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ALGEBRAIC TOPOLOGICAL RESULTS ON STANLEY REISNER RINGS

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ABSTRACT

This is a paper on Stanley-Reisner rings, "$k[\Delta]$". For Hilbert series, we prove, that $k[\Delta' \times \Delta''] \doteq k[\Delta'] \overline{\otimes} k[\Delta'']$ where k is a field and $\Delta' \times \Delta''$ denotes the "Simplicial Cartesian Product", while $k[\Delta'] \overline{\otimes} k[\Delta'']$ denotes a "Segre Product". Segre products together with $k[\Delta' \times \Delta'']$ are proved to be Koszul algebras if and only if both $k[\Delta']$ and $k[\Delta'']$ are Koszul algebras.
Modulo some minor modifications we give algebraic topological proofs for the claim that "the simplicial cartesian product" respectively "the join" of two simplicial complexes have the property of being Buchsbaum (Cohen-Macaulay, Gorensten) if and only if both factors possess the property in question. Our results generalize some related results of Baclawski on products of posets.

1. Hilbert Series

We will use the following definition of a simplicial complex;

Definition of an abstract Simplicial Complex. *An (abstract) simplicial complex Σ on a vertex set V is a collection (empty or non-empty) of finite subsets σ (empty or non-empty) of V satisfying*
(a) *If $v \in V$, then $v \in \Sigma$.*
(b) *If $\sigma \in \Sigma$ and $\tau \subset \sigma$ then $\tau \in \Sigma$.*

This definition doesn't differ from the classical except for the fact that it allows us to choose the non-empty collection containing nothing but the **empty subset** of V, i.e. $\{\emptyset\}$, as a simplicial complex. All simplicial complexes, in this paper, will, if nothig else is said, be finite, i.e. they have finite vertex sets.

Given two simplicial complexes Δ' and Δ'' with vertex sets $V_{\Delta'} := \{v'_1, \ldots, v'_a\}$ and $V_{\Delta''} := \{v''_1, \ldots, v''_b\}$ resp. where all the vertices belong to a common "universe" W. [5] defines in Def. 8.8 p. 67 "The Cartesian (Simplicial) Product, $\Delta' \times \Delta''$, of Δ' and Δ''", and shows in Lemma 8.9 p. 68 that $\Delta' \times \Delta''$ triangulates $|\Delta'| \times |\Delta''|$. By definition we have that $V_{\Delta' \times \Delta''} = \{(v'_1, v''_1), \ldots, (v'_a, v''_b)\}$, where we put $w_{i,j} := (v'_i, v''_j)$. Simplexes in $\Delta' \times \Delta''$ are sets $\{w_{i_0,j_0}, w_{i_1,j_1}, \ldots, w_{i_k,j_k}\}$, with $w_{i_s,j_s} \neq w_{i_{s+1},j_{s+1}}$ and $v'_{i_0} \leq v'_{i_1} \leq \ldots \leq v'_{i_k}$ ($v''_{j_0} \leq v''_{j_1} \leq \ldots \leq v''_{j_k}$) where $v'_{i_0}, v'_{i_1}, \ldots, v'_{i_k}$ ($v''_{j_0}, v''_{j_1}, \ldots, v''_{j_k}$) is a sequence of vertices, with repetitions possible, which constitutes a simplex in Δ' (Δ'').

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Definition. *A subset $s \subset \mathbf{W} \supset V_\Delta$ is said to be a* **non-simplex** *of Δ, denoted $s \not\in^{n} \Delta$, if, when we regard s as the full complex on its vertices, we have that $s \notin \Delta$ but $(\bar{s})^{((dim\ s)-1)} \subset \Delta$. (The last condition says that the $((dim\ s)-1)$-dimensional skeleton of the simplicial complex consisting of all proper faces of s is a subcomplex of Δ.) For a simplex $\delta = \{v_{i_1}, \ldots, v_{i_k}\}$ we define m_δ to be the squarefree monic monomial $m_\delta := 1_k \cdot v_{i_1} \cdots \cdot v_{i_k} \in k[\mathbf{W}]$ where $k[\mathbf{W}]$ is the graded polynomial algebra on the variable set $\mathbf{W}$. In particular $m_\emptyset = 1_k$. We let $k[\Delta] := k[\mathbf{W}]/\mathbf{I}_\Delta$ where $\mathbf{I}_\Delta := \left(\{m_\delta \mid \delta \not\in^{n} \Delta\}\right)$. $k[\Delta]$ is called the "face ring" or the "Stanley-Reisner ring" of Δ.*

Example 1. The *join* $\Delta_1 * \Delta_2 = \{\delta_1 \cup \delta_2 | \delta_i \in \Delta_i\ (i = 1, 2)\}$, $V_{\Delta_1} \cap V_{\Delta_2} = \emptyset$, gives

$$k[\Delta_1 * \Delta_2] = \frac{k[\mathbf{W}]}{(\{m_\delta |\ \delta \not\in^{n} \Delta_1 * \Delta_2\})} = \frac{k[\mathbf{V}_{\Delta_1} \cup \mathbf{V}_{\Delta_2} \cup (\mathbf{W} \setminus (\mathbf{V}_{\Delta_1} \cup \mathbf{V}_{\Delta_2}))]}{(\{m_\delta |\ [\delta \not\in^{n} \Delta_1 \vee \delta \not\in^{n} \Delta_2] \wedge [\delta \notin \Delta_1 * \Delta_2]\})} =$$

$$= \frac{k[\mathbf{V}_{\Delta_1} \cup \mathbf{V}_{\Delta_2} \cup (\mathbf{W} \setminus (\mathbf{V}_{\Delta_1} \cup \mathbf{V}_{\Delta_2}))]}{(\{m_\delta \in k[\mathbf{V}_{\Delta_1}] |\ \delta \not\in^{n} \Delta_1\} \cup \{m_\delta \in k[\mathbf{V}_{\Delta_2}] |\ \delta \not\in^{n} \Delta_2\} \cup [\mathbf{W} \setminus \mathbf{V}_{\Delta_1} \cup \mathbf{V}_{\Delta_2}])} \tilde{=}$$

$$\tilde{=} k[\Delta_1] \otimes k[\Delta_2],$$

as was, originally, shown in [7]. The last equality follows from [17] Theor. 35 p. 184.

Definition. *A graded k-algebra R is called a graded k-***standard** *algebra, if it is* finitely generated (over k) *by $x_1, \ldots, x_n \in [R]_1$.*

Definition. *A graded k-algebra R is called a Segre product of R_1 and R_2 over k, denoted by $R = \sigma_k(R_1, R_2)$ or $R = \sigma(R_1, R_2)$, if for every $p \in \mathbf{N}$ $[R]_p = [R_1]_p \otimes [R_2]_p$.*

Definition. *i. The trivial Segre product, $R_1 \mathbf{0} R_2$, is equipped with the trivial product, i.e. every product of elements, both of which lacks field term, equals 0.*

ii. The "canonical" Segre product, $R_1 \underline{\otimes} R_2$, is equipped with a product induced by extending (distributively and linearly) the following operation defined on simple homogeneous elements: If $m_1' \otimes m_1'' \in [R_1 \underline{\otimes} R_2]_\alpha$ and $m_2' \otimes m_2'' \in [R_1 \underline{\otimes} R_2]_\beta$ then $(m_1' \otimes m_1'')(m_2' \otimes m_2'') := m_1' m_2' \otimes m_1'' m_2'' \in [R_1 \underline{\otimes} R_2]_{\alpha+\beta}$.

iii. The "canonical" generator-order sensitive Segre product, $R_1 \bar{\otimes} R_2$, of two graded k-standard algebras R_1 and R_2 presupposes the existence of a uniquely defined partially ordered minimal set of generators of R_1 (R_2) in $[R_1]_1$ ($[R_2]_1$) and is equipped with a product induced by extending (distributively and linearly) the following operation defined on simple homogeneous elements, each of which, now are presumed to be written, in product form, as an increasing chain of the specified linearly ordered generators: If $(m_{11} \otimes m_{21}) \in [R_1 \bar{\otimes} R_2]_\alpha$ and $(m_{12} \otimes m_{22}) \in [R_1 \bar{\otimes} R_2]_\beta$ then $(m_{11} \otimes m_{21})(m_{12} \otimes m_{22}) := ((m_{11} m_{12} \otimes m_{21} m_{22}) \in [R_1 \bar{\otimes} R_2]_{\alpha+\beta}$ if by "pairwise" permutations, $(m_{11} m_{12}, m_{21} m_{22})$ can be made *into a chain in the product ordering, and 0 otherwise. Here, (x, y) is a pair in $(m_{11}m_{12}, m_{21}m_{22})$ if x occupy the same position as y counting from left to right in $m_{11}m_{12}$ and $m_{21}m_{22}$ respectively.*

Definition. *The Hilbert series of a graded k-algebra $R = \bigoplus_{i\geq 0} R_i$ is*
$\mathbf{Hilb}_R(t) := \sum_{i\geq 0}(\mathrm{H}(R,i))t^i := \sum_{i\geq 0}(\dim_k R_i)t^i$, (H:= *The Hilbert function*).

Note: 1. (*cf.*[14] p. 39-40) Every Segre product of R_1 and R_2 is module-isomorphic by definition and so, they all have the same Hilbert series.
2. If R_1 and R_2 are graded k-**standard** algebras, generated (over k) by $x_1, \ldots, x_n \in [R_1]_1, y_1, \ldots, y_m \in [R_2]_1$, resp., then $R_1 \underline{\otimes} R_2$ and $R_1 \bar{\otimes} R_2$ are generated by $(x_1 \otimes y_1), \ldots, (x_n \otimes y_m)$, and $\dim R_1 \bar{\otimes} R_2 = \dim R_1 \underline{\otimes} R_2 = \dim R_1 + \dim R_2 - 1$ (dim:=Krull-dim, cf.[9] p. 201 Th. (4.2.3)). $R_1 \mathbf{0} R_2$ is not finitely generated, so, non-Noetherian.
3. It is quite plain to define a generator-relation sensitive Segre product, of two graded k-algebras R_1 and R_2, enclosing all the above cases.

For a given monomial m, let Supp(m) be the squarefree monomial defined by putting every non-zero exponent (of the variables) in m equal to 1. Let $p_1(w_{i,j}) = p_1((v'_i, v''_j)) := v'_i$ and $p_2(w_{i,j}) := v''_j$ and if Supp$(m) := w_{i_1,j_1} \cdots w_{i_k,j_k}$ put $[\overline{p_t}(m)] := \{p_t(w_{i_1,j_1}), \ldots, p_t(w_{i_k,j_k})\}$ and $p_t(m) := p_t(w_{i_1,j_1}) \cdots p_t(w_{i_k,j_k})$ for $t = 1, 2$.

