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A note on Delaunay diagonal flips

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Abstract

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Suppose we wish to transform an arbitrary triangulation of a point set into its Delaunay triangulation. A *Delaunay diagonal flip* replaces the common edge of two abutting triangles with the opposite diagonal if the resulting triangles would locally satisfy the Delaunay empty-circle condition. We show that $\Theta(n^2)$ Delaunay diagonal flips are necessary and sufficient to transform any triangulation into the Delaunay triangulation.

1. Introduction

The Voronoi diagram and its geometric dual, the Delaunay triangulation, are structures central to computational geometry [9]. Because of the widespread utility of these structures, it is important to find algorithms to compute them that are as efficient and as simple as possible. Both of these structures can be computed on n sites in worst-case optimal time $O(n \log n)$ using divide-and-conquer [10,6] or sweepline [3] algorithms; random-incremental algorithms with expected running time $O(n \log n)$ are also known [2,5]. However, all of these algorithms are in practice somewhat complicated, and it would be a valuable contribution to devise a simpler algorithm that is equally efficient.

A particularly simple Delaunay triangulation algorithm is the diagonal flipping algorithm. Given a set of n sites, the diagonal flipping algorithm first computes an arbitrary triangulation of the sites and then transforms it into the Delaunay triangulation. The first step is relatively straightforward. To perform the transformation, one uses Delaunay diagonal flips. Suppose sites a, b, c, d are triangulated into two triangles abc and acd, the two triangles together form a convex quadrilateral, and locally the Delaunay empty-circle condition is violated, that is, d is in the circumcircle of triangle abc (equivalently, b is in the circumcircle of acd). Then the Delaunay diagonal flip replaces ac with bd, yielding triangles abd and bcd. This method seems to have been first suggested by Lawson [8]. It is not hard to show, see Theorem 1 below, that at most $O(n^2)$ Delaunay diagonal flips are necessary to transform any triangulation into the Delaunay triangulation.

Empirical evidence suggests that the diagonal flipping algorithm is a reasonable algorithm in practice [4], performing much better than this $O(n^2)$ bound, though not quite as well as more complex algorithms. One might hope to show that the flipping algorithm could be implemented to require less than $O(n^2)$ time even in the worst case. For example, one might hope to show that there is always some order of Delaunay diagonal flips that requires only $O(n)$ flips. If, say, the ordering could be found in time $O(\log n)$ per flip, an $O(n \log n)$ algorithm would result. Unfortunately, the main result below, Theorem 2, shows that in some cases $\Omega(n^2)$ Delaunay diagonal flips are required in the worst case, no matter how the flips are

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ordered. Hence this approach will not lead to an improved worst-case time bound.

The proofs

Throughout this note we assume that unless otherwise specified all point sets are in general position, that is, no three points are collinear and no four points are cocircular. Removing this assumption does not change the results, though more care would be needed in definitions and proofs.

Let S be a set of points in the plane, called *sites*. A *triangulation* of S is a set of segments connecting pairs of sites in S so that segments intersect only at endpoints, all convex hull segments are in the set, and all interior faces are triangles. When convenient we also think of a triangulation as the set of triangles not containing sites in their interiors. By planarity a triangulation on n sites has at most $3n$ segments; since each segment appears in at most two triangles, there are at most $2n$ triangles in a triangulation.

Let C_{abc} be the circle through a, b, c together with its interior. A triangulation is *Delaunay* if for every triangle abc in the triangulation, C_{abc} contains no additional sites. It is not hard to see that the vertices of any convex quadrilateral can be labeled a, b, c, d in clockwise order so that $d\in C_{abc}$, $b\in C_{acd}$, $c\notin C_{abd}$, and $a\notin C_{bcd}$. Diagonal bd is the *Delaunay diagonal* of $abcd$. Triangulation T' results from T by a *Delauney diagonal flip* (henceforth: flip) if

$$T'=T\cup\{bd\}-\{ac\},$$

where bd is the Delaunay diagonal of convex quadrilateral $abcd$. The following proposition implies that flips are always possible in non-Delaunay triangulations.

