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Segre products and Rees algebras of face rings

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1 Introduction

There has been a great deal of research on how homological properties of face rings (Stanley-Reisner rings), such as being Buchsbaum, Cohen-Macaulay (C-M), and Gorenstein, are reflected in topological properties of their associated simplicial complexes and vice versa. One aim of this paper is to investigate the ascent and descent of these homological properties when taking the Segre product of two face rings. As the Rees algebra of an algebra A can be viewed as a Segre product, our results are valid in particular for Rees algebras of face rings. We also show that the Rees algebra of a face ring $k[\Delta]$ has lots in common with the face ring of the closed cylinder of Δ, with an appropriate triangulation. The paper is organized as follows. In Section 2 we give some basic facts on the Segre product of two graded algebras. We interprete the Rees algebra $\mathcal{R}(A)$ of an algebra A as the Segre product of A and $k[t_1, t_2]$, and show that $\mathcal{R}(A)$ is a Koszul algebra (see next section for the definition) if and only if A is. In Section 3 we restrict to face rings. We derive criteria for the Segre product of two face rings to be Buchsbaum, C-M, and Gorenstein, respectively. This implies corresponding criteria for the Rees algebra. In Section 4 we compare the Rees algebra of a face ring $k[\Delta]$ with the face ring of the cylinder of Δ.

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2 Preliminaries.

We begin by giving some definitions and results in the general setting of graded algebras. A ***standard graded k-algebra*** is an algebra $A = \oplus_{i\geq 0} A_i$, which is not necessarily commutative, where $A_0 = k$ is a field, A is generated by A_1, and A_1 is a finite dimensional vector space. The graded maximal ideal $\oplus_{i>0} A_i$ will be denoted by m. We will use the term ***algebra*** as shorthand for a standard graded k-algebra. We define the Segre product of two algebras A and B to be $A \underline{\otimes} B = \oplus_{i\geq 0} A_i \otimes_k B_i$, where the multiplication is induced by $a_1 \otimes b_1 \cdot a_2 \otimes b_2 = a_1 a_2 \otimes b_1 b_2$. We define the Rees algebra of an algebra A to be $\mathcal{R}(A) = \oplus_{i\geq 0} m^i$. The first result was, in the commutative case, already noted in [H], p. 123.

Proposition 1 *If A is an algebra, then $\mathcal{R}(A) \simeq A \underline{\otimes} k[t_1, t_2]$ as graded algebras.*

Proof. Let $A = k < X_1, \ldots, X_n > /I = k < x_1, \ldots, x_n >$, where $k < X_1, \ldots, X_n >$ is the free associative algebra and x_i is the image of X_i. Then

$$\mathcal{R}(A) \simeq k < x_1, \ldots, x_n, x_1 t, \ldots, x_n t >,$$

where t is a central variable of degree 0. Now

$$A \underline{\otimes} k[t_1, t_2] \simeq k < x_1 t_1, \ldots, x_n t_1, x_1 t_2, \ldots, x_n t_2 >,$$

with t_1 and t_2 central of degree 0. The mapping $A \underline{\otimes} k[t_1, t_2] \to \mathcal{R}(A)$ induced by $t_1 \mapsto 1, t_2 \mapsto t$ is obviously an isomorphism.

The ***Hilbert series*** of an algebra A is $\mathrm{Hilb}_A(t) = \sum_{i\geq 0} \dim_k A_i t^i$.

Proposition 2 *If A is an algebra, then $\mathrm{Hilb}_{\mathcal{R}(A)}(t) = \frac{d}{dt}(t\,\mathrm{Hilb}_A(t))$.*

Proof. Let $\mathrm{Hilb}_A(t) = \sum_{i\geq 0} h_i t^i$. Since $\mathrm{Hilb}_{k[t_1,t_2]}(t) = \sum_{i\geq 0}(i+1)t^i$, we have

$$\mathrm{Hilb}_{\mathcal{R}(A)} = \sum_{i\geq 0}(i+1)h_i t^i = \frac{d}{dt}(t \sum_{i\geq 0} h_i t^i) = \frac{d}{dt}(t\mathrm{Hilb}_A(t)).$$

An algebra A is called a ***Koszul algebra*** if $(\mathrm{Tor}_i^A(k,k))_j = 0$ for $i \neq j$. This condition is equivalent to $P_A(t)\,\mathrm{Hilb}_A(-t) = 1$, see e.g. [B-F], where $P_A(t) = \sum_{i\geq 0} \dim_k \mathrm{Tor}_i^A(k,k) t^i$.

Proposition 3 *An algebra A is a Koszul algebra if and only if $\mathcal{R}(A)$ is a Koszul algebra.*

Proof. $\mathcal{R}(A) \simeq A \underline{\otimes} k[t_1, t_2]$, so if A is Koszul then $\mathcal{R}(A)$ is Koszul by [B-F], Theorem 4(b), since $k[t_1, t_2]$ is Koszul. By the same techniques as in the proof of this theorem it follows that if $A \underline{\otimes} B$ is Koszul, A not artinian, then B is Koszul.

