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COMBINATORIAL VECTOR FIELDS

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§0 Introduction.

In this paper we introduce the notion of a combinatorial dynamical system on any CW complex. Earlier in [Fo2] and [Fo3], we presented the idea of a combinatorial vector field (see also [Fo1] for the one-dimensional case), and studied the corresponding Morse Theory. Equivalently, we studied the homological properties of gradient vector fields (these terms were defined precisely in [Fo2], see also §2 of this paper). In this paper we broaden our investigation and consider general combinatorial vector fields.

We first study the homological properties of such vector fields, generalizing the Morse Inequalities of [Fo2]. We then introduce various zeta functions which keep track of the closed orbits of the corresponding flow, and prove that these zeta functions, initially defined only on a half plane, can be analytically continued to meromorphic functions on the entire complex plane. Lastly, we review the notion of Reidemeister Torsion of a CW complex (introduced in [Re], [Fr]) and show that the torsion is equal to the value at $z = 0$ of one of the zeta functions introduced earlier. Much of this paper can be viewed as a combinatorial analogue of the work on smooth dynamical systems presented in [P-P], [Fra], [Fri1, 2] and elsewhere.

We now give a more detailed overview of the contents of the paper. In §1 we review the notion, introduced in [Fo2] (see also [St] and [Du] for earlier related work), of a combinatorial vector field on a CW complex. In this introduction we will present all definitions in the special case of a simplicial complex, thereby avoiding some minor difficulties.

Let M be a simplicial complex, with K the set of simplices. If σ and τ are simplices of M, we write $\sigma^{(p)}$ if $\dim(\sigma) = p$, and $\sigma < \tau$ if σ lies in the boundary of τ.

Definition. A *combinatorial vector field* on M is a map

$$V : K \longrightarrow K \cup \{0\}$$

such that

(i) If $V(\sigma) \neq 0$, then $\dim V(\sigma) = \dim(\sigma) + 1$, and $\sigma < V(\sigma)$.
(ii) If $V(\sigma) = \tau \neq 0$, then $V(\tau) = 0$
(iii) For all $\sigma \in K$, $\#V^{-1}(\sigma) \leq 1$.

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It is useful to consider this definition pictorially. Given such a map V, and a $\sigma \in K$ with $V(\sigma) \neq 0$, draw an arrow on M whose tail begins at σ, and which extends into $V(\sigma)$. See Figure 0.1 for a simple example involving a 2-dimensional complex.

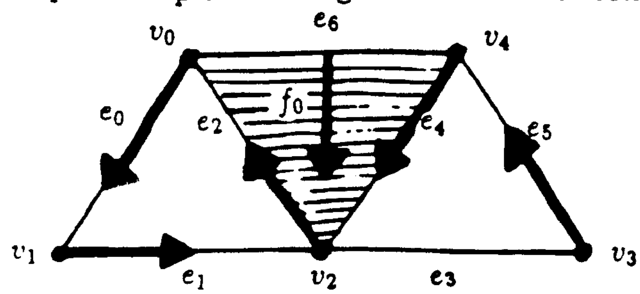

FIGURE 0.1

In the language of such pictures, property (i) of V implies that σ is always a face of $V(\sigma)$ so such an arrow is possible, property (ii) implies that if τ is the head of an arrow, then it can not also be the tail of an arrow, and property (iii) implies that every simplex σ is the head of at most 1 arrow (it also follows from the definition that σ can be the tail of at most 1 arrow).

Thus, for each simplex $\sigma^{(p)}$, there are precisely 3 disjoint possibilities:

(i) σ is the head of an arrow ($\sigma \in \text{Image}(V)$).
(ii) σ is the tail of an arrow ($V(\sigma) \neq 0$).
(iii) σ is neither the head nor the tail of any arrow ($V(\sigma) = 0$ and $\sigma \notin \text{Image}(V)$).

If $\sigma^{(p)}$ falls into category (iii) we say σ is a *rest point of V of index p*. For example, in Figure 0.1, the edge e_3 is a rest point of index 1.

The next step is to define the combinatorial analogue of the flow lines of V. Define a *V-path of index p* to be a sequence

(0.1) $$\gamma : \sigma_0^{(p)}, \tau_0^{(p+1)}, \sigma_1^{(p)}, \tau_1^{(p+1)}, \dots, \tau_{r-1}^{(p+1)}, \sigma_r^{(p)}$$

such that for all $i = 0, 1, \dots, r-1$

(i) $\tau_i = V(\sigma_i)$
(ii) $\sigma_i \neq \sigma_{i+1} < \tau_i$.

Say γ is a closed path (of length r) if $\sigma_0 = \sigma_r$. A V-path of index 1 is illustrated in Figure 0.2.

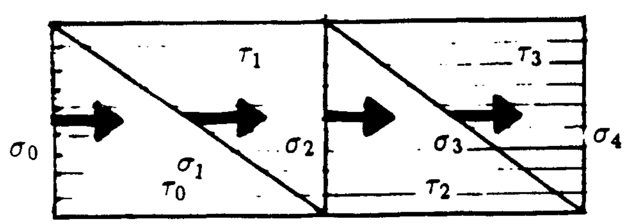

FIGURE 0.2

In Figure 0.1, the sequence

$$v_0, e_0, v_1, e_1, v_2, e_2, v_0$$

is a closed path of length 3 and index 0.

We define an equivalence relation on the set of closed paths by declaring 2 paths γ and $\tilde{\gamma}$ to be equivalent if $\tilde{\gamma}$ is the result of varying the starting point of γ. For example, if $\sigma_0 = \sigma_r$, then the path γ in (0.1) is closed and equivalent to

$$\tilde{\gamma} = \sigma_1^{(p)}, \tau_1^{(p+1)}, \ldots, \sigma_{r-1}^{(p)}, \tau_{r-1}^{(p+1)}, \sigma_0^{(p)}, \tau_0^{(p+1)}, \sigma_1^{(p)}.$$

An equivalence class of closed paths will be called a *closed orbit*. If γ is a closed path, then $[\gamma]$ denotes the corresponding orbit.

Given a combinatorial vector field V on M, we define the *chain recurrent set* $\mathcal{R}$ to be the set of simplices which are either rest points of V or are contained in some non-trivial closed path. The chain recurrent set can be decomposed into a disjoint union of *basic sets*

$$\mathcal{R} = \bigcup_i \Lambda_i$$

where 2 simplices σ, $\tau \in \mathcal{R}$ belong to the same basic set if and only if there is a closed V-path which contains both σ and τ.

The basic sets control the topology of M. This idea can be made precise. Denote by $\overline{\Lambda}_i$ the closure of Λ_i in M and $\dot{\Lambda}_i = \overline{\Lambda}_i - \Lambda$. Let $\mathbf{F}$ be a field. Define the Morse numbers of M with respect to $\mathbf{F}$ by

$$m_i = \sum_{\substack{\text{basic} \\ \text{sets } \Lambda_j}} \dim_{\mathbf{F}} H_i(\overline{\Lambda}_j, \dot{\Lambda}_j, \mathbf{F}).$$

Let

$$b_i = \dim_{\mathbf{F}} H_i(M, \mathbf{F})$$

denote the Betti numbers of M. In §3 we prove the generalized Strong Morse Inequalities

$$m_k - m_{k-1} + \cdots \pm m_0 \geq b_k - b_{k-1} + \cdots \pm b_0 \quad \forall \; k = 0, 1, 2, \ldots$$

which imply the Weak Morse Inequalities

$$m_k \geq b_k \quad \forall \; k = 0, 1, 2, \ldots$$
$$m_0 - m_1 + \cdots \pm m_n = b_0 - b_1 + \cdots \pm b_n$$

(where $n = \dim M$). The main tool is the existence of combinatorial Lyapunov functions, which is established in §2. See [Fra] for a discussion of these ideas in the context of smooth dynamical system.

In §5 we begin our study of zeta functions. We start with the simplest zeta function

$$\zeta(z) = \exp \sum_{r=1}^{\infty} \frac{z^n}{r} p_r$$

where

$$p_r = \#\{\text{closed } V\text{-paths of length } r\}.$$

If we make the change of variables

$$Z(s) = \zeta(e^{-s})$$

we can rewrite this zeta function in the suggestive form

$$Z(s) = \prod_{\substack{\text{prime closed} \\ \text{orbits } [\gamma]}} (1 - e^{-s|[\gamma]|})^{-1}$$

where a closed orbit is prime if it is not a multiple repeat of a shorter orbit, and $|[\gamma]|$ denotes the length of γ. We prove that ζ has an analytic continuation by proving that

$$\zeta(z) = \det(1 - zA)^{-1}$$

where A is a finite dimensional matrix arising from a subshift of finite type.

To construct more sophisticated zeta functions, we need to define some new ingredients. The first is the notion of the multiplicity of a closed path. If γ is a closed path, then its multiplicity $m(\gamma)$ is defined to be 1 or -1 (if M is a simplicial complex, all integers are possible if M is a general CW complex) depending on whether the $(p+1)$-dimensional band formed by the simplices in γ is orientable or not (see §4 for precise definitions).

We can also keep track of the homotopy classes of the closed orbits by choosing a unitary representation

$$\theta : \pi_1(M) \longrightarrow U(m)$$

(where $U(m)$ denotes the $m \times m$ unitary matrices) and "twisting the zeta function by θ (see §5 for details). In Theorem 5.12 we prove the existence of an analytic continuation of this twisted zeta function $\zeta(z, \theta)$. The main idea of the proof is to relate $\zeta(z, \theta)$ to $\det(I - z\Phi)$ where

$$\Phi : C_*(M, \theta) \longrightarrow C_*(M, \theta)$$

is the map on chains induced by the combinatorial *flow* along the vector field V. This notion of a combinatorial flow map was introduced in [Fo2], and is extended to the present context in §4.

In §6 we consider the Reidemeister Torsion of M with respect to the representation θ, $T(M, \theta)$, a combinatorial invariant of the pair (M, θ) (see [Re], [Fr]). The main result of this section is that if, restricted to each basic set Λ_i, θ is acyclic (i.e., $H_*(\overline{\Lambda}_i, \dot{\Lambda}_i, \theta) = 0$) then the torsion is equal to a special value of the zeta function $\zeta(z, \theta)$. Namely,

$$T(M, \theta) = \zeta(1, \theta) = Z(0, \theta) \tag{0.2}$$

(evaluating ζ at 1 requires the analytic continuation established in §5). Such a formula was proved for large classes of smooth flows in [Fri1] and [Fri2]. Our proof of (0.2) involves a combinatorial version of the deformation of the deRham complex introduced in [Wi] (see also [Fo3]). We think that this proof will provide clues for generalizing this type of formula to other settings.

In §7 we examine our results in the special case of combinatorial Morse-Smale vector fields.

§1 Preliminaries.

In this section we present the definitions of our objects of study. More extensive discussions of these topics can be found in [Fo2] and [Fo3].

Let M be a finite CW complex (see [L-W] for definitions and basic properties of CW complexes and regular CW complexes) and let K denote the set of open cells of M, with K_p the cells of dimension p. The notation $\sigma^{(p)}$ will indicate that σ is a cell of dimension p. To indicate relationships between cells, we write $\tau > \sigma$ (or $\sigma < \tau$) if $\sigma \neq \tau$ and $\sigma \subset \overline{\tau}$, where $\overline{\tau}$ is the closure of τ, and we say σ is a face of τ. We write $\tau \geq \sigma$ if either $\tau = \sigma$ or $\tau > \sigma$.

Suppose $\sigma^{(p)}$ is a face of $\tau^{(p+1)}$. Let B be a closed ball of dimension $p+1$, and

$$h : B \longrightarrow M$$

the characteristic map for τ, so that, in particular, h is a continuous map that maps interior(B) homeomorphically onto τ.

Definition 1.1. Say $\sigma^{(p)}$ is a *regular face* of $\tau^{(p+1)}$ if

(i) $h : h^{-1}(\sigma) \longrightarrow \sigma$ is a homeomorphism

(ii) $\overline{h^{-1}(\sigma)}$ is a closed p-ball.

Otherwise we say σ is an *irregular face* of τ.

We note that if M is a regular CW complex (and hence if M is a simplicial complex or a polyhedron) then all faces are regular. Of crucial importance is the following property. Suppose $\sigma^{(p)}$ is a regular face of $\tau^{(p+1)}$. Choose an orientation for each cell in M and consider σ and τ as elements in the cellular chain groups $C_p(M, \mathbf{Z})$ and $C_{p+1}(M, \mathbf{Z})$, respectively. Then

$$\langle \partial\tau, \sigma \rangle = \pm 1 \tag{1.1}$$

where $\langle \partial\tau, \sigma \rangle$ is the incidence number of τ and σ (for a proof see Corollary V.3.6 of [L-W]).

Definition 1.2. A *combinatorial vector field* is a map

$$V : K \longrightarrow K \cup \{0\}$$

satisfying

(1) For each p, $V(K_p) \subseteq K_{p+1} \cup \{0\}$

(2) For each $\sigma^{(p)} \in K_p$, either $V(\sigma) = 0$ or σ is a regular face of $V(\sigma)$.

(3) If $\sigma \in \text{Image}(V)$ then $V(\sigma) = 0$

(4) For each $\sigma^{(p)} \in K_p$

$$\#\{v^{(p-1)} \in K_{p-1} \mid V(v) = \sigma\} \leq 1.$$

(When M is a simplicial complex such objects have previously been considered under a different name, in [Du] and [Sta]).

Definition 1.3. Let V be a combinatorial vector field on M. We say $\sigma^{(p)}$ is a *zero (or rest point) of V of index p* if

$$V(\sigma) = 0$$

and

$$\sigma \notin \text{Image}(V).$$

Note that if $\sigma^{(p)}$ is a zero of V, then it necessarily has index p.

To proceed further, we need the notion of a flowline of a combinatorial vector field.

Definition 1.4. Let V be a combinatorial vector field. A *V-path of index p* from $\sigma^{(p)}$ to $\tilde{\sigma}^{(p)}$ is a sequence

$$\gamma : \sigma = \sigma_0^{(p)},\ \tau_0^{(p+1)},\ \sigma_1^{(p)},\ \tau_1^{(p+1)}, \ldots, \tau_{r-1}^{(p+1)},\ \sigma_r^{(p)} = \tilde{\sigma}$$

such that for each $i = 0, 1, \ldots, r-1$

(i) $V(\sigma_i) = \tau_i$
(ii) $\tau_i > \sigma_{i+1} \neq \sigma_i$.

Say γ is a *closed path* if $\sigma = \tilde{\sigma}$, and γ is *non-stationary* if $r > 0$.

These definitions immediately imply the following lemma.

Lemma 1.5. *Let γ be a non-stationary closed V-path, and suppose $\sigma^{(p)} \in \gamma$. Then*

(i) *either $V(\sigma) \neq 0$ or $\sigma \in$ Image(V)*
(ii) *if $V(\sigma) \neq 0$ then index$(\gamma) = p$*
(iii) *if $\sigma \in$ Image(V) then index$(\gamma) = p + 1$.*

§2 Chain Recurrent Sets and Lyapunov Functions.

Let M be a finite CW complex with a combinatorial vector field V. In this section we define the sets, distinguished by V, which "carry" the homology of M (in a sense which will be made precise in §3). First we define the chain recurrent set $\mathcal{R}$ (see [Fra] for the definition in the case V is a smooth vector field on a smooth manifold M).

Definition 2.1. Say $\sigma^{(p)} \in K$ is an element of the chain recurrent set $\mathcal{R}$ if either

(i) σ is a rest point of V

or

(ii) there is a non-stationary closed V-path γ with $\sigma \in \gamma$. (Note that γ must have index either $p-1$ or p).

The chain recurrent set $\mathcal{R}$ naturally decomposes into disjoint "minimal" recurrent sets Λ_i, defined by the following equivalence relation.

Definition 2.2. Given $\sigma, \tau \in \mathcal{R}$, say $\sigma \sim \tau$ if there is a non-trivial closed path γ with $\sigma \in \gamma$ and $\tau \in \gamma$.

