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Combinatorial Analogs of Brouwer's Fixed-Point Theorem on a Bounded Polyhedron

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In this paper, we present a combinatorial theorem on a bounded polyhedron for an unrestricted integer labelling of a triangulation of the polyhedron, which can be interpreted as an extension of the Generalized Sperner lemma. When the labelling function is dual-proper, this theorem specializes to a second theorem on the polyhedron, that is, an extension of Scarf's dual Sperner lemma. These results are shown to be analogs of Brouwer's fixed-point theorem on a polyhedron, and are shown to generalize other combinatorial theorems on bounded polyhedra as well. The paper also contains a pseudomanifold construction for a polyhedron and its dual that may be of interest to researchers in triangulations based on primal and dual polyhedra.

1. INTRODUCTION

In an article published in 1928, Emanuel Sperner demonstrated a purely combinatorial lemma on the n-simplex that implied the fixed-point theorem of Brouwer for continuous functions. The connection between combinatorial theorems and topological theorems was further investigated by Tucker [24], who developed a combinatorial lemma that implied the antipodal point theorems of Borsuk and Ulam and of Lusternik and Schnirelman [19]. Kuhn [15] and Fan [5] later examined combinatorial results on the n-cube that imply Brouwer's fixed-point theorem.

With the development of fixed-point computation algorithms stemming from Scarf's seminal work [21], there has been a resurgence of research in combinatorial analogs of Brouwer's theorem. Such analogs of Brouwer's theorem on the simplex include Scarf's "dual" Sperner lemma [22], the Generalized Sperner lemma [11], and of course, the original Sperner lemma [23]. Analogs of Brouwer's theorem on the cube include a pair of dual lemmas presented in [6], one of which is analogous to the constructive algorithm in van der Laan and Talman [17]. Recently, these combinatorial results have been extended to simplotopes (see Freund [7] and van der Laan *et al.* [18]), for which the simplex and cubical theorems are special cases.

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In this paper, we present a combinatorial theorem on a bounded polyhedron for an unrestricted labelling of a triangulation of the polyhedron, which can be interpreted as an extension of the Generalized Sperner lemma. This theorem is the main theorem of Section 3, Theorem 1. When the labelling function is dual-proper, Theorem 1 specializes to a second combinatorial theorem on the polyhedron, that is an extension of Scarf's dual Sperner lemma. These results are shown in Section 3, and their relationship to other results on bounded polyhedra are also shown in Section 3.

Section 4 addresses extensions and limitations of Theorem 1. We show how the geometric representation of a polyhedron can affect the implications of Theorem 1. We also address the issue of an extension of Sperner's lemma to a bounded polyhedron. We present such an extension as Theorem 4 of the section. However, the proof of Theorem 4 is based on Brouwer's theorem; it is an open question whether a purely combinatorial proof of Theorem 4 can be demonstrated.

Section 5 is devoted to a combinatorial proof of Theorem 1. As part of this proof, we present a pseudomanifold construction for a polyhedron and its dual (Lemma 3) that may be of interest to researchers in triangulations based on primal and dual polyhedra.

2. NOTATION

Let R^n denote real n-dimensional space, and define e to be the vector of 1's, namely $e = (1, ..., 1)$. Let $x \cdot y$ and $x \cdot y$ denote inner and outer product, respectively. Let $\emptyset$ denote the empty set, and let $|S|$ denote the cardinality of a set S. For two sets S, T, let $S\backslash T = \{x \mid x \in S,\ x \notin T\}$, and let $S \Delta T = \{x \mid x \in S \cup T,\ x \notin S \cap T\}$. If $x \in S$, we denote $S\backslash\{x\}$ by $S\backslash x$ to ease the notational burden. Let $v^0, ..., v^m$ be vectors in R^n. If $v^0, ..., v^m$ are affinely independent, i.e., if the matrix

$$\begin{bmatrix} v^0 & \cdots & v^m \\ 1 & \cdots & 1 \end{bmatrix}$$

has rank $(m+1)$, then the convex hull of $v^0, ..., v^m$, denoted $\langle v^0, ..., v^m \rangle$, is said to be a *real m-dimensional simplex*, or more simply an m-simplex. If $\sigma = \langle v^0, ..., v^m \rangle$ is an m-simplex and $\{v^{i_0}, ..., v^{i_k}\}$ is a nonempty subset of $\{v^0, ..., v^m\}$, then $\tau = \langle v^{i_0}, ..., v^{i_k} \rangle$ is a k-*face* or *face* of σ.

Let $V = \{v^0, \ldots, v^k\}$ be a finite subset of vectors in R^n. The set V is said to be in *general position* if each subset of V containing at most $n+1$ members is affinely independent.

Let $\mathscr{P}$ be a *cell* in R^n, i.e., a nonempty bounded polyhedron in R^n. Let T be a finite collection of m-simplices σ together with all of their faces. T is a finite *triangulation of* $\mathscr{P}$ if

(i) $\bigcup_{\sigma \in T} \sigma = \mathscr{P}$.

(ii) $\sigma, \tau \in T$ imply $\sigma \cap \tau$ is a face of σ and of τ.

(iii) If σ is an $(m-1)$-simplex of T, σ is a face of at most two m-simplices of T.

An *abstract complex* consists of a set of vertices K^0 and a set of finite subsets of K^0, denoted K, such that

(i) $v \in K^0$ implies $\{v\} \in K$, and

(ii) $x \subset y \in K$ implies $x \in K$.

Note that the empty set $\varnothing$ is an allowable member of a complex K. An element x of K is called an *abstract simplex*, or more simply a simplex. If $x \in K$ and $|x| = n+1$, then x is called an n-simplex, where $|\cdot|$ denotes cardinality. Technically, an abstract complex is defined by the pair (K^0, K). However, since the set K^0 is implied by K, it is convenient to denote the complex by K alone. An abstract complex K is said to be finite if K^0 is finite.

An n-dimensional *pseudomanifold*, or more simply an n-pseudomanifold, where $n \geq 1$, is a complex K such that

(i) $x \in K$ implies there exists $y \in K$ with $|y| = n+1$ and $x \subset y$, and

(ii) if $x \in K$ and $|x| = n$, then there are at most two n-simplices of K that contain x.

Let K be an n-pseudomanifold, where $n \geq 1$. The boundary of K, denoted ∂K, is defined to be the set of simplices $x \in K$ such that x is contained in an $(n-1)$-simplex $y \in K$, and y is a subset of exactly one n-simplex of K.

Let $\mathscr{P}$ be an m-cell in R^n, and let T be a finite triangulation of $\mathscr{P}$. For each nonempty face τ of each m-simplex σ of T, define $\bar{\tau} = \{v \mid v$ is a vertex of $\tau\}$. Then the collection $K = \{\bar{\tau} \mid \tau$ is a nonempty face of a simplex of $T\}$ is an m-pseudomanifold, and is called the m-pseudomanifold *corresponding* to T.

If A and b are a matrix and a vector, let A_i and b_i denote the ith *row* and component of A and b, respectively, and let A_β and b_β denote the submatrix and subvector of A and b corresponding to the *rows* and components of A and b indexed by β, respectively.

3. The Main Theorem

Consider a bounded polyhedron $\mathscr{P}$ of the form

$$\mathscr{P} = \{x \in R^n \mid Ax \leq b\}, \tag{1}$$

where A and b are a given $(m \times n)$-matrix and m-vector, respectively. Let T be a finite triangulation of $\mathscr{P}$, let K^0 denote the set of vertices of T, and let K be the pseudomanifold corresponding to T. Let $M = \{1, \ldots, m\}$ be the set of constraint row indices, and let $L(\cdot)\colon K^0 \to M$ be a labelling function that assigns a constraint row index i to each vertex v of K^0. Our interest lies in ascertaining the combinatorial implications of such a labelling function, under boundary conditions or not, in the spirit of and as a generalization of other combinatorial theorems on bounded polyhedra [5–7, 10, 15, 17, 18, 22, 24, 25]. Toward this goal, it will be convenient to make the following assumption regarding $\mathscr{P}$:

Assumption A. $\mathscr{P}$ is bounded, *solid* (i.e., $\dim \mathscr{P} = n$), and *centered* (i.e., $\mathscr{P}$ contains the origin in its interior). The system of inequalities (1) has no *redundant* constraints; i.e., every row of (A, b) corresponds to a facet of $\mathscr{P}$, and the rows of (A, b) have been *scaled* so that each $b_i = 1$, $i = 1, \ldots, n$ (i.e., $b = e$).

It should be noted that any n-dimensional bounded polyhedron $\mathscr{P}$ can be orthogonally transformed and translated so that it satisfies Assumption A, without disturbing the combinatorial structure of $\mathscr{P}$. Some of the components of Assumption A will be relaxed later on, in Section 4.

Let $\mathscr{P}^* = \{y \in R^n \mid y = \lambda A,\ \lambda \geq 0,\ \lambda \cdot b = 1\}$. Then $\mathscr{P}^*$ is bounded, solid, and contains the origin in its interior. Furthermore, $\mathscr{P}^*$ can alternately be described as $\mathscr{P}^* = \{y \in R^n \mid y \cdot x \leq 1$ for all $x \in \mathscr{P}\}$, whereby $\mathscr{P}^*$ is seen to be the *polar* of $\mathscr{P}$ (see [20]) and $\operatorname{int} \mathscr{P}^* = \{y \in R^n \mid y \cdot x < 1$ for any $x \in \mathscr{P}\}$. $\mathscr{P}^*$ is also a combinatorial dual of $\mathscr{P}$; i.e., there is a one-to-one inclusion reversing mapping from the k-faces of $\mathscr{P}$ to the $(n-k-1)$-faces of $\mathscr{P}^*$ (see [13]).

Because (1) has no redundant constraints, each row of A is a vertex of $\mathscr{P}^*$. Furthermore, every point $y \in \mathscr{P}^*$ can be expressed as a convex combination of $(n+1)$ extreme points of $\mathscr{P}^*$, i.e., $(n+1)$ rows of A. A point $y \in \mathscr{P}^*$ is called a *regular* point of $\mathscr{P}^*$ if y cannot be expressed as a convex combination of n or fewer rows of A. Because $\mathscr{P}$ is bounded, $\mathscr{P}^*$ is solid, and so almost every point in $\mathscr{P}^*$ is a regular point of $\mathscr{P}^*$, i.e., the set of points in $\mathscr{P}^*$ that are not regular is a set of measure zero, and $\mathscr{P}^*$ has positive measure. Figure 1 illustrates the above remarks. In the figure, y^1 is a regular point, and y^2 is not a regular point. The circled numbers on the

$$\mathscr{X} = \{x \in R^2 \mid Ax \le b\}, \text{ where } A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \\ -1 & 2 \\ -1 & -2 \\ 3 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

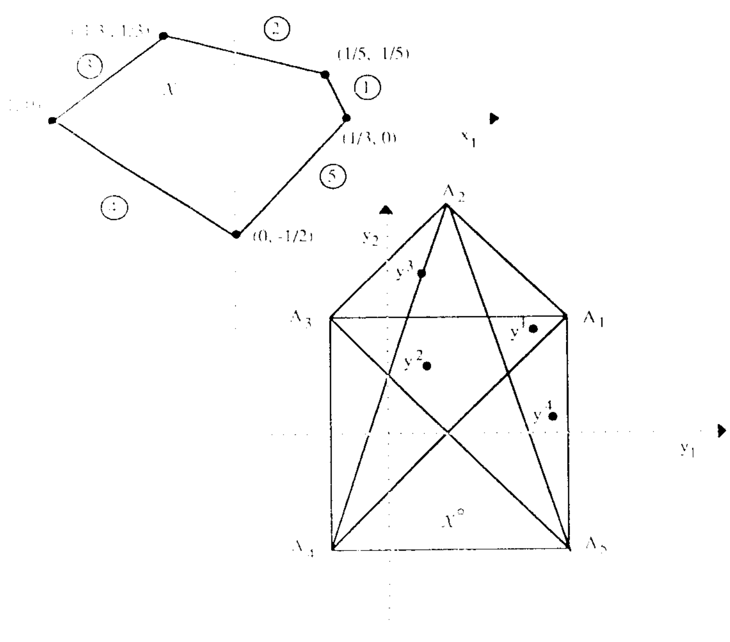

FIGURE 1

boundary of $\mathscr{X}$ in the figure indicate the row constraint index for the facets indicated.