Remark. We note that Koszul algebras have gained new interest in connection with quantum groups, see [M].

Corollary 1 *If A is a Koszul algebra, then*

$$P_{\mathcal{R}(A)}(t) = \frac{(P_A(t))^2}{P_A(t) - tP'_A(t)}.$$

Proof. $P_{\mathcal{R}(A)}(t) = \frac{1}{\mathrm{Hilb}_{\mathcal{R}(A)}(-t)} = \frac{1}{\frac{d}{dt}(t\,\mathrm{Hilb}_A(-t))} = \frac{1}{\frac{d}{dt}(\frac{t}{P_A(t)})} = \frac{(P_A(t))^2}{P_A(t)-tP'_A(t)}.$

From now on we restrict to commutative algebras, hence an algebra will henceforth be shorthand for a standard graded *commutative* k-algebra, where k is an infinite field.

An algebra is called *generalized C-M* if the local cohomology modules (A) are of finite length for $i < \dim A$. To avoid special cases we assume from now on that our algebras have dimension at least two. The following proposition was proven in [S-V], Proposition I.4.10.

Proposition 4 *If A and B are algebras of dimension ≥ 2, then $A \underline{\otimes} B$ is generalized C-M if and only if A and B are both generalized C-M.*

3 The case of face rings.

Now we restrict further to face rings (or Stanley-Reisner rings). If Δ is a simplicial complex on vertices $\{X_1, \ldots, X_n\}$, we denote by $k[\Delta]$ the face ring of Δ, i.e. $k[X_1, \ldots, X_n]/I$, where I is generated by all squarefree monomials which correspond to non-faces of Δ. Recall that we consider only the case $\dim k[\Delta] \geq 2$, i.e. $\dim \Delta \geq 1$. A good general reference to face rings is [St].

Proposition 5 *The following conditions are equivalent*
(i) $k[\Delta] \underline{\otimes} k[\Delta']$ is a Buchsbaum ring.
(ii) $k[\Delta] \underline{\otimes} k[\Delta']$ is generalized C-M.
(iii) $k[\Delta]$ and $k[\Delta']$ are both Buchsbaum rings.

Proof. That (i) implies (ii) is trivial. That (ii) implies (iii) follows from Proposition 4 and [Sch], Theorem 6.2.1 (see also [S-V], Lemma II.2.4). Finally that (iii) implies (i) follows from [S-V], Lemma II.2.4 and Proposition II.2.10.

Proposition 6 *$k[\Delta] \underline{\otimes} k[\Delta']$ is a C-M ring if and only if both $k[\Delta]$ and $k[\Delta']$ are C-M rings and both Δ and Δ' are acyclic (i.e. $\tilde{H}_i(\Delta; k) = \tilde{H}_i(\Delta'; k) = 0$ for all i).*

Proof. Let $R = k[\Delta], R' = k[\Delta'], d = \dim R, d' = \dim R'$ and let the graded maximal ideals of R and R' be m and m', respectively. Assume that $R \underline{\otimes} R'$ is a C-M ring. By Proposition 5 we have that R and R' are Buchsbaum rings. Hence the local cohomology modules $H^i_m(R), 1 \leq i \leq d-1$ and $H^i_{m'}(R'), 1 \leq i \leq d'-1$, are concentrated in degree 0. Moreover $H^0_m(R) = H^0_{m'}(R') = 0$. Hence

$$[H^0(R)]_0 = k \oplus H^1_m(R)$$

and

$$[H^0(R')]_0 = k \oplus H^1_{m'}(R').$$

Using the Künneth formula we get

$$[H^i_{m\underline{\otimes}m'}(R\underline{\otimes}R')]_0 = ([H^i_m(R)]_0 \otimes k) \oplus \cdots$$

for $2 \le i \le d$. Moreover, from the C-M property of $R\underline{\otimes}R'$ we also have for $2 \le i \le d-1$

$$H^i_m(R) = [H^i_m(R)]_0 \tag{1}$$

and

$$[H^d_m(R)]_0 = 0. \tag{2}$$

Moreover, from the exact sequence (0) we have

$$[H^0(R\underline{\otimes}R')]_0 \simeq [R\underline{\otimes}R']_0 \simeq k$$

Using the Künneth formula again we obtain

$$[H^0(R\underline{\otimes}R')]_0 \simeq [H^0(R)]_0 \otimes [H^0(R')]_0 \simeq (k \oplus [H^1_m(R)]_0) \otimes (k \oplus [H^1_{m'}(R')]_0)$$

$$\simeq k \oplus (k \otimes [H^1_{m'}(R')]_0) \oplus (k \otimes [H^1_m(R)]_0) \oplus \cdots$$