Theorem 1. $k[\Delta' \times \Delta''] \cong k[\Delta'] \bar{\otimes} k[\Delta'']$, *where the isomorphism is a graded k-algebra isomorphism of degree zero.*

Proof. If $k[\Delta'] = k[v'_1, \ldots, v'_a]/I_{\Delta'}$ and $k[\Delta''] = k[v''_1, \ldots, v''_b]/I_{\Delta''}$ we know (cf. the above Note) that the Segre product $k[\Delta'] \bar{\otimes} k[\Delta'']$, is generated by $\{(v'_1 \otimes v''_1), \ldots, (v'_a \otimes v''_b)\}$ where v'_i (resp. v''_j) now, by abuse of language, stands for the image of v'_i (resp. v''_j) under the quotient map. Let's define an algebra homomorphism $h : k[w_{1,1}, \ldots, w_{a,b}] \longrightarrow k[\Delta'] \bar{\otimes} k[\Delta'']$ by giving the values of h on the generators by $h(w_{i,j}) := v'_i \otimes v''_j$. Put $\mathbf{I}_{k[\Delta'] \bar{\otimes} k[\Delta'']} := \ker h$. $V_{\Delta'}$ and $V_{\Delta''}$ has been given total orderings and $V_{\Delta' \times \Delta''} = V_{\Delta'} \times V_{\Delta''}$ received thereby the product ordering. Now, $\mathbf{I}_{k[\Delta'] \bar{\otimes} k[\Delta'']}$ is generated by $C' := \{w_{i,j} w_{k,l} | \ i < k \wedge j > l\}$ and $D' := \{\mathbf{w} = w_{i_1,j_1} \cdots w_{i_k,j_k} | \ [\overline{p_1}(\mathbf{w})] \notin \Delta' \text{ or } [\overline{p_2}(\mathbf{w})] \notin \Delta''\}$. On the other hand, $I_{\Delta' \triangle \Delta''}$ ([5] defines the "$\triangle$-product" in Def. 8.1 p. 66) is generated by D', and deleting the non-chains, we see that also $I_{\Delta' \times \Delta''}$ is generated by $C' \cup D'$. ■

Corollary. $\mathbf{Hilb}_{k[\Delta'] \bar{\otimes} k[\Delta'']}(t) = \mathbf{Hilb}_{k[\Delta' \times \Delta'']}(t) = \mathbf{Hilb}_{k[\Delta'] \underline{\otimes} k[\Delta'']}(t)$. ■

Example 2. Let d', d'' and d denote the Krull-dimension (=(the usual simplicial dimension)+1) of $k[\Delta'], k[\Delta'']$ and $k[\Delta' \times \Delta'']$ respectively and let f'_i, f''_j, f_k stand for the number of simplices of simplicial dimension i, j, k in Δ', Δ'' and $\Delta' \times \Delta''$ respectively. [13] p. 63 Th. 1.4 gives the following formula for the Hilbert function of these "standard graded algebras", resp., where now $d - 1 = (d' - 1) + (d'' - 1)$. E.g. $H(k[\Delta' \times \Delta''], m) = \sum_{k=0}^{d-1} f_k \binom{m-1}{k}$ when $m > 0$ and $H(k[\Delta' \times \Delta''], 0) = 1$. $H(k[\Delta' \times \Delta''], m) = H(k[\Delta'] \underline{\otimes} k[\Delta''], m)$ by the corollary which, by the definition of the Segre product, implies $\sum_{k=0}^{d-1} f_k \binom{m-1}{k} = \left(\sum_{i=0}^{d'-1} f'_i \binom{m-1}{i}\right)\left(\sum_{j=0}^{d''-1} f''_j \binom{m-1}{j}\right)$.

Identifying coefficients w.r.t. the exponents of m, from the highest, downwards, i.e. $f_{(d'-1)+(d''-1)} = f'_{d'-1} f''_{d''-1} \binom{(d'-1)+(d''-1)}{d'-1}$. Next, $f_{(d'-1)+(d''-1)-1} = \ldots$ and so on. By Example 1 above, analogously for the "f-vector" of $\Delta' * \Delta''$.

Let's define an algebra homomorphism $g : k[w_{1,1}, \ldots, w_{a,b}] \longrightarrow k[\Delta'] \underline{\otimes} k[\Delta'']$ by giving the values of g on the generators by $g(w_{i,j}) := v'_i \otimes v''_j$. Put $\mathbf{I}_{k[\Delta'] \underline{\otimes} k[\Delta'']} := \ker g$. Put $C := \{w_{i,j} w_{k,l} - w_{i,l} w_{k,j} \mid i < k \wedge j > l\}$ and

$$D := \Big\{ \mathbf{w} = w_{i_1,j_1} \cdot \ldots \cdot w_{i_k,j_k} \;\Big|\; [\mathbf{w} \text{ is a chain}] \wedge \Big[\Big[[\overline{p_1}(\mathbf{w})] \overset{n}{\notin} \Delta' \Big] \wedge \Big[[\overline{p_2}(\mathbf{w})] \in \Delta'' \Big] \wedge \Big[{}^{i_1 < \cdots < i_k}_{j_1 \leq \cdots \leq j_k} \Big] \Big] \vee$$
$$\vee \Big[\Big[[\overline{p_1}(\mathbf{w})] \in \Delta' \Big] \wedge \Big[[\overline{p_2}(\mathbf{w})] \overset{n}{\notin} \Delta'' \Big] \wedge \Big[{}^{i_1 \leq \cdots \leq i_k}_{j_1 < \cdots < j_k} \Big] \Big] \vee \Big[\Big[[\overline{p_1}(\mathbf{w})] \overset{n}{\notin} \Delta' \Big] \wedge \Big[[\overline{p_2}(\mathbf{w})] \overset{n}{\notin} \Delta'' \Big] \wedge \Big[{}^{i_1 < \cdots < i_k}_{j_1 < \cdots < j_k} \Big] \Big] \Big] \Big\}.$$

Lemma. (a) *D', above, is reduced to a minimal generating set D'', for $I_{\Delta' \triangle \Delta''}$, in*

$$D'' := \Big\{ \mathbf{w} = w_{i_1,j_1} \cdot \ldots \cdot w_{i_k,j_k} \;\Big|\; \Big[\Big[[\overline{p_1}(\mathbf{w})] \overset{n}{\notin} \Delta' \Big] \wedge \Big[[\overline{p_2}(\mathbf{w})] \in \Delta'' \Big] \wedge \Big[{}^{p_1(w_{i,j}) \neq p_1(w_{k,l})}_{\text{if } i \neq k} \Big] \Big] \vee \Big[\Big[[\overline{p_1}(\mathbf{w})] \in \Delta' \Big] \wedge$$
$$\wedge \Big[[\overline{p_2}(\mathbf{w})] \overset{n}{\notin} \Delta'' \Big] \wedge \Big[{}^{p_2(w_{i,j}) \neq p_2(w_{k,l})}_{\text{if } j \neq l} \Big] \Big] \vee \Big[\Big[[\overline{p_1}(\mathbf{w})] \overset{n}{\notin} \Delta' \Big] \wedge \Big[[\overline{p_2}(\mathbf{w})] \overset{n}{\notin} \Delta'' \Big] \wedge \Big[{}^{p_1(w_{i,j}) \neq p_1(w_{k,l}) \text{ if } i \neq k}_{p_2(w_{i,j}) \neq p_2(w_{k,l}) \text{ if } j \neq l} \Big] \Big] \Big\}.$$

(b) *$C \cup D$ ($C' \cup D$) is a minimal generating set for $\mathbf{I}_{k[\Delta'] \underline{\otimes} k[\Delta'']}$ ($I_{\Delta' \times \Delta''} = \mathbf{I}_{k[\Delta'] \overline{\otimes} k[\Delta'']}$) and a reduced Gröbner basis for $\mathbf{I}_{k[\Delta'] \underline{\otimes} k[\Delta'']}$ ($I_{\Delta' \times \Delta''} = \mathbf{I}_{k[\Delta'] \overline{\otimes} k[\Delta'']}$).*

Proof. (a) Enlarge D'' to E' by changing $[\overline{p_i}(\mathbf{w})]] \in \Delta^*$ to $\neg [[\overline{p_i}(\mathbf{w})] \overset{n}{\notin} \Delta^*]$, $(i = 1, 2; \;\; * = {}', {}'')$, i.e. $E' :=$

$$\Big\{ \mathbf{w} = w_{i_1,j_1} \cdot \ldots \cdot w_{i_k,j_k} \;\Big|\; \Big[\Big[[\overline{p_1}(\mathbf{w})] \overset{n}{\notin} \Delta' \Big] \wedge \Big[\neg \Big[[\overline{p_2}(\mathbf{w})] \overset{n}{\notin} \Delta'' \Big] \Big] \wedge \Big[{}^{p_1(w_{i,j}) \neq p_1(w_{k,l})}_{\text{if } i \neq k} \Big] \Big] \vee \Big[\Big[\neg \Big[[\overline{p_1}(\mathbf{w})] \overset{n}{\notin} \Delta' \Big] \Big] \wedge$$
$$\wedge \Big[[\overline{p_2}(\mathbf{w})] \overset{n}{\notin} \Delta'' \Big] \wedge \Big[{}^{p_2(w_{i,j}) \neq p_2(w_{k,l})}_{\text{if } j \neq l} \Big] \Big] \vee \Big[\Big[[\overline{p_1}(\mathbf{w})] \overset{n}{\notin} \Delta' \Big] \wedge \Big[[\overline{p_2}(\mathbf{w})] \overset{n}{\notin} \Delta'' \Big] \wedge \Big[{}^{p_1(w_{i,j}) \neq p_1(w_{k,l}) \text{ if } i \neq k}_{p_2(w_{i,j}) \neq p_2(w_{k,l}) \text{ if } j \neq l} \Big] \Big] \Big\}$$

Simple logic tells us that $(E') = (D')$, but what about $(D'') \stackrel{?}{=} (E')$?
Take $\mathbf{w} \in E' \setminus D''$, i.e. $\mathbf{w}$ is in E' due to some of the two first sets of set-quantifiers, separated by an "$\vee$", defining E'. Say that $\mathbf{w} \in E'$ due to the first set of set-quantifiers defining E'. Then $[[\overline{p_1}(\mathbf{w})] \overset{n}{\notin} \Delta']$, $[\neg [[\overline{p_2}(\mathbf{w})] \overset{n}{\notin} \Delta'']$ and $\big[{}^{p_1(w_{i,j}) \neq p_1(w_{k,l})}_{\text{if } i \neq k} \big]$ and $[\overline{p_2}(\mathbf{w})] \notin \Delta''$. Now, start deleting "vertices" from the monomial $\mathbf{w}$, systematically, so that the projection $[\overline{p_2}(\mathbf{w})]$ doesn't become an element in Δ''. $\mathbf{w}$ is a product of pairs in $V_{\Delta'} \times V_{\Delta''}$ so, sooner or later we can't do any more deletions, and this is so exactly when $[\overline{p_2}(*)]$ has become an non-simplex of Δ''. Call this final monomial $\mathbf{w}'$. Now $\mathbf{w}' | \mathbf{w}$ and $\mathbf{w}' \neq \mathbf{w}$ and $[\overline{p_1}(\mathbf{w})] \in \Delta'$, $[\overline{p_2}(\mathbf{w})] \overset{n}{\notin} \Delta''$ and $\big[{}^{p_2(w_{i,j}) \neq p_2(w_{k,l})}_{\text{if } j \neq l} \big]$, since otherwise we chould have performed additional deletions, so $\mathbf{w}' \in D''$ and since $\mathbf{w}'$ is a factor in $\mathbf{w}$ we conclude that $(D'') = (E')$, so $(D'') = (D')$. □
(b) For the same reasons as in (a), changing $[\overline{p_i}(\mathbf{w})] \in \Delta^*$ to $\neg [[\overline{p_i}(\mathbf{w})] \overset{n}{\notin} \Delta^*]$ $(i = 1, 2; \; * = {}', {}'')$, won't change the generated ideal, and now, the statement on "generating set" is readily deduced, as is the "minimality" by actually checking that every single generator is necessary, to make the "generating set"-statement

remain true. The "Gröbner basis"-statement is trivilly true for the monomially generated ideals, within parenthesis.
Now, let

$$w_{a,1} < w_{a-1,1} < \cdots < w_{1,1} < w_{a,2} < \cdots < w_{1,2} < \cdots < w_{a,b} < \cdots < w_{1,b}$$

be the order of the variables and extend this ordering to all monomials in the degree lexicographical way. Then $\mathrm{In}(w_{i,j}w_{k,l} - w_{i,l}w_{k,j}) = w_{i,j}w_{k,l}$ $(i < k \wedge j > l)$ which, remebering **(a)**, implies

$$(\mathrm{In}[C \cup D])=(\mathrm{In}[C] \cup D)=(C' \cup D)=(C' \cup D'')=(C' \cup D')=(\mathrm{In}[C \cup D']).$$

Since we now have $(\mathrm{In}[C \cup D]) = (\mathrm{In}[C \cup D'])$, and $(C \cup D) = (C \cup D')$, we can draw the following conclusion:

If $C \cup D'$ is a Gröbner basis for $(C \cup D')$, so is $C \cup D$.