For a subset $\alpha \subset M$, define $S_\alpha = \{y \in R^n \mid y = \lambda_\alpha A_\alpha, \lambda_\alpha \geq 0, \lambda_\alpha \cdot b_\alpha = 1\}$; i.e., S_α is the convex hull of the rows of A indexed over α. We have $S_\alpha = \mathscr{X}^\circ$ for $\alpha = M$, and $S_\alpha \subset \mathscr{X}^\circ$ for all $\alpha \subset M$. For every $y \in \mathscr{X}^\circ$ define $G_y = \{\alpha \subset M \mid y \in S_\alpha\}$. Then G_y consists of the row index sets of vertices of cells S_α that contain the point y. Referring to Fig. 1 again, we see that G_{y^1}

consists of the four sets $\{1, 3, 4\}$, $\{1, 3, 5\}$, $\{1, 2, 4\}$, and $\{1, 2, 5\}$, plus all other subsets of M that contain one of these four sets. Likewise, the minimal members of G_{y^2} are $\{1, 2, 5\}$, $\{1, 3, 5\}$, and $\{1, 4, 5\}$. Regarding G_{y^3}, the minimal members of G_{y^3} are $\{1, 2, 3\}$, $\{2, 4\}$, and $\{2, 3, 5\}$.

Now let T be a finite triangulation of $\mathscr{X}$, let K be the pseudomanifold corresponding to T, and let $L(\cdot)\colon K \to M$ be a labelling function from K, the set of vertices of K, to M, the set of constraint row indices of $\mathscr{X}$. For a simplex $\sigma \in K$, let $L(\sigma) = \{i \in M \mid i = L(v)$ for some $v \in \sigma\}$. For a given subset S of $\mathscr{X}$, define $C(S) = \{i \in M \mid A_i x = b_i$ for all $x \in S\}$. For a point $x \in \mathscr{X}$, define $C(x) = C(\{x\})$. The mapping $C(\cdot)$ identifies the "carrier" hyperplanes of the set S or point x.

With the above notation in hand, we can state our main theorem:

THEOREM 1. *Let $\mathscr{X}$ be a polyhedron that satisfies Assumption A. Let T be a finite triangulation of $\mathscr{X}$, let K be the pseudomanifold corresponding to T, and let $L(\cdot)\colon K \to M$ be a labelling function. Then*

(i) *for any regular point $y \in \mathscr{X}^\circ$, there are an odd number of simplices $\sigma \in K$ such that $(L(\sigma) \cup C(\sigma)) \in G_y$, and hence at least one.*

(ii) *for any point $y \in \operatorname{int} \mathscr{X}^\circ$, there is at least one simplex $\sigma \in K$ such that $(L(\sigma) \cup C(\sigma)) \in G_y$.*

To illustrate the theorem, let us continue with the example of Fig. 1. Figure 2 shows a triangulation T of $\mathscr{X}$ and a labelling of K. Regarding y^1, a regular point of $\mathscr{X}^\circ$, there are five simplices σ of K for which $(L(\sigma) \cup$

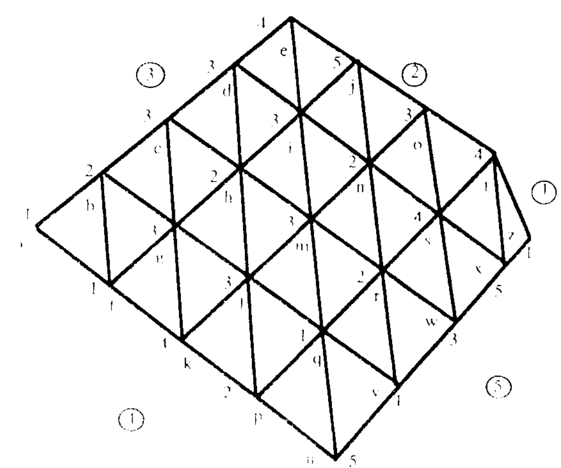

FIGURE 2

$C(\sigma)) \in G_{y^3} = \{\{1, 3, 4\}, \{1, 3, 5\}, \{1, 2, 4\}, \{1, 2, 5\}\}$, namely $\{t\}$, $\{w, v\}$, $\{a\}$, $\{f, g, k\}$, and $\{p, q, u\}$. Note that $L(\{w, v\}) = \{1, 3\}$, $C(\{w, v\}) = \{5\}$, and hence $(L(\{w, v\}) \cup C(\{u, v\})) = \{1, 3, 5\} \in G_{y^3}$. Regarding y^4, there are three simplices $\sigma \in K$ for which $(L(\sigma) \cup C(\sigma)) \in G_{y^4} = \{\{1, 2, 5\}, \{1, 3, 5\}, \{1, 4, 5\}\}$, namely $\{p, q, u\}$, $\{w, v\}$, and $\{x, t, z\}$. In the case of the pentagon $\mathcal{X}$ in Fig. 1, Theorem 1 actually makes eleven assertions about the oddness of certain instances of labels, one assertion for each of the eleven regions composing $\mathcal{X}$.

The assertions of Theorem 1 do not depend on any special restrictions of the labelling $L(\cdot)$ on the boundary of $\mathcal{X}$. If we restrict the labelling $L(\cdot)$ on the boundary of $\mathcal{X}$, we can obtain a stronger form of Theorem 1. A labelling $L(\cdot): K^0 \to M$ is called *dual-proper* if $L(v) \in C(v)$ for all $v \in \partial\mathcal{X}$, $v \in K^0$. If $L(\cdot)$ is dual-proper, $L(v)$ must index a binding constraint at v if v lies on the boundary of $\mathcal{X}$. This restriction was first introduced by Scarf [22] for the simplex. The denotation here is consistent with the notion of a dual-proper labelling as used in [7]. A triangulation T of $\mathcal{X}$ is said to be *bridgeless* if for each $\sigma \in T$, the intersection of all faces of $\mathcal{X}$ that meet σ is nonempty. This concept is illustrated in Fig. 3, for $n = 2$. In the figure, each of the simplices σ_1, σ_2, and σ_3 fails the intersection property. Essentially if T is bridgeless, then no simplex σ of T meets too many faces of $\mathcal{X}$ that are disparate.

If $L(\cdot)$ is dual-proper and T is bridgeless, we have the following stronger version of Theorem 1:

THEOREM 2. *Let $\mathcal{X}$ be a polyhedron that satisfies Assumption A. Let T be a finite triangulation of $\mathcal{X}$ and let K be the pseudomanifold corresponding to T. Let $K(\cdot): K^0 \to M$ be a labelling function on K^0. If $L(\cdot)$ is dual-proper and T is bridgeless, then:*

(i) *for any regular point $y \in \mathcal{X}$, there are an odd number of simplices $\sigma \in K$ such that $L(\sigma) \in G_y$, and hence at least one;*

(ii) *for any point $y \in \text{int}\,\mathcal{X}$, there is at least one simplex $\sigma \in K$ such that $L(\sigma) \in G_y$.*

Theorem 2 can be deduced from Theorem 1 as follows:

Proof of Theorem 2. Assuming Theorem 1 is true, it suffices to show that for each $y \in \text{int}\,\mathcal{X}$, if $(L(\sigma) \cup C(\sigma)) \in G_y$, then $C(\sigma) = \phi$. Suppose not. Then there exists $\bar{\sigma} \in K$ such that $(L(\bar{\sigma}) \cup C(\bar{\sigma})) \in G_y$ and $C(\bar{\sigma}) \neq \emptyset$. Because $C(\bar{\sigma}) \neq \emptyset$, $\bar{\sigma} \in \partial\mathcal{X}$, whereby each vertex v of $\bar{\sigma}$ must satisfy $L(v) \in C(v)$. If $L(v) = i$, then v, and hence $\bar{\sigma}$, meets the facet F_i defined by $F_i = \{x \in \mathcal{X} \mid A_i x = b_i\}$. Therefore $\bar{\sigma}$ meets every facet F_i for $i \in L(\bar{\sigma})$. Furthermore, $\bar{\sigma}$ meets every facet F_i for $i \in C(\bar{\sigma})$. Denoting $\alpha = (L(\bar{\sigma}) \cup C(\bar{\sigma}))$, we have $\bar{\sigma}$ meets F_i for every $i \in \alpha$. Thus $\bigcap_{i \in \alpha} F_i \neq \emptyset$,

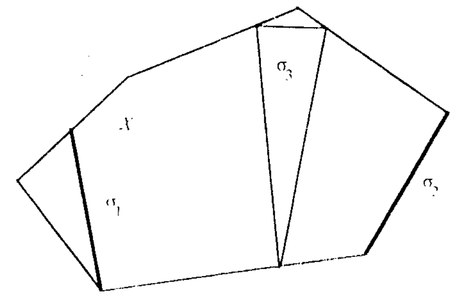

FIG. 3. Cases where the intersection of the faces that meet σ are empty.

because T is bridgeless. Let $\bar{x} \in \bigcap_{i \in \alpha} F_i$, i.e. $A_\alpha \bar{x} = b_\alpha$. Since $\alpha \in G_y$, there exists $\lambda_\alpha \geqq 0$ for which $b_\alpha \cdot \lambda_\alpha = 1$ and $y = \lambda_\alpha A_\alpha$. However, $y \cdot \bar{x} = \lambda_\alpha A_\alpha \bar{x} = \lambda_\alpha b_\alpha = 1$. However, this implies that $y \notin \text{int}\,\mathcal{X}$, a contradiction, and so the theorem is proved. ∎

Theorems 1 and 2 (without the oddness assertion) are equivalent to the fixed-point theorem of L. E. J. Brouwer [2], stated below:

BROUWER'S THEOREM ON A BOUNDED POLYHEDRON. *Let $\mathcal{X}$ be a non-empty bounded polyhedron, and let $f(\cdot): \mathcal{X} \to \mathcal{X}$ be a continuous function. Then there exists a fixed point of $f(\cdot)$, i.e. a point $x^* \in \mathcal{X}$ such that $f(x^*) = x^*$.*

In order to demonstrate the equivalence of Theorems 1 and 2 to Brouwer's theorem, we will use the following lemma, which relates the equivalance of polyhedral representations under projective transformation.

PROJECTIVE TRANSFORMATION LEMMA. *Let $\mathcal{X} = \{x \in R^n \mid Ax \leqslant b\}$ be a polyhedron that satisfies Assumption A, and let $\mathcal{X}^0 = \{y \in R^n \mid y = \lambda A, \lambda > 0, b \cdot \lambda = 1\}$. For any given $y \in \text{int}\,\mathcal{X}^0$, then the set $\mathcal{X}' = \{x' \in R^n \mid (A - e\,y)x' \leqslant b\}$ is combinatorially equivalent to $\mathcal{X}$, and $\mathcal{X}'^0 = \mathcal{X}^0 - y$. The projective transformation $g(x) = x/(1 - y \cdot x)$ maps faces of $\mathcal{X}$ onto the faces of $\mathcal{X}'$ and is inclusion preserving. Furthermore, T is a triangulation of $\mathcal{X}$ if and only if T' is a triangulation of $\mathcal{X}'$, where T' is the collection of simplices $\sigma' = g(\sigma)$ for every $\sigma \in T$.* ∎

See [13] for properties of polyhedra under projective transformation.