Hence

$$[H^1_m(R)]_0 = [H^1_{m'}(R')]_0 = 0. \tag{3}$$

From (1) and (3) it follows that R is a C-M ring. By Reisner's criterion [R], Theorem 1, we have that $\tilde{H}_i(\Delta; k) = 0$ for $i < \dim \Delta = d-1$. Moreover, by a result of Hochster (not published by him, see [G], Theorem 2) we have $[H^d_m(R)]_0 \simeq \tilde{H}_{d-1}(\Delta; k)$. From (2) it follows that Δ is acyclic. By symmetry it follows that R' is also a C-M ring and that Δ' is acyclic. For the converse, we can again use the Künneth formula to compute the local cohomology of $R\underline{\otimes}R'$ in order to show that $R\underline{\otimes}R'$ is a C-M ring. This implication was proven by [S-V], Theorem I.4.6(i) (see the remark below).

Remark. By the result of Hochster mentioned above, it follows that $k[\Delta]$ is a C-M ring and Δ is acyclic if and only if $k[\Delta]$ is a proper C-M ring in the sense of Chow (see e.g. [S-V], Corollary I.4.8).

To formulate the next proposition we need the following definition. For an algebra A of dimension d, let $a(A) = \max\{t; [H^d_m(A)]_t \neq 0\}$, see [G-W], Definition 3.14.

Proposition 7 *$k[\Delta]\underline{\otimes}k[\Delta']$ is Gorenstein if and only if $k[\Delta]$ and $k[\Delta']$ both are Gorenstein and $a(k[\Delta]) = a(k[\Delta']) < 0$.*

Proof. Suppose that $k[\Delta]\underline{\otimes}k[\Delta']$ is Gorenstein. By Proposition 6, we know that $k[\Delta]$ and $k[\Delta']$ are C-M. Hence the result follows immediately from [G-W], Theorem 4.4.9. For the converse we first note that $k[\Delta]$ is C-M and $a(k[\Delta]) < 0$ if and only if $k[\Delta]$ is C-M and Δ is acyclic, see the proof of Proposition 7. Hence, by Proposition 6, we get that $k[\Delta]\underline{\otimes}k[\Delta']$ is C-M. By [G-W], Corollary 4.3.3 it follows that $k[\Delta]\underline{\otimes}k[\Delta']$ is Gorenstein.

Corollary 2 *Let $k[\Delta]^{\underline{\otimes}m} = k[\Delta]\underline{\otimes}\cdots\underline{\otimes}k[\Delta]$ (m times).*

(i) $k[\Delta]^{\underline{\otimes}m}$ is Buchsbaum if and only if so is $k[\Delta]$.

(ii) $k[\Delta]^{\underline{\otimes}m}$ is C-M (Gorenstein, respectively) if and only if so is $k[\Delta]$ and Δ is acyclic.

Proof. By Proposition 5 we have that $k[\Delta]^{\underline{\otimes}m}$ is generalized C-M if and only if $k[\Delta]$ is generalized C-M. Moreover, in this case we also have that the graded module $H^i_{m^{\underline{\otimes}m}}(k[\Delta]^{\underline{\otimes}m})$ is concentrated in degree 0 for $i < \dim(k[\Delta]^{\underline{\otimes}m})$, and that depth $k[\Delta]^{\underline{\otimes}m} > 0$. By the Künneth formula we obtain $a(k[\Delta]^{\underline{\otimes}m}) = a(k[\Delta])$. Following the same lines as in the proofs of Propositions 5, 6, and 7 we can conclude by induction.

Example. If Δ is a sphere of any dimension, it follows from [G-W], Theorem 4.3.1, that $k[\Delta]^{\underline{\otimes}m}$ is quasi-Gorenstein, i.e. the canonical module of $k[\Delta]^{\underline{\otimes}m}$ is isomorphic to $k[\Delta]^{\underline{\otimes}m}$ up to some shift. By Corollary 2, $k[\Delta]^{\underline{\otimes}m}$ is not C-M, but Buchsbaum. Hence this gives an example of a Buchsbaum, not C-M, quasi-Gorenstein algebra.

Since the Rees algebra of $k[\Delta]$ is isomorphic to $k[\Delta]\underline{\otimes}k[\Delta']$, where the complex $\Delta' = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$, we also have:

Corollary 3 *(i) $\mathcal{R}(k[\Delta])$ is Buchsbaum if and only if so is $k[\Delta]$.*

(ii) $\mathcal{R}(k[\Delta])$ is C-M if and only if so is $k[\Delta]$ and Δ is acyclic.

(iii) $\mathcal{R}(k[\Delta])$ is Gorenstein if and only if so is $k[\Delta]$ and $a(k[\Delta]) = -2$.

Remark. A central point in proving the results of this section is the properties of local cohomology modules of face rings given in [S-V], Lemma II.2.5. Graded F-pure rings (in case of characteristic $p > 0$) and graded rings having a presentation of relative F-pure type (in case of characteristic 0) enjoy the same properties, see [H-R], Propositions 2.4 and 4.7. Hence all the results of this section are also true for such rings R, only replacing the condition "Δ is acyclic" by the condition "R is a proper C-M ring".