So, now it is enough to show that the S-polynomial $\mathrm{S}(f,g)$ is reduced to 0 by $C \cup D'$ (cf. [3] p. 191 Th. 6.2). If $f, g \in C$ this follows from [15], and if $f, g \in D'$ this is trivial, since D' consists of monomials only. It only remains the case when $f \in C, g \in D'$. Let $f = w_{i,j}w_{k,l} - w_{i,l}w_{k,j}$. Then $\mathrm{S}(f,g) = -w_{i,l}w_{k,j}g/\mathrm{GCD}(g, \mathrm{In}(f))$. Since $[\overline{p_i}(\mathrm{S}(f,g))] = [\overline{p_i}(g)]$, $\mathrm{S}(f,g)$ is an element in D', and so, the claim follows. That $C \cup D$ is a *reduced Gröbner basis* is now a trivial consequence of the fact that $C \cup D$ is a is a minimal generating set for $\mathbf{I}_{k[\Delta'] \underline{\otimes} k[\Delta'']}$. ∎

Definition. *The Poincare' series, $P_R(t)$, of a graded k-algebra R is*

$$P_R(t) = \sum_{i=0}^{\infty} (\dim_k \mathrm{Tor}_i^R(k,k)) t^i.$$

Definition. *A graded k-algebra R is called a Koszul algebra if*

$$\mathrm{Tor}_{i,j}^R(k,k) = 0 \text{ if } i \neq j, \text{ or, equivalently, } P_R(t)\mathrm{Hilb}_R(-t) = 1, \text{ see}^1.$$

Theorem 2. *If Δ and Δ' both are of dimension ≥ 0 we have*
$k[\Delta \times \Delta']$ *Koszul* $\Longleftrightarrow$ $k[\Delta] \underline{\otimes} k[\Delta']$ *Koszul* $\Longleftrightarrow$ $k[\Delta]$ *and* $k[\Delta']$ *both Koszul.*

Proof. If $k[\Delta \times \Delta']$ is Koszul we have by [1] 16 p. 87 that $I_{\Delta' \times \Delta''}$ is generated by polynomials of degree ≤ 2. Hence D above, consists of monomials of degree=2, so all non-simplices of Δ and Δ' are 1-simplices. Hence by [1] 17(b) p. 87 $k[\Delta]$ and $k[\Delta']$ are both Koszul, which by [1] Th. 4(b) p. 91 gives that $k[\Delta] \underline{\otimes} k[\Delta'] = k[w_{1,1}, \ldots, w_{a,b}]/\mathbf{I}_{k[\Delta'] \underline{\otimes} k[\Delta'']}$ is Koszul. Hence by [1] 16 p. 87 $\mathbf{I}_{k[\Delta'] \underline{\otimes} k[\Delta'']}$ is generated in degree 2, so D above consists of monomials of degree=2, and so $I_{\Delta' \times \Delta''}$ is generated by monomials of degree=2. By [1] 17(b) p. 87 $k[\Delta \times \Delta']$ is Koszul, so every implication is actually an equivalence. ∎

Theorem 3. $k[\Delta]$ *and* $k[\Delta']$ *both Koszul* $\Longrightarrow$ $P_{k[\Delta \times \Delta']}(t) = P_{k[\Delta] \underline{\otimes} k[\Delta']}(t)$.

Proof. $k[\Delta]$, $k[\Delta']$ both Koszul $\Rightarrow$ $k[\Delta \times \Delta']$, $k[\Delta] \underline{\otimes} k[\Delta']$ both Koszul by Theorem 2. Hence by Hence by[1] 16(5) p. 87 $\mathrm{P}_{k[\Delta \times \Delta']}(t) = (\mathbf{Hilb}_{k[\Delta' \times \Delta'']}(-t))^{-1}$ and $\mathrm{P}_{k[\Delta] \underline{\otimes} k[\Delta'](t)} = (\mathbf{Hilb}_{k[\Delta'] \underline{\otimes} k[\Delta'']}(-t))^{-1}$. So, $\mathrm{P}_{k[\Delta \times \Delta']}(t) = \mathrm{P}_{k[\Delta] \underline{\otimes} k[\Delta']}(t)$ by the Corollary. ∎

2. Pseudomanifolds

We will use "homogeneous n-dimensional" and "equidimensional" to denote that all the maximal simplices of a simplicial complex have the same dimension.

Definition 1. *An n-dimensional pseudomanifold is a simplicial complex* $\mathbf{K}$ *such that*
(α) $\mathbf{K}$ *is homogeneously n-dimensional.*
(β) *Every $(n-1)$-simplex of* $\mathbf{K}$ *is the face of* <u>*at most two*</u> *n-simplices of* $\mathbf{K}$.
(γ) *If s and s' are n-simplices in* $\mathbf{K}$, *there is a finite sequence $s = s_0, s_1, \ldots s_m = s'$ of n-simplices in* $\mathbf{K}$ *such that $s_i \cap s_{i+1}$ is an $(n-1)$-simplex for $0 \leq i < m$.*

The boundary of an n-dimensional pseudomanifold $\mathbf{K}$, *denoted* $\dot{\mathbf{K}}$ *or Bd*$\mathbf{K}$, *is the subcomplex of* $\mathbf{K}$ *generated by the set of $(n-1)$-simplices in* $\mathbf{K}$ *which are faces of exactly one n-simplex in* $\mathbf{K}$. *If* $\dot{\mathbf{K}} = \emptyset$, $\mathbf{K}$ *is said to be an n-dimensional pseudomanifold without boundary.*

A simplicial complex fulfilling condition (γ) is said to be *strongly connected.*

Definition 2. Σ *is a homology n-sphere over the coefficient ring* $\mathbf{A}$ *if for every $s \in \Sigma$:*

$$\tilde{H}_i(\Sigma;\mathbf{A}) = \begin{cases} \mathbf{A} & \text{if } i = n \\ 0 & \text{if } i \neq n. \end{cases} \quad \text{and} \quad \tilde{H}_{i-\dim s-1}(\mathrm{Lk}_\Sigma s^{\dagger};\mathbf{A}) = \begin{cases} \mathbf{A} & \text{if } i = n \\ 0 & \text{if } i \neq n \end{cases}.$$

Note. The first fomula in Def. 2 is a special case of the second for $s = \emptyset$, and if Σ is a homology n-sphere then it is homogeneously n-dimensional and so $\dim\mathrm{Lk}_\Sigma s = n - \dim s - 1$.

In dimension ≥ 1 there is no essential difference between these definitions above and the classical ones. In dimension ≤ 0 some differences occur however, this is due to the fact that the empty set is a simplex in our definition of simplicial complex.

Definition 3. *An n-dimensional pseudomanifold S is said to be orientable if, when* $\dim\Sigma \geq 1$, $H_n(S, \mathrm{Bd}S; \mathbf{A}) \doteq \mathbf{A}$ *and nonorientable otherwise (i.e. when equal to 0).*

Else: $\emptyset$ is an nonorientable pseudomanifold without boundary.

$\{\emptyset\}$ is an orientable pseudomanifold without boundary of dimension -1.

A zero-dimensional simplicial complex with only one vertex, denoted $\bullet$, is an orientable pseudomanifold with the boundary $\{\emptyset\}$.

A zero-dimensional simplicial complex with exactly two vertices, denoted $\bullet\bullet$, is a (strongly connected) orientable pseudo-manifold without boundary.

We have: A 1-dimensional complex with exactly two vertices v_1 and v_2 is an orientable pseudomanifold with the boundary $\{\{v_1\}, \{v_2\}, \emptyset\}$.
An equidimensional one-dimensional simplicial complex with exactly three vertices is an orientable pseudo-manifold that either triangulates a line and then its boundary is **generated** by its "end-vertices" or it triangulates a circle and then it has no boundary.

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† for def. of "Lk", see next page.

3. Buchsbaum, Cohen-Macaulay and Gorenstein complexes

We first recall some well-known definitions. The "link of a simplex $\sigma \in \Sigma$" is defined by

$$\mathrm{Lk}_{\Sigma}\sigma = \{\tau \in \Sigma | [\sigma \cap \tau = \emptyset] \wedge [\sigma \cup \tau \in \Sigma]\}.$$

The "*closed star* of a simplex δ in a simplicial complex Σ", "$\overline{\mathrm{st}}_{\Sigma}\delta$", is given by $\overline{\mathrm{st}}_{\Sigma}\delta := \{\tau \in \Sigma \mid \tau \cup \delta \in \Sigma\}$, and $v \in V_{\Sigma}$ is called a "cone point in Σ" if $\overline{\mathrm{st}}_{\Sigma}\bar{v} = \Sigma$. $\mathrm{core}\Sigma := \{\sigma \in \Sigma \mid \sigma$ contains no cone points$\}$. The set of cone points of a simplicial complex Σ constitutes a simplex in Σ that we call δ_{Σ}, and we see that if Σ has nothing but cone points as vertices then $\mathrm{core}\Sigma = \{\emptyset\}$, so we have that $[\mathrm{core}\Sigma \neq \emptyset] \iff [\Sigma \neq \emptyset]$) and moreover we get $\Sigma = (\mathrm{core}\Sigma) * \delta_{\Sigma}$ and $\mathrm{core}\Sigma = \mathrm{Lk}_{\Sigma}\delta_{\Sigma}$.

A simplicial complex Σ is said to be Buchsbaum (Bbm), (Cohen-Macaulay (C-M), Gorenstein (Gor)) complex over k if the face ring $k[\Sigma]$ is Bbm (C-M, Gor. resp.). We won't use the ring theoretic definitions of Bbm, C-M, Gor. resp. and therefore we won't write them out. Instead we will use the following three theorems, taken from [13], as "definitions".

Proposition 1. *(Schenzel) ([13] Prop. 8.1) Let Σ be a finite simplicial complex and let k be a field. Then the following are equivalent:*
(i) *Σ is Buchsbaum over k.*
(iii) *For all $\sigma \in \Sigma, \sigma \neq \emptyset$ and $i < \dim(\mathrm{Lk}_{\Sigma}\sigma)$, $\tilde{H}_i(\mathrm{Lk}_{\Sigma}\sigma; k) = 0$.*
(iv) *For all $x \in |\Sigma|$, and $i < \dim \Sigma$, $H_i(|\Sigma|, |\Sigma| \setminus \{x\}; k) = 0$.*

Proposition 2. *([13] Prop. 4.3, Cor. 4.2) Let Σ be a finite simplicial complex and let k be a field. Then the following are equivalent:*
(i) *Σ is Cohen-Macaulay over k.*
(ii) *(Reisner) For all $\sigma \in \Sigma$ and all $i < \dim(\mathrm{Lk}_{\Sigma}\sigma)$, $\tilde{H}_i(\mathrm{Lk}_{\Sigma}\sigma; k) = 0$.*
(iii) (Munkres) For all $x \in |\Sigma|$ and all $i < \dim \Sigma$, $\tilde{H}_i(|\Sigma|; k) = H_i(|\Sigma|, |\Sigma| \setminus \{x\}; k) = 0$.

