3:rd paper

BINARY OPERATIONS on STANLEY-REISNER RINGS

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ABSTRACT

Starting from a set of simplicial complexes all of whose vertices lies in a common "universe" $\mathbf{W}$, and then defining the concept of "non-simplices" relative to this universe, we reach, after some "local modifications", equivalences between the most common binary operations on simplicial complexes and Stanley-Reisner Rings i.e. $\mathbf{R}[\Delta'\hat{*}\Delta'']\doteq\mathbf{R}[\mathbf{W}]/(\mathbf{I}_{\Delta'}\cap\mathbf{I}_{\Delta''})$, where $\hat{*}$ is an exstension of the classical join to the case of possible common vertices, and $\mathbf{R}[\Delta'\cap\Delta'']\doteq\mathbf{R}[\mathbf{W}]/(\mathbf{I}_{\Delta'}+\mathbf{I}_{\Delta''})$, and changing the "universe" to $\mathbf{W}\times\mathbf{W}$ we have $\mathbf{R}[\Delta'\times\Delta'']\doteq\mathbf{R}[\Delta']\overline{\otimes}\mathbf{R}[\Delta'']$ where $\overline{\otimes}$ is a tempered Segre Product. Moreover, for a finite universe,, we get an isomorphism on distributive lattices, induced by the "Stanley-Reisner Ring assignment functor".

Definition. *An (abstract) simplicial complex Δ on a vertex set $\mathbf{V}_\Delta$ is a collection (empty or non-empty) of finite subsets δ* (**empty** *or non-empty) of $\mathbf{V}_\Delta$ satisfying*
(a) *If $v\in\mathbf{V}_\Delta$, then $v\in\Delta$.*
(b) *If $\delta\in\Delta$ and $\tau\subset\delta$ then $\tau\in\Delta$.*

Definition. *A subset $s\subset\mathbf{W}\supset V_\Delta$ is said to be a* **non-simplex** *of Δ, denoted $s\not\in^{n}\Delta$, if $s\notin\Delta$ but $(\bar{s})^{(\dim s)-1}\subset\Delta$ (i.e. the $(\dim s-1)$-dimensional skeleton of the simplicial complex consisting of all proper faces of s is a subcomplex of Δ). For a simplex $\delta=\{v_{i_1},\dots,v_{i_k}\}$ we define m_δ to be the squarefree monic monomial $m_\delta:=1_{\mathbf{R}}\cdot v_{i_1}\cdot\cdot v_{i_k}\in\mathbf{R}[\mathbf{W}]$ where $\mathbf{R}[\mathbf{W}]$ is the graded polynomial algebra on the variable set $\mathbf{W}$ over the commutative ring $\mathbf{R}$. In particular $m_\emptyset=1_{\mathbf{R}}$. Let $\mathbf{R}[\Delta]:=\mathbf{R}[\mathbf{W}]/\mathbf{I}_\Delta$ where $\mathbf{I}_\Delta:=(\{m_\delta\mid\delta\not\in^{n}\Delta\})$. $\mathbf{R}[\Delta]$ is called the "face ring" or "Stanley-Reisner ring" of Δ over $\mathbf{R}$. Frequently $\mathbf{R}=\mathbf{k}$, a field.*

Note. **i.** Since $\emptyset\in\Delta$ for every simplicial complex $\Delta\neq\emptyset$ we see that for every element $v\in\mathbf{W}\setminus\mathbf{V}_\Delta$, $\{v\}$ is a "non-proper" non-simplex of Δ (i.e. $[v\notin\mathbf{V}_\Delta\neq\emptyset]\Leftrightarrow[\{v\}\not\in^{n}\Delta\neq\emptyset]$).

ii. In particular, if $\Delta=\{\emptyset\}$, then the set of non-simplices equals $\mathbf{W}$, and so $\mathbf{R}[\{\emptyset\}]=\mathbf{R}$.