Proof of Theorem 1 (without the Oddness Assertion) from Brouwer's Theorem. Let $\mathcal{X}$, T, $L(\cdot)$, and K be given as in Theorem 1. Let $y \in \text{int}\,\mathcal{X}^0$

be given, define $\mathscr{X}'$ and T' as in the projective transformation lemma, let K' be the pseudomanifold corresponding to T', and define $L'(v') = L(g^{-1}(v'))$ for $v' \in K^{\prime 0}$. For each $v' \in K^{\prime 0}$, define $h(v') = A_{L'(v')} - y$, and extend $h(\cdot)$ in a piecewise-linear manner over all of $\mathscr{X}'$. Define $f(x') = \arg\min_{z' \in \mathscr{X}'} \|z' - x' + h(x')\|_2$, where $\|\cdot\|_2$ denotes the Euclidean norm. Because $h(\cdot)$ is continuous, $f(\cdot)$ is continuous and so contains a fixed point $\bar{x}'$. Let $\bar{\sigma}'$ be the smallest simplex σ' in T' that contains $\bar{x}'$, and let $\gamma = L(\bar{\sigma}')$, $\beta = C(\bar{\sigma}')$, and $\alpha = \gamma \cup \beta$. Then the Karush–Kuhn–Tucker conditions state that $\bar{x}' - \bar{x}' + h(\bar{x}') = -\bar{\lambda}_\beta(A - e \cdot y)_\beta$, for some $\lambda_\beta \geqslant 0$. Furthermore, $h(x') = \bar{\lambda}_\gamma(A - e \cdot y)_\gamma$ for some particular $\bar{\lambda}_\gamma \geqslant 0$, $\bar{\lambda}_\gamma \cdot e_\gamma = 1$. Therefore, $\bar{\lambda}_\beta(A - e \cdot y)_\beta + \bar{\lambda}_\gamma(A - e \cdot y)_\gamma = 0$, whereby $\lambda_\alpha(A - e \cdot y)_\alpha = 0$ has a nonnegative and nonzero solution. Upon rescaling the multipliers $\bar{\lambda}_\alpha$ so that they sum to unity, we have $\bar{\lambda}_\alpha A = y$, $\bar{\lambda}_\alpha \geqslant 0$, $\bar{\lambda}_\alpha \cdot e_\alpha = 1$. Thus $\alpha \in G_y$ and $(L(\bar{\sigma}') \cup C(\bar{\sigma}')) = \alpha$, whereby the simplex $\bar{\sigma} \in T$ defined by $\bar{\sigma} = g^{-1}(\bar{\sigma}')$ has $(L(\bar{\sigma}) \cup C(\bar{\sigma})) = \alpha \in G_y$, proving the result. ∎

The construction of the function $f(\cdot)$ was introduced by Eaves [3] to convert the stationary-point problem of $h(\cdot)$ to a fixed-point problem on $f(\cdot)$.

Proof of Brouwer's Theorem from Theorem 1. Let $\mathscr{X}$ be a polyhedron that satisfies Assumption A, and let $f(\cdot)\colon \mathscr{X} \to \mathscr{X}$ be a continuous function. Let T be a finite triangulation of $\mathscr{X}$ and let K be the pseudomanifold corresponding to T. Let $L(\cdot)$ be a labelling function on K^0 defined so that $L(v)$ equals any index i for which $A_i(v - f(v)) \geqslant 0$ and is maximum over all rows. Because $\mathscr{X}$ is bounded, such an index i must exist. Now let $y = 0$. Then $y \in \operatorname{int} \mathscr{Y}$. From Theorem 1, there exists a simplex $\sigma \in K$ such that $(L(\sigma) \cup C(\sigma)) \in G_y$.

Now consider an infinite sequence of triangulations T^j, the maximum diameter of whose simplices goes to zero as $j \to \infty$. Then there exists a sequence of simplices σ^j such that $(L(\sigma^j) \cup C(\sigma^j)) \in G_y$. Let $\bar{x}$ be any accumulation point of the sequence of σ^j, and let δ and β be any accumulation point of the appropriate subsequence of $L(\sigma^j)$ and $C(\sigma^j)$, respectively. Also, let $\gamma = C(\bar{x})$. Then, from the continuity $f(\cdot)$, $A_i(\bar{x} - f(\bar{x})) \geqslant 0$ for all $i \in \delta$, $A_i\bar{x} = 1$ for all $i \in \beta$, and $\beta \subset \gamma$, and so $A_i(\bar{x} - f(\bar{x})) \geqslant 0$ for all $i \in \delta \cup \beta$. Let $\alpha = \delta \cup \beta$. Because $\alpha \in G_y$, there exists $\lambda_\alpha \geqslant 0$, with $e_\alpha \lambda_\alpha = 1$ and $\lambda_\alpha A_\alpha = 0$. Thus $\lambda_\alpha A_\alpha(\bar{x} - f(\bar{x})) = 0$. If there exists an index $i \in \delta$ for which $\lambda_i > 0$, then $A_i(\bar{x} - f(\bar{x})) = 0$, and hence $A(\bar{x} - f(\bar{x})) = 0$, because the indices i were chosen maximally; this implies that $\bar{x} = f(\bar{x})$, because $\mathscr{X}$ is bounded. It thus remains to show that $\lambda_i > 0$ for some $i \in \delta$. Assuming the contrary, then $\lambda_\delta = 0$, and hence $\lambda_\alpha A_\alpha = \lambda_\beta A_\beta = 0$, $\beta \subset C(\bar{x})$, and $e_\beta \cdot \lambda_\beta = 1$. But if this were true then $0 = y \cdot \bar{x} = \lambda_\beta A_\beta \bar{x} = \lambda_\beta \cdot e_\beta = 1$, a contradiction. Thus $\lambda_\delta \neq 0$, and the derivation is proved. ∎

Relation of Theorems 1 and 2 to Other Combinatorial Results on Bounded Polyhedra

In this subsection, we show how various other combinatorial labelling results on bounded polyhedra can be derived as specific instances of Theorems 1 and 2. We first examine how Theorem 1 specializes to a strong form of a recent theorem due to Yamamoto, for labellings of a triangulation of a bounded polyhedra. We next show how Theorems 1 and 2 specialize to some well-known combinatorial theorems on the simplotope and the simplex, including the Generalized Sperner lemma [11] and Scarf's dual Sperner lemma [22].

A Theorem of Yamamoto

Let $\mathscr{X}$ be a polyhedron satisfying Assumption A, let T be a triangulation of $\mathscr{X}$, let K be the pseudomanifold corresponding to $\mathscr{X}$, and let $L(\cdot)\colon K^0 \to M$ be a given labelling function. If $\bar{x}$ is a nondegenerate extreme point of $\mathscr{X}$, then $C(\bar{x}) = \beta$ for some subset β of M with cardinality n. Furthermore, S_β then is a facet of $\mathscr{X}^*$ and is an $(n-1)$-simplex. Let $\bar{y}$ be an element of rel int S_β and let y be any regular point of $\mathscr{X}^*$ lying sufficiently close to $\bar{y}$. Then the minimal members of G_y consist of all sets of the form $\beta \cup \{j\}$, where $j \in M \setminus \beta$. From Theorem 1, then, there exist an odd number of simplices $\sigma \in K$ such that $L(\sigma) \cup C(\sigma) \supseteq \beta \cup \{j\}$ for some $j \in M \setminus \beta$. Thus there are an odd number of simplicies $\sigma \in K$ with the property that $L(\sigma) \cup C(\sigma)$ properly contains $C(\bar{x})$. This last statement is a stronger version of a recent result due to Yamamoto:

THEOREM 17 OF YAMAMOTO [25]. *Let $\mathscr{X} = \{x \in R^n \mid Ax \leqslant b\}$ be a bounded polyhedron, let T be a triangulation of $\mathscr{X}$, let K be the pseudomanifold corresponding to T, and let $L(\cdot)\colon K^0 \to M$ be a labelling function defined on the row indices M of the constraint matrix. Then if $\bar{x}$ is a nondegenerate extreme point of $\mathscr{X}$, there exists at least one simplex $\sigma \in K$ with the property that $L(\sigma) \cup C(\sigma)$ properly contains $C(\bar{x})$.*

Thus Yamamoto's theorem can be seen as an instance of Theorem 1, and his result can be made stronger. Indeed, as the previous remarks state, there are an *odd* number of such simplices σ with the indicated labelling property.

Combinatorial Theorems on the Simplex and Simplotope

We now show how Theorems 1 and 2 specialize to known results on the simplex and the simplotope. The three major combinatorial results on the simplex, namely Sperner's lemma [23], Scarf's dual Sperner lemma [22], and the Generalized Sperner lemma [11], all assert the existence of an odd number of simplices with certain label configurations. However, when these three results are extended to the cube and simplotope, the oddness assertion disappears, and the dimension of the specially labelled simplices of interest is reduced (see [7, 18]). The inability to assert that there are an odd number of specially labelled simplices stems from the constructive proofs of these simplotope theorems. Herein, by casting the simplex and simplotope theorems as instances of Theorems 1 and 2 for particular values of $\bar{y} \in \mathscr{S}^*$, we will see that the oddness assertion holds on the simplex precisely because $\bar{y}$ is a regular point in $\mathscr{S}^*$, and the oddness assertion on the simplotope (and hence the cube) does not hold, precisely because $\bar{y}$ is not a regular point in $\mathscr{S}^*$.

Let $S^n = \{x \in R^n \mid x \geq e, -e \cdot x \leq 1\}$. Then S^n is an n-dimensional simplex. By defining

$$A^n = \begin{bmatrix} I \\ -e^t \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} e \\ 1 \end{bmatrix}$$

we can write S^n as $S^n = \{x \in R^n \mid A^n x \leq b\}$. Let T be a triangulation of S^n, K the pseudomanifold corresponding to T, and $L(\cdot): K^0 \to M$, where $M = \{1, ..., m\} = \{1, ..., n+1\}$, because $m = n+1$. For $\mathscr{S} = S^n$, the set $\mathscr{S}^* = \{y \mid y = \lambda A^n, \lambda \geq 0, e \cdot \lambda = 1\}$ is an n-simplex that contains the origin, and any $y \in \operatorname{int} \mathscr{S}^*$ is a regular point in $\mathscr{S}^*$. In particular, $\bar{y} = 0$ is a regular point in $\mathscr{S}^*$, and $G_{\bar{y}} = \{M\} = \{\{1, ..., n+1\}\}$. Because S^n satisfies Assumption A, we can apply Theorem 1, and assert that there are an odd number of simplices $\sigma \in K$ with the property that $(L(\sigma) \cup C(\sigma)) \in G_{\bar{y}}$, i.e. $L(\sigma) \cup C(\sigma) = \{1, ..., n+1\}$. This is precisely the Generalized Sperner lemma [11], and is seen to follow as a specific instance of Theorem 1.

Now suppose that the labelling $L(\cdot)$ is dual-proper: i.e., for each $v \in \partial S^n$, $L(v) = i$ must be chosen so that $A_i v = b_i$. Furthermore, suppose that no simplex of T meets every facet F_i of S^n, where $F_i = \{x \in S^n \mid A_i x = b_i\}$, $i = 1, ..., n+1$. Then it can be shown that for any simplex $\bar{\sigma}$ of T, the intersection of all faces of S^n that meet $\bar{\sigma}$ is nonempty; i.e., T is bridgeless, whereby the conditions of Theorem 2 are satisfied. Thus there exist an odd number of simplices $\sigma \in K$ such that $L(\sigma) \in G_{\bar{y}}$, i.e., $L(\sigma) = \{1, ..., n+1\}$. This latter result is precisely Scarf's dual Sperner lemma [22], and it is seen to follow as a specific instance of Theorem 2.

We now turn our attention to theorems on the simplotope. A simplotope S is defined to be the cross-product of n simplices, $S = S^{m_1} \times \cdots \times S^{m_p}$, where, for simplicity, we will assume that each $m_j \geq 1$, $j = 1, ..., p$. Any point $x \in S$ is a vector in R^N, where $N = \sum_{j=1}^{p} m_j$, and x can be written as $x = (x^1; ...; x^p)$, where each each $x^j \in R^{m_j}$, $j = 1, ..., p$, and x is the concatenation of the n vectors x^j, $j = 1, ..., p$. Defining A^n as above, let us define A as the $(N+p) \times (N)$ matrix,

$$A = \begin{bmatrix} A^{m_1} & & 0 \\ & \ddots & \\ 0 & & A^{m_p} \end{bmatrix},$$

where A^n is as described previously.