Lemma 2.3. *The relation $\sim$ does, in fact, define an equivalence relation.*

Proof. The relation $\sim$ is clearly reflexive and symmetric. It remains to check transitivity. Suppose $\sigma_1 \sim \sigma_2$ and $\sigma_2 \sim \sigma_3$. Then there are closed orbits

$$\gamma_1 : \sigma, \ldots, \sigma_1, \ldots, \sigma_2, \ldots, \sigma$$
$$\gamma_2 : \tilde{\sigma}, \ldots, \sigma_2, \ldots, \sigma_3, \ldots, \tilde{\sigma}$$

We must show that $\sigma_1 \sim \sigma_3$, i.e., that there is a closed path which contains both σ_1 and σ_3. We can simply "splice" γ_2 into γ_1. That is, the sequence

$$\gamma : \sigma, \ldots, \sigma_1, \ldots, \sigma_2, \ldots, \sigma_3, \ldots, \tilde{\sigma}, \ldots, \sigma_2, \ldots, \sigma$$

is a non-stationary closed path which contains both σ_1 and σ_3. $\square$

Let $\Lambda_1, \ldots, \Lambda_k$ denote the equivalence classes of $\mathcal{R}/\sim$. The Λ_i's are called *basic sets*. Each Λ_i consists of either a single rest point of V, or is a union of non-stationary closed paths, each of which has the same index. We write $\Lambda_i^{(p)}$ if Λ_i consists of a rest point of index p or a union of closed paths of index p.

In [Fo2] we proved that if there are no non-stationary closed paths, then V is the combinatorial gradient vector field of a combinatorial Morse function (see [Fo2] for definitions). Of course, if V has closed paths, then it cannot be the gradient of a function. However, one can still find a "Morse-type" f, called a *Lyapunov function*, which is constant on each basic set, and has the property that, away from the chain recurrent set, V is the gradient of f.

We emphasize that when we speak of a combinatorial function f on M, we mean that f assigns a single number to each cell. That is, f is actually a function on K.

The following theorem is a combinatorial version of a theorem of Conley [Co] (Theorem 1.2 in [Fra]).

Theorem 2.4. *There is a function $f : K \to \mathbf{R}$ such that*

(i) *if $\sigma^{(p)} \notin \mathcal{R}$ and $\tau^{(p+1)} > \sigma$ then*

$$\begin{cases} f(\sigma) < f(\tau) & \text{if } \tau \neq V(\sigma) \\ f(\sigma) \geq f(\tau) & \text{if } \tau = V(\sigma) \end{cases}$$

(ii) *if $\sigma^{(p)} \in \mathcal{R}$ and $\tau^{(p+1)} > \sigma$ then*

$$\begin{cases} f(\sigma) = f(\tau) & \text{if } \sigma \sim \tau \\ f(\sigma) < f(\tau) & \text{if } \sigma \not\sim \tau. \end{cases}$$

Remark. Part (i) implies that if $\sigma_1^{(p)} \notin \mathcal{R}$ and

$$\sigma_1^{(p)},\ \tau_1^{(p+1)},\ \sigma_2^{(p)}$$

is a V-path, then

$$f(\sigma_1) \geq f(\tau_1) > f(\sigma_2).$$

That is, off the chain recurrent set, f decreases along V-paths.

Part (ii) implies that if

$$\gamma : \sigma_1^{(p)},\ \tau_0^{(p+1)},\ \sigma_1^{(p)}, \ldots, \sigma_{r-1}^{(p)},\ \tau_{r-1}^{(p+1)},\ \sigma_r^{(p)} = \sigma_0^{(p)}$$

is any closed path, then

$$f(\sigma_0) = f(\tau_0) = f(\sigma_1) = \cdots = f(\sigma_{r-1}) = f(\tau_{r-1})$$

so that f is constant on each basic set.

Proof of Theorem 2.4. For each $\sigma^{(p)} \in K$ define

$$\begin{aligned} d(\sigma) = \max\{s \mid \exists\ &V\text{-path} \\ \sigma \ =\ &\sigma_0^{(p)},\ \tau_0^{(p+1)},\ \sigma_1^{(p)}, \ldots, \tau_{r-1}^{(p+1)},\ \sigma_r^{(p)} \end{aligned}$$

such that the σ_i's include elements from exactly s distinct equivalence classes$\}$

$$D \ =\ \max_{\sigma \in K} d(\sigma).$$

We now define the function f. For any $\sigma^{(p)} \in K$

(i) if σ is a rest point of V, set

$$f(\sigma) \ =\ p$$

(ii) if $V(\sigma) \neq 0$, set

$$f(\sigma) = p + \frac{d(\sigma)}{2D}$$

so that (since $d(\sigma) \geq 1$)

$$p < f(\sigma) \leq p + \frac{1}{2}$$

(iii) if $\sigma \in \text{Image}(V)$ then there is a unique $v^{(p-1)}$ with $V(v) = \sigma$. Since $V(v) \neq 0$, $f(v)$ was defined in (ii), so we can set

(2.1) $$f(\sigma) = f(v)$$

so that

(2.2) $$p - 1 \leq f(\sigma) \leq p - \frac{1}{2}.$$

We must now check that f satisfies the desired properties.

First suppose $\sigma^{(p)} \notin \mathcal{R}$ and $\tau^{(p+1)} > \sigma$. If $V(\sigma) = \tau$ then $f(\sigma) \geq f(\tau)$ (in fact $f(\sigma) = f(\tau)$) by (2.1). If $V(\sigma) \neq \tau$ we consider the 3 possibilities:

(i) If τ is rest, then

$$f(\tau) = p + 1 > p + \frac{1}{2} \geq f(\sigma).$$

(ii) If $V(\tau) \neq 0$ then

$$f(\tau) > p + 1 > p + \frac{1}{2} \geq f(\sigma).$$

(iii) If $\tau \in \text{Image}(V)$ then there is a unique $\tilde{\sigma}^{(p)} \neq \sigma$ with $V(\tilde{\sigma}) = \tau$. Since $\sigma \notin \mathcal{R}$, σ is not rest so either $V(\sigma) \neq 0$ or $\sigma \in \text{Image}(V)$. If $\sigma \in \text{Image}(V)$ we learn from (2.1) and (2.2)

$$f(\tau) = f(\tilde{\sigma}) \geq p > p - \frac{1}{2} \geq f(\sigma).$$

If $V(\sigma) \neq 0$ and $\gamma : \sigma, \ldots$ is any V-path beginning at σ, then $\tilde{\gamma} : \tilde{\sigma}, \tau, \sigma, \ldots$ is a V-path beginning at $\tilde{\sigma}$. Moreover, since $\sigma \notin \mathcal{R}$, σ is not an element of any closed V-path. Thus $\tilde{\sigma}$ is not equivalent to any element of γ. This implies

$$d(\tilde{\sigma}) \geq d(\sigma) + 1.$$

Therefore

$$f(\tau) = f(\tilde{\sigma}) > f(\sigma).$$

We now consider the case $\sigma \in \mathcal{R}$. Suppose

$$\gamma : \sigma_0^{(p)},\ \tau_0^{(p+1)},\ \sigma_1^{(p)}, \ldots, \sigma_r^{(p)} \ = \ \sigma_0$$

is a non-stationary closed V-path. Then for each i and j, $0 \leq i,\ j \leq r-1$

$$d(\sigma_i) \ = \ d(\sigma_j)$$

since, using segments of γ, any V-path beginning at σ_i can be extended backwards to begin at σ_j, and vice versa. Thus

$$f(\sigma_i) \ = \ f(\sigma_j).$$

Since, for each i, $V(\sigma_i) = \tau_i$,

$$f(\tau_i) \ = \ f(\sigma_i)$$

so that

$$f(\sigma_0) \ = \ f(\tau_0) \ = \ f(\sigma_1) \ = \ \cdots .$$

Therefore

$$\sigma \sim \tau \Longrightarrow f(\sigma) \ = \ f(\tau).$$

Now suppose $\sigma^{(p)} \in \mathcal{R}$ and $\tau^{(p+1)} > \sigma$, $\tau \not\sim \sigma$. As in the case $\sigma \notin \mathcal{R}$, if

(i) τ is rest, or (ii) $V(\tau) \neq 0$ then

$$f(\tau) \geq p+1 > p + \frac{1}{2} \geq f(\sigma).$$

(ii) If $\tau \in \text{Image}(V)$ then, since $\tau \not\sim \sigma$, $\tau \neq V(\sigma)$ so there is a unique $\tilde{\sigma}^{(p)} \neq \sigma$ with $V(\tilde{\sigma}) = \tau$

a) If σ is rest or $\sigma \in \text{Image}(V)$

$$f(\tau) \ = \ f(\tilde{\sigma}) \ > \ p \ \geq \ f(\sigma).$$

b) If $V(\sigma) \neq 0$ and

$$\gamma : \sigma, \ldots$$

is any V-path beginning at σ, then

$$\tilde{\gamma} : \tilde{\sigma},\ \tau,\ \sigma, \ldots$$

is a V-path beginning at $\tilde{\sigma}$. Moreover, $\tilde{\sigma}$ is not equivalent to any element of γ, since otherwise σ and τ would be contained in a non-stationary closed path, which contradicts $\tau \not\sim \sigma$. Thus

$$d(\tilde{\sigma}) \ \geq \ d(\sigma) + 1$$

which implies

$$f(\tau) \ = \ f(\tilde{\sigma}) \ > \ f(\sigma). \qquad \square$$

§3 The Morse Inequalities.

In this section we use the Lyapunov function of Theorem 2.4 to present Morse inequalities relating the homology of the basic sets to the homology of the underlying complex. As an introduction, we briefly review the standard Morse inequalities (proved in [Fo2]), which form a special case of the main result of this section. Suppose V has no non-stationary closed paths, so that $\mathcal{R}$ consists only of rest points of V. Let m_k denote the number of rest points of index k. Let $\mathbf{F}$ denote a field and

$$b_k = \dim_{\mathbf{F}} H_k(M, \mathbf{F}) \tag{3.1}$$

the k'th Betti number of M. In [Fo2] we proved the Strong Morse Inequalities

$$m_k - m_{k-1} + \cdots \pm m_0 \geq b_k - b_{k-1} + \cdots \pm b_0 \quad \forall\, k = 0, 1, 2, \ldots$$

and hence the corollaries the Weak Morse Inequalities

$$\begin{aligned} m_k \;&\geq\; b_k \qquad \forall\, k = 0, 1, 2, \ldots \\ m_0 - m_1 + m_2 - \cdots \;&=\; b_0 - b_1 + b_2 - \cdots \qquad (\equiv \chi(M)). \end{aligned}$$

Before stating our main result, we present some notation. Let $A \subset K$ be a set of cells of M. By $\overline{A}$ we denote the subcomplex of M consisting of all cells in A, as well as all of their faces, i.e.,

$$\overline{A} = \bigcup_{\sigma \in A} \bigcup_{v \leq \sigma} v.$$

Although this is not quite standard notation, we define $\dot{A}$ to be the union of the cells of $\overline{A}$ which are not in A, i.e.,

$$\dot{A} = \bigcup_{\substack{\sigma \subseteq \overline{A} \\ \sigma \notin A}} \sigma.$$

Choose a coefficient field F and define

$$m_k = \sum_{\substack{\text{basic} \\ \text{sets } \Lambda_i}} \dim_{\mathbf{F}} H_k(\overline{\Lambda}_i, \dot{\Lambda}_i, \mathbf{F}). \tag{3.2}$$

Remark. 1) If index $\Lambda_i = p$, then

$$H_k(\overline{\Lambda}_i, \dot{\Lambda}_i, \mathbf{F}) = 0$$

unless

$$k = p \quad \text{or} \quad p + 1.$$

2) If Λ_i consists of a single rest point of index p (i.e., a p-cell $\sigma^{(p)}$) then

$$\begin{aligned} H_k(\overline{\Lambda}_i, \dot{\Lambda}_i, \mathbf{F}) &\cong H_k(\sigma^{(p)}, \dot{\sigma}^{(p)}, \mathbf{F}) \\ &\cong \begin{cases} \mathbf{F} & \text{if } p = k \\ 0 & \text{if } p \neq k \end{cases} \end{aligned}$$

so that, if V has no closed paths, m_k is precisely the number of rest points of index k.

Theorem 3.1 (The Morse Inequalities). *Let V be any combinatorial vector field on a finite CW complex M. Fix a coefficient field $\mathbf{F}$ and define b_k and m_k as in (3.1) and (3.2). Then*

(i) (The Strong Morse Inequalities)

$$m_k - m_{k-1} + \cdots \pm m_0 \geq b_k - b_{k-1} + \cdots \pm b_0 \quad \forall\, k = 0, 1, 2, \ldots$$

(ii) (The Weak Morse Inequalities)

$$\begin{aligned} m_k &\geq b_k \qquad \forall\, k = 0, 1, 2, \ldots \\ m_0 - m_1 + \cdots &= b_0 - b_1, + \cdots \end{aligned}$$

Proof. The proof uses the Lyapunov function $f : K \to \mathbf{R}$ of §2. It will be useful to modify the function slightly, to make it as injective as possible. Choose, for each $\sigma \in K$, a small positive number ϵ_σ, $0 < \epsilon_\sigma < |K|^{-1}$, such that for any σ, $\tau \in K$,

$$\epsilon_\sigma = \epsilon_\tau \iff \sigma = \tau \text{ or } \sigma,\ \tau \in \mathcal{R} \text{ and } \sigma \sim \tau.$$

Following the conventions and definitions of §2, define $f : K \to \mathbf{R}$ by

(i) if $\sigma^{(p)}$ is rest set

$$f(\sigma) = p + \epsilon_\sigma$$

(ii) if $V(\sigma^{(p)}) \neq 0$ set

$$f(\sigma) = p + \frac{d(\sigma)}{2D} + \epsilon_\sigma$$

(iii) if $V(\sigma) = \tau$ set

$$f(\tau) = f(\sigma).$$

This function is still a Lyapunov function, i.e., f satisfies the conclusions of Theorem 2.4. Moreover, for each $c \in \mathbf{R}$, if $f^{-1}(c) \neq \emptyset$ then either

(1) $f^{-1}(c) = \{\sigma, \tau\}$ for some $\sigma \notin \mathcal{R}$ and $\tau = V(\sigma)$
or
(2) $f^{-1}(c) = \Lambda$ for some basic set Λ.

For each $c \in \mathbf{R}$, define

$$M(c) = \bigcup_{f(\sigma) \leq c} \bigcup_{v \leq \sigma} v.$$

That is, $M(c)$ is the level subcomplex of M consisting of all cells σ with $f(\sigma) \leq c$, as well as all of their faces. For each $c \in \mathbf{R}$ define

$$\begin{aligned} m_k(c) &= \sum_{\substack{\text{basic sets} \\ \Lambda_i \text{ with} \\ f(\Lambda_i) \leq c}} \dim_{\mathbf{F}} H_k(\overline{\Lambda}_i, \dot{\Lambda}_i, \mathbf{F}) \\ b_k(c) &= \dim_{\mathbf{F}} H_k(M(c), \mathbf{F}). \end{aligned}$$

We will see that for each $c \in \mathbf{R}$, the sets $\{m_k(c)\}$, $\{b_k(c)\}$ satisfy the Morse inequalities. Choosing c large enough so that $M(c) = M$ yields the theorem.

The proof is by induction on c. The inequalities are trivially true for c small enough so that $M(c) = \emptyset$, since then $m_k(c) = b_k(c) = 0$ for all k.

We must now check that the inequalities remain true as c increases. We need only check what happens when c reaches one of the finitely many values in the image of f. Suppose $f^{-1}(c) \neq \emptyset$ and choose $b < c$ so that the interval (b, c) contains no values in the image of f. By the inductive hypothesis $\{m_k(b)\}$ and $\{b_k(b)\}$ satisfy the Morse inequalities. We need to see that $\{m_k(c)\}$ and $\{b_k(c)\}$ do also.

(1) Suppose $f^{-1}(c) = \{\sigma^{(p)}, \tau^{(p+1)}\}$ where $\sigma \notin \mathcal{R}$ and $V(\sigma) = \tau$. Since neither σ nor $\tau \in \mathcal{R}$,

$$m_k(b) \;=\; m_k(c) \qquad k = 0, 1, 2, \ldots .$$

We will see that

$$b_k(b) \;=\; b_k(c) \qquad k = 0, 1, 2, \ldots$$

so that the Morse inequalities remain true at c.