iii. If $\Delta=\emptyset$, then the set of non-simplices equals $\{\emptyset\}$, since $\emptyset\notin\emptyset$, and $\overline{(\emptyset)}^{((\dim\emptyset_o)-1)}=\overline{\{\emptyset\}}^{(-2)}=\emptyset\subset\emptyset$, implying $\mathbf{R}[\emptyset]=0=$"The trivial ring", since $m_\emptyset=1_{\mathbf{R}}$.

iv.
$$\mathbf{R}[\Delta]\doteq\frac{\mathbf{R}[\mathbf{V}_\Delta]}{(\{m_\delta\in\mathbf{R}[\mathbf{V}_\Delta]\mid\ \delta\not\in^{n}\Delta\})}\ .$$

In [Fo] p. 2 we gave a proof, working for arbitrary simplicial complexes, for the following graded $\mathbf{k}$-algebra isomorphism; $\mathbf{k}[\Delta'*\Delta'']\cong\mathbf{k}[\Delta']\otimes\mathbf{k}[\Delta'']$, but it is possible to improve the strength of this join related "Stanley-Reisner Ring assignment functor". To do so, we extend the definition of "join" to a join, denoted "$\hat{*}$", defined for all pairs of simplicial complexes over $\mathbf{W}$, i.e including joins of pairs with intersecting vertex sets. So, we make the following definition, where $[\delta]_\Delta:=\delta\cap V_\Delta$ and $\mathcal{P}$, the power set:

$$\Delta_1\hat{*}\Delta_2:=\{\delta_1\cup\delta_2\in\mathcal{P}(\mathbf{W})\,|\,[\delta_i\in\Delta_i,\ i=1,2.]\wedge\big[[[\delta_1\cup\delta_2]_{\Delta_1}\in\Delta_1]\vee[[\delta_1\cup\delta_2]_{\Delta_2}\in\Delta_2]\big]\}.$$

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"$\hat{*}$" is associative and (trivially) commutative. The associativity of "$\hat{*}$" is a trivial consequence of the following Lemma where $\mathbf{R}$ is an arbitrary ring, but where, as well as in the rest of this paper, we'll assume $\Delta_i \neq \emptyset$, since $\emptyset$, due to the fact that $\Delta \hat{*} \emptyset = \emptyset$ for all Δ's, doesn't fit into the lattice structure we are about to deduce:

Lemma. *If* $\Delta_i \neq \emptyset$ $i = 1, 2$, *then in* $\mathbf{R}[\mathbf{W}]$: **1.** $\mathbf{I}_{\Delta_1 \hat{*} \Delta_2} = \mathbf{I}_{\Delta_1} \cap \mathbf{I}_{\Delta_2}$. **2.** $\mathbf{I}_{\Delta_1 \cap \Delta_2} = \mathbf{I}_{\Delta_1} + \mathbf{I}_{\Delta_2}$.

Proof. 1. The following set generates $\mathbf{I}_{\Delta_1} \cap \mathbf{I}_{\Delta_2}$ as an ideal;
$\{m_\delta \in \mathbf{R}[\mathbf{W}] \,|\, [[\delta \subset V_{\Delta_1} \cup V_{\Delta_2}] \wedge [\delta_i \in \Delta_i,\ i = 1, 2.] \wedge [\exists \delta_1 \subset \delta; \delta_1 \overset{pr}{\notin} \Delta_1] \wedge [\exists \delta_2 \subset \delta; \delta_2 \overset{pr}{\notin} \Delta_2]] \vee$
$\vee [\delta = \{v\}; v \in \mathbf{W} \setminus (V_{\Delta_1} \cup V_{\Delta_2})]\}$ where "$\overset{pr}{\notin}$" stands for "is a <u>**proper**</u> non-simplex of" oppose to beeing just a non-vertex. This set determines (and is determined by) the following set of δ's, the complement of which is;
$\{\delta \in \mathcal{P}(\mathbf{W}) \,|\, [\delta \subset V_{\Delta_1} \cup V_{\Delta_2}] \wedge [[\nexists \delta_1 \subset \delta; \delta_1 \overset{pr}{\notin} \Delta_1] \vee [\nexists \delta_2 \subset \delta; \delta_2 \overset{pr}{\notin} \Delta_2]]\} = \Delta_1 \hat{*} \Delta_2$ □
2. The following set generates $\mathbf{I}_{\Delta_1} + \mathbf{I}_{\Delta_2}$ as an ideal;
$\{m_1 + m_2 \in \mathbf{R}[\mathbf{W}] \,|\, m_i \in \mathbf{I}_{\Delta_i}\} = \{m_\delta \,|\, \delta \in \Delta_1^c \cup \Delta_2^c\} = \{m_\delta \,|\, \delta \in (\Delta_1 \cap \Delta_2)^c\} = \mathbf{I}_{\Delta_1 \cap \Delta_2}$ ■