Then S can be described as $S = \{x \in R^N \mid Ax \leq b\}$ where $b \in R^{N+p}$ and $b = e$. Define $M = \{(j, k) \mid j = 1, ..., p, k = 1, ..., m_j + 1\}$. The rows of A can be indexed by the ordered pairs $(j, k) \in M$ where row (j, k) of A is in fact row number $(\sum_{i=1}^{j-1} (m_i + 1) + k)$ of A. Likewise, a vector $\lambda \in R^{N+p}$ will be indexed by the ordered pairs $(j, k) \in M$. Let T be a triangulation of S, let K be the pseudomanifold corresponding to T, and let $L(\cdot): K^0 \to M$ be a labelling function. For $\mathscr{S} = S$, $\mathscr{S}$ satisfies Assumption A, and so the conditions of Theorem 1 are met. We have $\mathscr{S}^* = \{y \in R^n \mid y = \lambda A, e \cdot \lambda = 1, \lambda \geq 0\}$ and $\bar{y} = 0 \in \mathscr{S}^*$. However, $\bar{y} = 0$ is not a regular point of $\mathscr{S}^*$. To see this, pick any one index j from among $j \in \{1, ..., p\}$.

Then set

$$\lambda_{(i,k)} = \begin{cases} 0 & \text{if } i \neq j \\ 1/(m_i + 1) & \text{if } i = j. \end{cases}$$

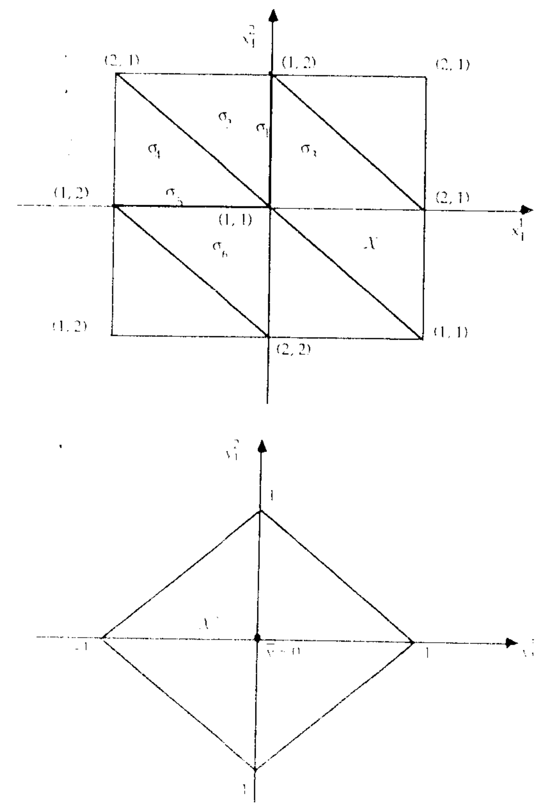

Figure 4

for each $(i, k) \in M$, and note that $\bar{\lambda} \geq 0$, $e \cdot \bar{\lambda} = 1$, and $\bar{\lambda} A = 0 = \bar{y}$. If we define $\alpha_j = \{(j, 1), \dots, (j, m_{j+1})\}$, we see that $0 \in S_{\alpha_j}$, but $|\alpha_j| = m_{j+1} < N + 1$, so long as $p > 1$. Thus $\bar{y} = 0$ is not a regular point of $\mathscr{P}^{\circ}$. Thus, by Theorem 1, we can only assert that there exists at least one simplex σ of K such that $(L(\sigma) \cup C(\sigma)) \in G_{\bar{y}}$. However, $G_{\bar{y}} = \{\alpha \subset M \mid \bar{y} \in S_\alpha\} = \{\alpha \subset M \mid \alpha = \alpha_j$ for some $j \in \{1, \dots, p\}\}$. Thus there exists a simplex σ of K such that $(L(\sigma) \cup C(\sigma)) \supseteq \{(j, 1), \dots, (j, m_{j+1})\}$ for some $j \in \{1, \dots, p\}$. This is precisely Theorem 1 of [7] or Lemma 3.2 of [18].

Figure 4 illustrates the theorem for $m_1 = m_2 = 1$ and $p = 2$. With $\bar{y} = 0$, $G_{\bar{y}} = \{\{(1, 1), (1, 2), (2, 1)\}, \{(1, 1), (1, 2), (2, 2)\}, \{(2, 1), (2, 2), (1, 1)\}, \{(2, 1), (2, 2), (1, 2)\}\}$. There are six simplices of S with $(L(\sigma) \cup C(\sigma)) \in G_{\bar{y}}$, namely $\sigma_1, \dots, \sigma_6$ in the figure. The set $\mathscr{P}^{\circ}$ is the convex hull of the points $(1, 0)$, $(-1, 0)$ $(0, 1)$, and $(0, -1)$, the diamond shown in the figure. As the figure shows, $\bar{y} = 0$ is not a regular point.

Suppose now that the labelling $L(\cdot): K^{\circ} \to M$ is dual proper, i.e., for each $v \in \partial S$, $L(v)$ must be chosen so that $A_i x = b_i$. Furthermore, suppose that no simplex $\sigma \in K$ meets each facet $F_{(j,k)} = \{x \in S \mid A_{(j,k)} x = b_{(j,k)}\}$, for all $(j, k) \in \alpha_j$, for any $j = 1, \dots, p$. Then it can be shown that the requirements of Theorem 2 are met. This being the case, the logic employed herein can be used to show that there exists a simplex $\sigma \in K$ such that $L(\sigma) \supseteq \alpha_j$ for some $j \in \{1, \dots, p\}$. This latter statement is precisely Theorem 2 of [7], and thus is a specific instance of Theorem 2 of this paper.

4. Limitations and Extensions of Theorem 1

Much of the beauty that lies in the classic combinatorial results that are analogs of topological theorems stems from the fact that the results are completely combinatorial in nature, and are independent of any particular geometric representation of the underlying polyhedron. For example, the Sperner lemma and Tucker's lemma are purely combinatorial statements about labelled pseudomanifolds whose boundaries have particular combinatorial properties, and yet these lemmas are precise analogs of theorems in continuous topology, namely Brouwer's fixed-point theorem and the Borsuk Ulam antipodal-point theorem, respectively. The other combinatorial theorems cited in the introduction to this paper all have this property as well.

The research that has led to the development of Theorem 1 was motivated by a desire to extend these other purely combinatorial results to the most general case: to present a purely combinatorial theorem for a bounded polyhedron that is completely independent of the geometric representation of the polyhedron, and is a combinatorial analog of Brouwer's fixed-point theorem. This section discusses the extent to which

this goal has been achieved, and presents open questions for further research.

Variance under Geometric Representation

In developing a general combinatorial theorem for a labelled triangulation of a bounded polyhedron $\mathscr{P}$, one of the aims is to state a result that is invariant under the geometric representation of the polyhedron, i.e., that is identical for all polyhedra of the same combinatorial type. Theorem 1, as stated, depends on the rows of the constraint matrix (A, b) of $\mathscr{P}$, and so appears to be dependent on the geometric representation of $\mathscr{P}$. In Theorem 1, $\mathscr{P}$ must satisfy the geometric conditions of Assumption A, namely that $\mathscr{P}$ is bounded, solid, centered, and that the rows of (A, b) have been scaled and contain no redundant constraints. Here we discuss the extent to which these assumptions can be relaxed without affecting the conclusions of Theorem 1.

The assumption that $\mathscr{P}$ is bounded is fundamental to Brouwer's theorem, as well as to the finite counting arguments to be developed in the proof of Theorem 1 in the next section, and so cannot be eliminated. Because redundant constraints do not contribute to either the geometric or combinatorial properties of a polyhedron, we retain the nonredundancy assumption to maintain the simplicity of the system, without detracting from the generality of the conclusions of Theorem 1. The assumptions that $\mathscr{P}$ is solid, centered, and scaled, can be eliminated, but the definition of $\mathscr{P}$ must then be changed.

Let us first consider the case when $\mathscr{P} = \{x \in R^n \mid Ax \leq b\}$ is solid but not centered and scaled. For any given $x^{*} \in \operatorname{int} \mathscr{P}$, $\mathscr{P}' = \{x \in R^n \mid Ax \leq b - Ax^{*}\}$ is just a translation of $\mathscr{P}$ by $-x^{*}$, and can alternatively be written as $\mathscr{P}' = \{x \in R^n \mid \bar{A}x \leq e\}$, where $\bar{A}_i = A_i/(b_i - A_i x^{*})$. $\mathscr{P}'$ now is centered and scaled, and so the assertions of Theorem 1 apply. In this case the set $\mathscr{P}'^{\circ} = \{y \in R^n \mid y = \lambda \bar{A}, e \cdot \lambda = 1, \lambda \geq 0\} = \{y \in R^n \mid y = \lambda A, \lambda \geq 0, \lambda \cdot (b - Ax^{*}) = 1\}$, and for $\alpha \subset M$, $S_\alpha = \{y \in R^n \mid y = \lambda_\alpha A_\alpha, \lambda_\alpha \geq 0, \lambda_\alpha \cdot (b - Ax^{*})_\alpha = 1\}$. Thus Theorem 1 (and hence Theorem 2) can be modifieed to include the case when $\mathscr{P}$ is not centered and scaled.

When $\mathscr{P}$ is neither solid nor centered and scaled, then the affine hull of $\mathscr{P}$ as well as a point $x^{*} \in \operatorname{rel\,int} \mathscr{P}$ can be determined, using the methodology in [8], for example, and $\mathscr{P}$ can be orthonormally transformed and translated to an equivalent combinatorial type $\mathscr{P}'$ that satisfies Assumption A. Theorem 1 can then be respecified through the transformed polyhedron $\mathscr{P}'$. Details of this generalization of Theorem 1 can be found in Theorem 3 of [9].

Theorem 1 can be stated more abstractly for any given bounded polyhedron, as follows. Let $\mathscr{P}$ only satisfy the boundedness assumption, and let $\mathscr{P}^{*}$ be *any* given polyhedron that is a combinatorial dual of $\mathscr{P}$. Let the vertices of $\mathcal{X}^\circ$ be the points $A_1, \ldots, A_m$ and let A be the matrix whose rows are given by these extreme points. Every face F of $\mathcal{X}$ corresponds to a unique dual face F° in $\mathcal{X}^\circ$, where $F^\circ = \{y \in \mathcal{X}^\circ \mid y = \lambda_\alpha A_\alpha$ for some $\lambda_\alpha \geqslant 0$ that satisfies $\lambda_\alpha \geqslant 0$ and $\lambda_\alpha e_\alpha = 1\}$ for some unique subset α of $M = \{1, \ldots, m\}$. Thus every face of $\mathcal{X}$ can be denoted by F_α for a particular index set $\alpha \subset M$. For any subset S of $\mathcal{X}$, let $C(S)$ be the index α of the smallest face F_α that contains S. For each $y \in \mathcal{X}^\circ$, the set G_y is defined by $G_y = \{\alpha \subset M \mid y = \lambda_\alpha A_\alpha, \lambda_\alpha \geqslant 0, e_\alpha \cdot \lambda_\alpha = 1$ has a solution for some $\lambda_\alpha\}$; and $y \in \mathcal{X}^\circ$ is a regular point in $\mathcal{X}^\circ$ if every set α composing G_y has at least $d+1$ elements, where $d = \dim \mathcal{X}^\circ$. With this notation in mind, we have the following generalization of Theorem 1.

THEOREM 3. *Let $\mathcal{X}$ be a bounded polyhedron, and let $\mathcal{X}^\circ$ be any given combinatorial dual of $\mathcal{X}$. Let T be a finite triangulation of $\mathcal{X}$ and let K be the pseudomanifold corresponding to T. Let $L(\cdot): K^\circ \to M$, where $M = \{1, \ldots, m\}$ indexes the vertices of $\mathcal{X}^\circ$. Then:*

(i) *If y is a regular point of $\mathcal{X}^\circ$, there exist an odd number of simplices $\sigma \in T$ with the property that $(L(\sigma) \cup C(\sigma)) \in G_y$, and hence at least one.*

(ii) *If $y \in \operatorname{rel\,int} \mathcal{X}^\circ$, then there exists at least one simplex $\sigma \in T$ with the property that $(L(\sigma) \cup C(\sigma)) \in G_y$.*

We will not prove Theorem 3 here. Its proof follows as a natural generalization of the structure of Theorem 1, as will be seen in the proof of Theorem 1 in the next section.