We first check that $\sigma \not\subseteq M(b)$ and $\tau \not\subseteq M(b)$, i.e., that there is no cell v with $f(v) \leq b$ and $v > \sigma$ or $v > \tau$. Note that

$$p \;\leq\; c \;=\; f(\sigma) \;=\; f(\tau) \;\leq\; p + \frac{3}{4}.$$

If $v > \sigma$ or $v > \tau$ then $\dim v \geq p + 1$. If $\dim v \geq p + 2$ then

$$f(v) \;\geq\; p + 1 \;>\; c.$$

If $\dim v = p + 1$ and $v > \sigma$, $v \neq \tau$ then by part (i) of Theorem 2.4

$$f(v) \;>\; f(\sigma) \;=\; c.$$

Thus,

$$\sigma \not\subseteq M(b), \quad \tau \not\subseteq M(b). \tag{3.3}$$

Now we prove that if $v \neq \sigma$ and $v < \tau$ then $f(v) < c$, so that $v \subseteq M(b)$. If $v < \tau$ then $\dim v \leq p$. If $\dim v \leq p - 1$ then

$$f(v) \;<\; p - \frac{1}{4} \;<\; c.$$

If $\dim v = p$, $v \neq \sigma$ and $v < \tau$ then $V(v) \neq \tau$, so by property (i) of Theorem 2.4

$$f(v) \;<\; f(\tau) \;=\; c.$$

Therefore

$$(\dot{\tau} - \sigma) \subseteq M(b). \tag{3.4}$$

Combining (3.3) and (3.4) we learn

$$M(c) = M(b) \bigcup_{\dot{\tau}-\sigma} \overline{\tau}.$$

By definition, σ is a regular face of τ, which implies that $M(b)$ is a deformation retract of $M(c)$, so that

$$b_k(c) = b_k(b) \qquad k = 0, 1, 2, \ldots .$$

(2) Suppose $f^{-1}(c) = \Lambda_i$ for some basic set Λ_i. Then

$$m_k(c) - m_k(b) = \dim_{\mathbf{F}} H_k(\overline{\Lambda}_i, \dot{\Lambda}_i, \mathbf{F}) \qquad k = 0, 1, 2, \ldots . \tag{3.5}$$

Following the argument of part (1) we can see that

$$\Lambda_i \cap M(b) = \emptyset$$

and

$$\dot{\Lambda}_i \subseteq M(b).$$

Thus, by excision,

$$H_k(M(c), M(b), \mathbf{F}) = H_k(\overline{\Lambda}_i, \dot{\Lambda}_i, \mathbf{F}) \qquad k = 0, 1, 2, \ldots . \tag{3.6}$$

The proof that (3.5) and (3.6) imply the desired Morse Inequalities is standard, and can be found on pages 28–31 of [Mi]. □

§4 Combinatorial Flows.

So far we have defined combinatorial vector fields and their associated paths, but we have not yet seen anything which would correspond to the smooth notion of a flow generated by a vector field. In this section we fill this gap.

Let $C_k(M, \mathbf{Z})$ denote the integer cellular chains on M, and, for each k,

$$\partial : C_k(M, \mathbf{Z}) \longrightarrow C_{k-1}(M, \mathbf{Z})$$

the usual boundary operator.

In §1, a combinatorial vector field was defined as a map of cells. We now modify the definition slightly and consider V as a map of oriented chains. Choose an orientation for each cell σ, and identify $-\sigma$ with σ given the opposite orientation. Given a combinatorial vector field V, we define a map

$$V_o : C_k(M, \mathbf{Z}) \longrightarrow C_{k+1}(M, \mathbf{Z}),$$

an oriented vector field, by setting

$$V_o(\sigma) = \begin{cases} 0 & \text{if } V(\sigma) = 0 \\ -\langle \partial\tau, \sigma \rangle \, \tau & \text{if } V(\sigma) = \tau \end{cases}$$

and extending linearly to $C_p(M, \mathbf{Z})$ (where $<,>$ is the canonical inner product on chains with respect to which the cells are orthonormal, so that $\langle \partial\tau, \sigma \rangle$ is the incidence number of τ and σ).

For example, suppose $V(\sigma) = \tau$. Then, since σ is a regular face of τ

$$\langle \partial\tau, \sigma \rangle = \pm 1$$

depending on the chosen orientations. Thus

$$\begin{aligned} \langle \partial V_o(\sigma), \sigma \rangle &= \langle \partial(-\langle \partial\tau, \sigma \rangle \, \tau), \sigma \rangle \\ &= -\langle \partial\tau, \sigma \rangle^2 = -1. \end{aligned}$$

To illustrate, if v_1 is a vertex and $V(v_1) = e$ then $V_o(v_1)$ is the edge e, oriented to be leaving v_1 (figure 4.1)

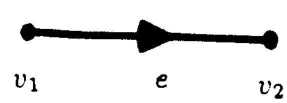

FIGURE 4.1

Lemma 4.1. $V_o^2 = 0$.

Proof. This follows from property (3) of Definition 1.2.

Since the vector field V uniquely determines V_o, and V_o uniquely determines V (simply ignore all orientations) we will, from now on, use the symbol V for both the original vector field and the induced map on chains. We do this to keep our notation consistent with that of [Fo2] and [Fo3]. We believe this will not cause confusion.

Definition 4.2. Define the (discrete time) flow

$$\Phi : C_k(M, \mathbf{Z}) \longrightarrow C_k(M, \mathbf{Z})$$

by

$$\Phi = 1 + \partial V + V\partial$$

i.e., for any oriented cell σ

$$\Phi(\sigma) = \sigma + \partial(V(\sigma)) + V(\partial\sigma).$$

For example, in figure 4.1,

$$\begin{aligned}\Phi(v_1) &= v_1 + \partial V(v_1) \\ &= v_1 + (v_2 - v_1) = v_2.\end{aligned}$$

A more complicated example is shown in figure 4.2, where the arrow indicates the vector field V, and e is the top edge oriented from left to right

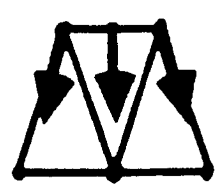

FIGURE 4.2

In figure 4.3 we calculate $\Phi(e)$.

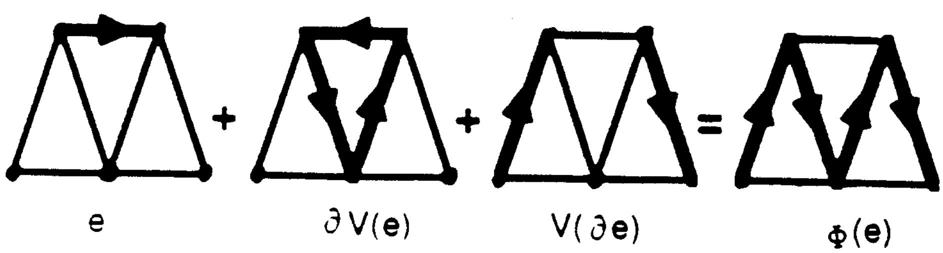

FIGURE 4.3

The main properties of the flow are listed in Theorem 4.3

Theorem 4.3. *(i)* $\Phi\partial = \partial\Phi$, $V\Phi = \Phi V$.

Choose an orientation for each cell. For any $\sigma \in K_p$ *write*

$$\Phi(\sigma) = \sum_{\bar{\sigma} \in K_p} c_{\sigma\bar{\sigma}} \bar{\sigma}.$$

(ii) If σ *is a rest point of* V *then*

$$c_{\sigma\sigma} = 1.$$

If σ *is not a rest point of* V *then*

$$c_{\sigma\sigma} = 0.$$

(iii) Let f *be a Lyapunov function for* V*. If* $\sigma \neq \bar{\sigma}$ *and* $c_{\sigma\bar{\sigma}} \neq 0$ *then*

$$f(\sigma) \geq f(\bar{\sigma})$$

and

$$f(\sigma) = f(\bar{\sigma}) \longleftrightarrow \sigma \sim \bar{\sigma}.$$

For a proof. see Theorem 6.4 of [Fo2].

Of great interest in later sections will be trace($\Phi^r_{(p)}$) for $r = 1, 2, 3, \ldots$. Our last goal in this section will be to express these traces in terms of V-paths. Express $\Phi^r_{(p)}$ as a matrix with respect to the basis of $C_p(M, \mathbf{Z})$ consisting of the oriented cells. Then

$$\text{trace}(\Phi^r_{(p)}) = \sum_{\sigma \in K_p} (\Phi^r_{(p)})_{\sigma\sigma}.$$

We first consider the trivial contributions to the trace

Theorem 4.4. *(i) If* $\sigma^{(p)} \notin \mathcal{R}$ *then for all* $r \geq 1$

$$(\Phi^r_{(p)})_{\sigma\sigma} = 0.$$

(ii) If $\sigma^{(p)}$ *is a rest point of* V *then for all* $r \geq 1$

$$(\Phi^r_{(p)})_{\sigma\sigma} = 1.$$

Proof. Let f be a Lyapunov function for V. From Theorem 4.3 (iii), for any $\sigma \in K$, $\Phi(\sigma)$ is the sum of cells $\bar{\sigma}$ with $f(\bar{\sigma}) \leq f(\sigma)$. Moreover, if $\sigma \notin \mathcal{R}$ then $f(\sigma)$ is the sum of cells $\bar{\sigma}$ with $f(\bar{\sigma}) < f(\sigma)$. Thus, inductively, if $\sigma \notin \mathcal{R}$ then for any $r > 1$

$$(\Phi^r)_{\sigma\bar{\sigma}} \neq 0 \Rightarrow f(\bar{\sigma}) < f(\sigma)$$

so that, in particular

$$(\Phi^r)_{\sigma\sigma} = 0.$$

Now suppose σ is rest, then by Theorem 4.1 (ii) and (iii),

$$\Phi(\sigma) = \sigma + \sum_{\substack{\bar{\sigma} \text{ such that} \\ f(\bar{\sigma}) < f(\sigma)}} c_{\sigma\bar{\sigma}} \bar{\sigma}$$

and the desired result follows by induction. □

It remains to find $(\Phi^r)_{\sigma\sigma}$ for $\sigma \in \mathcal{R}$ which is not rest. In order to state the result in this case, it is necessary to introduce the notion of the multiplicity of a V-path. Let

$$\gamma : \sigma_0^{(p)},\ \tau_0^{(p+1)},\ \sigma_1^{(p)},\ \ldots, \sigma_{r-1}^{(p)},\ \tau_{r-1}^{(p+1)},\ \sigma_r^{(p)}$$

denote a V-path of index p and length r. For simplicity, we assume for the moment that for each i, σ_{i+1} is a regular face of τ_i (by definition σ_i is a regular face of τ_i). Choose orientations for σ_0 and σ_r. The orientation on σ_0 induces an orientation on τ_0, such that with this orientation

$$\langle \partial\tau_0, \sigma_0 \rangle \ = \ -1.$$

This orientation of τ_0 induces an orientation on σ_1, such that with this orientation

$$\langle \partial\tau_0, \sigma_1 \rangle \ = \ 1.$$

Continuing in this fashion, an orientation on each cell induces an orientation on the next cell in turn (see figure 4.4 where a V-path of index 1 is indicated as well as an orientation on the initial edge σ_0 and the induced orientations on the remaining cells in the path).

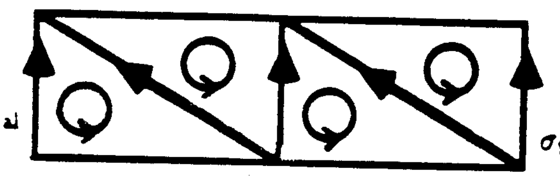

FIGURE 4.4

Thus, the initial orientation on σ_0 induces an orientation on σ_r. We define the multiplicity of γ, denoted by $m(\gamma)$, to be $+1$ if the induced orientation on σ_r is equal to the chosen orientation. We set $m(\gamma) = -1$ if the induced orientation is the opposite of the chosen orientation.

Choose an orientation for each cell of M. We observe, in the special case that all faces are regular, that the multiplicity can be expressed by the formula

$$m(\gamma) \ = \ \prod_{i=0}^{r-1} - \langle \partial\tau_i, \sigma_i \rangle \langle \partial\tau_i, \sigma_{i+1} \rangle .$$

We can use this formula to define the multiplicity in general, i.e., not assuming all faces are regular.

Definition 4.5. Choose an orientation for each cell of M. For any V-path

$$\gamma : \sigma_0^{(p)},\ \tau_0^{(p+1)},\ \sigma_1^{(p)},\ \ldots, \sigma_{r-1}^{(p)},\ \tau_{r-1}^{(p+1)},\ \sigma_r^{(p)}$$

define the multiplicity of γ by

$$m(\gamma) \ = \ \prod_{i=0}^{r-1} - \langle \partial\tau_i, \sigma_i \rangle \langle \partial\tau_i, \sigma_{i+1} \rangle .$$

In Lemma 4.4 we list some immediate implications of the definitions.

Lemma 4.6. *(i) The multiplicity of a V-path γ from σ to $\bar{\sigma}$ depends only on the chosen orientations of σ and $\bar{\sigma}$.*

(ii) If γ is a closed path, then $m(\gamma)$ is independent of all choices.

(iii) If γ_1 is a V-path from σ to $\bar{\sigma}$, and γ_2 is a V-path from $\bar{\sigma}$ to $\bar{\bar{\sigma}}$, then $\gamma_1 \circ \gamma_2$, i.e., the sequence of cells traced out by first following γ_1 from σ to $\bar{\sigma}$ and then following γ_2 from $\bar{\sigma}$ to $\bar{\bar{\sigma}}$, is a V-path from σ to $\bar{\bar{\sigma}}$ and

$$m(\gamma_1 \circ \gamma_2) \;=\; m(\gamma_1)m(\gamma_2).$$

Remark. If γ is a closed path of index p then one can think of the cells in γ as playing the role that the unstable directions along a closed orbit of index p of a hyperbolic flow play in the smooth case. With this analogy, at least when all faces are regular, $m(\gamma) = -1$ corresponds to what is called a "twisted orbit" in [Fra], and $m(\gamma) = 1$ is an "untwisted orbit."

For $\sigma, \bar{\sigma} \in K$ let $\mathcal{P}_r(\sigma, \bar{\sigma})$ denote the set of V-paths of length r from σ to $\bar{\sigma}$. We are now ready to proceed to the next step in our computation of the trace of Φ^r.

Theorem 4.7. *Suppose $\sigma \in \mathcal{R}$ and $V(\sigma) \neq 0$. Then*

$$(\Phi^r)_{\sigma\sigma} \;=\; \sum_{\gamma \in \mathcal{P}_r(\sigma,\sigma)} m(\gamma).$$

The proof will follow from 2 lemmas.

Lemma 4.8. *Suppose $V(\sigma) \neq 0$. Then*

$$(\Phi^r)_{\sigma\sigma} \;=\; (\widetilde{\Phi}^r)_{\sigma\sigma} \tag{4.1}$$

where

$$\widetilde{\Phi} \;=\; 1 + \partial V \;=\; \Phi - V\partial. \tag{4.2}$$

Proof. We first observe that for each r there is a linear map L_r such that

$$\Phi^r - \widetilde{\Phi}^r \;=\; VL_r. \tag{4.3}$$

This easily follows inductively from (4.2) and Theorem 4.3 (i).

Now (4.1) follows from (4.3) and the observation that

$$V(\sigma) \neq 0 \Rightarrow \sigma \notin \text{Image}(V). \qquad \square$$

Lemma 4.9. *Suppose $V(\sigma) \neq 0$. Then for all $\bar{\sigma} \in K$*

$$(\widetilde{\Phi})_{\sigma\bar{\sigma}} = \sum_{\gamma \in \mathcal{P}_1(\sigma,\bar{\sigma})} m(\gamma).$$

Proof. Suppose $V(\sigma) = \tau$ with τ oriented so that $\langle \partial\tau, \sigma \rangle = -1$. Then

$$\begin{aligned}
\widetilde{\Phi}\sigma &= \sigma + \partial V\sigma = \sigma + \partial\tau \\
&= \sigma + \sum_{\bar{\sigma}<\tau} \langle \partial\tau, \bar{\sigma} \rangle \bar{\sigma} \\
&= \sigma + \sum_{\bar{\sigma}<\tau} - \langle \partial\tau, \sigma \rangle \langle \partial\tau, \bar{\sigma} \rangle \bar{\sigma} \\
&= \sum_{\substack{\bar{\sigma}<\tau \\ \bar{\sigma}\neq\sigma}} - \langle \partial\tau, \sigma \rangle \langle \partial\tau, \bar{\sigma} \rangle \bar{\sigma}.
\end{aligned}$$

We now observe that if $\bar{\sigma} < \tau$, $\bar{\sigma} \neq \sigma$, then

$$\gamma : \sigma, \quad \tau, \quad \bar{\sigma}$$

is a V-path of length 1 with

$$m(\gamma) = - \langle \partial\tau, \sigma \rangle \langle \partial\tau, \bar{\sigma} \rangle$$

and every such path corresponds to precisely one such $\bar{\sigma}$. Thus

$$\widetilde{\Phi}\sigma = \sum_{\bar{\sigma}} \left[\sum_{\gamma \in \mathcal{P}_1(\sigma,\bar{\sigma})} m(\gamma) \right] \bar{\sigma}$$

or, in the language of matrices

$$(\widetilde{\Phi})_{\sigma\bar{\sigma}} = \sum_{\gamma \in \mathcal{P}_1(\sigma,\bar{\sigma})} m(\gamma). \qquad \square$$

Proof of Theorem 4.7. It follows inductively from Lemmas 4.4 and 4.7 that if $V(\sigma) \neq 0$ and $\bar{\sigma} \in K$ then for any $r > 0$

$$(\widetilde{\Phi}^r)_{\sigma\bar{\sigma}} = \sum_{\gamma \in \mathcal{P}_r(\sigma,\bar{\sigma})} m(\gamma).$$

Therefore, using Lemma 4.8

$$(\Phi^r)_{\sigma\sigma} = (\widetilde{\Phi}^r)_{\sigma\sigma} = \sum_{\gamma \in \mathcal{P}_r(\sigma,\sigma)} m(\gamma)$$

as desired. $\square$

Lastly, we need to find $(\Phi^r)_{\sigma\sigma}$ for $\sigma \in \mathcal{R}$, $\sigma \in \text{Image}\, V$. To this end we require a new definition.