"Δ^c" denotes the set-theoretic complement of Δ in $\mathcal{P}(\mathbf{W})$.
$\mathbf{I}_{\Delta_1} \cap \mathbf{I}_{\Delta_2}$ and $\mathbf{I}_{\Delta_1} + \mathbf{I}_{\Delta_2}$ are generated by a set (no restrictions on its cardinality) of squarefree monomials, if both $\mathbf{I}_{\Delta_1}$ and $\mathbf{I}_{\Delta_2}$ are. It is known that these squarefree monomially generated ideals form a distributive sublattice $(\mathcal{J}^\circ; \cap, +, \mathbf{R}[\mathbf{W}]_+)$, of the ordinary lattice structure on the set of ideals of the polynomial ring $\mathbf{R}[\mathbf{W}]$, with a counterpart, with reversed lattice order, called the squarefree monomial rings with unit, denoted $(\mathbf{R}^\circ[\mathbf{W}]; \cap, +, \mathbf{R})$. $\mathbf{R}[\mathbf{W}]_+$ is the unique homogeneous maximal ideal and zero element. We can use this structure to define a distributive lattice structure on $\mathbf{\Sigma}^\circ_{\mathbf{W}} :=$ "The set of non-empty simplicial complexes over $\mathbf{W}$", with $\{\emptyset\}$ as zero element and denoted $(\mathbf{\Sigma}^\circ_{\mathbf{W}}; \hat{*}, \cap, \{\emptyset\})$, as follows:

Definition. $\Delta_1 \geq \Delta_2$ *if* $\mathbf{I}_{\Delta_1} \leq \mathbf{I}_{\Delta_2}$.

Lemma. $\begin{cases} \text{lub}(\Delta_1, \Delta_2) = \Delta_1 \hat{*} \Delta_2 & (\text{since glb}(\mathbf{I}_{\Delta_1}, \mathbf{I}_{\Delta_2}) = \mathbf{I}_{\Delta_1} \cap \mathbf{I}_{\Delta_2}) \\ \text{glb}(\Delta_1, \Delta_2) = \Delta_1 \cap \Delta_2 & (\text{since lub}(\mathbf{I}_{\Delta_1}, \mathbf{I}_{\Delta_2}) = \mathbf{I}_{\Delta_1} + \mathbf{I}_{\Delta_2}) \end{cases}$ ■

Summing up, we get:

Theorem. *The "Stanley-Reisner Ring assignment functor" defines a monomorphism on distributive lattices from* $(\mathbf{\Sigma}^\circ_{\mathbf{W}}, \hat{*}, \cap, \{\emptyset\})$ *to* $(\mathbf{R}^\circ[\mathbf{W}], \cap, +, \mathbf{R})$, *which is an isomorphism for finite* $\mathbf{W}$. ■

Below, we'll use the notations from [B-H 2] § 7.1 p. 289 ff., but skip the simple verifications.

Definition. The Graded Hodge Algebra structure on a Stanley-Reisner Ring

H:=$\mathbf{W}$ (*The partial order on H is irrelevant for dicrete graded Hodge Algebras*).