Proposition 1. *If triangulation T is not Delaunay, then there are sites a, b, c, d so that abcd is a convex quadrilateral, bd is the Delaunay diagonal of abcd, and abc and acd are triangles in T.*

Proof. Choose d in the circumcircle of some triangle in T. Of all such triangles, choose abc so that d is as close as possible to ac, where sites a, b, c, d are labeled as the vertices of a quadrilateral in clockwise order. Since b and d are on opposite sides of ac, segment ac is not on the convex hull of the set of sites. Hence there is a site $e\neq b$ so that $ace\in T$. If $d=e$, then the lemma is satisfied, since quadrilateral $abcd$ is convex as $d\in C_{abc}$. If $d\neq e$, then e must be in the circumcircle of C_{abc}: if not, then d is in the circumcircle of C_{ace} and the segment perpendicular to ac from d to ac must intersect ae or ce, contradicting the choice of abc as triangle with circumcircle containing d and side closest to d. The lemma is satisfied with e replacing d. □

Proposition 2. *Suppose a, b, c, d form the vertices of a convex quadrilateral. If bd is the Delaunay diagonal of abcd, then*

$$C_{abd}\cup C_{bcd}\subseteq C_{abc}\cup C_{acd}$$

and

$$C_{abd}\cap C_{bcd}\subseteq C_{abc}\cap C_{acd}.$$

Proof. For arbitrary points x, y, z, let H^z_{xy} be the half-plane containing z with bounding line through x and y. Since $d\in C_{abc}$, $H^d_{ab}\cap C_{abd}\subseteq C_{abc}$; since $b\in C_{acd}$, $H^b_{ad}\cap C_{abd}\subseteq C_{acd}$. Since

$$C_{abd}=(H^d_{ab}\cap C_{abd})\cup(H^b_{ad}\cap C_{abd}),$$

$C_{abd}\subseteq C_{abc}\cup C_{acd}$. Similarly $C_{bcd}\subseteq C_{abc}\cup C_{acd}$. For the second part, notice

$$C_{abd}\cap C_{bcd}=(C_{abd}\cap H^c_{bd})\cup(C_{bcd}\cap H^a_{bd}).$$

Now $C_{abd}H^c_{bd}\subseteq H^b_{ad}\cap C_{abd}\subseteq C_{acd}$. Similarly $C_{bcd}\cap H^a_{bd}\subseteq C_{acd}$. Hence $C_{abd}\cap C_{bcd}\subseteq C_{acd}$. Similarly, $C_{abd}\cap C_{bcd}\subseteq C_{abc}$. □

Theorem 1. *Let S be a set of n sites. Then n^2 flips are sufficient to transform any triangulation into the Delaunay triangulation of S.*

Proof. By Proposition 1, if a triangulation is not Delaunay, then there is always a possible flip. We show that no sequence of flips has length more than n^2.

For T a triangulation of S, let $w(T)$ be the sum over all triangles t in T of the number of sites of S in the interior of C_t. Since there are at most $2n$ triangles in T and each circumcircle can contain at most n sites, $w(T)\leqslant 2n^2$. Clearly if T is Delaunay, then $w(T)=0$.

Suppose T' results from T by flipping the diagonal of $abcd$ from ac to bd, replacing triangles abc and acd by abd and bcd. We claim $w(T')\leqslant w(T)-2$. First note that circumcircles of triangles distinct from abc and acd are unaffected by the flip. Second, by Proposition 2, if $e \neq a, b, c, d$ and e is in one of C_{abd} or C_{bcd}, then e must have been in one of C_{abc} or C_{acd}. Similarly, if e is in both C_{abd} and C_{bcd}, then e must have been in both C_{abc} and C_{acd}. Finally, note $d \in C_{abc}$ and $b \in C_{acd}$ but $a \notin C_{bcd}$ and $c \notin C_{abd}$. □

The *lifting map* $\lambda: \mathbb{R}^2 \to \mathbb{R}^3$ is defined by

$$\lambda(x, y) = (x, y, x^2 + y^2) ;$$

$\lambda(\mathbb{R}^2)$ is a paraboloid of revolution about the z-axis. Using the lifting map it is possible to show the stronger result that once an edge is removed by a flip, it never reappears [4,7]. Of course, this fact immediately implies Theorem 1.