Definition 4.10. A *V-bridge* of index p and length r from $\sigma^{(p)}$ to $\tilde{\sigma}^{(p)}$ is a sequence

$$\gamma : \sigma \;=\; \sigma_0^{(p)},\; v_0^{(p-1)},\; \sigma_1^{(p)},\; v_1^{(p-1)}, \ldots, \sigma_{r-1}^{(p)},\; v_{r-1}^{(p-1)},\; \sigma_r^{(p)} \;=\; \tilde{\sigma}$$

satisfying for each $i = 0, 1, \ldots, r-1$

(i) $V(v_i) = \sigma_{i+1}$

(ii) $\sigma_{i+1} \neq \sigma_i > v_i$.

We define the multiplicity of a V-bridge γ, denoted $m(\gamma)$, by

$$m(\gamma) \;=\; \prod_{i=0}^{r-1} - \langle \partial\sigma_i, v_i \rangle \, \langle \partial\sigma_{i+1}, v_i \rangle \, .$$

For any σ, $\tilde{\sigma} \in K$ let $\mathcal{B}_r(\sigma, \tilde{\sigma})$ denote the set of V-bridges from σ to $\tilde{\sigma}$. As for V-paths, if γ_1 is a V-bridge from σ to $\tilde{\sigma}$, and γ_2 is a V-bridge from $\tilde{\sigma}$ to $\tilde{\tilde{\sigma}}$, then $\gamma_1 \circ \gamma_2$ is a V-bridge from σ to $\tilde{\tilde{\sigma}}$ and

$$m(\gamma_1 \circ \gamma_2) \;=\; m(\gamma_1) m(\gamma_2).$$

Theorem 4.11. *Suppose $\sigma \in \mathcal{R}$ satisfies $\sigma \in \text{Image}(V)$. Then*

$$(\Phi^r)_{\sigma\sigma} \;=\; \sum_{\gamma \in \mathcal{B}_r(\sigma,\sigma)} m(\gamma).$$

Proof. We only sketch the proof, as it is precisely dual to the proof of Theorem 4.7. We first note that

$$(\Phi^r)_{\sigma\sigma} \;=\; (\bar{\bar{\Phi}}^r)_{\sigma\sigma} \tag{4.4}$$

where

$$\bar{\bar{\Phi}} \;=\; 1 + V\partial \;=\; \Phi - \partial V.$$

This follows, analogously to the proof of Lemma 4.6, from the observations

(i) there is a linear map L_r such that

$$\Phi^r - \bar{\bar{\Phi}}^r \;=\; L_r V.$$

(ii) $\sigma \in \text{Image}(V) \Rightarrow V(\sigma) = 0$.

We then note that for all $\tilde{\sigma}$

$$\bar{\bar{\Phi}}_{\tilde{\sigma}\sigma} \;=\; \sum_{\gamma \in B_1(\tilde{\sigma},\sigma)} m(\gamma)$$

(which is proved as in Lemma 4.7) so that

$$\left(\bar{\bar{\Phi}}^r\right)_{\tilde{\sigma}\sigma} \;=\; \sum_{\gamma \in \mathcal{B}_r(\tilde{\sigma},\sigma)} m(\gamma). \tag{4.5}$$

Together, (4.4) and (4.5) yield the theorem. □

Adding together the results of Theorems 4.6, 4.7 and 4.11 we learn

$$\begin{aligned}\operatorname{tr}\Phi^r_{(p)} &= \#\,(\text{rest points of index } p) + \sum_{\substack{\sigma^{(p)}\in\mathcal{R}\\ V(\sigma)\neq 0}} \sum_{\gamma\in\mathcal{P}_r(\sigma,\sigma)} m(\gamma)\\ &+ \sum_{\substack{\sigma^{(p)}\in\mathcal{R}\\ \sigma\in\operatorname{Image}(V)}} \sum_{\gamma\in\mathcal{B}_r(\sigma,\sigma)} m(\gamma).\end{aligned}$$

To simplify this formula, let

$$c_p = \#\,\{\text{rest points of index } p\}$$

and let $\mathcal{P}_r^{(p)}$ denote the set of closed V-paths of length r and index p, i.e.,

$$\mathcal{P}_r^{(p)} = \bigcup_{\sigma^{(p)}\in K} \mathcal{P}_r(\sigma,\sigma).$$

We then have

Theorem 4.12.

$$\operatorname{tr}\Phi^r_{(p)} = c_p + \sum_{\gamma\in\mathcal{P}_r^{(p)}} m(\gamma) + \sum_{\gamma\in\mathcal{P}_r^{(p-1)}} m(\gamma).$$

Proof. We first observe that if $\sigma\notin\mathcal{R}$ or $V(\sigma)=0$, then $\mathcal{P}_r(\sigma,\sigma)=\emptyset$ for all $r>1$. Therefore

$$\bigcup_{\substack{\sigma^{(p)}\in\mathcal{R}\\ V(\sigma)\neq 0}} \mathcal{P}_r(\sigma,\sigma) = \bigcup_{\sigma^{(p)}\in K} \mathcal{P}_r(\sigma,\sigma) = \mathcal{P}_r^{(p)}$$

so that

$$\sum_{\substack{\sigma^{(p)}\in\mathcal{R}\\ V(\sigma)\neq 0}} \sum_{\gamma\in\mathcal{P}_r(\sigma,\sigma)} m(\gamma) = \sum_{\gamma\in\mathcal{P}_r^{(p)}} m(\gamma).$$

It remains to see that

$$\sum_{\substack{\sigma^{(p)}\in\mathcal{R}\\ \sigma\in\operatorname{Image}(V)}} \sum_{\gamma\in\mathcal{B}_r(\sigma,\sigma)} m(\gamma) = \sum_{\gamma\in\mathcal{P}_r^{(p-1)}} m(\gamma).$$

Let

$$\mathcal{B}_r^{(p)} = \bigcup_{\sigma^{(p)}\in K} \mathcal{B}_r(\sigma,\sigma).$$

If $\sigma \notin \mathcal{R}$ or $\sigma \notin \mathrm{Image}(V)$ then $\mathcal{B}_r(\sigma,\sigma) = \emptyset$. Therefore

$$\bigcup_{\substack{\sigma^{(p)}\in\mathcal{R} \\ \sigma^{(p)}\in \mathrm{Image}(V)}} \mathcal{B}_r(\sigma,\sigma) \;=\; \bigcup_{\sigma^{(p)}\in K} \mathcal{B}_r(\sigma,\sigma) \;=\; \mathcal{B}_r^{(p)}$$

so that

$$\sum_{\substack{\sigma^{(p)}\in\mathcal{R} \\ \sigma^{(p)}\in \mathrm{Image}(V)}} \; \sum_{\gamma\in\mathcal{B}_r(\sigma,\sigma)} m(\gamma) \;=\; \sum_{\gamma\in\mathcal{B}_r^{(p)}} m(\gamma). \tag{4.6}$$

Now suppose $\gamma \in \mathcal{B}_r^{(p)}$, then γ is a closed bridge of the form

$$\gamma : \sigma_0^{(p)},\ \nu_0^{(p-1)},\ \sigma_1^{(p)},\ldots,\nu_{r-1}^{(p-1)},\ \sigma_r^{(p)} \;=\; \sigma_0.$$

It follows immediately from the definition that

$$\tilde{\gamma} : \nu_0^{(p-1)},\ \sigma_1^{(p)},\ldots,\nu_{r-1}^{(p-1)},\ \sigma_r^{(p)} \;=\; \sigma_0^{(p)},\ \nu_0^{(p-1)}$$

is a closed V-path of length r and index $p-1$ satisfying

$$m(\tilde{\gamma}) \;=\; m(\gamma).$$

This establishes a 1–1 multiplicity preserving correspondence between closed bridges of length r and index p, and closed paths of length r and index $p-1$. Therefore

$$\sum_{\gamma\in\mathcal{B}_r^{(p)}} m(\gamma) \;=\; \sum_{\gamma\in\mathcal{P}_r^{(p-1)}} m(\gamma). \tag{4.7}$$

Substituting (4.6) into (4.7) completes the proof. □

§5 Zeta Functions.

Zeta functions have become a standard way to keep track of the closed orbits of a vector field. In this section we present analogues of the most commonly studied zeta functions.

We begin with the most straight forward zeta function. Define

$$\zeta(z) = \exp \sum_{r=1}^{\infty} \frac{z^n}{r} p_r \tag{5.1}$$

where

$$p_r = \sum_{k=0}^{n} \#\mathcal{P}_r^{(p)} = \#\{\text{closed } V\text{-paths of length } r\}.$$

Let

$$k = \#K = \#\{\text{cells in } M\}.$$

The trivial bound

$$p_r \leq k^r$$

shows that $\zeta(z)$ has a radius of convergence of at least k^{-1}. Our first goal is to make more precise statements about the analytic behavior of $\zeta(z)$.

Definition 5.1. A *V-step* is a V-path of length 1, i.e., a sequence

$$\gamma : \sigma_0^{(p)},\ \tau_0^{(p+1)},\ \sigma_1^{(p)}$$

for some p, where $V(\sigma_0) = \tau_0$ and $\sigma_0 \neq \sigma_1 < \tau_0$.

Define the origin of γ, $o(\gamma)$, and the terminus of γ, $t(\gamma)$, by

$$o(\gamma) = \sigma_0, \quad t(\gamma) = \sigma_1.$$

We observe that a sequence of V-steps

$$\gamma_0, \gamma_1, \ldots, \gamma_{r-1}$$

fit together to form a V-path of length r if and only if for each $i = 0, 1, \ldots, r-2$

$$t(\gamma_i) = o(\gamma_{i+1}).$$

Let S denote the set of V-steps, and let A denote the square matrix whose rows and columns are indexed by the elements of S, and where, for $\gamma_0, \gamma_1 \in S$

$$A_{\gamma_0,\gamma_1} = \begin{cases} 1 & \text{if } t(\gamma_0) = o(\gamma_1) \\ 0 & \text{if } t(\gamma_0) \neq o(\gamma_1). \end{cases}$$

It is then quite easy to see that

$$p_r = \operatorname{tr} A^r.$$

Theorem 5.2. *Where $\zeta(z)$ converges*

$$\zeta(z) = \det(I - zA)^{-1}.$$

Thus, $\det(I - zA)^{-1}$ provides an analytic continuation of $\zeta(z)$ to a meromorphic function on the entire complex plane.

The set $\{S, A\}$ is an example of a subshift of finite type. For a thorough analysis of more general zeta functions associated to such objects, see [P-P].

Before leaving this topic, we pause to rewrite $\zeta(z)$ in a very suggestive form. Let $\mathcal{P} = \bigcup_{r=1}^{\infty} \mathcal{P}_r$ denote the set of all non-stationary closed V-paths.

Definition 5.3. Say a closed path $\gamma \in \mathcal{P}$ is *prime* if γ is not the multiple cover of another path, i.e., if there is no closed path $\tilde{\gamma}$ such that

$$\gamma = \underbrace{\tilde{\gamma} \circ \tilde{\gamma} \circ \cdots \circ \tilde{\gamma}}_{\ell\text{-times}} = \tilde{\gamma}^{\ell}$$

for some $\ell > 1$. Let $\mathcal{P}_*$ denote the set of all prime closed paths, and $\mathcal{P}_*^{(p)}$ those with index p.

In our definition of $\mathcal{P}_r^{(p)}$, we distinguish between the closed paths

$$\gamma_0 : \sigma_0^{(p)},\ \tau_0^{(p+1)},\ \sigma_1^{(p)},\ \tau_1^{(p+1)}, \ldots, \tau_{r-1}^{(p+1)},\ \sigma_0^{(p)}$$

and

$$\gamma_1 : \sigma_1^{(p)},\ \tau_1^{(p+1)}, \ldots, \tau_{r-1}^{(p+1)},\ \sigma_0^{(p)},\ \tau_0^{(p+1)},\ \sigma_1^{(p)}$$

even though they trace out the same cells in the same order, differing only in the starting point. It seems natural to identify these 2 paths.

Definition 5.4. Say 2 closed paths $\gamma, \tilde{\gamma} \in \mathcal{P}_r^{(p)}$ represent the same *closed orbit (of index p)* if it is possible to write

$$\gamma = \gamma_1 \circ \gamma_2$$

for 2 (not necessarily closed) V-paths γ_1 and γ_2 such that

$$\tilde{\gamma} = \gamma_2 \circ \gamma_1.$$

If $\gamma \in \mathcal{P}_r^{(p)}$ we denote the corresponding orbit by $[\gamma]$. Denote by $\mathcal{O}_*$ the set of prime closed orbits, i.e.,

$$\mathcal{O}_* = \{[\gamma] \mid \gamma \in \mathcal{P}_*\}$$

and by $\mathcal{O}_*^{(p)}$ those of index p.

The following lemma is evident from the definitions.

Lemma 5.5. *If γ is a prime closed orbit of length r, then there are precisely r distinct closed paths in $\mathcal{P}_r^{(p)}$ which represent the closed orbit $[\gamma] \in \mathcal{O}_*$, each of which is prime.*

Let $|\gamma|$ denote the length of γ, i.e., if $\gamma \in \mathcal{P}_r$ then $|\gamma| = r$. For a closed orbit $[\gamma]$, let $||[\gamma]||$ denote the common length of the closed paths which represent $[\gamma]$, i.e.,

$$||[\gamma]|| = |\gamma|.$$

With these definitions and notations in hand, we can write

$$\begin{aligned}
\log \zeta(z) &= \sum_{r=1}^{\infty} \frac{z^r}{r} P_r = \sum_{\gamma \in \mathcal{P}} \frac{z^{|\gamma|}}{|\gamma|} \\
&= \sum_{\gamma \in \mathcal{P}_\bullet} \sum_{\ell=1}^{\infty} \frac{z^{|\gamma^\ell|}}{|\gamma^\ell|} = \sum_{\gamma \in \mathcal{P}_\bullet} \sum_{\ell=1}^{\infty} \frac{z^{\ell|\gamma|}}{\ell|\gamma|} \\
&= \sum_{\gamma \in \mathcal{P}_\bullet} \frac{1}{|\gamma|} \sum_{\ell=1}^{\infty} \frac{z^{\ell|\gamma|}}{\ell} = \sum_{[\gamma] \in \mathcal{O}_\bullet} \sum_{\ell=1}^{\infty} \frac{z^{\ell||[\gamma]||}}{\ell} \\
&= \sum_{[\gamma] \in \mathcal{O}_\bullet} -\log(1 - z^{||[\gamma]||})
\end{aligned}$$

so that

$$\zeta(z) = \prod_{[\gamma] \in \mathcal{O}_\bullet} (1 - z^{||[\gamma]||})^{-1}.$$

It is customary to change variables here and let $z = e^{-s}$.