One question that arises in light of Theorem 1 and Theorem 3 is whether the family of sets G_y is invariant, regardless of the geometric representation of $\mathcal{X}$ or $\mathcal{X}^\circ$. Stated another way, is it possible, given two combinatorially equivalent polyhedra $\mathcal{X}$ and $\bar{\mathcal{X}}$, to obtain dual polyhedra $\mathcal{X}^\circ$ and $\bar{\mathcal{X}}^\circ$, such that the set G_y arises for some $y \in \operatorname{int} \mathcal{X}^\circ$, but has no counterpart $G_{\bar{y}}$ for any $\bar{y} \in \operatorname{int} \bar{\mathcal{X}}^\circ$? If the answer to this question is no, then Theorem 1 is, in essence, completely independent of the geometric representation of the underlying polyhedra $\mathcal{X}$ and $\mathcal{X}^\circ$.

A partial answer of "yes" to the above question is given by the two dual polyhedra $\hat{\mathcal{X}}^\circ$ and $\bar{\mathcal{X}}^\circ$ shown in Fig. 5. Both are hexagons, and can be considered each as the dual of a polyhedron $\hat{\mathcal{X}}$ or $\bar{\mathcal{X}}$ which are combinatorially equivalent. The set $\hat{\mathcal{X}}^\circ$ gives rise to the index set $G_{\hat{y}}$ whose minimal members are $\{2, 3, 6\}$, $\{2, 4, 6\}$, $\{2, 3, 5\}$, $\{1, 4, 6\}$, $\{2, 4, 5\}$, $\{1, 3, 5\}$, $\{1, 4, 5\}$, and $\{1, 3, 6\}$. Such a set is not realizable in the hexagon $\bar{\mathcal{X}}^\circ$, however, and so the polyhedra $\hat{\mathcal{X}}$ and $\bar{\mathcal{X}}$, though combinatorially identical, give rise to different index sets for Theorem 1. This shows that the conclusions of Theorem 1 do indeed depend on the geometric representation of the underlying polyhedron $\hat{\mathcal{X}}$.

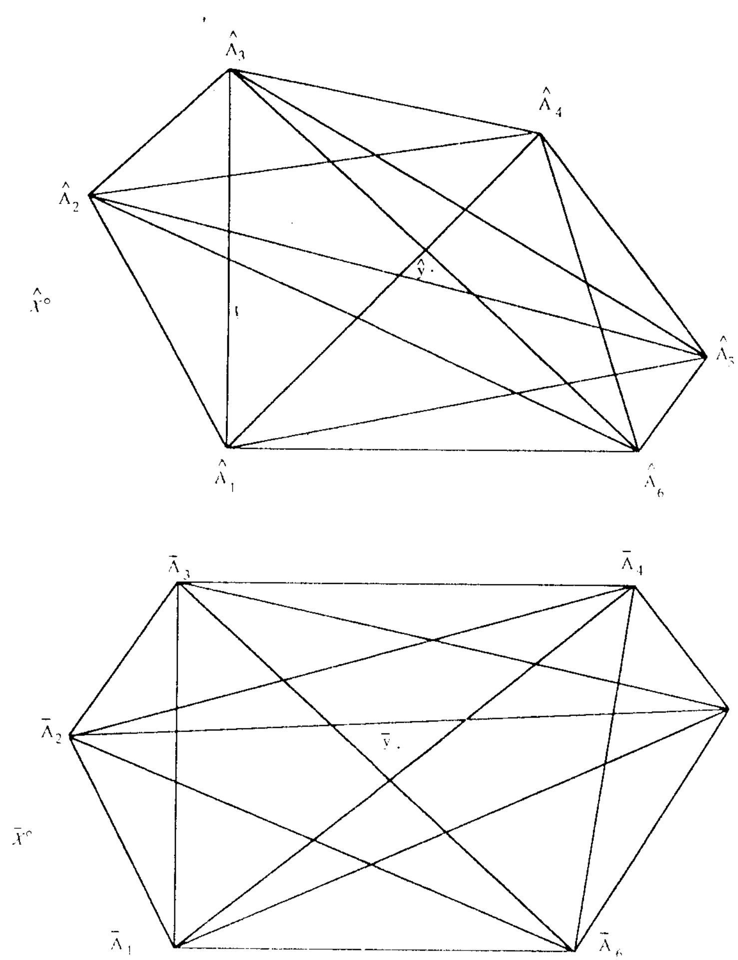

FIGURE 5

However, if the indices of the extreme points of $\bar{\mathcal{X}}^\circ$ are reordered appropriately, i.e., if we replace $\bar{A}_1$ by $\bar{A}_3$, $\bar{A}_3$ by $\bar{A}_1$, $\bar{A}_4$ by $\bar{A}_6$, and $\bar{A}_6$ by $\bar{A}_4$, then $G_{\hat{y}} = G_{\bar{y}}$ and the two dual polyhedra give rise to the same collection of index sets $G_{\hat{z}}$ (and $G_{\bar{z}}$) as $\hat{z}$ (and $\bar{z}$) varies over all points in $\hat{\mathcal{X}}^\circ$ (and $\bar{\mathcal{X}}^\circ$). This raises the question of the combinatorial structure of the collection of sets $\{G_z \mid z \in \mathcal{X}^\circ\}$. This set is not unique for combinatorially equivalent polyhedra $\mathcal{X}^\circ$. Are there, however, significant properties of the sets $\{G_z \mid z \in \mathcal{X}^\circ\}$ that are invariant for combinatorially equivalent polyhedra $\mathcal{X}^\circ$?

A Generalization of Sperner's Lemma

Theorems 1 and 2 have been shown to generalize combinatorial results on the simplex and the simplotope that have unrestricted or dual-proper labels, respectively. The Sperner lemma, and its extension to the simplotope [7, 17], is based on proper labels, and does not appear to be a specific instance of Theorems 1 or 2. Sperner's lemma can be derived from the Generalized Sperner lemma (see [6]) but this derivation fails to carry over to the simplotope. In the remainder of this section, we present a theorem that generalizes the results on the simplex and simplotope for proper labels, including Sperner's Lemma [23].

Let $\mathscr{X}$, T, and $L(\cdot)$ satisfy the assumptions of Theorem 1, and let $\mathscr{X}^{\circ} = \{y \in R^n \mid y = \lambda A, \lambda \geqslant 0, \lambda \cdot e = 1\}$. For any $y \in \operatorname{int} \mathscr{X}^{\circ}$, let $D_y = \{(\alpha, \beta) \in M \times M \mid \lambda_\beta A_\beta - \lambda_\alpha A_\alpha = y$ has a solution $\lambda_\beta, \lambda_\alpha$ such that $\lambda_\beta \geqslant 0$, $\lambda_\alpha \geqslant 0$, and $e_\alpha \cdot \lambda_\alpha + e_\beta \cdot \lambda_\beta = 1\}$. We have:

THEOREM 4. *Let $\mathscr{X} = \{x \in R^n \mid Ax \leqslant b\}$ satisfy Assumption A. Let T be a triangulation of $\mathscr{X}$, let K be the pseudomanifold corresponding to T, and let $L(\cdot): K^{\circ} \to M$ be a labelling function. Then if $y \in \operatorname{int} \mathscr{X}^{\circ}$, there exists at least one simplex $\sigma \in K$ with the property that $(L(\sigma), C(\sigma)) \in D_y$.*

Proof. Let $\mathscr{X}$, T, $L(\cdot)$, and K be given as in Theorem 4. Let $\bar{y} \in \operatorname{int} \mathscr{X}^{\circ}$ be given, and define $\mathscr{X}'$ and T' as in the projective transformation lemma, let K' be the pseudomanifold corresponding to T', and define $L'(v') = L(g^{-1}(v'))$ for $v' \in K'^{\circ}$, where $g(\cdot)$ is as defined in the projective transformation lemma. For each $v' \in K'^{\circ}$, define $h'(v') = -(A_{L'(v')} + \bar{y})$, and extend $h'(\cdot)$ in a piecewise-linear manner over all of $\mathscr{X}'$. Define $f'(x') = \arg\min_{z' \in \mathscr{X}'} \|z' - x' + h'(x')\|_2$, where $\|\cdot\|_2$ denotes the Euclidean norm. Because $h'(\cdot)$ is continuous, $f'(\cdot)$ is continuous and so contains a fixed point $\bar{x}'$. Let $\bar{\sigma}'$ be the smallest simplex σ' in T' that contains $\bar{x}'$, and let $\alpha = L(\bar{\sigma}')$, $\beta = C(\bar{\sigma}')$. Let $\bar{\sigma} = g^{-1}(\bar{\sigma}')$. Then $\alpha = L(\bar{\sigma})$ and $\beta = C(\bar{\sigma})$. The Karush–Kuhn–Tucker conditions state that $\bar{x}' - \bar{x}' + h'(\bar{x}') = -\bar{\lambda}_\beta(A - e\cdot\bar{y})_\beta$ for some $\bar{\lambda}_\beta \geqslant 0$. Furthermore, $h'(\bar{x}') = -\bar{\lambda}_\alpha A_\alpha - \bar{y}$ for some particular $\bar{\lambda}_\alpha \geqslant 0$, $e_\alpha \cdot \bar{\lambda}_\alpha = 1$. Therefore, $\bar{\lambda}_\beta A_\beta - \bar{\lambda}_\alpha A_\alpha = (e \cdot \bar{\lambda}_\beta + e \cdot \bar{\lambda}_\alpha)\bar{y}$. After normalizing the vectors $\bar{\lambda}_\beta$ and $\bar{\lambda}_\alpha$ so that the sum of the component of both vectors is one, we see that $(\alpha, \beta) = (L(\bar{\sigma}), C(\bar{\sigma})) \in D_y$. ∎

The proof of Theorem 1 using Brouwer's theorem, presented in Section 2, derives from the existence of an outward normal of the function h. The existence of an inward normal of $h(\cdot)$ is equivalent to the existence of a fixed point of $f(\cdot)$ (see Eaves [3]). When $y = 0$, the function $h'(\cdot)$ in the proof above is just $-h(\cdot)$ and the existence of an inward normal of $h(\cdot)$ is the same as the existence of an outward normal of $h'(\cdot)$.

To show that Sperner's lemma derives from Theorem 4, let S^n, A^n be defined as in Section 2, let T be a triangulation of S^n, let K be the pseudomanifold corresponding to T, and let $L(\cdot): K^{\circ} \to M$ be a labelling function, where $M = \{1, ..., n+1\}$. $L(\cdot)$ is said to be *proper* if for each $v \in K^{\circ}$, $L(v)$ is the index of an element of $M \setminus C(v)$, i.e.; $L(v)$ is the index of a nonbinding contraint of v, for $v \in K^{\circ}$. For $\mathscr{X} = S^n$, the set $\mathscr{X}^{\circ} = \{y \in R^n \mid y = \lambda A^n, \lambda \geqslant 0, e \cdot \lambda = 1\}$ is an n-simplex that contains the origin, and so $y = 0 \in \operatorname{int} \mathscr{X}^0$. The conditions of Theorem 4 are met, and so there exists a simplex $\sigma \in K$ with the property that $(L(\sigma), C(\sigma)) \in D_y$ for $y = 0$. Let $\alpha = L(\sigma)$, $\beta = C(\sigma)$; then there exists $\lambda_\alpha, \lambda_\beta$ such that $\lambda_\beta A^n_\beta = \lambda_\alpha A^n_\alpha$, $\lambda_\beta \geqslant 0$, $\lambda_\alpha \geqslant 0$, $e_\beta \cdot \lambda_\beta + e_\alpha \cdot \lambda_\alpha = 1$. Because $L(\cdot)$ is proper, $\alpha \cap \beta = \emptyset$. Note that for any $i, j \in M$, $i \neq j$, $A^n_i \cdot A^n_j \leqslant 0$. Thus $A^n_\beta (A^n_\alpha)^T \leqslant 0$ and so $0 \geqslant \lambda_\beta A^n_\beta (A^n_\alpha)^T \lambda_\alpha = (\lambda_\alpha A^n_\alpha)(\lambda_\alpha A^n_\alpha)^T \geqslant 0$ whereby $\lambda_\alpha A^n_\alpha = 0$; thus $\alpha = M = \{1, ..., n+1\}$, and so $L(\sigma) = \{1, ..., n+1\}$. This is precisely Sperner's lemma, without the oddness assertion.