4 Topology.

In this section we will show that there is a great similarity between the Rees ring of the face ring of Δ and the face ring of (a natural triangulation of) the cylinder of Δ, $\mathrm{Cyl}(\Delta)$. We start by defining what we mean by $\mathrm{Cyl}(\Delta)$. Let Δ be a simplicial complex on vertices $\{X_1, X_2, \ldots, X_n\}$ and let Δ' be a copy of Δ

on vertices $\{Y_1, Y_2, \ldots, Y_n\}$. Then our triangulation of Cyl(Δ) will have vertices $\{X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n\}$. For each maximal simplex $\{X_{i_1}, \ldots, X_{i_k}; i_1 < \cdots < i_k\}$ of Δ we let the maximal simplices of Cyl(Δ) be

$$\{\{X_{i_1}, \ldots, X_{i_l}, Y_{i_l}, \ldots, Y_{i_k}\}; i_1 < \cdots < i_k, 1 \le l \le k\}.$$

This obviously gives a triangulation of $|\Delta| \times [0,1]$, where we identify $|\Delta|$ with $|\Delta| \times \{0\}$ and $|\Delta'|$ with $|\Delta| \times \{1\}$. If $k[\Delta] = k[X_1, \ldots, X_n]/I$, then $k[\mathrm{Cyl}(\Delta)] = k[X_1 \ldots, X_n, Y_1, \ldots, Y_n]/I'$, where I' is generated by

$$\{Y_i X_j; i < j\}$$

and by

$$\{X_{j_1} \cdots X_{j_l} Y_{j_{l+1}} \cdots Y_{j_m}; j_1 < \cdots < j_m, 0 \le l \le m, X_{j_1} \cdots X_{j_m} \in I\}.$$

This means that if we extend the ordering $Y_1 < Y_2 < \cdots < Y_n < X_1 < X_2 \cdots < X_n$ lexicographically to all monomials in $k[X_1, \ldots X_n, Y_1, \ldots Y_n]$, we have that the generators of I' are the initial monomials of the natural generators

$$\{X_i Y_j - X_j Y_i; 1 \le i < j \le n\}$$

and

$$\{X_{j_1} \cdots X_{j_l} Y_{j_{l+1}} \cdots Y_{j_m}; j_1 < \cdots < j_m, 0 \le l \le m, X_{j_1} \cdots X_{j_m} \in I\}$$

of the defining ideal J for the Rees algebra $\mathcal{R}(k[\Delta]) = k[X_1, \ldots X_n, Y_1, \ldots Y_n]/J$.

Proposition 8 *Let Δ be a simplicial complex and let* Cyl(Δ) *be the naturally triangulated cylinder of Δ. Then*

(i) $\mathrm{Hilb}_{k[\mathrm{Cyl}(\Delta)]}(t) = \mathrm{Hilb}_{\mathcal{R}(k[\Delta])}(t)$.

(ii) $k[\mathrm{Cyl}(\Delta)]$ *is a Koszul algebra if and only if* $\mathcal{R}(k[\Delta])$ *is so.*

In case both are Koszul algebras, we get $P_{k[\mathrm{Cyl}(\Delta)]}(t) = P_{\mathcal{R}(k[\Delta])}(t)$.

Proof. We have shown before the proposition that the initial forms of the generators of J are the generators for I'. We will use that both rings are multihomogeneous, $\deg X_i = \deg Y_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ (the one in place i). Then it is easy to see that they have the same size in each homogeneous part, hence have the same Hilbert series. A Koszul algebra has only relations of degree two, so a face ring is a Koszul algebra if and only if all relations are of degree two, see e.g. [B-F]. Hence it is clear that $k[\mathrm{Cyl}(\Delta)]$ is Koszul if and only if $k[\Delta]$ is Koszul, and by Proposition 3, the same is true for $\mathcal{R}(k[\Delta])$. If the two algebras are Koszul, their Poincaré series coincide, since their Hilbert series do so.

Remark. We have shown that the generators of J in fact constitute a Gröbner basis of J and that $I' = \mathrm{in}(J)$. This means that if $k[\Delta]$ is Koszul, then $\mathcal{R}(k[\Delta])$ is a Poincaré-Birkhoff-Witt algebra (PBW-algebra), see [P].

In the proof of the next proposition we need the following definitions. Let σ be a simplex in Δ. The **closed star** of σ in Δ, $\overline{\mathrm{St}}(\sigma)$, is the union of all simplices in Δ

having σ as a face. The **link** of σ in Δ, $\mathrm{Lk}(\sigma)$, is the union of all simplices in Δ lying in $\overline{\mathrm{St}}(\sigma)$ that are disjoint from σ. The **join** $\Delta_1 * \Delta_2$ of two simplicial complexes Δ_1 and Δ_2 consists of all faces $\sigma_1 \cup \sigma_2$, where $\sigma_1 \in \Delta_1$ and $\sigma_2 \in \Delta_2$. Note that $\Delta_1 * \Delta_2$ is acyclic if Δ_1 or Δ_2 is acyclic.