Theorem 5.6. *Define* $Z(s) = \zeta(e^{-s})$. *Then*

$$Z(s) = \prod_{[\gamma] \in \mathcal{O}_\bullet} (1 - e^{-s||[\gamma]||})^{-1}.$$

We now consider a zeta function which keeps track of more information than simply the length of the closed orbits. We observe that the key ingredient in (5.1) is the quantity

$$p_r = \sum_{\gamma \in \mathcal{P}_r} 1. \tag{5.2}$$

One can create more "sophisticated" zeta functions by changing the 1 in (5.2) to a function of γ with more information. One natural choice is to consider the multiplicity. Define

$$\zeta_m(z) = \exp \sum_{r=1}^{\infty} \frac{z^r}{r} \sum_{\gamma \in \mathcal{P}_r} m(\gamma).$$

With our results of §4, it is simple to find an analytic continuation for $\zeta_m(z)$. By Theorem 4.10

$$\sum_{k=0}^{n} \operatorname{tr} \Phi_{(k)}^r = 2 \left(\sum_{\gamma \in \mathcal{P}_r} m(\gamma) \right) + \sum_{k=0}^{n} c_k$$

so that

$$\zeta_m(z) = \exp \left\{ \frac{1}{2} \sum_{k=0}^{n} \left[\sum_{r=1}^{\infty} \frac{z^r}{r} \operatorname{tr} \Phi_{(k)}^r \right] - \frac{1}{2} \left(\sum_{r=1}^{\infty} \frac{z^r}{r} \right) \left(\sum_{k=0}^{n} c_k \right) \right\}$$

which leads to the theorem

Theorem 5.7. *Where $\zeta_m(z)$ converges*

$$(\zeta_m(z))^2 = (1-z)^{\sum_{k=1}^{n} c_k} \prod_{k=0}^{n} \det\left(1 - z\Phi_{(k)}\right)^{-1}$$

so that the right hand side provides an analytic continuation of ζ_m^2 to a meromorphic function on the entire complex plane.

By Lemma 4.4, if closed paths γ and $\tilde{\gamma}$ represent the same closed orbit, then

$$m(\gamma) = m(\tilde{\gamma}).$$

For any closed orbit $[\gamma]$, we define its multiplicity $m([\gamma])$ to be the common multiplicity of the paths which represent $[\gamma]$, i.e.,

$$m([\gamma]) = m(\gamma).$$

Moreover, for any ℓ

$$m(\gamma^{\ell}) = (m(\gamma))^{\ell}.$$

Following the proof of Theorem 5.6, we prove

Theorem 5.8. *Define $Z_m(s) = \zeta_m(e^{-s})$. Then*

$$Z_m(s) = \prod_{[\gamma]\in\mathcal{O}_*} (1 - m([\gamma])e^{-s|[\gamma]|})^{-1}.$$

It will be useful in the next section to consider a zeta function which incorporates the index of a closed path. To that end, we introduce

$$\zeta_{m,i}(z) = \exp\sum_{r=1}^{\infty}\frac{z^r}{r}\left[\sum_{k=0}^{n}(-1)^k \sum_{\gamma\in\mathcal{P}_r^{(k)}} m(\gamma)\right].$$

From Theorem 4.10 we learn

$$\sum_{k=0}^{n}(-1)^{k+1}k\,\mathrm{tr}\,\Phi_{(k)}^r = \sum_{k=0}^{n}(-1)^k \sum_{\gamma\in\mathcal{P}_r^{(k)}} m(\gamma) + \sum_{k=0}^{n}(-1)^{k+1}kc_k$$

so that

Theorem 5.9. *Where $\zeta_{m,i}(z)$ converges*

$$\zeta_{m,i}(z) = (1-z)^{\sum_{k=0}^{n}(-1)^{k+1}kc_k} \prod_{k=0}^{n}\left[\det\left(1 - z\Phi_{(k)}\right)\right]^{(k+1)(-1)^k}$$

so that the right hand side provides an analytic continuation of $\zeta_{m,i}(z)$ to a meromorphic function on the entire complex plane.

The zeta function $\zeta_{m,i}(z)$ can also be expressed in terms of prime closed orbits

Theorem 5.10. *Let* $Z_{m,i}(s) = \zeta_{m,i}(e^{-s})$. *Then*

$$Z_{m,i}(z) = \prod_{[\gamma]\in\mathcal{O}_*} \left(1 - m(\gamma)e^{-s|\gamma|}\right)^{(-1)^{i([\gamma])}}$$

where $i([\gamma])$ *is the common index of the closed paths which represent* $[\gamma]$, *i.e.,*

$$i([\gamma]) = \text{index}(\gamma).$$

To proceed further, we now assume that M has a non-trivial fundamental group, which we denote by Γ. We can then associate to each closed path its homotopy class, and we can incorporate this data into the zeta function. One manner of carrying this out is to choose a unitary representation

$$\theta : \Gamma \longrightarrow U(m)$$

where $U(m)$ denotes the group of $m \times m$ complex unitary matrices. We now describe how to "twist" the zeta function by θ.

Let $\widetilde{M}$ denote the universal cover of M, $\widetilde{K}$ the cells of $\widetilde{M}$ and $\widetilde{K}_p$ the cells of $\widetilde{M}$ of dimension p. Let π denote the canonical projection

$$\pi : \widetilde{M} \longrightarrow M$$

as well as the induced map

$$\pi : \widetilde{K} \longrightarrow K.$$

The group Γ acts on $\widetilde{K}$. We denote the image of the action of $g \in \Gamma$ on a cell $\tilde{\sigma} \in \widetilde{K}$ by $g(\tilde{\sigma})$. The fundamental property of this action is

$$\forall\ \tilde{\sigma}, \tilde{\tau} \in \widetilde{K},\quad \pi(\tilde{\sigma}) = \pi(\tilde{\tau}) \longleftrightarrow \exists\ g \in \Gamma \text{ such that } g(\tilde{\sigma}) = \tilde{\tau}.$$

It is possible to lift the combinatorial vector field V on M to a combinatorial vector field $\widetilde{V}$ on $\widetilde{M}$. Namely, suppose $\sigma^{(p)}$, $\tau^{(p+1)} \in K$ satisfy $V(\sigma) = \tau$ and $\tilde{\sigma} \in \widetilde{K}$ satisfies $\pi(\tilde{\sigma}) = \sigma$. Then there is a unique $\tilde{\tau} \in \widetilde{K}$ with $\pi(\tilde{\tau}) = \tau$ and $\tilde{\tau} > \tilde{\sigma}$, so we set

$$\widetilde{V}(\tilde{\sigma}) = \tilde{\tau}.$$

Let

$$\gamma : \sigma^{(p)},\ \tau_0^{(p+1)}, \ldots, \sigma_r^{(p)} = \sigma_0^{(p)}$$

denote a closed V-path, and let $\tilde{\sigma}_0 \in \widetilde{K}$ denote any cell of $\widetilde{M}$ satisfying $\pi(\tilde{\sigma}_0) = \sigma_0$. Then there is a unique $\widetilde{V}$-path beginning at $\tilde{\sigma}_0$,

$$\tilde{\gamma} : \tilde{\sigma}_0^{(p)},\ \tilde{\tau}_0^{(p+1)}, \ldots, \tilde{\sigma}_r^{(p)}$$

covering γ, i.e., such that for each i

$$\pi(\tilde{\sigma}_i) = \sigma_i, \quad \pi(\tilde{\tau}_i) = \tau_i.$$

However, $\tilde{\gamma}$ need not be closed. It is certain only that

$$\pi(\tilde{\sigma}_0) = \pi(\tilde{\sigma}_r) = \sigma_0$$

so that there is a $g_\gamma \in \Gamma$ with

$$g_\gamma(\tilde{\sigma}_0) = \tilde{\sigma}_r.$$

We can now try to define a map from $\mathcal{P}$, the closed V-paths, to $U(m)$, which we also denote by θ, by setting

$$\theta(\gamma) = \theta(g_\gamma).$$

Unfortunately, this map is not well-defined, since our definition of g_γ required an initial choice of $\tilde{\sigma}_0$. Varying our choice of $\tilde{\sigma}_0$, say by replacing it by $h(\tilde{\sigma}_0)$ for some $h \in \Gamma$, has the effect of conjugating $\theta(\gamma)$ by $\theta(h)$. Therefore, although $\theta(\gamma)$ is not well-defined trace $(\theta(\gamma))$ is.

Writing tr for trace, we now define the twisted zeta function

$$\zeta(z,\theta) = \sum_{r=1}^{\infty} \frac{z^r}{r} \left[\sum_{k=0}^{n} (-1)^k \sum_{\gamma \in \mathcal{P}_r^{(k)}} m(\gamma) \operatorname{tr} \theta(\gamma) \right].$$

Note that if we let θ_0 denote the trivial representation

$$\theta_0 : \Gamma \longrightarrow U(1)$$

which maps every element of Γ to the number 1, then

$$\zeta(z,\theta_0) = \zeta_{m,i}(z).$$

In fact, we will derive an analytic continuation of $\zeta(z,\theta)$ by simply "twisting" the analytic continuation for $\zeta_{m,i}(z)$.

Let $C_p(\widetilde{M}, \mathbf{C}^m)$ denote the $\mathbf{C}^m$-valued p-chains on $\widetilde{M}$, i.e., objects of the form $\sum_{\tilde{\sigma} \in \tilde{K}_p} c_{\tilde{\sigma}} \tilde{\sigma}$, where $c_{\tilde{\sigma}} \in \mathbf{C}^m$. Each $g \in \Gamma$ induces a map

$$g_* : C_p(\widetilde{M}, \mathbf{C}^m) \longrightarrow C_p(\widetilde{M}, \mathbf{C}^m)$$

by

$$g_* \left(\sum_{\tilde{\sigma} \in \tilde{K}_p} c_{\tilde{\sigma}} \tilde{\sigma} \right) = \sum_{\tilde{\sigma} \in \tilde{K}_p} c_{\tilde{\sigma}} g(\tilde{\sigma}) = \sum_{\tilde{\sigma} \in \tilde{K}_p} c_{g^{-1}(\tilde{\sigma})} \tilde{\sigma}.$$

Any matrix $v \in U(m)$ also acts on $C_p(\widetilde{M}, \mathbf{C}^m)$, by

$$v\left(\sum_{\tilde{\sigma}\in\tilde{K}_p} c_{\tilde{\sigma}}\tilde{\sigma}\right) = \sum_{\tilde{\sigma}\in\tilde{K}_p} v(c_{\tilde{\sigma}})\tilde{\sigma}.$$

Denote by $C_p(M,\theta)$ the elements of $C_p(\widetilde{M}, \mathbf{C}^m)$ which transform under Γ via θ. That is

$$\begin{aligned} C_p(M,\theta) &= \{c \in C_p(\widetilde{M}, \mathbf{C}^m) \mid \forall\, g \in \Gamma\ g_*(c) = [\theta(g)](c)\} \\ &= \left\{\sum_{\tilde{\sigma}\in\tilde{K}_p} c_{\tilde{\sigma}}\tilde{\sigma} \mid \forall\, g \in \Gamma,\ \forall\, \tilde{\sigma} \in \tilde{K}_p,\ c_{g^{-1}(\tilde{\sigma})} = [\theta(g)](c_{\tilde{\sigma}})\right\}. \end{aligned}$$

The boundary operator

$$\partial : C_k(\widetilde{M}, \mathbf{C}^m) \longrightarrow C_{k-1}(\widetilde{M}, \mathbf{C}^m)$$

extends in a natural way to a map

$$\partial : C_k(\widetilde{M}, \mathbf{C}^m) \longrightarrow C_{k-1}(\widetilde{M}, \mathbf{C}^m)$$

which commutes with the action of Γ and hence preserves $C_*(M,\theta)$. Therefore, there is an induced boundary operator

$$\partial_\theta : C_k(M,\theta) \longrightarrow C_{k-1}(M,\theta).$$

Similarly, the lift $\tilde{V}$ to $\widetilde{M}$ of the combinatorial vector field V is invariant under Γ, i.e., $\forall \tilde{\sigma} \in \tilde{k}$ and $g \in \Gamma$

$$\tilde{V}(g(\tilde{\sigma})) = g(\tilde{V}(\tilde{\sigma})).$$

Modifying $\tilde{V}$ to act on oriented cells of $\widetilde{M}$, as described in §4, yields a map

$$\tilde{V} : C_k(\widetilde{M}, \mathbf{C}^m) \longrightarrow C_{k-1}(\widetilde{M}, \mathbf{C}^m)$$

which is invariant under the action of Γ, and hence induces a map

$$V_\theta : C_k(M,\theta) \longrightarrow C_{k-1}(M,\theta).$$

This allows us to define a flow

$$\Phi_\theta : C_*(M,\theta) \longrightarrow C_*(M,\theta)$$

by setting

$$\Phi_\theta = 1 + V_\theta\partial_\theta + \partial_\theta V_\theta.$$

Denote by $\Phi_{\theta(k)}$ the restriction of Φ_θ to $C_k(M,\theta)$. Following the proof of Theorem 4.10 in this twisted context yields

Theorem 5.11.

$$\operatorname{tr}(\Phi_{\theta(k)})^r = kc_k + \sum_{\gamma\in\mathcal{P}_r^{(p)}} m(\gamma)\operatorname{tr}\theta(\gamma) + \sum_{\gamma\in\mathcal{P}_r^{(p-1)}} m(\gamma)\operatorname{tr}\theta(\gamma).$$

Thus, generalizing Theorem 5.9 we learn

Theorem 5.12. *Where $\zeta(z,\theta)$ converges*

$$\zeta(z,\theta) = (1-z)^{\sum_{k=0}^{n}(-1)^{k+1}kc_k} \prod_{k=0}^{n} \left[\det(I - z\Phi_{\theta(k)})\right]^{(k+1)(-1)^k}$$

so that the right hand side provides an analytic continuation of $\zeta(z,\theta)$ to a meromorphic function on the entire complex plane.

To express $\zeta(z,\theta)$ in terms of closed orbits, rather than closed paths, we observe that if γ and $\tilde{\gamma}$ are closed paths which represent the same closed orbit, then $\theta(\gamma)$ and $\theta(\tilde{\gamma})$ are conjugate, so that

$$\operatorname{tr}\theta(\gamma) = \operatorname{tr}\theta(\tilde{\gamma}).$$

Thus, if $[\gamma] \in \mathcal{O}$ is any closed orbit, we can define $\operatorname{tr}\theta([\gamma])$ to be the common value of $\operatorname{tr}\theta(\gamma)$ for the paths γ which represent $[\gamma]$, i.e.,

$$\operatorname{tr}\theta([\gamma]) = \operatorname{tr}\theta(\gamma).$$

Precisely as in Theorem 5.10 we have

Theorem 5.13. *Let $Z(s,\theta) = \zeta(e^{-s},\theta)$. Then, where the series converge*

$$Z(s,\theta) = \prod_{\gamma\in\mathcal{O}_\bullet} \left[\det\left(1 - m(\gamma)\theta([\gamma])e^{-s|[\gamma]|}\right)\right]^{(-1)^{i([\gamma])}}.$$

§6 Reidemeister Torsion.

In this section we examine the Reidemeister torsion of a CW complex, and its relation to combinatorial flows. The study of this relationship finds its origins in [Fri1, 2] in which large classes of smooth flows on smooth manifolds were found in which the Reidemeister torsion of the underlying manifold could be expressed in terms of the closed orbits of the flow. In this section we show that this relationship holds for any combinatorial flow or any finite CW complex.

We begin by reviewing the notion of Reidemeister torsion. Let

$$\theta : \Gamma \to U(m)$$

be a unitary representation of the fundamental group Γ of M, and consider the twisted chain complex defined in §5

$$C_*(M,\theta) : C_n(M,\theta) \xrightarrow{\partial} C_{n-1}(M,\theta) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(M,\theta) \longrightarrow 0.$$

As is the case with the usual chain complex, $\partial^2 = 0$, so it is natural to define the homology

$$H_k(M,\theta) \equiv \frac{\text{Kernel}(\partial : C_k(M,\theta) \longrightarrow C_{k-1}(M,\theta))}{\text{Image}(\partial : C_{k+1}(M,\theta) \longrightarrow C_k(M,\theta))}.$$

We will assume, for simplicity, that θ is chosen such that the homology vanishes, i.e.,

$$H_k(M,\theta) = 0 \qquad \forall\, k.$$

This requires, in particular, that

$$\sum_{k=0}^{n} (-1)^k \dim H_k(M,\theta) = 0.$$

From standard linear algebra, this alternating sum is equal to

$$\sum_{k=0}^{n} (-1)^k \dim C_k(M,\theta) = \sum_{k=0}^{n} (-1)^k m(\#\text{ of cells of dimension } k).$$

Therefore, a necessary condition for such a θ to exist is

$$0 = \sum_{k=0}^{n} (-1)^k (\#\text{ of cells of dimension } k) \equiv \chi(M).$$

There is a natural inner product on each $C_k(M,\theta)$. To define this inner product, choose a lift $\sigma^* \in \tilde{K}$ for each cell $\sigma \in K$. For chains a, $b \in C_k(M,\theta)$ of the form

$$a = \sum_{\tilde{\sigma}^{(k)} \in \tilde{K}} \alpha_{\tilde{\sigma}} \tilde{\sigma}, \quad b = \sum_{\tilde{\sigma}^{(k)} \in \tilde{K}} \beta_{\tilde{\sigma}} \tilde{\sigma}$$

define

$$< a,b > = \sum_{\tilde{\sigma}^{(k)}\in K} (\alpha_{\sigma^*}, \beta_{\sigma^*})$$

where (,) is the standard inner product on $\mathbf{C}^m$. It is easy to check that, since the representation θ is unitary, this inner product is independent of the chosen lifts σ^*.