The logic used above can also be used to prove Theorem 3 of [7] (see also van der Laan and Talman [17]), which generalizes Sperner's lemma to the simplotope.

Theorem 4 does not contain an assertion about the oddness of the number of simplices under consideration. The basic constructs used to prove Theorem 1 combinatorially do not appear to carry over directly to the case of Theorem 4. It is an open question whether there exists a combinatorial proof of Theorem 4 which asserts the existence of an odd number of simplices $\sigma \in K$ for which $(L(\sigma), C(\sigma)) \in D_y$, when y is regular.

5. A Combinatorial Proof of Theorem 1

This section contains a combinatorial proof of Theorem 1. The ideas behind the proof derive from relatively straightforward concepts that are easy to follow in two dimensions. In higher dimensions, they become more encumbered due to the possible presence of degeneracy in $\mathscr{X}$. Hence, in order to motivate the proof along more intuitive lines, we start by showing an example of the proof in two dimensions. We then proceed to the more general case

Example of Proof in Two Dimensions

Let $\mathscr{X}$ and $\mathscr{X}^{\circ}$ be as shown in Fig. 1, let T and $L(\cdot)$ be as shown in Fig. 2, and let K be the pseudomanifold corresponding to T. Define $\hat{K}$ to be the pseudomanifold consisting of simplices $\sigma \in K$ "joined" with the indices of $C(\sigma)$, i.e.,

$$\hat{K} = \{\hat{\sigma} \mid \hat{\sigma} = (\sigma \cup C(\sigma)), \sigma \in K\}$$

and

$$\bar{K} = K \times \{1, ..., m\} = K \times M.$$

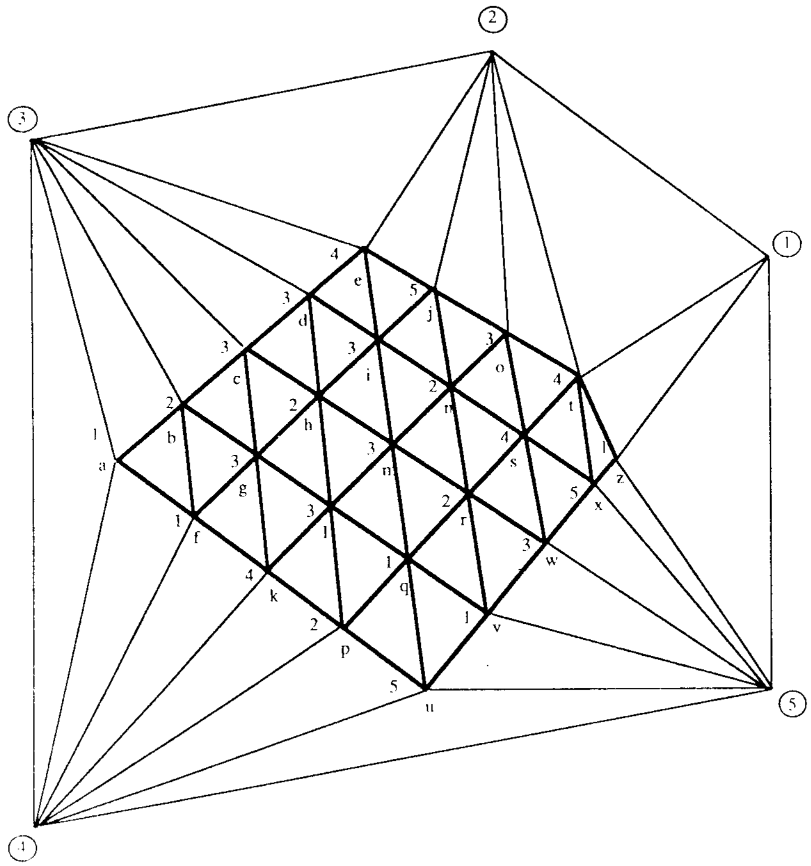

FIG. 6. The pseudomanifold $\bar{K}$.

The construction of $\bar{K}$ is shown in Fig. 6. Note that

$$\partial\bar{K} = \{\beta \mid \beta = C(x) \text{ for some } x \in K\}.$$

For each $i \in M$, extend $L(\cdot): K^0 \to M$ to $L(\cdot): \bar{K}^0 \to M$ by the association $L(i) = i$ for $i \in M$. For each $y \in \mathcal{T}$, let $\# G_y$ denote the number of simplices $\bar{\sigma} \in \bar{K}$ with the property that $L(\bar{\sigma}) \in G_y$. In order to prove Theorem 1, it suffices to show that $\# G_y$ is odd for all regular points $y \in \mathcal{T}$. Now let $\beta \subset M = \{1, \ldots, 5\}$ with $|\beta| = n = 2$. Let $R_\beta = \{\beta \cup \{j\}, j \in M, j \notin \beta\}$. For example, for $\beta = \{1, 3\}$, $R_\beta = \{\{1, 2, 3\}, \{1, 3, 4\}, \{1, 3, 5\}\}$. Let $\# R_\beta$ be the number of simplices $\bar{\sigma} \in \bar{K}$ with the property that $L(\bar{\sigma}) \in R_\beta$, and let q_β be the number of simplices $\bar{\sigma} \in \partial\bar{K}$ with the property that $L(\bar{\sigma}) = \beta$. A parity argument, first introduced by Kuhn [16], and later used by Gould and Tolle [12], states that the parity of $\# R_\beta$ and the parity of q_β are the same for any given β, with $|\beta| = n$. This implies, in particular, that

(i) if $\beta \in \partial\bar{K}$, $|\beta| = 2$, then $\# R_\beta$ is odd, and

(ii) if $\beta \notin \partial\bar{K}$, $|\beta| = 2$, then $\# R_\beta$ is even.

The first statement follows from the fact that if $\beta \in \partial\bar{K}$, then $L(\beta) = \beta$, and there is no other simplex $\bar{\sigma} \in \partial\bar{K}$ with $L(\bar{\sigma}) = \beta$. Thus $q_\beta = 1$, an odd number, whereby $\# R_\beta$ is odd. As an example, let $\beta = \{4, 5\}$. Note that $\beta \in \partial\bar{K}$. There are five simplices $\bar{\sigma} \in \bar{K}$ with $L(\bar{\sigma}) \in R_\beta$, namely $\{4, p, u\}$, $\{x, t, z\}$, $\{w, x, s\}$, $\{e, j, 2\}$, and $\{e, j, i\}$, an odd number. The second statement follows from the fact that if $\beta \notin \partial\bar{K}$, there can be no simplices $\bar{\sigma} \in \partial\bar{K}$ with $L(\bar{\sigma}) = \beta$. Thus $q_\beta = 0$, an even number, and hence $\# R_\beta$ is an even number. As an example, let $\beta = \{1, 4\}$, and hence $\beta \notin \partial\bar{K}$. There are four simplices $\bar{\sigma} \in \bar{K}$ with $L(\bar{\sigma}) \in R_\beta$, namely $\{a, 3, 4\}$, $\{f, g, k\}$, $\{x, t, z\}$, and $\{t, 1, 2\}$.

Now consider the set $\mathcal{T}$, shown in Fig. 7, subdivided into the eleven regions τ_k, $k = 1, \ldots, 11$. For any $y \in \operatorname{int} \tau_1$, y is a regular point of $\mathcal{T}$. Also, for any $y \in \tau_1$, $G_y = \{\{1, 4, 5\}, \{2, 4, 5\}, \{3, 4, 5\}\}$, i.e., $G_y = R_\beta$, where $\beta = \{4, 5\}$. Because $\beta \in \partial\bar{K}$, by (i) above, $\# R_\beta$ is odd, whereby $\# G_y$ is odd, because $R_\beta = G_y$. This proves Theorem 1 for all $y \in \operatorname{int} \tau_1$. For $y \in \operatorname{int} \tau_1$, those simplices $\bar{\sigma} \in \bar{K}$ for which $L(\bar{\sigma}) \in G_y$ are $\{p, u, 4\}$, $\{s, x, w\}$, $\{x, t, z\}$, $\{e, i, j\}$, and $\{e, j, 2\}$. The main fact that has been used is that all $y \in \operatorname{int} \tau_1$ are "sufficiently close" to the face $\langle A_4, A_5 \rangle$ so that $y \in \operatorname{int} \bigcap_{j \neq 4,5} \operatorname{conv} \langle A_4, A_5, A_j \rangle$, whereby $G_y = \{\{4, 5, j\} \mid j \neq 4, j \neq 5, j \in M\}$, i.e., $G_y = R_{\{4,5\}}$.

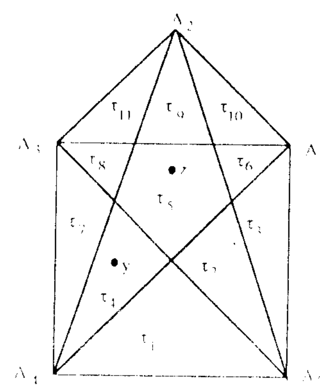

FIG. 7. The subdivided cell $\mathcal{T}$.

We next will show that if y and z are in the interior of adjacent regions τ_i and τ_j of $\mathcal{X}^{\circ}$, the parity of $\# G_y$ equals the parity of $\# G_z$. Since the parity of $\# G_y$ is odd for $y \in \operatorname{int} \tau_1$, then this will mean that the parity of $\# G_y$ is odd for $y \in \operatorname{int} \tau_k$, $k = 2, \ldots, 11$, proving assertion (i) of Theorem 1. Assertion (ii) follows from a closure argument.

Therefore, consider any two adjacent regions τ_i and τ_j, in $\mathcal{X}^{\circ}$, for example τ_4 and τ_5. For any $y \in \operatorname{int} \tau_4$ and $z \in \operatorname{int} \tau_5$, $G_y = \{\{1, 2, 4\}, \{2, 4, 5\}, \{2, 3, 4\}, \{1, 3, 4\}, \{3, 4, 5\}\}$, and $G_z = \{\{1, 2, 4\}, \{2, 4, 5\}, \{2, 3, 4\}, \{1, 3, 4\}, \{1, 3, 5\}, \{2, 3, 5\}\}$. Note that $G_y \Delta G_z = \{\{1, 3, 5\}, \{2, 3, 5\}, \{3, 4, 5\}\} = R_{\{3,5\}}$. Furthermore, the face of $\tau_4 \cap \tau_5$ that separates τ_4 from τ_5 is generated by the line segment $\langle A_3, A_5 \rangle$. It is no coincidence that the set $\{3, 5\}$ appears in each of the last two statements. Every simplex $\langle A_3, A_5, A_j \rangle$, $j \notin \{3, 5\}$, contains either τ_4 or τ_5 but not both. This shows that $R_{\{3,5\}} \subset G_y \Delta G_z$. But because the line segment $\langle A_3, A_5 \rangle$ is the unique line segment separating τ_4 from τ_5, then any α that lies in $G_y \Delta G_z$ must contain $\{3, 5\}$, i.e., $\alpha \supseteq \{3, 5\}$, whereby $G_y \Delta G_z \subseteq R_{\{3,5\}}$. Thus $G_y \Delta G_z = R_{\{3,5\}}$. For any collection D of subsets of M, let $\# D$ denote the number of simplices $\bar{\sigma} \in \bar{K}$ such that $L(\bar{\sigma}) \in D$. Note that

$$G_y = (G_y \setminus G_z) \cup (G_y \cap G_z),$$

whereby

$$\# G_y = \#(G_y \setminus G_z) + \#(G_y \cap G_z),$$

because these two sets are disjoint. Similarly, we have

$$\# G_z = \#(G_z \setminus G_y) + \#(G_y \cap G_z).$$

We obtain

$$\begin{aligned}
\# G_y - \# G_z &= \#(G_y \setminus G_z) - \#(G_z \setminus G_y) \\
&= \#(G_y \setminus G_z) + \#(G_z \setminus G_y) - 2\#(G_z \setminus G_y) \\
&= \#(G_y \Delta G_z) - 2\#(G_z \setminus G_y) \\
&= \# R_{\{3,5\}} - 2\#(G_z \setminus G_y).
\end{aligned}$$

However, $\# R_{\{3,5\}}$ is even, because $\{3, 5\} \notin \partial \bar{K}$. Therefore $\# G_y - \# G_z$ is even, i.e., $\# G_y$ and $\# G_z$ have the same parity. This completes the proof of Theorem 1 for the example of Figs. 1 and 2.