Proposition 3. *([13] Prop 5.1) Let Σ be a finite simplicial complex, k a field and $\Gamma := \mathrm{core}\Sigma$. Then the following are equivalent:*
(i) *Σ is Gorenstein over k.*
(ii) *For all $\sigma \in \Gamma$,* $\tilde{H}_i(\mathrm{Lk}_{\Gamma}\sigma; k) = \begin{cases} k & \text{if } i = \dim(\mathrm{Lk}_{\Gamma}\sigma) \\ 0 & \text{if } i \neq \dim(\mathrm{Lk}_{\Gamma}\sigma). \end{cases}$
(iii) *For all $x \in |\Gamma|$* $\tilde{H}_i(|\Gamma|; k) = H_i(|\Gamma|, |\Gamma| \setminus \{x\}; k) = \begin{cases} k & \text{if } i = \dim \Gamma \\ 0 & \text{if } i \neq \dim \Gamma. \end{cases}$
(iv) *Σ is C-M/k, and Γ is an orientable pseudomanifold without boundary.*
(v) *Either* **(1)** *$\Sigma = \{\emptyset\}$, $\bullet$, $\bullet\bullet$, or* **(2)** *Σ is Cohen-Macaulay over k, $\dim \Sigma \geq 1$, and the link of every $(\dim(\Sigma) - 2)$-face is either a circle or a line with two or three vertices and $\tilde{\chi}(\Gamma) = (-1)^{\dim \Gamma}$.*

The proofs of the equivalences given above are easily found through the references in [13], except for Prop. 3. (iv) where the reader is referred to -"a remark by Björner". Anders Björner has, on our request, kindly sent us a very enlightening algebraic topological proof that we now, with his permission, publish as an appendix.

Note. $\mathcal{R}[\Sigma]$, $\mathcal{R}$ a commutative ring, is defined in exactly the same way as $\mathbf{k}[\Sigma]$. Here some results on $\mathbf{Z}[\Sigma]$ and $\mathbf{Q}[\Sigma]$; (cf. [10] p. 181 Prop. (2.3)+(2.4) and p. 210 Prop. (6.6) and [11] p. 34, p. 37 and p. 44 Lemma 5 (+Remark 2), 8 and 11 resp.)
1) Σ Bbm (C-M, Gor) over the field $\mathbf{Q}$ of rational numbers $\Longleftrightarrow$ Σ Bbm (C-M, Gor) over some field $\mathbf{k} \Longleftrightarrow \Sigma$ Bbm (C-M, Gor) over the prime field $\mathbf{Z}_p$ of $\mathbf{k}$. $\Longleftrightarrow \Sigma$ Bbm (C-M, Gor) for all but at most finitely many prime fields $\mathbf{Z}_p$.
2) Σ Bbm (C-M, Gor) over $\mathbf{Z} \Longleftrightarrow \Sigma$ Bbm (C-M, Gor) over every prime field $\mathbf{Z}_p$.

4 Joins of Buchsbaum, Cohen-Macaulay and Gorenstein complexes

For $\mathbf{A}$ a field [4] Corollary 2.3 p. 126 † gives the following formula;

$$\tilde{H}_{i+1}(\Sigma_1 * \Sigma_2; \mathbf{A}) \doteq \bigoplus_{p+q=i} (\tilde{H}_p(\Sigma_1; \mathbf{A}) \otimes (\tilde{H}_q(\Sigma_2; \mathbf{A})). \qquad (1)$$

Directly from definition of "Link" we have that $\mathrm{Lk}_\Sigma \emptyset = \Sigma$ ("the missing link").

Lemma. *If* $V_{\Sigma_1} \cap V_{\Sigma_2} = \emptyset$ $(\Leftrightarrow \Sigma_1 \cap \Sigma_2 = \{\emptyset\})$ *then*

$$[\tau \in \Sigma_1 * \Sigma_2] \Longleftrightarrow [\exists! \ \sigma_i \in \Sigma_i \ (i = 1, 2) \text{ so that } \tau = \sigma_1 \cup \sigma_2], \quad \text{and}$$

$$\mathrm{Lk}_{\Sigma_1 * \Sigma_2}(\sigma_1 \cup \sigma_2) = (\mathrm{Lk}_{o\Sigma_1} \sigma_1) * (\mathrm{Lk}_{\Sigma_2} \sigma_2) \ . \qquad \blacksquare$$

which, when inserted in Eq. (1), gives

$$\tilde{H}_{i+1}(\mathrm{Lk}_{\Sigma_1 * \Sigma_2}(\sigma_1 \cup \sigma_2); \mathbf{A}) = \tilde{H}_{i+1}((\mathrm{Lk}_{\Sigma_1}\sigma_1) * (\mathrm{Lk}_{\Sigma_2}\sigma_2); \mathbf{A}) \doteq$$

$$\doteq \bigoplus_{p+q=i} (\tilde{H}_p(\mathrm{Lk}_{\Sigma_1}\sigma_1; \mathbf{A}) \otimes \tilde{H}_q(\mathrm{Lk}_{\Sigma_2}\sigma_2; \mathbf{A})). \qquad (2)$$

If σ_1 is a maximal simplex we have that $\mathrm{Lk}_{\Sigma_1}\sigma_1 = \{\emptyset\}$ and we get

$$\tilde{H}_{i+1}(\mathrm{Lk}_{\Sigma_1 * \Sigma_2}(\sigma_1 \cup \sigma_2); \mathbf{A}) = \tilde{H}_{i+1}(\mathrm{Lk}_{\Sigma_2}\sigma_2; \mathbf{A}). \qquad (3)$$

If $\sigma_1 = \emptyset$, we have

$$\tilde{H}_{i+1}(\mathrm{Lk}_{\Sigma_1 * \Sigma_2}\sigma_2; \mathbf{A}) \doteq \bigoplus_{p+q=i} (\tilde{H}_p(\Sigma_1; \mathbf{A}) \otimes \tilde{H}_q(\mathrm{Lk}_{\Sigma_2}\sigma_2; \mathbf{A})). \qquad (4)$$

If $\sigma_1 = \emptyset$ and σ_2 is a maximal simplex then

$$\tilde{H}_{i+1}(\mathrm{Lk}_{\Sigma_1 * \Sigma_2}\sigma_2; \mathbf{A}) \doteq \tilde{H}_{i+1}(\Sigma_1; \mathbf{A}). \qquad (5)$$

Moreover, the lemma above gives us the following equality:

$$\dim(\mathrm{Lk}_{\Sigma_1 * \Sigma_2}(\sigma_1 \cup \sigma_2)) = \dim(\mathrm{Lk}_{\Sigma_1}\sigma_1) + \dim(\mathrm{Lk}_{\Sigma_2}\sigma_2) + 1. \qquad (6)$$

Our theorems below are true also when $\mathbf{A}$ is interpreted not only as a field but also as e.g. $\mathbf{Z}$ by the Note above.

† Our use of the CW-complex related results from [4] is justified by [4] Th. 6.3 Th. 3.2 and Cor. 4.9 p. 26, p. 150 and p. 170 resp.

Theorem 4. *If* $\dim \Sigma_i \geq 0$, $i = 1, 2$ *then,*

$$\Sigma_1 * \Sigma_2 \textit{ Buchsbaum over } \mathbf{A} \Longleftrightarrow \Sigma_1, \Sigma_2 \textit{ both Cohen} - \textit{Macaulay over } \mathbf{A}.$$

Proof. ($\Leftarrow$) Put $n :=\dim(\mathrm{Lk}_{\Sigma_1}\sigma_1) + \dim(\mathrm{Lk}_{\Sigma_2}\sigma_2)$. Then:
If **(a)** $p + q = i < n$ then $p < \dim \mathrm{Lk}_{\Sigma_1}$ or $q < \dim \mathrm{Lk}_{\Sigma_2}$.
If **(b)** $p + q = i > n$ then $p > \dim \mathrm{Lk}_{\Sigma_1}$ or $q > \dim \mathrm{Lk}_{\Sigma_2}$.
Now, Eq. (2) gives the desired conclusion in case (a) and by dimension the result follows in case (b). □($\Leftarrow$)
($\Rightarrow$) If σ_1 is a maximal simplex in Σ_1 (i.e. $d_1 = s_1$) then: $\dim(\mathrm{Lk}_{\Sigma_1 * \Sigma_2}(\sigma_1 \cup \sigma_2) = (d_1 + d_2 + 1) - (s_1 + s_2 + 1) - 1 = d_2 - s_2 - 1 = \dim(\mathrm{Lk}_{\Sigma_2}\sigma_2)$ which by Eq. (2), Eq. (3) and Eq. (4) gives that Σ_2 (Σ_1 equivalently) is C-M/**A**, since we now also have $\sigma_2 = \emptyset$ as a legitimate option. ■

Theorem 5. *If* $\dim \Sigma_i \geq 0$ $(i = 1, 2)$ *then,*

$$\Sigma_1, \Sigma_2 \textit{ both Cohen} - \textit{Macaulay over } \mathbf{A} \Leftrightarrow \Sigma_1 * \Sigma_2 \textit{ Cohen} - \textit{Macaulay over } \mathbf{A}$$

Proof. Because of Theorem 4 it only remains to prove that

$$[\Sigma_1, \Sigma_2 \text{ both Cohen} - \text{Macaulay}/\mathbf{A}] \Rightarrow [\tilde{H}_i(\Sigma_1 * \Sigma_2; \mathbf{A}) = \emptyset \text{ if } i \neq d_1 + d_2 + 1]$$

but this follows from the original formula Eq. (1) on the previous page, and the dimension calculations in the end of the proof of the last theorem, putting $s_1 = s_2 = -1$. ■

Corollary. *If* $\dim \Sigma_i \geq 0$ $(i = 1, 2)$ *then,* $\Sigma_1 * \Sigma_2$ *Buchsbaum over* **A**$\Leftrightarrow$
$\Leftrightarrow$ Σ_1, Σ_2 *both Cohen-Macaulay over* **A**$\Leftrightarrow$ $\Sigma_1 * \Sigma_2$ *Cohen-Macaulay over* **A**. ■

Theorem 6. $\Sigma_1 * \Sigma_2$ *Gorenstein over* **A** $\Leftrightarrow$ Σ_1, Σ_2 *both Gorenstein over* **A**.

Proof. We have that

$$\begin{cases} \Sigma_1 * \Sigma_2 & = (\mathrm{core}(\Sigma_1 * \Sigma_2)) * v_1 * \cdots * v_k; \\ \Sigma_1 & = (\mathrm{core}(\Sigma_1)) * v'_1 * v'_2 * \cdots * v'_{k'}; \\ \Sigma_2 & = (\mathrm{core}(\Sigma_2)) * v''_1 * v''_2 * \cdots * v''_{k''}; \end{cases}$$

Where $\{v_1, v_2, \ldots, v_k\}$ ($\{v'_1, v'_2, \ldots, v'_{k'}\}$ resp. $\{v''_1, v''_2, \ldots, v''_{k''}\}$) are the cone points in $\Sigma_1 * \Sigma_2$ (Σ_1 resp. Σ_2).