Let

$$\partial^* : C_p(M,\theta) \longrightarrow C_{p+1}(M,\theta)$$

denote the adjoint of ∂ with respect to the inner product, and define the Laplacian

$$\Delta_\theta^{(p)} : \partial\partial^* + \partial^*\partial : C_p(M,\theta) \longrightarrow C_p(M,\theta).$$

It follows from standard linear algebra that for each p

$$\text{Kernel}(\Delta_\theta^{(p)}) \cong H_p(M,\theta)$$

so that, by assumption, each $\Delta_\theta^{(p)}$, $p = 0, 1, \ldots, n$, is invertible. We define the **Reidemeister** torsion of M with respect to the representation θ, $T(M,\theta)$, by the formula

$$T(M,\theta) \;=\; \prod_{p=0}^{n} (\text{Det}\,\Delta_\theta^p)^{(-1)^p \frac{p+1}{2}}.$$

This formula first appeared in [R-S] (see also [Fo3]). This unusual combination of determinants has the remarkable property of being a combinatorial invariant, that is, it is invariant under finite subdivisions of the cell structure ([Fr]).

The proofs in §2 can be generalized to accomodate the representation θ, so that the Morse inequalities, suitably interpreted, are still true. More precisely, define

$$m_p(\theta) \;=\; \sum_{\substack{\text{basic} \\ \text{sets } \Lambda_i}} \dim H_p(\overline{\Lambda}_i, \dot{\Lambda}_i, \theta)$$

where the relative homology $H_*(\overline{\Lambda}_i, \dot{\Lambda}_i, \theta)$ is defined in the natural way. Then for any θ (i.e., θ need not be acyclic) and any k

$$m_k(\theta) - m_{k-1}(\theta) + \cdots \pm m_0(\theta) \;\geq\; b_k(\theta) - b_{k-1}(\theta) + \cdots \pm b_0(\theta)$$

where

$$b_j(\theta) \;=\; \dim H_j(M,\theta).$$

We have already assumed that $b_k(\theta) = 0$ for all k. We now make the stronger assumption that

$$m_p(\theta) \;=\; 0 \qquad \forall p.$$

This requires, in particular, that V have no rest points. The reason for this assumption will become clear shortly (see Theorem 6.3).

We are now ready to state the main theorem of this section

Theorem 6.1. *Suppose V has no rest points and θ is chosen so that for each basic set Λ_i*

$$H_*(\overline{\Lambda}_i, \dot{\Lambda}_i, \theta) = 0$$

(this is equivalent to $m_p(\theta) = 0$ for all p). Then $\zeta(z,\theta)$ is analytic at $z = 1$ and

$$T(M,\theta) = \zeta(1,\theta).$$

Equivalently, $Z(s,\theta)$ is analytic at $s = 0$ and

$$T(M,\theta) = Z(0,\theta).$$

The remainder of this section is devoted to proving Theorem 6.1. The proof makes use of Witten's deformation of the chain complex $C_*(M,\theta)$ ([Wi], see also [Fo3] for the use of this technique in the combinatorial setting).

Let f be a Lyapunov function for V (see Theorem 2.4). Lift f to a function

$$\tilde{f} : \widetilde{K} \longrightarrow \mathbf{R}$$

by setting

$$\tilde{f}(\tilde{\sigma}) = f(\pi(\tilde{\sigma})) \qquad \forall\, \tilde{\sigma} \in \widetilde{K}.$$

Define a one parameter family of automorphisms

$$e^{t\tilde{f}} : C_p(M, \mathbf{C}^m) \longrightarrow C_p(M, \mathbf{C}^m)$$

as follows. If

$$a = \sum_{\tilde{\sigma} \in \widetilde{K}} \alpha_{\tilde{\sigma}} \tilde{\sigma} \in C_p(M, \mathbf{C}^m)$$

then

$$e^{t\tilde{f}}(a) = \sum_{\tilde{\sigma} \in \widetilde{K}} \alpha_{\tilde{\sigma}} e^{t\tilde{f}(\tilde{\sigma})} \tilde{\sigma}.$$

Define $e^{-t\tilde{f}}$ similarly. Since the maps $e^{\pm t\tilde{f}}$ commute with the action of Γ they preserve the spaces $C_k(M,\theta)$. Thus we can consider the one-parameter family of chain complexes

$$C_*(M,\theta,t) : C_n(M,\theta) \xrightarrow{\partial_t} C_{n-1}(M,\theta) \xrightarrow{\partial_t} \cdots \xrightarrow{\partial_t} C_0(M,\theta) \longrightarrow 0$$

where

$$\partial_t = e^{t\tilde{f}} \partial e^{-t\tilde{f}}.$$

Let us now work more explicitly. Let $e_1, \ldots, e_m$ denote the standard basis for $\mathbf{C}^m$. For any $\tilde{\sigma}^{(p)} \in \widetilde{K}$, and any $i \in \{1, 2, \ldots, m\}$ let $c_{\tilde{\sigma},i}$ denote the p-chain

$$c_{\tilde{\sigma},i} = \sum_{g \in \Gamma} [\theta(g)](e_i) g^{-1}(\tilde{\sigma}).$$

Then $c_{\tilde\sigma,i} \in C_p(M,\theta)$. For each cell $\sigma^{(p)} \in K_p$, choose a lift $\sigma^* \in \widetilde{K}$. Then

$$\{c_{\sigma^*,i}\}_{\substack{\sigma\in K_p \\ i=1,2,\dots,m}}$$

forms an orthonormal basis of $C_p(M,\theta)$. The boundary operator ∂ acts by

$$\begin{aligned}
\partial c_{\tilde\sigma,i} &= \sum_{g\in\Gamma}[\theta(g)](e_i)\partial g^{-1}(\tilde\sigma) \\
&= \sum_{g\in\Gamma}[\theta(g)](e_i)g^{-1}(\partial\tilde\sigma) \\
&= \sum_{g\in\Gamma}[\theta(g)](e_i)g^{-1}\left(\sum_{\tilde v^{(p-1)}\in\widetilde{K}}\langle\partial\tilde\sigma,\tilde v\rangle\,\tilde v\right) \\
&= \sum_{\tilde v^{(p-1)}\in\widetilde{K}}\langle\partial\tilde\sigma,\tilde v\rangle\sum_{g\in\Gamma}[\theta(g)](e_i)g^{-1}(\tilde v) \\
&= \sum_{\tilde v^{(p-1)}\in\widetilde{K}}\langle\partial\tilde\sigma,\tilde v\rangle\, c_{\tilde v,i}.
\end{aligned}$$

The operator ∂_t acts by

$$\partial_t c_{\tilde\sigma,i} = e^{t\tilde f}\partial e^{-t\tilde f}c_{\tilde\sigma,i} = \sum_{\tilde v^{(p-1)}\in\widetilde{K}}\langle\partial\tilde\sigma,\tilde v\rangle\, e^{t(\tilde f(\tilde v)-\tilde f(\tilde\sigma))}c_{\tilde v,i}. \tag{6.1}$$

Similarly, ∂^* and ∂_t^* (the adjoints of ∂ and ∂_t with respect to the inner product) act by

$$\begin{aligned}
\partial^* c_{\tilde\sigma,i} &= \sum_{\tilde\tau^{(p+1)}\in\widetilde{K}}\langle\partial\tilde\tau,\tilde\sigma\rangle\, c_{\tilde\tau,i} \\
\partial_t^* c_{\tilde\sigma,i} &= \sum_{\tilde\tau^{(p+1)}\in\widetilde{K}}\langle\partial\tilde\tau,\tilde\sigma\rangle\, e^{t(\tilde f(\tilde\sigma)-\tilde f(\tilde\tau))}c_{\tilde\tau,i}
\end{aligned}$$

so that

$$\begin{aligned}
\Delta_\theta^{(p)}(t)c_{\tilde\sigma,i} &\equiv (\partial_t\partial_t^* + \partial_t^*\partial_t)c_{\sigma,i} \\
&= \sum_{\tilde\sigma^{(p)}\in\widetilde{K}}\Bigg[\sum_{\tilde\tau^{(p+1)}\in\widetilde{K}}\langle\partial\tilde\tau,\tilde\sigma\rangle\,\langle\partial\tilde\tau,\tilde\sigma\rangle\, e^{t(\tilde f(\tilde\sigma)+\tilde f(\tilde\sigma)-2\tilde f(\tilde\tau))} \\
&\quad + \sum_{\tilde v^{(p-1)}\in\widetilde{K}}\langle\partial\tilde\sigma,\tilde v\rangle\,\langle\partial\tilde\sigma,\tilde v\rangle\, e^{t(2\tilde f(\tilde v)-\tilde f(\tilde\sigma)-\tilde f(\tilde\sigma))}\Bigg]\tilde\sigma.
\end{aligned}$$

For all $t \in \mathbf{R}$, the homology of the complex $C(M,\theta,t)$ is equal to the homology of the original complex $C(M,\theta)$, and hence is trivial by assumption. This implies that for each $t \in \mathbf{R}$

$$\Delta_\theta^p(t) : C_p(M,\theta) \longrightarrow C_p(M,\theta)$$

is invertible, and hence has non-zero determinant.

We define the torsion of the complex $C(M,\theta,t)$, denoted by $T(M,\theta,t)$ by

$$T(M,\theta,t) \;=\; \prod_{p=0}^{n} (\operatorname{Det} \Delta_\theta^p(t))^{(-1)^p \frac{p-1}{2}}.$$

Of crucial importance is the following lemma.

Lemma 6.2. *For all $t \in \mathbf{R}$*

$$T(M,\theta) \;=\; T(M,\theta,t).$$

Proof. Since $T(M,\theta) = T(M,\theta,0)$, it is sufficient to prove

$$\frac{d}{dt}T(M,\theta,t) \;=\; 0. \tag{6.2}$$

From Lemma 6.2 of [Fo3]

$$\frac{d}{dt}T(M,\theta,t) \;=\; \sum_{p=0}^{n}(-1)^p \operatorname{tr}\left[\left(\frac{d}{dt}e^{t\bar{f}}\right)e^{-t\bar{f}} : C_p(M,\theta) \longrightarrow C_p(M,\theta)\right].$$

Since

$$\left(\frac{d}{dt}e^{t\bar{f}}\right)e^{-t\bar{f}} \;=\; \bar{f}$$

maps each $c_{\sigma,i}$ to $\bar{f}(\sigma)c_{\sigma,i}$

$$\operatorname{tr}\left[\left(\frac{d}{dt}e^{t\bar{f}}\right)e^{-t\bar{f}} : C_p(M,\theta) \longrightarrow C_p(M,\theta)\right] \;=\; m\sum_{\sigma\in K_p} f(\sigma)$$

so that

$$\frac{d}{dt}T(M,\theta,t) \;=\; m\sum_{p=0}^{n}(-1)^p \sum_{\sigma\in K_p} f(\sigma).$$

We have assumed that V has no rest points. Therefore

$$K \;=\; \operatorname{Image}(V) \cup \{\sigma \in K \mid V(\sigma) \neq 0\}.$$

Moreover, if $V(\sigma) \neq 0$ then

$$f(\sigma) \;=\; f(V(\sigma)).$$

Thus

$$\begin{aligned}
\frac{d}{dt}T(M,\theta,t) &= m\sum_{p=0}^{n}(-1)^p\left[\sum_{V(\sigma^{(p)})\neq 0} f(\sigma) + \sum_{\substack{\sigma^{(p)}\in\\ \mathrm{Image}(V)}} f(\sigma)\right] \\
&= m\sum_{p=0}^{n}(-1)^p\left[\sum_{V(\sigma^{(p)})\neq 0} f(\sigma) - \sum_{\substack{\sigma^{(p+1)}\in\\ \mathrm{Image}(V)}} f(\sigma)\right] \\
&= m\sum_{p=0}^{n}(-1)^p\left[\sum_{V(\sigma^{(p)})\neq 0} f(\sigma) - \sum_{V(\sigma^{(p)})\neq 0} f(V(\sigma))\right] \\
&= m\sum_{p=0}^{n}(-1)^p\left[\sum_{V(\sigma^{(p)})\neq 0} f(\sigma) - f(V(\sigma))\right] \\
&= 0.
\end{aligned}$$

□

The main idea of the proof of Theorem 6.1 is to use Lemma 6.2 and to take the limit of $T(M,\theta,t)$ as $t \to \infty$. Consider the formula for the action of ∂_t given in (6.1). Since $\langle\partial\tilde{\sigma},\tilde{v}\rangle = 0$ unless $\tilde{v} < \tilde{\sigma}$ we can write

$$\partial_t c_{\tilde{\sigma},i} = \sum_{\tilde{v}<\tilde{\sigma}} \langle\partial\tilde{\sigma},\tilde{v}\rangle\, e^{t(\bar{f}(\tilde{v})-\bar{f}(\tilde{\sigma}))} c_{\tilde{v},i}.$$

We now observe that

$$\tilde{v} < \tilde{\sigma} \Rightarrow \pi(\tilde{v}) < \pi(\tilde{\sigma})$$

which implies

$$\bar{f}(\tilde{v}) - \bar{f}(\tilde{\sigma}) = f(\pi(\tilde{v})) - f(\pi(\tilde{\sigma})) \leq 0.$$

Thus, as $t \to \infty$

$$e^{t(\bar{f}(\tilde{v})-\bar{f}(\tilde{\sigma}))} \longrightarrow 0 \text{ or } 1.$$

In particular,

$$\partial_\infty \equiv \lim_{t\to\infty}\partial_t$$

exists. To simplify future formulas, we introduce some notation. If $v^{(p-1)}$, $\sigma^{(p)} \in K$, we will write

$$v \approx \sigma \quad (\text{or } \sigma \approx v)$$

if

$$v < \sigma \quad \text{and} \quad f(v) = f(\sigma).$$

Similarly, if $\tilde{v}^{(p-1)}, \tilde{\sigma}^{(p-1)} \in \widetilde{K}$, we will write

$$\tilde{v} \approx \tilde{\sigma}$$

if

$$\tilde{v} < \tilde{\sigma} \quad \text{and} \quad f(\tilde{v}) = f(\tilde{\sigma}).$$

Note that

$$\tilde{v} \approx \tilde{\sigma} \Rightarrow \pi(\tilde{v}) \approx \pi(\tilde{\sigma}).$$

With this notation,

(6.3)
$$\begin{aligned} \partial_\infty c_{\tilde{\sigma},i} &= \sum_{\tilde{v}^{(p-1)} \approx \sigma} \langle \partial\tilde{\sigma}, \tilde{v} \rangle \, c_{\tilde{v},i} \\ \partial^*_\infty c_{\tilde{\sigma},i} &= \sum_{\tilde{\tau}^{(p-1)} \approx \tilde{\sigma}} \langle \partial\tilde{\tau}, \tilde{\sigma} \rangle \, c_{\tilde{\tau},i}. \end{aligned}$$

Let

$$\Delta^{(p)}_\infty = \partial_\infty \partial^*_\infty + \partial^*_\infty \partial_\infty : C_p(M,\theta) \longrightarrow C_p(M,\theta).$$

The upshot is that if each Δ^p_∞ is invertible, then we can simply let $t \to \infty$ in (6.2) and

(6.4)
$$T(M,\theta) = \prod_{p=0}^{n} \left(\Delta^{(p)}_\infty \operatorname{Det} \right)^{(-1)^p \frac{p+1}{2}}.$$

With this in mind, we now take a closer look at the operators ∂_∞, ∂^*_∞, Δ_∞.