Proposition 9 *Let Δ be a simplicial complex. Then $k[\mathrm{Cyl}(\Delta)]$ is Buchsbaum if and only if $\mathcal{R}(k[\Delta])$ is Buchsbaum.*

Proof. We use the following two characterizations of Buchsbaum complexes, essentially due to Schenzel, see [St], Theorem 8.1. Let $X = |\Delta|$. Then we have that $k[\Delta]$ is Buchsbaum if and only if

$$H_i(X, X - x; k) = 0, \quad \text{for every} \quad x \in X \quad \text{and} \quad i < \dim X.$$

or

$$\tilde{H}_i(\mathrm{Lk}(\sigma); k) = 0, \quad \text{for every} \quad \sigma \in \Delta, \sigma \neq \emptyset, \quad \text{and every} \quad i < \dim(\mathrm{Lk}(\sigma)).$$

Let $Y = X \times [0,1]$. We know that $k[\Delta]$ is Buchsbaum if and only if $\mathcal{R}(k[\Delta])$ is Buchsbaum, see Corollary 3. Hence, what we need to prove is the following. The following two conditions are equivalent:

$$(i) \qquad H_i(X, X - x) = 0, \quad \text{for all} \quad x \in X, \quad \text{and all} \quad i < \dim X$$

and

$$(ii) \qquad H_i(Y, Y - y) = 0, \quad \text{for all} \quad y \in Y, \quad \text{and all} \quad i < \dim Y.$$

Assume (i). Triangulate X so that x is a vertex and let $U = \overline{\mathrm{St}}(x)$, which is a closed contractible neighborhood of x in X. Choose an ordering of the vertices, such that $x = X_n$ (the last one). Let $y = (x,t)$, and first assume that $t = 0$. Triangulate Y according to our description of this section. Let $W = \mathrm{Cone}(U) = \{(x,1)\} * U$. Then $W = \overline{\mathrm{St}}(y)$ is a closed contractible neighborhood of y. We have $H_i(Y, Y - y) = H_i(W, W - y)$, see [Mu], Lemma 35.1, and $H_i(W, W - y) = H_i(\overline{\mathrm{St}}(y), \mathrm{Lk}(y))$, [Mu], Lemma 35.4. The long exact sequence in reduced relative homology, [Mu], Theorem 23.3 yields,

$$\cdots \longrightarrow \tilde{H}_i(W) \longrightarrow H_i(W, \mathrm{Lk}(y)) \longrightarrow \tilde{H}_{i-1}(\mathrm{Lk}(y)) \longrightarrow \cdots$$

Obviously $\mathrm{Lk}(y) = \mathrm{Cone}(\mathrm{Lk}(x))$, and since both W and $\mathrm{Cone}(\mathrm{Lk}(x))$ are contractible, we get that $H_i(W, \mathrm{Lk}(y)) = 0$ and we are through. The same argument of course applies if $y = (x,1)$, so assume that $y = (x,t)$ with $t \in (0,1)$. For simplicity let $t = 1/2$. Triangulate $U \times [1/2, 1]$ according to our description and mirror this triangulation to $U \times [0, 1/2]$ to get a triangulation of Y. Let $W = s(U) = (\{(x,0)\} \cup \{(x,1)\}) * U$, the suspension of U. Then W is a closed contractible neighborhood of y. We use the long exact sequence in relative reduced homology again. Obviously $\mathrm{Lk}(y) = s(\mathrm{Lk}(x))$, and we have $\tilde{H}_i(W) = 0$ and $\tilde{H}_{i+1}(\mathrm{Lk}(y)) = \tilde{H}_{i+1}(s(\mathrm{Lk}(x))) = \tilde{H}_i(\mathrm{Lk}(x))$, [Mu] Theorem 25.4. It is well known that $\dim(\mathrm{Lk}(x)) = \dim(X)-1$ for each $x \in X$. Hence $\tilde{H}_i(\mathrm{Lk}(x)) = 0, i < \dim(X)-1$, so $\tilde{H}_i(\mathrm{Lk}(y)) = 0, i < \dim(X) = \dim(\mathrm{Lk}(y))$, and we are through.

Now assume (ii). Let $x \in X$ and $U = \overline{\mathrm{St}}(x)$ as before. Let $y = (x, 1/2)$. We get that $\tilde{H}_{i+1}(\mathrm{Lk}(y)) = \tilde{H}_i(\mathrm{Lk}(x)) = 0$ for $i < \dim(\mathrm{Lk}(x))$.