Since $\mathrm{core}\Sigma$ never has any cone points and the fact that "join" is commutative we get

$$\mathrm{core}(\Sigma_1 * \Sigma_2) = \mathrm{core}(\Sigma_1) * \mathrm{core}(\Sigma_2),\ k = k' + k''$$

and

$$\tilde{H}_{i+1}(\mathrm{Lk}_{\mathrm{core}\Sigma_1 * \Sigma_2}\tau; \mathbf{A}) = \tilde{H}_{i+1}(\mathrm{Lk}_{\mathrm{core}\Sigma_1 * \mathrm{core}\Sigma_2}\tau; \mathbf{A}) =$$

$$= \tilde{H}_{i+1}(\mathrm{Lk}_{\mathrm{core}\Sigma_1 * \mathrm{core}(\Sigma_2)}\tau_1 \cup \tau_2; \mathbf{A}) = \tilde{H}_{i+1}((\mathrm{Lk}_{\mathrm{core}\Sigma_1}\tau_1) * (\mathrm{Lk}_{\mathrm{core}\Sigma_2}\tau_2); \mathbf{A}) \stackrel{\sim}{=}$$

$$\cong \bigoplus_{p+q=i} (\tilde{H}_p(\mathrm{Lk}_{\mathrm{core}\Sigma_1}\tau_1; \mathbf{A}) \otimes (\tilde{H}_q(\mathrm{Lk}_{\mathrm{core}\Sigma_2}\tau_2; \mathbf{A}))).$$

Now, essentially the same reasoning as in the proof of the last theorem will, together with Prop. 3 (ii), give the result. ■

The original proof of Theorem 5 above was topological and given in P. F. Garst, Dissertation, Univ. of Wisconsin-Madison, 1979. The original proof of Theorem 6 above was ring theoretic and given in [7], together with a ring theoretic proof of Theorem 5.

5. Products of Buchsbaum and Cohen-Macaulay Complexes

Definition. *"The simplicial cartesian product of pairs" (cf. [5] p. 67),*

$$(\Sigma_1, \Delta_1) \times (\Sigma_2, \Delta_2) := (\Sigma_1 \times \Sigma_2, (\Sigma_1 \times \Delta_2) \cup (\Delta_1 \times \Sigma_2)) \tag{7}$$

and similarly for topological pairs (cf. [12] p. 234), and for join.

Note: There are homeomorphisms $|\Sigma_1| \times |\Sigma_2| \simeq |\Sigma_1 \times \Sigma_2|$ and $|\Sigma_1| * |\Sigma_2| \simeq |\Sigma_1 * \Sigma_2|$ (cf. [5] Lemma 8.9 p. 68. and [16] p. 99 respectively).

We will use Künneth's fomula from [12] Th. 10 p. 235 ; which, for a field as coefficient ring gives

$$H_q((X, A) \times (Y, B); k) \cong [H(X, A; k) \otimes H(Y, B; k)]_q. \tag{8}$$

The homeomorphism $|\Sigma_1| \times |\Sigma_2| \simeq |\Sigma_1 \times \Sigma_2|$ (mapping (x_1, x_2) to $(\widetilde{x_1, x_2})$) gives,

$$(|\Sigma_1|, |\Sigma_1| \setminus \{x_1\}) \times (|\Sigma_2|, |\Sigma_2| \setminus \{x_2\}) :=$$

$$= (|\Sigma_1| \times |\Sigma_2|, (|\Sigma_1| \times (|\Sigma_2| \setminus \{x_2\}) \cup (|\Sigma_1| \setminus \{x_1\}) \times |\Sigma_2|)) =$$

$$= (|\Sigma_1| \times |\Sigma_2|, |\Sigma_1| \times |\Sigma_2| \setminus \{(x_1, x_2)\}) \simeq (|\Sigma_1 \times \Sigma_2|, |\Sigma_1 \times \Sigma_2| \setminus \{(\widetilde{x_1, x_2})\}). \tag{9}$$

Our theorems below are true also when $\mathbf{A}$ is interpreted not only as a field but also as e.g. $\mathbf{Z}$ by the "Note" ending Ch. 3. Since $\Sigma \times \emptyset = \emptyset \times \Sigma = \emptyset$ and $\Sigma \times \{\emptyset\} = \{\emptyset\} \times \Sigma = \{\emptyset\}$ we leave these cases out, and because of the local character of the Buchsbaum condition, it isn't necessary to have any other restrictions in this case.

Theorem 7. *If* $\dim \Sigma_1 \geq 0 \leq \dim \Sigma_2$ *then*
$\Sigma_1 \times \Sigma_2$ *Buchsbaum over* $\mathbf{A} \iff \Sigma_1$ *and* Σ_2 *both Buchsbaum over* $\mathbf{A}$.

Proof. $\{|\Sigma_1| \times (|\Sigma_2| \setminus \{x_2\}), (|\Sigma_1| \setminus \{x_1\}) \times |\Sigma_2|$ is an excisive couple (cf. ADDENDUM p 18). Our claim now follows from Proposition 1(iv), and Eq. (8)-(9), remembering that $H_{\dim \Sigma}(|\Sigma|, \ |\Sigma| \setminus x; \mathbf{A}) = \mathbf{A}$ if $x \in \mathrm{Int}\sigma$ and σ is a maximal simplex in Σ. ■

Note. $[\dim \Sigma_1 \geq 1 \leq \dim \Sigma_2] \iff [\dim(\Sigma_1 \times \Sigma_2) > \max(\dim \Sigma_1, \dim \Sigma_2)]$.

Lemma. *When* $[\dim\Sigma_1 \geq 1 \leq \dim\Sigma_2]$ *then the following equivalences hold,*

$$[\tilde{H}_i(\Sigma_1 \times \Sigma_2; \mathbf{A}) = 0 \text{ for } i < \dim(\Sigma \times \Sigma_2)] \Longleftrightarrow$$
$$\Longleftrightarrow [\Sigma_1 \text{ and } \Sigma_2 \text{ both acyclic}] \Longleftrightarrow [\Sigma_1 \times \Sigma_2 \text{ acyclic}].$$

Proof. Use $[\dim\Sigma \geq 0 \Longrightarrow H_0(\Sigma; \mathbf{A}) \neq 0]$ in Eq. (8) remembering the last note. ■

Corollary 1. *When* $\dim\Sigma_i \geq 1$ *then,* $\Sigma_1 \times \Sigma_2$ *is never a homology sphere over* $\mathbf{A}$.

Proof. Follows from the last lemma and the definition of a homology sphere.

Theorem 8. *If* $\dim\Sigma_1 \geq 0 \leq \dim\Sigma_2$ *then*
(a) *If* $\dim(\Sigma_1) = 0 = \dim(\Sigma_2)$ *then* $\dim(\Sigma_1 \times \Sigma_2) = 0$ *and so, all three of them are C-M over* $\mathbf{A}$.
(b) *If only one of* Σ_1 *and* Σ_2 *is 0-dimensional, then* $\dim(\Sigma_1 \times \Sigma_2) \geq 1$ *and* $\Sigma_1 \times \Sigma_2$ *is C-M over* $\mathbf{A}$ *iff the 0-dimensional complex contains one and only one point (i.e. C-M over* $\mathbf{A}$ *and acyclic) and the other complex is C-M over* $\mathbf{A}$.
(c) *When* $\dim\Sigma_1 \geq 1 \leq \dim\Sigma_2$, *then* $\Sigma_1 \times \Sigma_2$ *is C-M over* $\mathbf{A} \Longleftrightarrow \Sigma_1$ *and* Σ_2 *are both C-M over* $\mathbf{A}$ *and both acyclic.*

Proof. (a) and (b) are trivial and (c) follows immediately from Theorem 7 above, Proposition 2(iii) and the last lemma. ■

From Propositions 1–3 we get, Σ Gorenstein over $\mathbf{A} \Longrightarrow \Sigma$ C-M over $\mathbf{A} \Longrightarrow \Sigma$ Bbm over $\mathbf{A}$, which gives us the following corollary to Theorem 8.

Corollary 2. *If* $\dim\Sigma_i \geq 1$ *then:* $[\Sigma_1 \times \Sigma_2$ *Gorenstein over* $\mathbf{A}] \Longrightarrow [\Sigma_1 \times \Sigma_2$ *has at least one cone point*$]$.

Proof. If $\Sigma_1 \times \Sigma_2$ lacks cone points, then $\Sigma_1 \times \Sigma_2 = \text{core}(\Sigma_1 \times \Sigma_2)$ which is a homology sphere over $\mathbf{A}$ by Proposition 3(iii), since $\Sigma_1 \times \Sigma_2$ is supposed to be Gorenstein, which is a contradiction to Corollary 1. ■

Since we can eliminate all cone points in a simplicial complex Σ, by refinements, we have:

Corollary 3. *The property of being Gorenstein is not (unlike C-M and Bbm) a topological property of* $|\Sigma|$. ■

Consider the following facts:

1. Using Baclawski's notation we make the following quote from [2];
   (a) p. 233, Prop. 3.3. *Let* P *be a poset. Then* P *is CM if and only if* $\Delta(P)$ *is CM and similarly for ACM* (:="Almost C-M"$\Longleftrightarrow$to our use of "Buchsbaum").
   (b) p. 249. Let P and Q be posets then; $\Delta(P \times Q)$ triangulates $|\Delta(P)| \times |\Delta(Q)|$.
2. The polytopes of a simplicial complex and its refinements are homeomorphic.
3. Σ being Buchsbaum or Cohen-Macaulay over $\mathbf{A}$ is a topological invariant of $|\Sigma|$. (Immediate from Proposition 1–2).

Items 1-3 together with Theorem 7 and Theorem 8 above make it possible to generalize Baclawski's Theorem 7.1. p. 249, in [2] on posets, to become

Theorem 9. *If $l(P) \geq 0 \leq l(Q)$ then;*

i. *$P \times Q$ ACM $\iff$ P and Q both ACM.*

ii. *$P \times Q$ CM $\iff$ $P \times Q$ both CM and acyclic or both antichains or one consists of a single element and the other is CM.* ■

6. Products and Joins of Pseudomanifolds

Let Δ_1 and Δ_2 be two arbitrary simplicial complexes with a linear order defined on their vertex sets V_1 resp. V_2. When the order-relation between two vertices is significant we let the subindexing be done in accordance with the appropriate ordering i.e. $v_i < v_j \Leftrightarrow i < j$.