Theorem 2. *For all n there is a set S_n of n sites and a triangulation T_0 of S_n so that $\Omega(n^2)$ flips are required to transform T_0 into the Delaunay triangulation of S_n.*

Proof. Without loss of generality assume n is even and set $m = n/2$. Choose m sites $\{l_1, ..., l_m\}$ on the segment from $(-1, 0)$ to $(-1, 1)$ uniformly spaced numbered from top to bottom and m sites $\{r_1, ..., r_m\}$ on the segment from $(1, -1)$ to $(1, 0)$ uniformly spaced numbered from bottom to top. See Figures 1 and 2. These two sets of sites have the property that if $i < j$ then $C_{l_i l_j r_k}$ contains exactly the l_p with $i \leq p \leq j$ and exactly the r_p with $p \geq k$.

We claim that the sites can be perturbed slightly preserving this property so that all sites lie on the convex hull and so that no three sites are collinear. We sketch a proof of this claim. The only slight difficulty is the first condition, that $C_{l_i l_j r_k}$ contains exactly

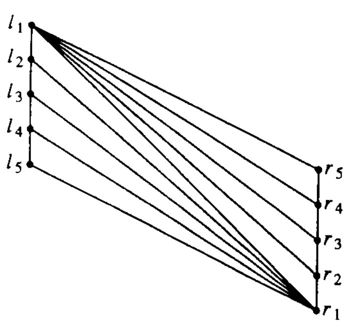

Figure 1. Triangulation T_0.

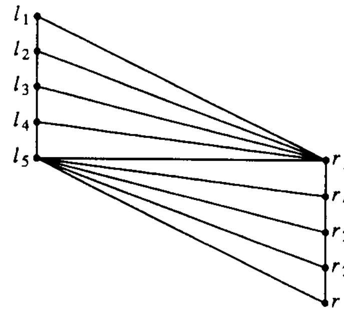

Figure 2. Delaunay triangulation T_D.

the l_p with $i \leq p \leq j$. If two sites are perturbed by at most ε, then the perpendicular bisector of the two sites moves only slightly, specifically it rotates by an angle $O(\varepsilon)$ about a point near the midpoint of the two sites, then translates by an amount $O(\varepsilon)$. The center of $C_{l_i l_j r_k}$ is the intersection of the perpendicular bisector of l_i and l_j, a horizontal line with y-value at most 1, and the perpendicular bisector of l_j and r_k, a line of slope at least 1 intersecting the x-axis at a point $x < 1$. This implies that the radius of $C_{l_i l_j r_k}$ is at most 3, a constant; hence the center of $C_{l_i l_j r_k}$ moves by at most $O(\varepsilon)$ if each of l_i, l_j and r_k is perturbed by at most ε. Now if $p \neq i, j$ then $C_{l_i l_j r_k}$ lies at least $\Omega(1/n^2)$ to one side or the other of l_p on the horizontal line through l_p. Hence if we choose ε small compared with $1/n^2$, membership of l_p inside $C_{l_i l_j r_k}$ does not change.

We let S_n be the perturbed set of sites. Clearly, these sites also satisfy the property that if $i < j$ then $C_{r_i r_j l_k}$ contains exactly the r_p with $i \leq p \leq j$ and exactly the l_p with $p \geq k$.