Proposition 10 *Let Δ be a simplicial complex. Then $k[\mathrm{Cyl}(\Delta)]$ is C-M if and only if $\mathcal{R}(k[\Delta])$ is C-M.*

Proof. We use the following characterization of C-M complexes, due to Reisner and Munkrees, see [St], Corollary 4.2 and Theorem 4.3. Let $X = |\Delta|$. Then $k[\Delta]$ is C-M if and only if $\tilde{H}(X;k) = H_i(X, X-x;k) = 0$ for every $x \in X$ and $i < \dim X$. Let $Y = X \times [0,1]$. We know that $k[\Delta]$ is C-M and Δ acyclic if and only if $\mathcal{R}(k[\Delta])$ is C-M, see Corollary 3. In view of the preceding proposition, we only have to note that $\tilde{H}_i(Y) = 0$ for $i < \dim Y = \dim X + 1$ if and only if $\tilde{H}_i(X) = 0$ for $i \leq \dim X$.

Corollary 4 *Let $\mathrm{Cyl}^m(\Delta) = \mathrm{Cyl}(\mathrm{Cyl}(\cdots(\mathrm{Cyl}(\Delta))\cdots))$ (m times). Then we get that $k[\mathrm{Cyl}^m(\Delta)]$ is Buchsbaum (C-M, respectively) if and only if $k[\Delta]$ is Buchsbaum (C-M, respectively).*

For the Gorenstein case the situation is more complicated. This reflects the fact that being Gorenstein is not a topological condition. It is a necessary condition for $k[\mathrm{Cyl}(\Delta)]$ to be Gorenstein that $\mathcal{R}(k[\Delta])$ is Gorenstein. This follows from the spectral sequence of Gräbe, [G2] Satz 4.1 (see also the proof of Proposition 12 below). The condition is however not sufficient. If $k[\Delta]$ is Gorenstein with $a(k[\Delta]) = -2$, it depends on the ordering of the variables if $k[\mathrm{Cyl}(\Delta)]$ is Gorenstein or not. Let, for example, Δ_1 be the Gorenstein simplicial complex on $\{X_1, X_2, X_3, X_4\}$ with maximal faces $\{X_1, X_2, X_3\}$ and $\{X_2, X_3, X_4\}$. Let $k[X_1, \ldots, X_4] = k[X]$ and $k[X_1, \ldots, X_4, Y_1, \ldots, Y_4] = k[X, Y]$. Then $k[\Delta_1] = k[X]/(X_1X_4)$, hence $a(k[\Delta_1]) = -2$, and

$$k[\mathrm{Cyl}(\Delta_1)] = k[X,Y]/(Y_1X_2, Y_1X_3, Y_1X_4, Y_2X_3, Y_2X_4, Y_3X_4, X_1X_4, X_1Y_4, Y_1Y_4),$$

which is not Gorenstein. But for the simplicial complex Δ_2 on $\{X_1, X_2, X_3, X_4\}$ with maximal faces $\{X_1, X_2, X_4\}$ and $\{X_1, X_3, X_4\}$ (which just corresponds to a renumbering of the vertices), then $k[\Delta_2] = k[X]/(X_2X_3)$, and

$$k[\mathrm{Cyl}(\Delta_2)] = k[X,Y]/(Y_1X_2, Y_1X_3, Y_1X_4, Y_2X_3, Y_2X_4, Y_3X_4, X_2X_3, X_2Y_3, Y_2Y_3),$$

which is Gorenstein. The first example gives an algebra $\mathcal{R}(k[\Delta_1]) = k[X,Y]/J$ for which a minimal system of generators for the defining ideal J constitutes a Gröbner basis for J, so that the first Betti numbers for the algebra and its "associated graded" are equal,

$$b_1(k[X,Y]/J) = b_1(k[X,Y]/\mathrm{in}(J)),$$

but $k[X,Y]/J$ and $k[X,Y]/\mathrm{in}(J)$ have different higher Betti numbers,

$$b_i(k[X,Y]/J) \neq b_i(k[X,Y]/\mathrm{in}(J)), i > 1.$$

We will however show that if we choose a good ordering of the variables, then $k[\mathrm{Cyl}(\Delta)]$ is Gorenstein if and only if $\mathcal{R}(k[\Delta])$ is Gorenstein.

We need a preliminary result. Let Δ be a complex on $V = \{X_1, \dots, X_n\}$. We recall the definition of the **core** of Δ. At first we set $\mathrm{Core}(V) = \{X_i \in V; \overline{\mathrm{St}}(\{X_i\}) \neq V\}$. Then we define $\mathrm{Core}(\Delta)$ to be $\Delta_{\mathrm{Core}(V)}$, the full subcomplex of Δ on $\mathrm{Core}(V)$.