A simplicial complex Δ can always be uniquely described as a union,

$$\Delta = \bigcup_{\delta^m \in \Delta} \overline{\delta^m} \qquad \text{where the “}m\text{” } in\ \delta^m \text{ denotes “maximal simplex”.}$$

One way to represent the maximal simplices in the Simplicial Cartesian Product, denoted $\Delta_1 \times \Delta_2$, of two simplicial complexes Δ_1 and Δ_2, is by the use of what we will call the *representation matrices*, denoted $M_{\delta_1^m \times \delta_2^m}$, the entries of which are $x_{ij} := (v_{1i}, v_{2j})$, where $v_{1i} \in V_{\overline{\delta_1^m}}$ and $v_{2j} \in V_{\overline{\delta_2^m}}$ and where $[x_{ij} < x_{kl}] \Leftrightarrow [v_{1i} < v_{1k}\ and\ v_{2j} < v_{2l}]$ (equivalent to $[i < k$ and $j < l])$. The definition of x_{ij} implies that $V_{\Delta_1 \times \Delta_2} = V_{\Delta_1} \times V_{\Delta_2}$ is given the product ordering. $M_{\delta_1^m \times \delta_2^m} =$

$$= \begin{pmatrix} w_{0,0} & w_{0,1} & \cdots & w_{0,q} \\ w_{1,0} & w_{1,1} & \cdots & w_{1,q} \\ \cdot & \cdot & \cdots & \cdot \\ \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \cdots & \cdot \\ w_{p,0} & w_{p,1} & \cdots & w_{p,q} \end{pmatrix} = \begin{pmatrix} (v_{1,i_0}, v_{2,j_0}) & (v_{1,i_0}, v_{2,j_1}) & \cdots & (v_{1,i_0}, v_{2,j_q}) \\ (v_{1,i_1}, v_{2,j_0}) & (v_{1,i_1}, v_{2,j_1}) & \cdots & (v_{1,i_1}, v_{2,j_q}) \\ \cdot & \cdot & \cdots & \cdot \\ \vdots & \vdots & \ddots & \vdots \\ \cdot & \cdot & \cdots & \cdot \\ (v_{1,i_p}, v_{2,j_0}) & (v_{1,i_p}, v_{2,j_1}) & \cdots & (v_{1,i_p}, v_{2,j_q}) \end{pmatrix}$$

where $\delta_1^m = [v_{1,i_0}, v_{1,i_1}, \ldots, v_{1,i_p}]$ and $\delta_2^m = [v_{2,j_0}, v_{2,j_1}, \ldots, v_{2,j_q}]$ are maximal simplices in Δ_1 resp. Δ_2. **This product order** on the vertices of $\Delta_1 \times \Delta_2$ induced from the linear orderings on V_{Δ_1} and V_{Δ_2} **is a partial order** on $V_{\Delta_1 \times \Delta_2}$ that, in particular, induces a linear order on the vertices of every simplex $\tau \in \overline{\delta_1^m} \times \overline{\delta_2^m}$ for every maximal simplex $\delta_1^m \in \Delta_1$ ($\delta_2^m \in \Delta_2$), **which,** when the elements in V_{Δ_1} and V_{Δ_2} are numbered in accordance with the linear ordering, respectively, **can be expressed as**

$$[(v_{1,i}, v_{2,j}) < (v_{1,k}, v_{2,l})] \Leftrightarrow [[(i,j) \neq (k,l)]\ and\ [i \leq k\ and\ j \leq l]].$$

Visualized in $M_{\delta_1^m \times \delta_2^m}$, we get $[(v_{1,i}, v_{2,j}) < (v_{1,k}, v_{2,l})] \Leftrightarrow [(v_{1,k}, v_{2,l})$ does **not** lie above or to the left of $(v_{1,i}, v_{2,j})] \Leftrightarrow [(v_{1,k}, v_{2,l})$ is in the closed lower right rectangular sector with $(v_{1,i}, v_{2,j})$ as upper left corner]. As any other simplicial complex $\Delta_1 \times \Delta_2$ can be written in the form $\Delta_1 \times \Delta_2 = \bigcup_{\tau^m \in \Delta_1 \times \Delta_2} \overline{\tau^m}$ which gives us

$$(\bigcup_{\delta_1^m \in \Delta_1} \overline{\delta_1^m}) \times (\bigcup_{\delta_2^m \in \Delta_2} \overline{\delta_2^m}) = \Delta_1 \times \Delta_2 = \bigcup_{\tau^m \in \Delta_1 \times \Delta_2} \overline{\tau^m} = \bigcup_{\delta_i^m \in \Sigma_i} (\overline{\delta_1^m} \times \overline{\delta_2^m}).$$

Let Σ_1 and Σ_2 be simplicial complexes and let ∇ stand either for the join "$*$", or for the simplicial product, "$\times$". [We put the join-related restrictions within brackets].

Lemma 1. **A)** *If $d_i := \dim \Sigma_i \geq 0$ $(i = 1, 2)$ $[\Sigma_1 \neq \emptyset \neq \Sigma_2]$ then,*
$\Sigma_1 \nabla \Sigma_2$ is equidimensional $\Longleftrightarrow$ Σ_1 and Σ_2 are both equidimensional.
B) *Let the dimension of every maximal simplex in Σ_i be ≥ 1 $(i = 1, 2)$ $[\Sigma_1 \neq \emptyset \neq \Sigma_2]$* then; *Every submaximal face of $\Sigma_1 \nabla \Sigma_2$ lies in at most (exactly) two maximal* faces *of $\Sigma_1 \nabla \Sigma_2$ $\Longleftrightarrow$ The same conditions are true for both Σ_1 and Σ_2.*
C) *If $d_i := \dim \Sigma_i \geq 0$ $(i = 1, 2)$ $[\Sigma_1 \neq \emptyset \neq \Sigma_2]$ then,*
$\Sigma_1 \nabla \Sigma_2$ is strongly connected $\Longleftrightarrow$ Σ_1 and Σ_2 are both strongly connected.

Proof. We leave out the proof for the join case since it, in spirit, is the same considerations concerning facets and subfacets as in the product case, and the only difference is that now everything works totally without complications.
A) follows directly from the construction of the simplicial cartesian product.

B) ($\Longleftarrow$) Take a maximal face τ of $\Sigma_1 \times \Sigma_2$. We have to show that any submaximal face σ in τ belongs to at most (exactly) one maximal face in $\Sigma_1 \times \Sigma_2$ besides τ. Using the representation matrix for τ, we first suppose that the deleted vertex is in the lower right corner or the upper left corner. Then the projections of σ give a maximal face in one (say the first) factor, and a submaximal face σ_2 in the other factor. There are at most (exactly) two maximal faces of Σ_2 containing σ_2. These give at most (exactly) two maximal faces of $\Sigma_1 \times \Sigma_2$ (coming from different representation matrices) containing σ. The same reasoning applies if the deleted vertex gives rise to a vertical or horizontal jump, so we can suppose that we have a diagonal jump. Then both projections give maximal faces, hence any maximal face containing σ must belong to the same representation matrix as τ. There are exactly two such maximal faces.
($\Longrightarrow$) Let $\sigma_1 = (x_1, \ldots, x_m)$ be a submaximal face of Σ_1. Take any maximal face τ_2 in Σ_2. Each maximal face τ_1 of Σ_1 that contains σ_1 gives, together with τ_2, rise to a representing matrix containing

$$\sigma = \{(x_1, y_0), (x_2, y_0), \ldots, (x_m, y_0), (x_m, y_1), \ldots, (x_m, y_n)\}$$

as a submaximal face. But there are at most (exactly) two maximal faces of $\Sigma_1 \times \Sigma_2$ containing σ. This gives the claim.

C)($\Longrightarrow$) Let τ_1, τ_1' be maximal faces of Σ_1. Take any maximal face τ_2 in Σ_2. Let τ be any maximal face of $\Sigma_1 \times \Sigma_2$ in the representation matrix coming from the pair

(τ_1, τ_2), and let τ' be any maximal face of $\Sigma_1 \times \Sigma_2$ coming from the pair (τ_1', τ_2). Then there is a sequence of maximal faces in $\Sigma_1 \times \Sigma_2$ connecting τ with τ' and with a submaximal face as intersection of two consecutive faces in the sequence. The projection on the first factor gives a sequence (perhaps with repetitions) of maximal faces of Σ_1 which fulfills the wanted condition.

($\Longleftarrow$) Let τ_1 and τ_2 be any maximal faces of $\Sigma_1 \times \Sigma_2$, and let τ_1' (τ_1'') and τ_2' (τ_2'') be their projections to Σ_1 (Σ_2). We can easily get a sequence within the representation matrix for τ_1, to reach the maximal face τ_1^* which belongs to the first row and last column in the same matrix. Now τ_1'' and τ_2'' are connected to each other in Σ_2 by a sequence. We first describe an ideal situation when the last vertex in the chain from τ_1'' to τ_2'' remains the same through the whole chain. Then we get a corresponding sequence of maximal faces of "first row-last column"-faces of different representation matrices, until we reach the representation matrix of the pair (τ_1', τ_2''). Then we use the sequence in Σ_1 between τ_1' and τ_2'. If the first vertex in this chain remains the same throughout, we get a corresponding sequence of "first row-last column"-faces of different representation matrices until we reach the representation matrix of the pair (τ_2', τ_2''). Finally we continue within that matrix to reach τ_2.
To make the algorithm complete we make switches from the "first row-last column"-face in a representation matrix to the "first column-last row"-face in the same matrix and vice versa when it is necessary to change the last (first) vertex in the projected chains, and then go on to do this switch back and forth. ■

We now use Lemma 1 to prove some classical results, for $\mathbf{A} = k$ or $\mathbf{Z}$.

Theorem 10. *When* $d_i := \dim \Sigma_i \geq 0$ $(i = 1, 2)$ *and* $\Sigma_i \neq \bullet\bullet$ $[\Sigma_1 \neq \emptyset \neq \Sigma_2]$ *then we have the following equivalences:*

10.1 $\Sigma_1 \nabla \Sigma_2$ *is a* $(d_1 + d_2\ [+1])$*-pseudomanifold* $\Longleftrightarrow$ Σ_i *is a* d_i*-pseudomanifold*

10.2
$$\mathrm{Bd}(\Sigma_1 \nabla \Sigma_2) = ((\mathrm{Bd}\Sigma_1) \nabla \Sigma_2) \cup (\Sigma_1 \nabla (\mathrm{Bd}\Sigma_2))$$

10.3 $\Sigma_1 \nabla \Sigma_2$ *is orientable over* $\mathbf{A} \Longleftrightarrow \Sigma_1, \Sigma_2$ *are both orientable over* $\mathbf{A}$.

Proof. (10.1) follows from Lemma 1, which also works for $\Sigma_i = \bullet$. □
(10.2) Combining a maximal simplex from one simplicial complex with a submaximal simplex in the boundary of the other simplicial complex, it's then quite plain to show that the totality of all such combinations constitute the boundary of $\Sigma_1 \nabla \Sigma_2$ and that this totality is the very set on the right hand side of (10.2) is self-evident.
(10.3) (The "$\times$-case") First, let $\mathbf{A} = k$. To prove the statement on orientability, we use Eq. (8) with $(X, A) := (\Sigma_1, \mathrm{Bd}\Sigma_1)$ and $(Y, B) := (\Sigma_2, \mathrm{Bd}\Sigma_2)$, and thereby immediately get a proof of our claim, since by the definition of "$\times$" for pairs, cf. Eq. (7) and (10.2), we have that: $(\Sigma_1, \mathrm{Bd}\Sigma_2) \times (\Sigma_1, \mathrm{Bd}\Sigma_2) = (\Sigma_1 \times \Sigma_2, (\Sigma_1 \times$

$\mathrm{Bd}\Sigma_2) \cup (\mathrm{Bd}\Sigma_1 \times \Sigma_2)) = (\Sigma_1 \times \Sigma_2, \mathrm{Bd}(\Sigma_1 \times \Sigma_2))$, and so $H_{\dim(\Sigma_1 \times \Sigma_2)}((\Sigma_1, \mathrm{Bd}\Sigma_1) \times (\Sigma_2, \mathrm{Bd}\Sigma_2); k) \hat{=} H_{\dim \Sigma_1}(\Sigma_1, \mathrm{Bd}\Sigma_1; k) \otimes H_{\dim \Sigma_2}(\Sigma_2, \mathrm{Bd}\Sigma_2; k)$. Björner's Remark 1 in the Appendix, on orientable pseudomanifolds, implies: $\Sigma_1 \nabla \Sigma_2$ is an orientable pseudomanifold over $\mathbf{Z} \iff \Sigma_1, \Sigma_2$ both orientable pseudomanifolds over $\mathbf{Z}$.
(10.3) (The "$*$-case") Björner's Remark 1 in the Appendix, on the absolutness of the notion of orientability of pseudomanifolds can also be described in the following way; (Supposing that char$(\mathbf{A}) \neq 2$ because char$(\mathbf{A}) = 2$ makes (10.3) trivially true.)

