrm{Tor}(M, \Phi).$$

Thus the theorem is proved once we see that

$$\frac{d}{dt} \log T(C'(M, \Phi, t)) = 0. \tag{6.5}$$

By Lemma 6.2

$$\frac{d}{dt} \log T(C'(M, \Phi, t)) = \frac{1}{2} \sum_{p=0}^{n} (-1)^{p+1} \left[\mathrm{Tr}(\theta_{1,p}(t) + \theta_{1,p}^{*}(t)) + \mathrm{Tr}(\theta_{2,p}(t) + \theta_{2,p}^{*}(t))\right] \tag{6.6}$$

where

$$\theta_1(t) : C_p(M, \Phi) \longrightarrow C_p(M, \Phi)$$

is given by

$$\theta_1(t) = \left(\frac{d}{dt} U_1(t)\right) U_1^{-1}(t)$$

and

$$\theta_2(t) : W_p(\Phi, t) \longrightarrow W_p(\Phi, t)$$

is given by

$$\theta_2(t)) = \left(\frac{d}{dt} U_2(t)\right) U_2^{-1}(t).$$

Now

$$\mathrm{Tr}\, \theta_{1,p} = \mathrm{Tr}\, \theta_{1,p}^{*} = k \sum_{\sigma^{(p)} \in K_p(M)} f(\sigma).$$

However, each non-critical p-cell of M can be paired with a non-critical cell τ of dimension either $p+1$ or $p-1$, satisfying

$$f(\sigma) = f(\tau)$$

so that the contributions from these cells cancel in the alternating sum (6.6). Thus,

$$\frac{1}{2} \sum_{p=0}^{n} (-1)^{p+1} \mathrm{Tr}(\theta_{1,p}(t) + \theta_{1,p}^{*}(t)) = k \sum_{p=0}^{n} (-1)^{p+1} \sum_{\substack{\sigma^{(p)} \in K_p(M) \\ \sigma \text{ critical}}} f(\sigma). \tag{6.7}$$

On the other hand, it follows directly from the definition of U_2 that

$$\mathrm{Tr}\, \theta_{2,p}(t) = \mathrm{Tr}\, \theta_{2,p}^{*}(t) = -k \sum_{\substack{\sigma^{(p)} \in K_p(M) \\ \sigma \text{ critical}}} f(\sigma)$$

so that

$$\frac{1}{2} \sum_{p=0}^{n} (-1)^{p+1} \mathrm{Tr}(\theta_{2,p}(t) + \theta_{2,p}^{*}(t)) = -k \sum_{p=0}^{n} (-1)^{p+1} \sum_{\substack{\sigma^{(p)} \in K_p(M) \\ \sigma \text{ critical}}} f(\sigma). \tag{6.8}$$

Combining (6.7) and (6.8) yields (6.5) and hence proves the theorem. $\square$

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