Lemma 6.3. *If $\sigma \notin \mathcal{R}$ then for any lift σ^* of σ, and any $i \in \{1, 2, \ldots, n\}$*

$$\Delta_\infty c_{\sigma^*,i} = c_{\sigma^*,i}.$$

Proof. Either $V(\sigma) \neq 0$ or $\sigma \in \text{Image}(V)$. Suppose $V(\sigma) \neq 0$ and let $p =$ dimension (σ). Since $\sigma \notin \text{Image}(V)$, for all $v^{(p-1)} < \sigma$, $f(v) < f(\sigma)$. Therefore there is no $v^{(p-1)}$ with $v \approx \sigma$. This implies there is no $\tilde{v}^{(p-1)}$ with $\tilde{v} \approx \sigma^*$. Hence

$$\partial_\infty c_{\sigma^*,i} = 0.$$

If $V(\sigma^*) = \tilde{\tau}^{(p+1)}$ then $\sigma^* \approx \tilde{\tau}$ and there is no other $(p+1)$-cell $\tilde{\tau}$ with $\tilde{\tau} \approx \sigma^*$. Hence,

$$\partial^*_\infty c_{\sigma^*,i} = \langle \partial\tilde{\tau}, \sigma^* \rangle \, c_{\tilde{\tau},i}$$

so that

$$\Delta_\infty c_{\sigma^*,i} = \langle \partial\tilde{\tau}, \sigma^* \rangle \, \partial_\infty c_{\tilde{\tau},i}.$$

Similarly, $\sigma^* \approx \tilde{\tau}$ and there is no other such p-face of $\tilde{\tau}$, so

$$\partial_\infty c_{\tilde{\tau},i} = \langle \partial\tilde{\tau}, \sigma^* \rangle \, c_{\sigma^*,i}$$

and

$$\Delta_\infty c_{\sigma^*,i} = \langle \partial\tilde{\tau}, \sigma^* \rangle^2 c_{\sigma,i} = c_{\sigma,i}$$

($\langle \partial\tilde{\tau}, \sigma^* \rangle = \pm 1$ since σ^* is a regular face of $\tilde{\tau}$).

On the other hand, if $\sigma \in \text{Image}(V)$, say $\sigma = V(v^{(p-1)})$, then $\sigma^* = V(\tilde{v})$ for some $\tilde{v} \in \tilde{k}$ with $\pi(\tilde{v}) = v$. Similar arguments show that

$$\begin{aligned} \partial_\infty c_{\sigma^*,i} &= \langle \partial\sigma^*, \tilde{v} \rangle c_{\tilde{v},i}, \quad \partial^*_\infty c_{\sigma^*,i} = 0 \\ \partial^*_\infty c_{\tilde{v},i} &= \langle \partial\sigma^*, \tilde{v} \rangle c_{\sigma^*,i} \end{aligned}$$

so that

$$\Delta_\infty c_{\sigma^*,i} = \langle \partial\sigma^*, \tilde{v} \rangle^2 c_{\sigma^*,i} = c_{\sigma^*,i}. \qquad \square$$

It follows from Lemma 6.2 that

$$\text{Det}\,\Delta^{(p)}_\infty = \prod_{\substack{\text{basic} \\ \text{sets } \Lambda_i}} \text{Det}\,\Delta^{(p)}_\infty \Big|_{C_p(\Lambda_i,\theta)} . \tag{6.5}$$

Suppose $\sigma^{(p)} \in \Lambda_i$, and $v^{(p-1)} < \sigma$. Then $f(v) \leq f(\sigma)$ and

$$f(v) = f(\sigma) \longleftrightarrow v \approx \sigma \longleftrightarrow v \in \Lambda_i.$$

Therefore, if $\tilde{v}^{(p-1)} < \sigma^*$ then $\tilde{f}(\tilde{v}) \leq \tilde{f}(\sigma^*)$ and

$$\tilde{f}(\tilde{v}) = \tilde{f}(\sigma^*) \longleftrightarrow \tilde{v} \approx \sigma^* \longleftrightarrow \pi(\tilde{v}) \in \Lambda_i.$$

Hence, we can rewrite (6.3) as

$$\partial_\infty c_{\sigma,i} = \sum_{\pi(\tilde{v}^{(p-1)}) \in \Lambda_i} \langle \partial\sigma^*, \tilde{v} \rangle c_{\tilde{v},i}.$$

This differential also appears from another point of view. For each p there is a canonical isomorphism

$$C_p(\overline{\Lambda}_i, \dot{\Lambda}_i, \theta) \cong C_p(\Lambda_i, \theta). \tag{6.6}$$

Namely, map $\sum_{\pi(\tilde{\sigma}) \in \overline{\Lambda}_i} c_{\tilde{\sigma}} \tilde{\sigma}$ to $\sum_{\pi(\tilde{\sigma}) \in \Lambda_i} c_{\tilde{\sigma}} \tilde{\sigma}$, i.e., simply ignore the terms corresponding to cells not in Λ_i. Via this isomorphism, the differential

$$\partial : C_p(\overline{\Lambda}_i, \dot{\Lambda}_i, \theta) \longrightarrow C_{p-1}(\overline{\Lambda}_i, \dot{\Lambda}_i, \theta)$$

induces a differential on $C_p(\Lambda_i, \theta)$ which is precisely ∂_∞. From this discussion, and Lemma 6.2, we learn

Theorem 6.3. *The following statements are equivalent*

(i) *For each basic set* Λ_i

$$H_*(\overline{\Lambda}_i, \dot{\Lambda}_i, \theta) = 0.$$

(ii) *For each basic set* Λ_i *the complex*

$$C(\Lambda_i, \theta, \infty) : C_n(\Lambda_i, \theta) \xrightarrow{\partial_\infty} C_{n-1}(\Lambda_i, \theta) \xrightarrow{\partial_\infty} \cdots \xrightarrow{\partial_\infty} C_0(\Lambda_i, \theta) \longrightarrow 0$$

is exact.

(iii) *The complex*

$$C(M, \theta, \infty) : 0 \longrightarrow C_n(M, \theta) \xrightarrow{\partial_\infty} C_{n-1}(M, \theta) \xrightarrow{\partial_\infty} \cdots \xrightarrow{\partial_\infty} C_0(M, \theta) \longrightarrow 0$$

is exact.

If any one (and hence all) of these conditions hold then we have

$$\begin{aligned} T(M, \theta) &= \prod_{p=0}^{n} \left(\mathrm{Det}\, \Delta_\infty^{(p)}\right)^{(-1)^p \frac{p+1}{2}} \\ &= \prod_{\substack{\text{basic} \\ \text{sets } \Lambda_i}} \left[\prod_{p=0}^{\infty} \left(\mathrm{Det}\, \Delta_\infty^{(p)} \Big|_{C_p(\Lambda_i, \theta)} \right)^{(-1)^p \frac{p+1}{2}} \right] \\ &= \prod_{\substack{\text{basic} \\ \text{sets } \Lambda_i}} T(\Lambda_i, \theta) \end{aligned}$$

where $T(\Lambda_i, \theta)$ *refers to the torsion of the complex* $C(\Lambda_i, \theta, \infty)$, *which can be thought of as the torsion of* X_i *with respect to the representation of the fundamental group of* Λ *induced by* θ *and the imbedding* $\Lambda_i \hookrightarrow M$.

We continue to assume that the conditions stated in Theorem 6.3 hold, and we now take a closer look at

$$\mathrm{Det}\, \Delta_\infty^{(p)} \Big|_{C_p(\Lambda_i, \theta)}$$

for some basic set Λ_i. If Λ_i has index q, then $C_p(\Lambda_i, \theta)$ is non-zero only for $q = p$ or $p - 1$, so we must consider

$$\begin{aligned} \Delta_\infty^{(p)} &: C_p(\Lambda_i^{(p)}, \theta) \longrightarrow C_p(\Lambda_i^{(p)}, \theta) \\ \Delta_\infty^{(p+1)} &: C_{p+1}(\Lambda_i^{(p)}, \theta) \longrightarrow C_{p+1}(\Lambda_i^{(p)}, \theta). \end{aligned}$$

Lemma 6.4.

(i) $$\operatorname{Det} \Delta_\infty^{(p)} \Big|_{C_p(\Lambda_i^{(p)},\theta)} = \left| \operatorname{Det} R_i \partial V \Big|_{C_p(\Lambda_i^{(p)},\theta)} \right|^2$$

(ii) $$\operatorname{Det} \Delta_\infty^{(p+1)} \Big|_{C_{p+1}(\Lambda_i^{(p)},\theta)} = \left| \operatorname{Det} R_i V \partial \Big|_{C_{p+1}(\Lambda_i^{(p)},\theta)} \right|^2$$

where

$$R_i : C_*(M,\theta) \longrightarrow C_*(\Lambda_i,\theta)$$

is the natural projection.

Proof. If $\sigma^{(p)} \in \Lambda_i^{(p)}$, then there is no $\nu^{(p-1)}$ with $\nu \approx \sigma$ so for any i

$$\partial_\infty c_{\sigma^*,i} = 0$$

and

$$\Delta_\infty c_{\sigma^*,i} = \partial_\infty \partial_\infty^* c_{\sigma^*,i} = \sum_{\tilde\tau^{(p+1)} \approx \sigma^*} \langle \partial\tilde\tau, \sigma^* \rangle \sum_{\tilde\sigma^{(p)} \approx \tilde\tau} \langle \partial\tilde\tau, \tilde\sigma \rangle \, c_{\tilde\sigma,i}.$$

We note that $\tilde\tau^{(p+1)} \approx \sigma^*$, if and only if $\tilde\tau > \sigma^*$ and $\tilde\tau = \pm V(\tilde\sigma)$ for some $\tilde\sigma \in \widetilde{K}$ with $\pi(\tilde\sigma) = \Lambda_i$ (such a $\tilde\sigma$ is necessarily unique).

On the other hand

$$V^* \partial^* c_{\sigma^*,i} = \sum_{\tilde\tau > \sigma^*} \langle \partial\tilde\tau, \sigma^* \rangle \, V^*(c_{\tilde\tau,i})$$

where $V(\tilde\sigma) = \pm\tilde\tau$, so that $V^*(\tilde\tau) = -\langle \partial\tilde\tau, \tilde\sigma \rangle \, \tilde\sigma$. Hence

$$R_i V^* \partial^* c_{\sigma^*,i} = -\sum_{\tilde\tau \approx \tilde\sigma} \langle \partial\tilde\tau, \sigma^* \rangle \, V^*(c_{\tilde\tau,i}).$$

For $\tilde\tau \in \operatorname{Image}(V)$

$$\tilde\tau = VV^*$$

so for each i

$$VV^*(\sigma_{\tilde\tau,i}) = \sigma_{\tilde\tau,i}.$$

Thus

$$VR_i V^* \partial^* c_{\sigma,i} = \sum_{\tilde\tau \approx \tilde\sigma} \langle \partial\tilde\tau, \sigma^* \rangle \, c_{\tilde\tau,i}$$

and

$$\begin{aligned} \partial V R_i V^* \partial^* c_{\sigma^*,i} &= \sum_{\tilde\tau \approx \sigma^*} \langle \partial\tilde\tau, \sigma^* \rangle \sum_{\tilde\sigma < \tilde\tau} \langle \partial\tilde\tau, \tilde\sigma \rangle \, c_{\tilde\sigma,i} \\ R_i \partial V R_i V^* \partial^* c_{\sigma^*,i} &= \sum_{\tilde\tau \approx \sigma^*} \langle \partial\tilde\tau, \sigma^* \rangle \sum_{\tilde\sigma \approx \tilde\tau} \langle \partial\tilde\tau, \tilde\sigma \rangle \, c_{\sigma^*,i} \\ &= \Delta_\infty c_{\sigma^*,i}. \end{aligned}$$

Therefore,

$$\begin{aligned}
\operatorname{Det} \Delta_\infty \Big|_{C_p(\Lambda_i,\theta)} &= \operatorname{Det} R_i \partial V R_i V^* \partial^* R_i \Big|_{C_p(\Lambda_i,\theta)} \\
&= \operatorname{Det} (R_i \partial V R_i)(R_i \partial V R_i)^* \Big|_{C_p(\Lambda_i,\theta)} \\
&= \left| \operatorname{Det} R_i \partial V R_i \Big|_{C_p(\Lambda_i,\theta)} \right|^2 .
\end{aligned}$$

Since, restricted to $C_p(\Lambda_i, \theta)$, R_i is the identity

$$\left| \operatorname{Det} R_i \partial V R_i \Big|_{C_p(\Lambda_i,\theta)} \right| = \left| \operatorname{Det} R_i \partial V \Big|_{C_p(\Lambda_i,\theta)} \right|$$

which completes the proof of (i).

The identity (ii) is proved by a similar argument. □

Before completing the proof of Theorem 6.1 we observe that if $\sigma^{(p)} \in \Lambda_i^{(p)}$ then

$$V\partial\sigma^{(p)} = \sum_{f(\sigma') < f(\sigma)} \alpha_{\sigma'}\sigma'.$$

Thus

$$V\partial c_{\sigma^*,i} = \sum_{\bar{f}(\bar{\sigma}) < \bar{f}(\sigma^*)} \alpha_{\bar{\sigma}} c_{\bar{\sigma},i}$$

so that

$$R_i V \partial c_{\sigma^*,i} = 0.$$

Hence, restricted to $C_p(\Lambda_i^{(p)}, \theta)$

$$R_i \partial V = R_i(\partial V + V\partial) = -R_i(1 - \Phi).$$

Similarly, restricted to $C_{p+1}(\Lambda_i^{(p)}, \theta)$

$$R_i V \partial = R_i(\partial V + V\partial) = -R_i(1 - \Phi).$$

Therefore, Lemma 6.4 can be restated as

Lemma 6.5.

(i) $$\operatorname{Det} \Delta_\infty^{(p)} \Big|_{C_p(\Lambda_i^{(p)},\theta)} = \left| \operatorname{Det} R_i(1 - \Phi) \Big|_{C_p(\Lambda_i^{(p)},\theta)} \right|^2$$

(ii) $$\operatorname{Det} \Delta_\infty^{(p+1)} \Big|_{C_{p+1}(\Lambda_i^{(p)},\theta)} = \left| \operatorname{Det} R_i(1 - \Phi) \Big|_{C_{p+1}(\Lambda_i^{(p)},\theta)} \right|^2 .$$

The next step is to recombine the maps $R_i(1 - \Phi)$ into a single map.

Lemma 6.6. *For each p*

$$\operatorname{Det} 1 - \Phi \Big|_{C_p(M,\theta)} = \prod_{\substack{\text{basic} \\ \text{sets } \Lambda_i}} \operatorname{Det} R_i(1-\Phi) \Big|_{C_p(\Lambda_i,\theta)} .$$

Proof. Theorem 4.3 implies that for any $\sigma \in \Lambda_i$

$$(1-\Phi)c_{\sigma^*,i} = R_i(1-\Phi)c_{\sigma^*,i} + \sum_{\bar{f}(\tilde{\sigma}) < \bar{f}(\sigma^*)} \alpha_{\tilde{\sigma}} c_{\tilde{\sigma},i}$$

and if $\sigma \notin \mathcal{R}$ then

$$(1-\Phi)c_{\sigma^*,i} = c_{\sigma^*,i} + \sum_{\bar{f}(\tilde{\sigma}) < \bar{f}(\sigma^*)} \alpha_{\tilde{\sigma}} c_{\tilde{\sigma},i}.$$

Thus, ordering the p-cells so that $f(\sigma_1^{(p)}) \leq f(\sigma_2^{(p)}) \leq \cdots$ puts the operator $(1-\Phi)$ in block upper triangular form with diagonal blocks

$$\begin{cases} 1 & \text{if } \sigma \notin \mathcal{R} \\ R_i(1-\Phi) & \text{if } \sigma \in \Lambda_i. \end{cases}$$

Therefore,

$$\operatorname{Det}(1-\Phi) \Big|_{C_p(M,\theta)} = \prod_{\substack{\text{basic} \\ \text{sets } \Lambda_i}} \operatorname{Det} R_i(1-\Phi) \Big|_{C_p(\Lambda_i,\theta)} . \qquad \square$$

Proof of Theorem 6.1. From Theorem 6.3, if $H_*(\bar{\Lambda}_i, \dot{\Lambda}_i, \theta) = 0$ for each basic set Λ_i, then for each p $\Delta_\infty^{(p)} \Big|_{C_p(M,\theta)}$ is invertible, so combining (6.5), and Lemmas 6.5 and 6.6

$$\begin{aligned} 0 \neq \operatorname{Det} \Delta_\infty^{(p)} \Big|_{C_p(M,\theta)} &= \prod_{\substack{\text{basic} \\ \text{sets } \Lambda_i}} \operatorname{Det} \Delta_\infty^{(p)} \Big|_{C_p(\Lambda_i,\theta)} \\ &= \left| \prod_{\substack{\text{basic} \\ \text{sets } \Lambda_i}} \operatorname{Det} R_i(1-\Phi) \Big|_{C_p(\Lambda_i,\theta)} \right|^2 \\ &= \left| \operatorname{Det}(1-\Phi) \Big|_{C_p(M,\theta)} \right|^2 . \end{aligned}$$

Therefore, from (6.4) and Theorem 5.12

$$\begin{aligned} 0 \neq T(M,\theta) &= \prod_{p=0}^{n} \left(\operatorname{Det} \Delta_\infty^{(p)} \right)^{(-1)^p \frac{p+1}{2}} \\ &= \left| \prod_{p=0}^{n} \left(\operatorname{Det}(1-\Phi) \Big|_{C_p(M,\theta)} \right)^{(-1)^p p+1} \right| \\ &= | \zeta(1,\theta) | \end{aligned}$$

as was to be shown. $\square$

§7 Combinatorial Morse-Smale Vector Fields.