The important facts leading to the proof that $\# G_y$ and $\# G_z$ have the same parity if y and z are interior to adjacent regions τ_i and τ_j of $\mathcal{X}^{\circ}$ are as follows: If τ_i and τ_j are adjacent, there is a unique index set β such that

the $(n-1)$-simplex $S_\beta = \{y \in R^n \mid y = \lambda_\beta A_\beta,\ \lambda_\beta \geq 0,\ e \cdot \lambda_\beta = 1\}$ separates τ_i from τ_j. Furthermore, S_β cannot lie on $\partial \mathcal{X}^{\circ}$, whereby $\beta \notin \partial \bar{K}$. Finally, $G_y \Delta G_z = R_\beta$. Therefore $\# G_y - \# G_z = \# R_\beta - 2\#(G_y \setminus G_z)$, which is an even number.

Proof of Theorem 1. Let $\mathcal{X}$, T, $L(\cdot)$, and K be as given in Theorem 1. Because of the possibility that $\mathcal{X}^{\circ}$ is not a simplical polytope (i.e., some facet of $\mathcal{X}^{\circ}$ is not an $(n-1)$-simplex), we will perturb $\mathcal{X}^{\circ}$ in order to obtain a simplical polytope. Let $\mathcal{X}'$ be the simplical polytope obtained by *pulling* the vertices of $\mathcal{X}^{\circ}$ to general position (see [13, Chap. 5.2] for properties of the pulling procedure). A point $z \in \mathcal{X}'$ is a *regular point* of $\mathcal{X}'$ if z cannot be expressed as the convex combination of n or fewer vertices of $\mathcal{X}'$ and $z \in \mathcal{X}'$ is a *very regular point* of $\mathcal{X}'$ if z cannot be expressed as the affine combination of n or fewer vertices of $\mathcal{X}'$. (In two dimensions, all regular points are very regular. In higher dimensions, this need not be the case.) For any point $z \in \mathcal{X}'$, define G'_z to be the index sets of vertices of $\mathcal{X}'$ containing z in their convex hulls.

LEMMA 1. *Let y be a regular point of $\mathcal{X}^{\circ}$. Then there exists a very regular point $z \in \mathcal{X}'$ such that $G'_z = G_y$.*

Proof. This is evident if suitably small perturbations are used during the pulling procedure. ∎

Let $J = \{\alpha \subset M \mid \alpha = C(x)$ for some $x \in \mathcal{X}\}$. We call a member α of J an *admissible* index set. For each admissible α, S_α is a face of $\mathcal{X}^{\circ}$, and, in fact, J can alternatively be defined (dually) as $J = \{\alpha \subset M \mid S_\alpha$ is a face of $\mathcal{X}^{\circ}\}$. Given an admissible α, $F_\alpha = \{x \in X \mid A_\alpha x = b_\alpha\}$ is a face of $\mathcal{X}$, S_α is the face of $\mathcal{X}^{\circ}$ corresponding dually to F_α, and if $k = \dim F_\alpha$, then $n - k - 1 = \dim S_\alpha$.

Let K_α be the restriction of K to the face F_α, with vertices K_α. Then K_α is the k-pseudomanifold corresponding to the restriction of the triangulation T to the face F_α, where $k = \dim F_\alpha$. Furthermore $\sigma \in \partial K_\alpha$ if and only if $\sigma \in K_\alpha$ and α is a proper subset of $C(\sigma)$.

Because the vertices of $\mathcal{X}'$ are in general position, all faces of $\mathcal{X}'$ are simplices; and because the vertices of $\mathcal{X}'$ are obtained by pulling the vertices of $\mathcal{X}^{\circ}$, the faces of $\mathcal{X}'$ correspond in a natural way to a triangulation of the boundary of $\mathcal{X}^{\circ}$ without introducing any new vertices. Let K' be the $(n-1)$-pseudomanifold corresponding to this triangulation of $\mathcal{X}^{\circ}$. Let K'_α denote the restriction of this pseudomanifold to the face S_α of $\mathcal{X}^{\circ}$, where α is admissible. If $\dim F_\alpha = k$, then K'_α is an $(n-k-1)$-pseudomanifold, and $\beta \in \partial K'_\alpha$ if and only if $\beta \in K'_\alpha$ and β is a subset of some admissible set γ that is a proper subset of α. The above remarks are summarized in the following:

LEMMA 2. *For each $\alpha \in J$, let $k = \dim F_\alpha$. Then*

(1) *K_α is a k-pseudomanifold.*

(2) *$\sigma \in \partial K_\alpha$ if and only if $\sigma \in K_\alpha$ and α is a proper subset of $C(\sigma)$.*

(3) *K'_α is an $(n-k-1)$ pseudomanifold.*

(4) *$\beta \in \partial K'_\alpha$ if and only if $\beta \in K'_\alpha$ and β is a subset of some admissible set γ that is a proper subset of α.*

With the aid of Lemma 2, we are in a position to construct the n-dimensional version of the pseudomanifold $\bar K$ constructed in the proof of Theorem 1 in two dimensions.

Define

$$\bar K = K \cup M$$

and

$$\bar K = \{\bar\sigma \in \bar K \mid \bar\sigma = \sigma \cup \beta, \text{ where } \sigma \in K, \text{ and } \beta \in K' \text{ and } \beta \subseteq C(\sigma)\}.$$

Our main pseudomanifold construction is:

LEMMA 3. *$\bar K$ is an n-pseudomanifold, and $\partial \bar K = \{\beta \subset M \mid \beta \in K'\}$.*

Proof. Clearly $\bar K$ is closed under subsets. Let $\sigma \cup \beta \in \bar K$, and let $\alpha = C(\sigma)$. Let $k = \dim F_\alpha$. Then there exists a simplex $\bar\sigma \in K_\alpha$ with $\bar\sigma \supseteq \sigma$ and $|\bar\sigma| = k+1$. Also, since $\beta \in K'_\alpha$, there exists $\bar\beta \in K'_\alpha$ with $\bar\beta \supseteq \beta$ and $|\bar\beta| = n-k$. Thus $\sigma \cup \beta \subseteq \bar\sigma \cup \bar\beta \in \bar K$, and $|\bar\sigma \cup \bar\beta| = k+1+n-k = n+1$. Thus every member of $\bar K$ is contained in an n-simplex in $\bar K$. It remains to show that every $(n-1)$-simplex of $\bar K$ is contained in at most two n-simplices of $\bar K$.

Let $\sigma \cup \beta \in \bar K$ be an n-simplex in $\bar K$, and let $\alpha = C(\sigma)$, $k = \dim F_\alpha$. Then by the above remarks, $|\sigma| = k+1$ and $|\beta| = n-k$. Any $(n-1)$-simplex of $\sigma \cup \beta$ will either be of the form $\sigma \cup \beta \setminus v$ where $v \in \sigma$ or $\sigma \cup \beta \setminus i$ where $i \in \beta$. We first will consider the former case. We have three subcases:

(i) $\sigma \setminus v \notin \partial K_\alpha$. Then $C(\sigma \setminus v) = \alpha$, by Lemma 2. If $\sigma \cup \beta \setminus v \cup \{i\} \in \bar K$ for some $i \in M$, then $|\beta \cup \{i\}| = n-k+1$, and $\beta \cup \{i\} \in K'_\alpha$, which is a contradiction since K'_α is an $(n-k-1)$-pseudomanifold. Thus any other n-simplex of $\bar K$ which contains $\sigma \cup \beta \setminus v$ must be of the form $\sigma \cup \beta \setminus v \cup \{w\}$ for some $w \in \bar K$, $w \neq v$, and since $\sigma \setminus v \cup \{w\}$ is a k-simplex of K_α and $\sigma \setminus v \notin \partial K_\alpha$, such a w exists and is unique.

(ii) $\sigma \setminus v \in \partial K_\alpha$ and $\sigma \setminus v = \emptyset$. In this case $k = 0$, $F_\alpha = \{v\}$, K'_α is an $(n-1)$-pseudomanifold, and $|\beta| = n$. Thus there is no $i \in M$, $i \notin \beta$, for which $\beta \cup \{i\} \in K'$. Also, there is no element $w \neq v$, $w \in K$, for which $\{w\} \cup \beta \in K$. Thus $\sigma \cup \beta \setminus \{v\} = \beta \in \partial \bar K$.

(iii) $\sigma \setminus v \in \partial K_\alpha$ and $\sigma \setminus v \neq \emptyset$. Thus there can be no $w \in K$, $w \neq v$, for which $\sigma \cup \beta \setminus v \cup \{w\} \in \bar K$. Since $\sigma \setminus v \in \partial K_\alpha$, $C(\sigma \setminus v) = \bar\alpha$ contains α as a proper subset, and $\beta \subset \alpha \subseteq \bar\alpha$, which means that $\beta \in \partial K'_\alpha$, by Lemma 2. We must have $\dim F_\gamma = \dim F_\alpha - 1 = k-1$, and so K'_γ is an $(n-k)$-pseudomanifold. Therefore, there is a unique index $i \in M$ for which $\beta \cup \{i\} \in K'_\gamma$, and $\sigma \cup \beta \setminus v \cup \{i\} \in \bar K$.

We now consider the case when the $(n-1)$-simplex of $\sigma \cup \beta$ is of the form $\sigma \cup \beta \setminus i$ for some $i \in \beta$. We have two subcases:

(i) $\beta \setminus i \notin \partial K'_\alpha$. In this instance, there is a unique index $j \in M$, $j \neq i$, for which $\beta \setminus i \cup \{j\} \in K'_\alpha$, and hence $\sigma \cup \beta \setminus i \cup \{j\} \in \bar K$. Note that we cannot have $\sigma \cup \beta \setminus i \cup \{w\} \in \bar K$ for any $w \in K$. For if this were so, then $\sigma \cup \{w\} \in K$, and $\sigma \cup \{w\} \subset F_\gamma$ for some γ which is a proper subset of α. But since $\beta \setminus i \in K'_\gamma$, by Lemma 2, $\beta \setminus i \in \partial K'_\alpha$, a contradiction.

(ii) $\beta \setminus i \in \partial K'_\alpha$. In this case, there can be no $j \in M$, $j \neq i$, for which $\sigma \cup \beta \setminus i \cup \{j\} \in \bar K$. However, since $\beta \setminus i \in \partial K'_\alpha$, there exists a proper subset γ of α that is admissible, for which $\beta \setminus i \in K'_\gamma$, and hence $\dim F_\gamma = k-1$. Thus $\sigma \in \partial K_\alpha$, and there exists a unique vertex $v \in K$ such that $\sigma \cup \{v\} \in K_\gamma$. Thus $\sigma \cup \beta \setminus i \cup \{v\} \in \bar K$. ∎

The construction of $\bar K$ by essentially aligning faces of $\mathscr{T}$ with the dual faces of $\mathscr{T}$ resembles the construction of an antiprism in Broadie [1], but is combinatorial in nature and so does not depend on the geometric projection property in his work. The construction of $\bar K$ is also closely related to the construction of a primal-dual pair of subdivided manifolds, as in Kojima and Yamamoto [14], although $\bar K$ is combinatorial while the primal-dual pair of manifolds is not. An alternative construction of $\bar K$ that uses a lexicographic rule for constructing the triangulation of $\partial\mathscr{T}$ is offered in the appendix of [9].

We now extend $L(\cdot)\colon K \to M$ to $L(\cdot)\colon \bar K \to M$, by defining $L(i) = i$ for $i \in M$.