Proposition 11 *If Δ is a Gorenstein complex on $V = \{X_1, \dots, X_n\}$, then*

$$a(k[\Delta]) = \#\,\mathrm{Core}(V) - n. \tag{5}$$

Proof. Let $d = \dim k[\Delta]$. We use that if $\mathrm{Hilb}_{k[\Delta]}(t) = p(t)/(1-t)^d$, then $a(k[\Delta]) = \deg(p) - d$ (this follows e.g. from [G-W], Remark 3.16), and that $\mathrm{Hilb}_{k[\Delta]}(t) = 1 + f_0 t/(1-t) + \dots + f_{d-1} t^d/(1-t)^d$, where f_i is the number of i-simplices in Δ, see [F], Lemma 6. This gives that $a(k[\Delta]) = 0$ if and only if the alternating sum $f_{d-1} - f_{d-2} + \dots + (-1)^{d-1} f_0 + (-1)^d \neq 0$, which is equivalent to that the alternating sum of $\dim \tilde{H}_i(\Delta)$ is not zero. This in turn is true if and only if $\tilde{H}_{d-1}(\Delta) \neq 0$, since $\tilde{H}_i(\Delta) = 0$ for $i < d-1$. Using [St], Theorem 5.1(c) we see that $\tilde{H}_{d-1}(\Delta) \neq 0$ if $\{X_1, \dots, X_n\} = \mathrm{Core}(V)$. The claim follows by factoring out variables which do not belong to $\mathrm{Core}(\Delta)$. This operation increases the value of both sides in (5) with one for each variable, and does not affect the condition of being Gorenstein.

Remark. Proposition 11 makes it possible to classify each face ring of dimension ≤ 4, whose Rees algebra is Gorenstein. We know that $k[\Delta] = k[\mathrm{Core}\,\Delta][Y_1, Y_2]$, where $k[\mathrm{Core}\,\Delta]$ is Gorenstein of dimension $\dim(k[\Delta]) - 2$, and with $a(k[\mathrm{Core}\,\Delta]) = 0$. So if $\dim(k[\Delta]) = 2$, there is only one possibility, namely $k[X_1, X_2]$. If $\dim(k[\Delta]) = 3$, then $k[\Delta]$ must have Hilbert series $(1+t)/(1-t)^3$, hence $k[\mathrm{Core}\,\Delta]$ has Hilbert series $(1+t)/(1-t)$, which gives only $k[X_1, X_2, Y_1, Y_2]/(X_1X_2)$. If $\dim(k[\Delta]) = 4$, then $\mathrm{Hilb}_{k[\Delta]}(t) = (1 + mt + t^2)/(1-t)^4$ $(m \geq 1)$, hence $\mathrm{Hilb}_{k[\mathrm{Core}\,\Delta]}(t) = (1 + mt + t^2)/(1-t)^2$. Then

$$k[\Delta] = k[X_1, X_2, X_3, Y_1, Y_2]/(X_1X_2X_3)$$

or (for $m > 1$)

$$k[X_1, \dots, X_{m+2}, Y_1, Y_2]/(X_iX_j; |i-j| \neq 0, 1 \pmod{m+2}).$$

Other examples, of higher dimension, are the complete intersections

$$k[X_1, \dots, X_n, Y_1, Y_2]/(X_1 \cdots X_{i_1}, X_{i_1+1} \cdots X_{i_2}, \dots, X_{i_k+1} \cdots X_n)$$

where $1 \leq i_1 < i_2 < \dots < i_k < n$. The next proposition will provide lots of further examples.

Proposition 12 *Let Δ be a simplicial complex. Then $k[\mathrm{Cyl}(\Delta)]$ is Gorenstein if and only if $\mathcal{R}(k[\Delta])$ is Gorenstein and the ordering of the variables is such that $\mathrm{Core}(V) = \{X_2, \dots, X_{n-1}\}$.*

Proof. Suppose that $k[\mathrm{Cyl}(\Delta)]$ is Gorenstein. We will use the following characterization of Gorenstein complexes, see [St] Theorem 5.1(b). The ring $k[\Delta]$ is Gorenstein if and only if for each face σ in $\mathrm{Core}(\Delta)$ we have

$$\tilde{H}_i(\mathrm{Lk}_{\mathrm{Core}(\Delta)}\,\sigma; k) = 0 \quad \text{if} \quad i < \dim(\mathrm{Lk}_{\mathrm{Core}(\Delta)}\,\sigma),$$

and

$$\tilde{H}_i(\mathrm{Lk}_{\mathrm{Core}(\Delta)}\,\sigma; k) \simeq k \quad \text{if} \quad i = \dim(\mathrm{Lk}_{\mathrm{Core}(\Delta)}\,\sigma).$$