From condition (γ) in the definition of a pseudomanifold we have the following; If s and s' are n-simplices in $\mathbf{K}$, there is a finite sequence $s = s_0, s_1, \ldots s_m = s'$ of n-simplices in $\mathbf{K}$ such that $s_i \cap s_{i+1}$ is an $(n-1)$-simplex for $0 \leq i < m$. We can suppose that the sequence contains no repetitions. Now, if s_i is an oriented simplex then there is one and only one orientation possible for s_{i+1} to make the coefficient for $\delta s_i - \delta s_{i+1} = 0$. A pseudomanifold is orientable iff for any closed sequence (i.e. $s = s'$) without repetitions the orientation of $s' = s$ induced by the condition $\delta s_i - \delta s_{i+1} = 0$ remains the same as the the original one.

$(\Longrightarrow)$ If one of the factors, say Σ_1, is non-orientable then there is an orientation-switching sequence $s_1 = s_{10}, s_{11}, \ldots s_{1m} = s_1'$, then for an arbitrary simplex $s_2 \in \Sigma_2$, $s_1 \cup s_2 = s_{10} \cup s_2, s_{11} \cup s_2, \ldots s_{1m} \cup s_2 = s_1' \cup s_2$ is an orientation-switching sequence in $\Sigma_1 * \Sigma_2$. $\square(\Longrightarrow)$

$(\Longleftarrow)$ If there is orientation-switching sequence in $\Sigma_1 * \Sigma_2$, its projections down on the factors are, perhaps after deleting repetitions, sequences in the factors, and they can't both be orientation-switching, since then the union whould not be orientation-switching, a contradiction. So, exactly one of the projections is orientation-switching, implying that at least one of the factors are non-orientable.■

7. The Simplicial Cartesian Product of Gorenstein Complexes

The cone points in a simplicial complex Σ are characterized in the following lemma:

Lemma 2. *$v \in V_\Delta$ is a cone point in $\Delta \iff v$ is a vertex in every maximal simplex in Δ.*

Theorem 10 imply the following lemma:

Lemma 3. coreΣ_i $(i = 1, 2)$ *are (orientable) pseudomanifolds* iff core$(\Sigma_1 \times \Sigma_2)$ *is one.*

Lemma 4. *If $\dim\Delta_i \geq 1$ $(i = 1, 2)$, then $\Delta_1 \times \Delta_2$ has at most two cone points, and both Δ_1 and Δ_2 has at least as many cone points as $\Delta_1 \times \Delta_2$, and*

$$\text{(i)} \quad \begin{bmatrix} w := (v_1, v_2) \text{ is a cone} \\ \text{point in } \Delta_1 \times \Delta_2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} v_i \text{ is a cone point in } \Delta_i \ (i = 1, 2) \text{ and either} \\ \text{are both } v_1 \text{ and } v_2 \text{ minimal elements or are} \\ \text{both maximal elements in } V_{\Delta_1},\ V_{\Delta_2}. \end{bmatrix}.$$

If any side of the last equivalence is valid, and if both Δ_1 and Δ_2 are pseudomanifolds then,

$$\left[\begin{array}{c}\Delta_1 \times \Delta_2 \text{ has } \textbf{exactly} \text{ one cone} \\ \text{point and } \mathrm{Bd}(\mathrm{core}(\Delta_1 \times \Delta_2)) = \emptyset\end{array}\right] \Longleftrightarrow \left[\begin{array}{c}\Delta_i \text{ has } \textbf{exactly} \text{ one cone point} \\ \text{and } \mathrm{Bd}(\mathrm{core}(\Delta_i)) = \emptyset\end{array}\right] \tag{10}$$

(ii) $$\left[\begin{array}{c} w_1 := (v_{11}, v_{21}) \text{ and} \\ w_2 := (v_{12}, v_{22}) \\ \text{are different cone} \\ \text{points in } \Delta_1 \times \Delta_2 \end{array}\right] \Longleftrightarrow \left[\begin{array}{c} v_{ij}\ (i,j = 1,2) \text{ are all different cone points} \\ \text{in the respective simplicial complex and} \\ \text{(say) } v_{11},\ v_{21} \text{ are minimal elements and} \\ v_{12},\ v_{22} \text{ are maximal elements in } V_{\Delta_1}\ V_{\Delta_2} \\ \text{respectively (or vice versa).} \end{array}\right]$$

If any side of the last equivalence is valid, and if both Δ_1 and Δ_2 are pseudomanifolds then,

$$[(\mathrm{Bd}(\mathrm{core}(\Delta_1 \times \Delta_2)) = \emptyset] \Longleftrightarrow \left[\begin{array}{c}\Delta_i \text{ has } \textbf{exactly} \text{ two cone points} \\ \text{and } \mathrm{Bd}(\mathrm{core}(\Delta_i)) = \emptyset\end{array}\right]. \tag{11}$$

Proof. $\Rightarrow$ For a vertex $w \in V_{\Delta_1 \times \Delta_2}$ to be a cone point in $\Delta_1 \times \Delta_2$ it must lie in the upper left corner in **every** "representation matrix", or it must lie in the lower right corner in **every** "representation matrix", since we are to allow, as simplices, any **vertex chain** in the product ordering that comes out of simplices in the **simplicial product** $\Delta_1 \triangle \Delta_2$ (c.f. [5] Def. 8.1 + Def. 8.2 + Lemma 8.9 p. 66, 67, and 68 resp.). So there can't be more than two cone points in $\Delta_1 \times \Delta_2$. By the definition of the product ordering this also gives the statements on the presumptive maximality/minimality of the coordinate cone points.

To prove that both Δ_1 and Δ_2 have at least as many cone points as $\Delta_1 \times \Delta_2$ we first note that if $\Delta_1 \times \Delta_2$ doesn't have any cone points at all, then there is nothing to show.

If $w = (v', v'')$ is a cone point in $\Delta_1 \times \Delta_2$ then v', v'' are cone points in Δ_1 resp. Δ_2, so if $\Delta_1 \times \Delta_2$ has exactly one cone point, say $w = (v', v'')$, then v', v'' are cone points in Δ_1 resp. Δ_2, and we're done.

If $\Delta_1 \times \Delta_2$ has exactly two cone points, say $w = (v_{11}, v_{21})$ and $w = (v_{12}, v_{22})$ then all the v's are cone points in Δ_1 resp. Δ_2 and where now either $v_{11} \neq v_{12}$ or $v_{21} \neq v_{22}$, say $v_{21} \neq v_{22}$. Suppose that say $v_{11} = v_{12}$, then (v_{11}, v_{21}) and $w = (v_{11}, v_{22})$ are the cone points in $\Delta_1 \times \Delta_2$. Since $\dim\Delta_i \geq 1$ (i=1,2) there is at least one more vertex from Δ_1 as first coordinate in some of the entries in some of the representation matrices. But since $w = (v_{11}, v_{21})$, say, occupies the upper left corner and $w = (v_{11}, v_{22})$ the lower left corner, or vice versa, in all the representation matrices, there is no room for any other vertex from Δ_1, which is a contradiction, and we're done since the opposite implications are trivial. □

Since $\dim\Delta_i \geq 1$ (i=1,2) we know that core($\Delta_1 \times \Delta_2) \neq \emptyset$ and so we can use Theorem 10, and so since $\Delta_1 \times \Delta_2 =$ (core($\Delta_1 \times \Delta_2)) * (\overline{w_1} * \overline{w_2})$, supposing $w_1 \neq w_2$ are cone points, its boundary can now, by Theorem 10, be written $\mathrm{Bd}(\Delta_1 \times \Delta_2) =$

$= (\text{Bd}(\text{core}(\Delta_1 \times \Delta_2))) * (\overline{w_1} * \overline{w_2}) \cup (\text{core}(\Delta_1 \times \Delta_2)) * (\text{Bd}(\overline{w_1} * \overline{w_2})) =$
$= (\text{Bd}(\text{core}(\Delta_1 \times \Delta_2)) * (\overline{w_1} * \overline{w_2})) \cup ((\text{core}(\Delta_1 \times \Delta_2)) * (\{\{w_1\}, \{w_2\}, \emptyset_o\})) =$
$(\text{Bd}(\text{core}(\Delta_1 \times \Delta_2)) * \overline{w_1} * \overline{w_2}) \cup (\text{core}(\Delta_1 \times \Delta_2) * \{w_1\}) \cup (\text{core}(\Delta_1 \times \Delta_2) * \{w_2\})$
which tells us that

$$[\{w_1, w_2\} \not\subset \text{Bd}(\Delta_1 \times \Delta_2)] \iff [\text{Bd}(\text{core}(\Delta_1 \times \Delta_2)) = \emptyset]. \tag{12}$$

We know, from above, that Δ_1 and Δ_2 both have **at least** two cone points, so we can construct sub-complexes $\mathbf{K}_1$ and $\mathbf{K}_2$ of Δ_1 and Δ_2 respectively, defined so that $\mathbf{K}_i$ contains all simplices in Δ_i that does not contain either of the two cone points v_{i1} or v_{i2} (i=1,2). Since v_{i1} and v_{i2} are different cone points in Δ_i we have that: $\Delta_i = \mathbf{K}_i * (\bar{v}_{i1} * \bar{v}_{i2})$ and therefore, observing that $\mathbf{K}_i \neq \emptyset$, we get by Theorem 10: $\text{Bd}(\Delta_i) = ((\text{Bd}\mathbf{K}_i) * (\bar{v}_{i1} * \bar{v}_{i2})) \cup (\mathbf{K}_i * (\{\{v_{i1}\}, \{v_{i2}\}, \emptyset\}))$ $(i = 1, 2)$.
We now calculate $\text{Bd}(\Delta_1 \times \Delta_2)$ using these equalities: $\text{Bd}(\Delta_1 \times \Delta_2) = [(\text{Bd}(\Delta_1)) \times \Delta_2] \cup [\Delta_1 \times (\text{Bd}(\Delta_2))] = [((\text{Bd}\mathbf{K_1}) * (\bar{v}_{11} * \bar{v}_{12})) \cup (\mathbf{K_1} * (\{\{v_{11}\}, \{v_{12}\}, \emptyset\})) \times \Delta_2] \cup$ $\cup [\Delta_1 \times (((\text{Bd}\mathbf{K_2}) * (\bar{v}_{21} * \bar{v}_{22})) \cup (\mathbf{K_2} * (\{\{v_{21}\}, \{v_{22}\}, \emptyset\})))]$ which tells us that

$$[\{w_1, w_2\} \notin \text{Bd}(\Delta_1 \times \Delta_2)] \iff [(\text{Bd}\mathbf{K_1} = \emptyset \textbf{ and } \text{Bd}\mathbf{K_2} = \emptyset]. \tag{13}$$

Eq. (12) together with Eq. (13) gives:

$$[\text{Bd}(\text{core}(\Delta_1 \times \Delta_2)) = \emptyset] \iff [(\text{Bd}\mathbf{K_1} = \emptyset \textbf{ and } \text{Bd}\mathbf{K_2} = \emptyset].$$

But $\text{Bd}\mathbf{K}_i = \emptyset$ is possible only if $\mathbf{K}_i = \text{core}\Delta_i$ which implies $\text{Bd}(\text{core}\Delta_i) = \emptyset$, and the very last sentence read in "the opposite direction" together with Eq. (13) gives us Eq. (11).