For each $\beta \subset M$, $|\beta| = n$, define $R_\beta = \{\beta \cup \{j\} \mid j \in M \setminus \beta\}$. For any collection D of subsets of M, let $\# D$ denote the number of simplices $\bar\sigma \in \bar K$ with the property that $L(\bar\sigma) \in D$. We have the following result:

LEMMA 4. *Let $\beta \subset M$ with $|\beta| = n$.*

(a) *If $\beta \in \partial \bar K$, then $\# R_\beta$ is odd, and*

(b) *if $\beta \notin \partial \bar K$, then $\# R_\beta$ is even.*

Proof. Let q_β be the number of simplices σ of $\partial \bar K$ with the property that $L(\bar\sigma) = \beta$. A simplex $\bar\sigma$ for which $L(\bar\sigma) = \beta$ is called β-complete, and a simplex $\bar\sigma$ is called β-very-complete if $L(\bar\sigma) \in R_\beta$, i.e., if β is a proper subset of

$L(\bar{\sigma})$. Then every β-complete n-simplex contains exactly two β-complete $(n-1)$-simplices, and every β-very complete n-simplex contains exactly one β-complete $(n-1)$-simplex. Hence the parity of the β-very-complete n-simplices equals the parity of β-complete boundary $(n-1)$-simplices; i.e., the parity of q_β equals the parity of $\# R_\beta$. If $\beta \in \partial\bar{K}$ and $|\beta| = n$, then $L(\beta) = \beta$ and $q_\beta = 1$, whereby $\# R_\beta$ is odd. If $\beta \notin \partial\bar{K}$, then $q_\beta = 0$, and hence $\# R_\beta$ is even. ∎

(The parity argument used above was first introduced by Kuhn [16] and later used by Gould and Tolle [12].)

We return now to the perturbed dual polyhedron $\mathscr{X}'$, whose vertices we denote by $v^1, \ldots, v^m$. Let $\beta \subset M$ with $|\beta| = n$ be given. Because the vertices are in general position, there is a unique hyperplane H_β that passes through all vertices v^i, where $i \in \beta$. The set of all such hyperplanes H_β as β ranges over all n-element subsets of M induces a piecewise linear subdivision of $\mathscr{X}'$ (see Eaves [4]). Let $\tau_1, \ldots, \tau_p$ be the collection of n-cells comprising this subdivision. The following is an elementary consequence of the above remarks.

LEMMA 5. *A point $z \in \mathscr{X}'$ is very regular if and only if $z \in \operatorname{int} \tau_i$ for some $i \in \{1, \ldots, p\}$. If z and w are both interior to the same n-cell τ_i of the subdivision, then $G_z = G_w$. Furthermore, if z and w lie in the interior of adjacent n-cells of the subdivision of $\mathscr{X}'$, then either $G'_w = G'_z$ or $G'_w \Delta G'_z = R_\beta$ for some n-element set $\beta \subset M$, and $\beta \notin \partial\bar{K}$.*

Figure 7 illustrates this lemma in two dimensions. We will need two additional intermediary results before we prove Theorem 1.

LEMMA 6. *For each $(n-1)$-simplex β of $\partial\bar{K}$, there exists some very regular point $z \in \mathscr{X}'$ for which $G'_z = R_\beta$.*

Proof. Let F be the convex hull of the vertices v^i of $\mathscr{X}'$, $i \in \beta$. Then by definition, $\beta \in \partial\bar{K}$ if and only if F is a facet of $\mathscr{X}'$. Let w be any point in rel int F, and let z be any point in int $\mathscr{X}'$ sufficiently close to w. Then $G_z = R_\beta$. ∎

LEMMA 7. *Let y be a regular point of $\mathscr{X}$. There is a one-to-one correspondence between simplices $\bar{\sigma} \in \bar{K}$ that satisfy $L(\bar{\sigma}) \in G_y$ and simplices $\sigma \in K$ that satisfy $(L(\sigma) \cup C(\sigma)) \in G_y$.*

Proof. Let $\bar{\sigma} = \sigma \cup \beta$ satisfy $L(\bar{\sigma}) \in G_y$. By definition, $L(\bar{\sigma}) = L(\sigma) \cup L(\beta) = L(\sigma) \cup \beta$. Since $L(\sigma) \cup \beta \in G_y$, then $L(\sigma) \cup C(\sigma) \in G_y$, because $\beta \subset C(\sigma)$ and G_y is closed under supersets.

Conversely, let $\sigma \in K$ such that $L(\sigma) \cup C(\sigma) \in G_y$. Let F be the smallest face of $\mathscr{X}$ containing σ and let F^* be the corresponding face of the boundary of $\mathscr{X}^*$ under polarity. Let $\alpha = C(\sigma)$, let $L = L(\sigma)$, and let $\gamma = L \cup \alpha$. Then $F^* = S_\alpha$. Also, let $k = \dim F$. Because $y \in \operatorname{int} S_\gamma$, $\dim S_\gamma = n$. However, $\dim S_\alpha = n - k - 1$, and $\dim S_L \leqslant |\sigma| = (\dim \sigma) + 1 \leqslant (\dim F) + 1 = k + 1$. Therefore $\dim S_\gamma \leqslant \dim S_\alpha + \dim S_L \leqslant n - k - 1 + k + 1 = n$, whereby $\dim S_L = k + 1$, and $|L| = k + 1$. This means in turn that S_γ is a $(k-1)$-fold pyramid over S_α (see [13, Chap. 4.2]). The restriction of K' to S_α induces the triangulation of S_α whose maximal simplices correspond to maximal simplices in K'_α, and hence are of the form S_β, where $\beta \in K'_\alpha$, $|\beta| = n - k$, and $\beta \subseteq \alpha$. The pseudomanifold K'_α induces a triangulation of S_γ whose maximal simplices are $S_{L \cup \beta}$ for every maximal index set $\beta \in K'_\alpha$. Because y is a regular point of $\mathscr{X}^*$, y lies in the interior of exactly one maximal simplex $S_{L \cup \beta}$ of S_γ, and hence $L \cup \beta \in G_y$. Thus $\sigma \cup \beta \in \bar{K}$ and we have $\bar{\sigma} = \sigma \cup \beta$ and $L(\bar{\sigma}) \in G_y$. ∎

We now have:

Proof of Theorem 1. Let y be a given regular point of $\mathscr{X}^*$. From Lemma 7, it suffices to show that $\# G_y$ is odd. From Lemma 1, it suffices to show that $\# G'_z$ is odd for every very regular point $z \in \mathscr{X}'$. We prove this last statement as follows.

Let $\beta \in \partial\bar{K}$ with $|\beta| = n$. Let z be the very regular point described in Lemma 6. Then $G'_z = R_\beta$ and by Lemma 4, $\# R_\beta$ is odd, and hence $\# G'_z$ is odd. Now let τ_i be the unique n-cell of the piecewise-linear subdivision of $\mathscr{X}'$ that contains z. Then by Lemma 5, $G'_w = G'_z$ for all $w \in \operatorname{int} \tau_i$. Now let τ_j be another n-cell of the subdivision of $\mathscr{X}'$ that is adjacent to τ_i, and let w now lie in int τ_j. Then, by Lemma 5, either $G'_w = G'_z$, whereby $\# G'_w$ is odd, or $G'_w \Delta G'_z = R_\gamma$ for some $\gamma \notin \partial\bar{K}$, $|\gamma| = n$. We have $G'_z = (G'_z \setminus G'_w) \cup (G'_z \cap G'_w)$, whereby

$$\# G'_z = \# (G'_z \setminus G'_w) + \# (G'_z \cap G'_w).$$

Similarly, $\# G'_w = \# (G'_w \setminus G'_z) + \# (G'_w \cap G'_z)$, and so

$$\begin{aligned} \# G'_w + \# G'_z &= \# (G'_w \setminus G'_z) + \# (G'_z \setminus G'_w) + 2\, \# (G'_z \cap G'_w) \\ &= \# (G'_w \Delta G'_z) + 2\, \# (G'_z \cap G'_w) \\ &= \# R_\gamma + 2\, \# (G'_z \cap G'_w). \end{aligned}$$

However, by Lemma 4, $\# R_\gamma$ is even, and so $\# G'_w$ is odd, as it has the same parity as $\# G'_z$. Proceeding in like fashion over all adjacent n-cells τ_j of the subdivision of $\mathscr{X}'$, we see that $\# G'_z$ is odd for all very regular points of $\mathscr{X}'$, completing the proof. ∎

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References

1. M. N. Broadie, A theorem about antiprisms, *Linear Algebra Appl.* **66** (1985), 99–111.
2. L. E. J. Brouwer, Über Abbildung von Mannigfaltigkeiten, *Math. Ann.* **71** (1910), 97–115.
3. B. C. Eaves, On the basic theorem of complementarity, *Math. Programming* **1** (1971), 168–75.
4. B. C. Eaves, A short course in solving equations with *PL* homotopies, *SIAM–AMS Proc.* **9** (1976), 73–143.
5. K. Fan, Combinatorial properties of certain simplicial and cubical vertex maps, *Arch. Math.* **11** (1960), 368–377.
6. R. M. Freund, Variable dimension complexes, part II: A unified approach to some combinatorial lemmas in topology, *Math. Oper. Res.* **9** (1984), 498–509.
7. R. M. Freund, Combinatorial theorems on the simplotope that generalize results on the simplex and cube, *Math. Oper. Res.* **11** (1986), 169–179.
8. R. W. Freund, R. Roundy, and M. J. Todd, Identifying the set of Always-Active Constraints in a System of Linear Inequalities by a Single Linear Program," Sloan Working Paper No. 1674-85, MIT, July, 1985."
9. R. M. Freund, "Combinatorial Analogs of Brouwer's Fixed-Point Theorem on a Bounded Polyhedron," Sloan Working Paper No. 1720-85, MIT, October, 1985.
10. D. Gale, The game of hex and the Brouwer fixed point theorem, *Amer. Math. Monthly*, **86**, 10.
11. C. B. Garcia, A hybrid algorithm for the computation of fixed points, *Management Sci.* **22**, No. 5 (1976), 606–613.
12. F. J. Gould and J. W. Tolle, A unified approach to complementarity in optimization, *Discrete Math.* **7** (1974), 225–271.
13. B. Grünbaum, "Convex Polytopes," Wiley, New York, 1967.
14. M. Kojima and Y. Yamamoto, Variable dimension algorithms: Basic theory, interpretations and extensions of some existing methods, *Math. Programming* **24** (1982a), 177–215.
15. H. W. Kuhn, Some combinatorial lemmas in topology, *IBM J. Res. Develop.* **4** (1980), 518–524.
16. H. W. Kuhn, Simplicial approximation of fixed points, *Proc. Nat. Acad. Sci. U.S.A.* **61** (1968), 1238–1242.
17. G. van der Laan and A. J. J. Talman, On the computation of fixed points in the product space of unit simplices and an application to noncooperative *N* person games, *Math. Oper. Res.* **7**, No. 1 (1982), 1–13.
18. G. van der Laan, A. J. J. Talman, and L. Van der Heyden, Variable dimension algorithms for unproper Labellings, *Mat. Oper. Res.*, in press.
19. S. Lefschetz, "Introduction to topology," Princeton Univ. Press, Princeton, NJ, 1949.
20. R. T. Rockafellar, "Convex Analysis," Princeton Univ. Press, Princeton, NJ, 1970.
21. H. Scarf, The approximation of fixed points of a continuous mapping, *SIAM J. Appl. Math.* **15**, No. 5 (1967a), 1328–1343.
22. H. Scarf, The computation of equilibrium prices. An exposition, *in* "Handbook of Mathematical Economics, (K. J. Arrow and M. D. Intrilligator, Eds), Vol. II, North-Holland, Amsterdam, 1982.
23. E. Sperner, Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes, *Abh. Math. Sem. Univ. Hamburg* **6**.
24. A. W. Tucker, Some topological properties of disk and sphere, "Proceedings of the First Canadian Mathematical Congress," pp. 285–309, Montreal, 1945.
25. Y. Yamamoto, "Orientability of a Pseudomanifold and Generalization of Sperner's Lemma," Report No. 86411-OR, Institut für Ökonometrie und Operations Research, Rheinische Friedrich-Wilhelms-Universität, Bonn, 1986.