In this section we briefly examine the special case of combinatorial Morse-Smale flows, in which the results of the previous section can be made more explicit. Compare this section with Chapter 8 of [Fra].

Definition 7.1. A combinatorial vector field V on a finite CW complex is a *combinatorial Morse-Smale vector field* if the chain recurrent set $\mathcal{R}$ consists only of rest points and pairwise disjoint closed orbits.

Equivalently, V is a Morse-Smale vector field if every pair of distinct prime closed V-paths is disjoint.

Suppose V is a combinatorial Morse-Smale vector field. Let c_p denote the number of rest points of index p, and A_p the number of prime closed V-orbits of index p. The simplest set of Morse inequalities is the following

Theorem 7.1. *For any coefficient field* $\mathbf{F}$, *and any* k

$$A_k + c_k - c_{k-1} + c_{k-2} - \cdots \pm c_0 \;\geq\; b_k(\mathbf{F}) - b_{k-1}(\mathbf{F}) + \cdots \pm b_0(\mathbf{F})$$

where

$$b_p(\mathbf{F}) \;=\; \dim_{\mathbf{F}}(M, \mathbf{F}).$$

Proof. Suppose $[\gamma]$ is a prime closed orbit of index p, represented by

$$\gamma : \sigma^{(p)} \;=\; \sigma_0^{(p)},\ \tau_0^{(p+1)}, \ldots, \sigma_0^{(p)}$$

so that in particular, $V(\sigma) = \tau_0$. We define a new vector field V' on M by setting, for all $v \in K$

$$V'(v) \;=\; \begin{cases} V(v) & \text{if } v \neq \sigma \\ 0 & \text{if } v = \sigma. \end{cases}$$

Then V' has one fewer closed orbit than V, since $[\gamma]$ is no longer a V'-path. Moreover, the rest points of V' are precisely the rest points of V along with $\sigma^{(p)}$ and $\tau_0^{(p+1)}$.

We can continue in this fashion, killing each closed orbit, one at a time, each time creating 2 rest points. The end result is a vector field V^* with no closed orbits, and with $m_p^* = C_p + A_p + A_{p-1}$ rest points of index p. The Strong Morse Inequalities proved in [Fo2] (and reviewed in §3) imply that for any k

$$m_k^* - m_{k-1}^* + \cdots \pm m_0^* \;\geq\; b_k(\mathbf{F}) - b_{k-1}(\mathbf{F}) + \cdots \pm b_0(\mathbf{F})$$

which is equivalent to the desired result. □

It is possible to work more precisely by taking into account the multiplicities of the closed orbits. Let $\Lambda_i^{(p)}$ denote a basic set consisting of the cells in a single closed path

$$\gamma^{(p)} : \sigma_0^{(p)},\ \tau_0^{(p+1)}, \ldots, \sigma_{r-1}^{(p)}, \tau_{r-1}^{(p)},\ \sigma_r^{(p)} \ = \ \sigma_0^{(p)}.$$

The set Λ_i contributes

$$\dim H_*(\overline{\Lambda}_i, \dot{\Lambda}_i, \mathbf{F})$$

to the Morse numbers of V (see Theorem 3.1). This contribution is determined in Theorem 7.3.

Theorem 7.3. *Let Λ_i denote a basic set consisting of the single closed path $\gamma^{(p)}$ as above, and let*

$$m(\gamma) \ = \ \prod_{i=0}^{r-1} - \langle \partial\tau_i, \sigma_i \rangle \langle \partial\tau_i, \sigma_{i+1} \rangle$$

denote the multiplicity of γ. Then

$$\begin{aligned}
H_k(\overline{\Lambda}_i, \dot{\Lambda}_i, \mathbf{Z}) &\cong 0 \quad \text{if } k \neq p,\ p+1 \\
H_{p+1}(\overline{\Lambda}_i, \dot{\Lambda}_i, \mathbf{Z}) &\cong \begin{cases} \mathbf{Z} & \text{if } m(\gamma) = 1 \\ 0 & \text{otherwise} \end{cases} \\
H_p(\overline{\Lambda}_i, \dot{\Lambda}_i, \mathbf{Z}) &\cong \mathbf{Z}/(1 - m(\gamma))\mathbf{Z}.
\end{aligned}$$

Proof. For each k,

$$C_k(\overline{\Lambda}_i, \dot{\Lambda}_i, \mathbf{Z}) \ \cong \ C_k(\Lambda_i, \mathbf{Z})$$

where $C_k(\Lambda_i, \mathbf{Z})$ denotes the integer k-chains spanned by the k-cells of Λ_i, i.e.,

$$\begin{aligned}
C_k(\Lambda_i, \mathbf{Z}) &= 0 \qquad k \neq p,\ p+1 \\
C_{p+1}(\Lambda_i, \mathbf{Z}) &= \left\{ \sum_{i=0}^{r-1} \alpha_i \tau_i \mid \alpha_i \in \mathbf{Z} \right\} \\
C_p(\Lambda_i, \mathbf{Z}) &= \left\{ \sum_{i=0}^{r-1} \beta_i \sigma_i \mid \beta_i \in \mathbf{Z} \right\}.
\end{aligned}$$

Thus, $H_*(\overline{\Lambda}_i, \dot{\Lambda}_i, \mathbf{Z})$ is the homology of the complex

$$0 \longrightarrow C_n(\Lambda_i, \mathbf{Z}) \xrightarrow{\overline{\partial}} C_{n-1}(\Lambda_i, \mathbf{Z}) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} C_0(\Lambda_i, \mathbf{Z}) \longrightarrow 0$$

where $\overline{\partial}$ is the induced differential. In particular

$$H_k(\overline{\Lambda}_i, \dot{\Lambda}_i, \mathbf{Z}) \ = \ 0 \qquad \text{if } k \neq p,\ p+1$$

and $H_p(\overline{\Lambda}_i, \dot{\Lambda}_i, \mathbf{Z})$, $H_{p+1}(\overline{\Lambda}_i, \dot{\Lambda}_i, \mathbf{Z})$ is the homology of the complex

$$0 \longrightarrow C_{p+1}(\Lambda_i, \mathbf{Z}) \xrightarrow{\overline{\partial}} C_p(\Lambda_i, \mathbf{Z}) \longrightarrow 0$$

where $\overline{\partial}$ is the induced differential given by

$$\overline{\partial}\tau_i = \langle \partial\tau_i, \sigma_i \rangle \sigma_i + \langle \partial\tau_i, \sigma_{i+1} \rangle \sigma_{i+1}.$$

Note that the condition that V be a Morse-Smale vector field implies that if $j \neq i,\ i+1$ σ_j is not a face of τ_i.

We first find

$$H_{p+1}(\overline{\Lambda}_i, \dot{\Lambda}_i, \mathbf{Z}) = \text{Kernel}(\overline{\partial}).$$

Since

$$\overline{\partial}\left(\sum_{i=0}^{r-1} \alpha_i \tau_i\right) = \sum_{i=0}^{r-1} \left(\alpha_i \langle \partial\tau_i, \sigma_i \rangle + \alpha_{i-1} \langle \partial\tau_{i-1}, \sigma_i \rangle\right) \sigma_i$$

(where all subscripts are $\mod r$,

$$\sum_{i=0}^{r-1} \alpha_i \tau_i \in \text{Kernel}(\overline{\partial})$$

if and only if for each i

(7.1) $$\alpha_i \langle \partial\tau_i, \sigma_i \rangle + \alpha_{i-1} \langle \partial\tau_{i-1}, \sigma_i \rangle = 0.$$

We now recall that $V(\sigma_i) = \tau_i$ implies that σ_i is a regular face of τ_i so that $\langle \partial\tau_i, \sigma_i \rangle = \pm 1$. Thus, we can rewrite (7.1) as

(7.2) $$\alpha_i = \alpha_{i-1} \left(- \langle \partial\tau_i, \sigma_i \rangle \langle \partial\tau_{i-1}, \sigma_i \rangle\right).$$

The equation (7.2) allows us to solve for each α_i recursively, once we have chosen α_0. The only restriction is that the α_i must be periodic with period r, which will hold if and only if $\alpha_r = \alpha_0$. From (7.2)

$$\begin{aligned}
\alpha_1 &= \alpha_0 \left(- \langle \partial\tau_1, \sigma_1 \rangle \langle \partial\tau_0, \sigma_1 \rangle\right) \\
\alpha_2 &= \alpha_1 \left(- \langle \partial\tau_2, \sigma_2 \rangle \langle \partial\tau_1, \sigma_2 \rangle\right) \\
&= \alpha_0 \left(- \langle \partial\tau_1, \sigma_1 \rangle \langle \partial\tau_0, \sigma_1 \rangle\right) \left(- \langle \partial\tau_2, \sigma_2 \rangle \langle \partial\tau_1, \sigma_2 \rangle\right) \\
&\vdots \\
\alpha_r &= \alpha_0 \left(- \langle \partial\tau_1, \sigma_1 \rangle \langle \partial\tau_0, \sigma_1 \rangle\right) \cdots \left(- \langle \partial\tau_r, \sigma_r \rangle \langle \partial\tau_{r-1}, \sigma_r \rangle\right) \\
&= \alpha_0 m(\gamma).
\end{aligned}$$

Therefore $\alpha_r = \alpha_0 m(\gamma)$, so $\alpha_r = \alpha_0$ has a non-zero solution if and only if $m(\gamma) = 1$ in which case the solutions are generated freely over $\mathbf{Z}$ by the single solution resulting from setting $\alpha_0 = 1$. This proves

$$H_{p+1}(\overline{\Lambda}_i, \dot{\Lambda}_i, \mathbf{Z}) \cong \begin{cases} \mathbf{Z} & \text{if } m(\gamma) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We now find

$$H_p(\overline{\Lambda}_i, \Lambda_i, \mathbf{Z}) \cong \frac{C_p(\Lambda_i, \mathbf{Z})}{\text{Image}(\overline{\partial})}.$$

We note that

$$\sum_{i=0}^{r-1} \beta_i \sigma_i \in \text{Image}(\gamma)$$

if and only if there is a $(p+1)$-chain $\sum_{i=0}^{r-1} \alpha_i \tau_i$ with

$$\overline{\partial}\left(\sum_{i=0}^{r-1} \alpha_i \tau_i\right) = \sum_{i=0}^{r-1} \beta_i \sigma_i$$

which holds if and only if

$$\alpha_i \langle \partial\tau_i, \sigma_i \rangle + \alpha_{i-1} \langle \partial\tau_{i-1}, \sigma_i \rangle = \beta_i \qquad \forall\, i.$$

This leads to the recursive relationship

(7.3) $$\alpha_i = \beta_i \langle \partial\tau_i, \sigma_i \rangle + \alpha_{i-1} \left(- \langle \partial\tau_i, \sigma_i \rangle \langle \partial\tau_{i-1}, \sigma_i \rangle\right).$$

Fixing the β_i, once we have chosen α_0, (7.3) enables us to solve for each α_i in turn, with the only restriction that we must have $\alpha_r = \alpha_0$. From (7.3)

$$\begin{aligned}
\alpha_1 &= \beta_1 \langle \partial\tau_1, \sigma_1 \rangle + \alpha_0 \left(- \langle \partial\tau_1, \sigma_1 \rangle \langle \partial\tau_0, \sigma_1 \rangle\right) \\
\alpha_2 &= \beta_2 \langle \partial\tau_2, \sigma_2 \rangle + \alpha_1 \left(- \langle \partial\tau_2, \sigma_2 \rangle \langle \partial\tau_1, \sigma_2 \rangle\right) \\
&= \beta_2 \langle \partial\tau_2, \sigma_2 \rangle + \beta_1 \left(- \langle \partial\tau_2, \sigma_2 \rangle \langle \partial\tau_1, \sigma_2 \rangle\right) \langle \partial\tau_1, \sigma_1 \rangle \\
&\qquad + \alpha_0 \left(- \langle \partial\tau_1, \sigma_1 \rangle \langle \partial\tau_0, \sigma_1 \rangle\right) \left(- \langle \partial\tau_2, \sigma_2 \rangle \langle \partial\tau_1, \sigma_2 \rangle\right) \\
&\ \vdots \\
\alpha_r &= \beta_r \langle \partial\tau_r, \sigma_r \rangle + \cdots + \alpha_0 m(\gamma).
\end{aligned}$$

Therefore, the condition $\alpha_r = \alpha_0$ is equivalent to

(7.4) $$\alpha_0 (1 - m(\gamma)) = \beta_r \langle \partial\tau_r, \sigma_r \rangle + \cdots$$

where the right hand side is linear in the β_i's. Of significance is the fact that $\langle \partial\tau_r, \sigma_r \rangle$, the coefficient of β_r, is equal to ± 1. Let

$$f(\beta_1, \ldots, \beta_r)$$

denote the right hand side of (7.4). Then (7.4) can be solved for α_0 if and only if

$$f(\beta_1, \ldots, \beta_r) \in (1 - m(\gamma))\mathbf{Z}.$$

Thus

$$\text{Image}(\overline{\partial}) = \left\{ \sum_{i=0}^{r-1} \beta_i \sigma_i \mid \beta_i \in \mathbf{Z},\ f(\beta_1, \ldots, \beta_r) \in (1 - m(\gamma))\mathbf{Z} \right\}$$

so that

$$\begin{aligned} H_p(\overline{\Lambda}_i, \dot{\Lambda}_i, \mathbf{Z}) &\cong \frac{\mathbf{Z}^r}{\{(\beta_1, \ldots, \beta_r) \in \mathbf{Z}^r \mid f(\beta_1, \ldots, \beta_r) \in (1 - m(\gamma))\mathbf{Z}\}} \\ &\cong \mathbf{Z}/(1 - m(\gamma))\mathbf{Z} \end{aligned}$$

where it is in this last isomorphism that we use the fact that the coefficient of β_r is ± 1. □

For example, working over $\mathbf{R}$, we find

$$H_k(\overline{\Lambda}_i, \dot{\Lambda}_i, \mathbf{R}) = 0 \qquad \text{if } k \neq p,\ p+1$$

$$H_{p+1}(\overline{\Lambda}_i, \dot{\Lambda}_i, \mathbf{R}) = \begin{cases} \mathbf{R} & \text{if } m(\gamma) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$H_p(\overline{\Lambda}_i, \dot{\Lambda}_i, \mathbf{R}) = \begin{cases} \mathbf{R} & \text{if } m(\gamma) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

In this case we can apply Theorem 3.1 to get a more refined version of Theorem 7.2.

Corollary 7.4. *Let A'_p denote the number of closed orbits of V which have index p and multiplicity 1, so that, in particular, $A'_p \leq A_p$. Then for each k*

$$A'_k - c_k - c_{k-1} + \cdots \pm c_0 \leq b_k(\mathbf{R}) - b_{k-1}(\mathbf{R}) + \cdots \pm b_0(\mathbf{R}).$$

We observe that if M is a regular CW complex, then every face is regular, so for every closed path γ

$$m(\gamma) = \pm 1.$$

Therefore, if γ has index p

$$H_p(\overline{\Lambda}_i, \dot{\Lambda}_i, \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } m(\gamma) = 1 \\ \mathbf{Z}/2\mathbf{Z} & \text{if } m(\gamma) = -1 \end{cases}$$

and we see that a crucial factor is whether the coefficient field $\mathbf{F}$ has characteristic 2 or not.

We observe that V is a Morse-Smale vector field if and only if V has only finitely many distinct prime closed orbits. Thus, the zeta functions introduced in §5 are finite products, and there is no need for analytic continuations. In particular, in this context Theorem 5.13 becomes

Theorem 7.5. *Suppose V is a Morse-Smale vector field. Let Γ denote the fundamental group of M, and*

$$\theta : \Gamma \longrightarrow U(m)$$

a unitary representation. Suppose that for each prime closed orbit $[\gamma]$

$$\det(I - m([\gamma])\theta([\gamma])) \neq 0.$$

Then

$$H_*(M, \theta) = 0$$

and

$$T(M, \theta) = \prod_{[\gamma] \in \mathcal{O}^*} [\det(I - m([\gamma])\theta([\gamma]))]^{(-1)^{i([\gamma])}}.$$

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