Denote the set of variables in $\mathrm{Cyl}(\Delta)$ by $W = \{X_1, \dots, X_n, Y_1, \dots, Y_n\}$. Our proof is based on a calculation of links of points in $\mathrm{Core}(W)$. If $X_1 \in \mathrm{Core}(V)$, then $X_1 \in \mathrm{Core}(W)$. Clearly also $Y_1 \in \mathrm{Core}(W)$. Now we have

$$\mathrm{Lk}_{\mathrm{Core}(\mathrm{Cyl}(\Delta))}\{Y_1\} = \{X_1\} * \sigma,$$

where σ is a subcomplex with vertices in $\{Y_2, \dots, Y_n\}$. This join is acyclic, hence we have a contradiction to the Gorenstein property of $k[\mathrm{Cyl}(\Delta)]$, and we conclude that $X_1 \notin \mathrm{Core}(V)$. Analogously, by considering $\mathrm{Lk}_{\mathrm{Core}(\mathrm{Cyl}(\Delta))}\{X_n\}$ (if $X_n \in \mathrm{Core}(V)$), we get that $X_n \notin \mathrm{Core}(V)$. Now it easily follows that $\mathrm{Core}(W) = \{X_2, \dots, X_n, Y_1 \dots, Y_{n-1}\}$. Further, if $X_i \notin \mathrm{Core}(V)$, where $1 < i < n$, then it is easy to see that

$$\mathrm{Lk}_{\mathrm{Core}(\mathrm{Cyl}(\Delta))}\{Y_1\} = \{Y_i\} * \sigma,$$

for some σ. Again this join is acyclic, which gives a contradiction. Hence we get that $\mathrm{Core}(V) = \{X_2, \dots, X_{n-1}\}$. If $\sigma = \{X_{i_1}, \dots, X_{i_l}; i_1 < \cdots < i_l\}$ is a face of $\mathrm{Core}(\Delta)$, then we set $\tau = \{X_{i_1}, \dots, X_{i_l}, Y_{i_l}\}$ which obviously is a face of $\mathrm{Core}(\mathrm{Cyl}(\Delta))$. It is easy to check that $\mathrm{Lk}_{\mathrm{Core}(\mathrm{Cyl}(\Delta))}\,\tau \simeq \mathrm{Lk}_{\mathrm{Core}(\Delta)}\,\sigma$. By the criterion above it then follows that $k[\Delta]$ is Gorenstein. By Corollary 3(iii) and Proposition 11, we get that $\mathcal{R}(k[\Delta])$ is Gorenstein.

For the other implication we will use the following characterization of Gorenstein complexes, due to Björner, see [St] Theorem 5.1(d). We have that $k[\Delta]$ is Gorenstein if and only if $k[\Delta]$ is C-M and $\mathrm{Core}(\Delta)$ is an orientable pseudomanifold.

Suppose that $\mathcal{R}(k[\Delta])$ is Gorenstein and that $\mathrm{Core}(V) = \{X_2, \dots, X_{n-1}\}$. It was noted above that

$$\mathrm{Core}(W) = \{X_2, \dots, X_n, Y_1, \dots, Y_{n-1}\}.$$

We must show that $k[\mathrm{Cyl}(\Delta)]$ is C-M, and that $\mathrm{Core}(\mathrm{Cyl}(\Delta))$ is an orientable pseudomanifold. That $k[\mathrm{Cyl}(\Delta)]$ is C-M follows from Proposition 10. It is not hard to see that $\mathrm{Core}(\mathrm{Cyl}(\Delta))$ is $\mathrm{Cyl}(\mathrm{Core}(\Delta))$ with its top joined to Y_1 and its bottom joined to X_n, which is homotopic to $s(\mathrm{Core}(\Delta))$, the suspension of $\mathrm{Core}(\Delta)$, with suspension points X_n and Y_1, (smash $\mathrm{Cyl}(\mathrm{Core}(\Delta))$ to $\mathrm{Core}(\Delta)$).
This gives that

$$\tilde{H}_i(\mathrm{Core}(\mathrm{Cyl}(\Delta))) = \tilde{H}_i(s(\mathrm{Core}(\Delta))) = \tilde{H}_{i-1}(\mathrm{Core}(\Delta)) = 0,$$

if $i < \dim(\mathrm{Core}(\mathrm{Cyl}(\Delta))) = \dim(\mathrm{Core}(\Delta)) + 1$, and that

$$\tilde{H}_i(\mathrm{Core}(\mathrm{Cyl}(\Delta))) = k,$$

if $i = \dim(\mathrm{Core}(\mathrm{Cyl}(\Delta)))$. It follows that $\mathrm{Core}(\mathrm{Cyl}(\Delta))$ is orientable. We have that $\mathrm{Core}(\mathrm{Cyl}(\Delta))$ is a pseudomanifold if and only if each face of submaximal dimension dim(Core(Cyl(Δ))) − 1 lies in exactly two faces of maximal dimension in Core(Cyl(Δ)). That this is true, follows directly from our definition of the cylinder and our description of Core(Cyl(Δ)).

Corollary 5 *Suppose that* $k[\Delta]$ *is a Gorenstein face ring on* $V = \{X_1, \ldots, X_n\}$ *with* $\mathrm{Core}(V) = \{X_2, \ldots, X_{n-1}\}$. *Then* $k[\mathrm{Cyl}^m(\Delta)]$ *is Gorenstein.*

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