Σ *has* $\{index\ of\ m_\delta \mid \delta \overset{pr}{\notin} \Delta\}$ *as its set of generators.*

$(\mathbf{H}_2)$ $m_\delta = 0$ *for every* $\delta \overset{pr}{\notin} \Delta$

Dealing with monomially generated ideals only, we get the following results; $\mathbf{I}_{\Delta_1 \hat{*} \Delta_2} = \mathbf{I}_{\Delta_1} \cap \mathbf{I}_{\Delta_2} = (\{m \mid m = \text{Lcm}(m_{\delta_1}, m_{\delta_2}),\ \delta_i \overset{pr}{\notin} \Delta_i,\ i = 1, 2\})$ and $\mathbf{I}_{\Delta_1 \cap \Delta_2} = \mathbf{I}_{\Delta_1} + \mathbf{I}_{\Delta_2} = (\{m_\delta \mid \delta \overset{pr}{\notin} \Delta_1 \vee \delta \overset{pr}{\notin} \Delta_2\})$. Now, deleting all elements in the generator sets, resp. having other elements as proper factors, we'll get a presentation of the Hodge Algebra structure on $R[\Delta_1 \hat{*} \Delta_2]$ resp. $R[\Delta_1 \cap \Delta_2]$

Definition. *A graded* $\mathbf{R}$*-algebra* R *is called a Segre product of* R_1 *and* R_2 *over* $\mathbf{R}$*, denoted by* $R = \sigma_{\mathbf{R}}(R_1, R_2)$ *or* $R = \sigma(R_1, R_2)$*, if for every* $p \in \mathbf{N}$, $[R]_{\overline{p}} = [R_1]_p \otimes_{\mathbf{R}} [R_2]_p$. *The "canonical Segre product", denoted* $R_1 \underline{\otimes}_{\mathbf{R}} R_2$*, is equipped with componentwise multiplication.*

Given a monomial m, let $\mathrm{Supp}(m)$ be the squarefree monomial defined by putting every non-zero exponent (of the variables) in m equal to 1. Put $p_1(w_{\lambda,\mu}) = p_1((v'_\lambda, v''_\mu)) := v'_\lambda$, $p_2(w_{\lambda,\mu}) := v''_\mu$ and if $\mathrm{Supp}(m) := w_{\lambda_1,\mu_1} \cdots w_{\lambda_k,\mu_k}$ put $[\overline{p_t}(m)] := \{p_t(w_{\lambda_1,\mu_1}), \ldots, p_t(w_{\lambda_k,\mu_k})\}$ and $p_t(m) := p_t(w_{\lambda_1,\mu_1}) \cdots p_t(w_{\lambda_k,\mu_k})$ for $t = 1, 2$. (The same notations when $w_{\lambda,\mu} := v_\lambda \otimes v_\mu$.)

Put $C' := \{w_{\lambda,\mu} w_{\nu,\xi} \mid \lambda < \nu \wedge \mu > \xi\}$ where $w_{\lambda,\mu} := v_\lambda \otimes v_\mu$ with $(v_\lambda, v_\mu) \in \mathbf{W} \times \mathbf{W}$ and where the subindices reflects the linear ordering, now presupposed on $\mathbf{W}$ and

$$D := \Big\{ \mathbf{w} = w_{\lambda_1,\mu_1} \cdot \ldots \cdot w_{\lambda_k,\mu_k} \;\Big|\; [\mathbf{w} \text{ is a chain}] \wedge \Big[\Big[[[\overline{p_1}(\mathbf{w})] \notin^{n} \Delta_1] \wedge [[\overline{p_2}(\mathbf{w})] \in \Delta_2] \wedge \Big[{\lambda_1 < \cdots < \lambda_k \atop \mu_1 \leq \cdots \leq \mu_k} \Big] \Big] \vee$$