Eq. (10) is proved in the same way as Eq. (11), only with the important difference that $\{w_1, w_2\}$ in Eq. (12) and Eq. (13) is changed to $\{w\}$. ∎

Corollary 2 to Theorem 8 together with Lemma 4 above give us:

Lemma 5. $\Delta_1 \times \Delta_2$ *Gorenstein (with* $|V_{\Delta_i}| > 1$) $\Rightarrow \Delta_1 \times \Delta_2$ *has one or two cone points.*

The last three lemmas, in particular Eq. (10) + Eq. (11), and Prop. 3 (iv) now, for $\mathbf{A} = k$ or $\mathbf{Z}$, give us the following theorem:

Theorem 11. *Let* Δ_1 *and* Δ_2 *be two arbitrary simplicial complexes with* $\dim\Delta_i \geq 1$ *(i=1,2) and a linear order defined on their vertex sets* $V_{\Delta_1}, V_{\Delta_2}$ *respectively, then*

(I) $\Delta_1 \times \Delta_2$ Gorenstein over $\mathbf{A}$

is equivalent to the disjunction of the following two statements

(II) $\begin{cases} \Delta_i \text{ are both Gorenstein over } \mathbf{A} \text{ with exactly one cone point } v_i, \\ i = 1, 2 \text{ which either both are minimal } \textbf{or} \text{ both are maximal.} \end{cases}$

(III) $\begin{cases} \Delta_i \text{ are both Gorenstein over } \mathbf{A} \text{ with exactly two cone points } v_{ij}, 1 \leq i, \\ j \leq 2 \text{ where } v_{i1} \text{ are minimal elements in } V_{\Delta_i}, \text{and } v_{i2} \text{ are maximal in } V_{\Delta_i}. \end{cases}$

$$\left(\text{i.e. (I)} \iff \text{(II)} \vee \text{(III)}\right)$$ ∎

Note. The cases when either of the complexes have dimension less then 1 can easily be analysed using Prop. 3. (iv) and (v).

APPENDIX: Characterization of Gorenstein Complexes

A. Björner, Nov. 13-92

Let Δ be a simplicial complex, $\dim\Delta = d-1$. Let $\mathbf{k}$ be a field or $\mathbf{k} = \mathbf{Z}$.

Theorem. *Suppose Δ is* **C-M**$_{/\mathbf{k}}$, *and let Γ=coreΔ. Then*
Δ is Gorenstein$_{/\mathbf{k}}$ $\Longleftrightarrow$ every subfacet is contained in two facets of Γ and $\tilde{\chi} = (-1)^{\dim\Gamma}$
$\Longleftrightarrow$ *Γ is a pseudomanifold* † *and $\tilde{\chi} = (-1)^{\dim\Gamma}$.*

Proof. The first $\Longrightarrow$ follows from Thm. 5.1(b) of [13]. The second $\Longrightarrow$ follows from the fact that (C-M)-complexes are strongly connected.

Assume now for simplicity that $\Delta = \Gamma$. Recall that if Δ is a pseudomanifold then either

$$\tilde{H}_{d-1}(\Delta;\mathbf{k})\cong\mathbf{k}$$

and there is a unique $(d-1)$-cycle with support Δ^{d-1} (i.e. all $(d-1)$-faces), Δ is then **orientable over k**, or

$$\tilde{H}_{d-1}(\Delta;\mathbf{k}) = 0 \qquad\qquad (\Delta \text{ is non}-\text{orientable}).$$

Lemma 1. *Let Δ be a C-M$_{/\mathbf{k}}$ pseudomanifold. Then*

$$\tilde{\chi} = (-1)^{d-1} \Longleftrightarrow \exists\ (d-1)\text{−cycle} \sum_{\sigma\in\Delta^{d-1}} a_\sigma\cdot\sigma \text{ such that } a_\sigma = \pm 1 \text{ for all } \sigma\in\Delta^{d-1}.$$

Proof. C-M $\Longrightarrow$ $\tilde{\beta}_i(\Delta) = 0$, $i < d-1$ $\Longrightarrow$ $\tilde{\chi}(\Delta) = \sum(-1)^i\tilde{\beta}_i = (-1)^{d-1}\tilde{\beta}_{d-1}$. Hence, $\tilde{\chi}(\Delta) = (-1)^{d-1} \Longleftrightarrow \tilde{\beta}_{d-1}(\Delta) \Longleftrightarrow \tilde{H}_{d-1}(\Delta,\mathbf{k})\cong\mathbf{k}$. But this last condition is that Δ is orientable over $\mathbf{k}$, which means the existence of such a $(d-1)$-cycle.

Lemma 2. *Let Δ be a C-M$_{/\mathbf{k}}$ pseudomanifold. Then*
(1) Lk(σ) is a C-M$_{/\mathbf{k}}$ pseudomanifold, $\forall\sigma\in\Delta$.
(2) If Δ is orientable, then so is also Lk(σ), $\forall\sigma\in\Delta$.

Proof. (1) C-M is inherited by links, so in particular all links are strongly connected. The pseudomanifold property is then also inherited by links.
(2) Take a non-zero $(d-1)$-cycle $\sum_{\sigma\in\Delta^{d-1}} a_\sigma\cdot\sigma$. Fix $\tau\in\Delta$ and let $\Delta_\tau^{d-1} = \{\sigma\in\Delta^{d-1} | \sigma\supseteq\tau\}$. Then $\sum_{\sigma\in\Delta_\tau^{d-1}} a_\sigma\cdot\sigma\setminus\tau$ is a non-zero $(d-1-|\tau|)$-cycle in Lk(τ).

Now we can finish the proof of the theorem. Suppose Δ is a C-M$_{/\mathbf{k}}$ pseudomanifold and $\tilde{\chi}(\Delta) = (-1)^{d-1}$. Theorem 5.1 (b) of [13] shows that what we must prove is

$$(*) \qquad \tilde{H}_i(\text{Lk}\sigma,\mathbf{k})\cong\begin{cases}\mathbf{k}, & i = \dim(Lk\sigma)\\ 0, & i < \dim(\text{Lk}\sigma)\end{cases}$$

† "pseudomanifold" here is equivalent to our use of "pseudomanifold without boundary" in the article.

But Lemma 2 shows that $\mathrm{Lk}(\sigma)$ is an orientable C-M$_{/\mathbf{k}}$ pseudomanifold, which implies $(*)$.

Remark 1. For a pseudomanifold the word "orientable" has an absolute meaning. Since only $\{+1,-1\}$-coefficients are used in the fundamental cycle (Lemma 1) we get orientable$_{/\mathbf{k}}$, $\mathrm{char}(\mathbf{k}) \neq 2 \Longrightarrow$ orientable$_{/\mathbf{Z}} \Longrightarrow$ orientable$_{/\mathbf{k}}$, any field $\mathbf{k}$. Hence, for any field $\mathbf{k}$ of characteristic $\neq 2$ we have orientable$_{/\mathbf{k}} \Longleftrightarrow$ orientable$_{/\mathbf{Z}}$. But *every* pseudomanifold is orientable over $\mathbf{Z}_2$, so $\mathrm{char}(k)=2$ must be excluded. (This is not clearly expressed in [13].)

Remark 2. Reisner gave the first example of a Stanley-Reisner ring $\mathbf{k}[\Delta]$ whose Cohen-Macaulayness depends on $\mathrm{char}(\mathbf{k})$. This Δ is a pseudomanifold with 6 vertices and triangulates $\mathbf{RP}^2$. Is there a similar tangible example of a Δ such that the Gorensteinness of $\mathbf{k}[\Delta]$ depends on $\mathrm{char}(\mathbf{k})$?

————*————

ADDENDUM: $\{|\Sigma_1|\times(|\Sigma_2|\setminus\{x_2\}),(|\Sigma_1|\setminus\{x_1\})\times|\Sigma_2|\}$ is an excisive couple.

Definition. $(X,A)\times(Y,B) := (X\times Y, X\times B\cup A\times Y)$.

We get the following theorem from [12] p. 235. (Theorem 3. comes from [12] p. 188.)

Theorem. *If $\{X\times B, A\times Y\}$ is an excisive couple in $X\times Y$ and G and G' are modules over a principal ideal domain R such that $\mathrm{Tor}_1^R(G,G')=0$, there is a functorial short exact sequence*

$$0 \to [H(X,A;G)\otimes H(Y,B;G')]_q \xrightarrow{\mu'} H_q((X,A)\times(Y,B);G\otimes G') \longrightarrow$$
$$\longrightarrow [\mathrm{Tor}_1^R(H(X,A;G),H(Y,B;G')]_{q-1} \longrightarrow 0$$

and this sequence is split. –"In particular, if the right hand side vanishes (which always happens if R is a field) then the cross product, μ', is an isomorphism".

Theorem 3. *If $\mathbf{X_1}\cup\mathbf{X_2} = \mathrm{Int}_{\mathbf{X_1}\cup\mathbf{X_2}}\mathbf{X_1}\cup\mathrm{Int}_{\mathbf{X_1}\cup\mathbf{X_2}}\mathbf{X_2}$, then $\{\mathbf{X_1},\mathbf{X_2}\}$ is an excisive couple.*

Which we now use to prove,

Lemma. *$\{|\Sigma_1|\times(|\Sigma_2|\setminus\{x_2\}),(|\Sigma_1|\setminus\{x_1\})\times|\Sigma_2|\}$ is an excisive couple.*

Proof. $|\Sigma_1|\times(|\Sigma_2|\setminus\{x_2\}) = (|\Sigma_1|\times|\Sigma_2|)\setminus(|\Sigma_1|\times\{x_2\}) = ((|\Sigma_1|\times|\Sigma_2|)\setminus\{x_1,x_2\})\setminus((|\Sigma_1|\times\{x_2\})\setminus\{x_1,x_2\})$ but $(|\Sigma_1|\times\{x_2\})\setminus\{x_1,x_2\} = ((|\Sigma_1|\times|\Sigma_2|)\setminus\{x_1,x_2\})\cap(|\Sigma_1|\times\{x_2\})$ and $|\Sigma_1|\times\{x_2\}$ is closed in $|\Sigma_1|\times|\Sigma_2|$ implying that $(|\Sigma_1|\times\{x_2\})\setminus\{x_1,x_2\}$ is closed in $(|\Sigma_1|\times|\Sigma_2|)\setminus\{x_1,x_2\}$ so, $|\Sigma_1|\times(|\Sigma_2|\setminus\{x_2\})$ is open in $(|\Sigma_1|\times|\Sigma_2|)\setminus\{x_1,x_2\}$, i.e.
$\mathrm{Int}_{(|\Sigma_1|\times|\Sigma_2|)\setminus\{x_1,x_2\}}(|\Sigma_1|\times(|\Sigma_2|\setminus\{x_2\})) = |\Sigma_1|\times(|\Sigma_2|\setminus\{x_2\})$ so $(|\Sigma_1|\times|\Sigma_2|)\setminus\{x_1,x_2\} = =\mathrm{Int}_{(|\Sigma_1|\times|\Sigma_2|)\setminus\{x_1,x_2\}}(|\Sigma_1|\times(|\Sigma_2|\setminus\{x_2\}))\cup\ \mathrm{Int}_{(|\Sigma_1|\times|\Sigma_2|)\setminus\{x_1,x_2\}}((|\Sigma_1|\setminus\{x_1\})\times|\Sigma_2|)$, and we're through since $(|\Sigma_1|\times(|\Sigma_2|\setminus\{x_2\}))\cup((|\Sigma_1|\setminus\{x_1\})\times|\Sigma_2|) = (|\Sigma_1|\times|\Sigma_2|)\setminus\{x_1,x_2\}$. Now, [Sp] Th. 3. p. 188 gives the result. ∎

Acknowledgement: The work on this paper was initiated in late 1991 when Ralf Fröberg pointed out to me that some of the results in his joint work [8] with Lê Tuan Hoa entitled "Segre products and Rees algebras of face rings" probably were to lend themselves to generalizations. I like to thank Anders Björner for his kindness of sending me the proof included in this paper as " APPENDIX", and Ralf Fröberg, my teacher, for all the support he's been given me during the time of preparation of this paper.

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