Let T_0 be the triangulation of S_n consisting of the segments of the convex hull and the segments $\{l_i r_1\}$ and $\{l_1 r_i\}$, $i = 1, ..., m$. See Figure 1. Triangles of the form $l_i l_{i+1} r_m$ and $r_i r_{i+1} l_m$ have empty circumcircles, so the Delaunay triangulation T_D of S_n consists of the segments of the convex hull and the segments $\{l_i r_m\}$ and $\{l_m r_i\}$, $i = 1, ..., m$. See Figure 2. We show that $(m-1)^2$ flips are necessary to transform T_0 into T_D.

We claim that the only possible flip is if $l_{i+1} r_{j+1}$ replaces $l_i r_j$ as the diagonal of $l_i l_{i+1} r_j r_{j+1}$. We use as induction hypothesis the claim that all non-convex-hull segments go between the l's and the r's. To see this claim, suppose $l_i r_j$ is about to be replaced. Since all non-convex-hull segments go from the l's to the r's, the remaining vertices of the quadrilateral with diagonal $l_i r_j$ must be two of $l_{i-1}, l_{i+1}, r_{j-1}, r_{j+1}$. The two vertices cannot be l_{i-1} and l_{i+1}, since $C_{l_{i-1}l_{i+1}r_j}$ contains l_i. Similarly the two vertices cannot be r_{j-1} and r_{j+1} since $C_{l_i r_{j-1} r_{j+1}}$ contains r_j, nor can they be l_{i-1} and r_{j-1}, since $C_{l_{i-1}l_i r_{j-1}}$ contains r_j. Hence the two vertices must be l_{i+1} and r_{j+1}.

We show that any sequence of flips transforming T_0 into T_D has exactly $(m-1)^2$ flips. For T any intermediate triangulation, let

$$\nu(T) = \sum_{l_i r_j \in T} i+j\,.$$

Clearly we have

$$\begin{aligned}\nu(T_D)-\nu(T_0) &= 2\left(\sum_{k=1}^{m}(k+m)-2m\right)\\ &\quad-\left(2\sum_{k=1}^{m}(k+1)-2\right)\\ &=2(m-1)^2\,.\end{aligned}$$

Since ν goes up by 2 per flip, $(m-1)^2$ flips are necessary to transform T_0 into T_D. □

Discussion

The example of Figure 1 is essentially as bad as it could be. It consists of two sets of points, separated by a single diagonal. Each of the two sets of points is originally triangulated with its Delaunay triangulation. Yet still $\Omega(n^2)$ flips are necessary to transform the individual Delaunay triangulations into the joint triangulation. Also notice that the sites of Figure 1 form the vertices of a convex polygon; Aggarwal et al. [1] show that its Voronoi diagram can be computed in linear time. Hence the initial triangulation seems more to get in the way than to be useful.

It would still be desirable to have a local rule that would transform an arbitrary triangulation into the Delaunay triangulation with only $O(n)$ steps. For example, the triangulation of Figure 1 can be transformed into the triangulation of Figure 2 with $3m-5$ diagonal flips: first triangulate so all triangles contain l_m (this requires $2m-3$ diagonal flips), then use an additional $m-2$ flips to obtain the triangles containing r_m. Of course, the first $m-2$ diagonal flips are not Delaunay. Is there some (easily computed) rule that would produce this sequence of flips?

There is some empirical evidence that Delaunay diagonal flips work well in practice. Is it the case that Delaunay diagonal flips work quickly for 'most' or 'random' triangulations? Alternatively, is there some (easy) way of choosing the original triangulation so that only a few Delaunay diagonal flips will be needed to obtain the Delaunay triangulation? Guibas et al. [5] consider adding points one by one, updating the triangulation to the Delaunay triangulation using Delaunay diagonal flips after each addition. They show that the expected number of flips is linear, where the expectation is taken over the permutation used to order insertion, with each permutation equally likely. Their algorithm is slightly complicated, since they need to maintain a point-location data structure to determine the triangle containing the next point to be inserted. Possibly there is a simple variant of their algorithm that avoids the need for the point-location data structure.

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