$$\vee \Big[[[\overline{p_1}(\mathbf{w})] \in \Delta_1] \wedge [[\overline{p_2}(\mathbf{w})] \notin^{n} \Delta_2] \wedge \Big[{\lambda_1 \leq \cdots \leq \lambda_k \atop \mu_1 < \cdots < \mu_k} \Big] \Big] \vee \Big[[[\overline{p_1}(\mathbf{w})] \notin^{n} \Delta_1] \wedge [[\overline{p_2}(\mathbf{w})] \notin^{n} \Delta_2] \wedge \Big[{\lambda_1 < \cdots < \lambda_k \atop \mu_1 < \cdots < \mu_k} \Big] \Big] \Big] \Big\}.$$

Definition. *The Graded Hodge Algebra structure for the Canonical Segre Product of two Stanley-Reisner rings, denoted* $\mathbf{R}[\Delta'] \underline{\otimes} \mathbf{R}[\Delta'']$.
H:=$\mathbf{W} \times \mathbf{W}$ *equipped with the product order.*
Σ *has* $\{\textit{index of } m \mid m \in C' \cup D\}$ *as its set of generators.*

$$(\mathbf{H}_2) \quad g := \begin{cases} w_{\lambda,\xi} w_{\nu,\mu} & \textit{if } g = w_{\lambda,\mu} w_{\nu,\xi} \in C' \\ 0 & \textit{if } g \in D. \end{cases}$$

By [Fo] p. 4 lemma and [B-H] p. 291 Prop. 7.1.3 this Hodge Algebra is graded algebra isomorphic to the canonical Segre Product of two Stanley-Reisner Rings, but it's not discrete and so, not that of a Stanley-Reisner Ring, but this is fixed by "the discrete algebra on the same data as the Canonical Segre Product" (c.f. [B-H 2] p. 292), denoted $\mathbf{R}[\Delta'] \bar{\otimes} \mathbf{R}[\Delta'']$, so we just change the last "$(\mathbf{H}_2)$" to become - "All elements in $C' \cup D$ are put to 0". In [Fo] p. 4 lemma, we gave a proof, working for arbitrary simplicial complexes, for the following graded $\mathbf{k}$-algebra isomorphism; $\mathbf{k}[\Delta' \times \Delta''] \cong \mathbf{k}[\Delta'] \bar{\otimes} \mathbf{k}[\Delta'']$, but in this proof, it turns out that the role played by the coefficient field $\mathbf{k}$, is totally irrelevant to the success of the proof method, so we actually proved that $\mathbf{R}[\Delta' \times \Delta''] \cong \mathbf{R}[\Delta'] \bar{\otimes}_{\mathbf{R}} \mathbf{R}[\Delta'']$, for any coefficient ring $\mathbf{R}$. Note: $C' \cup D = \{m_\delta \mid \delta \notin^{n} \Delta' \times \Delta''\}$ under the identification $v_\lambda \otimes v_\mu \leftrightarrow (v_\lambda, v_\mu)$.
[Fo] p. 5 Th. 2 resp. [Fo] p. 10-11 Th's 7-8 combined with [F-H] p. 3371 Prop. 5-6 gives

Proposition. $\mathbf{R}[\Delta'] \underline{\otimes} \mathbf{R}[\Delta'']$ *is Koszul (Buchsbaum, Cohen-Macaulay) if and only if* $\mathbf{R}[\Delta'] \bar{\otimes}_{\mathbf{R}} \mathbf{R}[\Delta''] \cong \mathbf{R}[\Delta' \times \Delta'']$ *is.*

Necessary and sufficent conditions for the Gorenstein version of the last proposition is easily extracted by combining [Fo] p. 17 Th. 11 and [F-H] p. 3372 Prop. 7 where $-a(k[\Delta])$ for Cohen-Macaulay rings should be interpreted as the number of cone points in Δ.

For completeness, we quote from [E] p. 251 the following ASL-construction for a Stanley-Reisner ring. Now, the partial ordering on H is vital, since it fully determines Σ.

Definition. The ASL-structure on a Stanley-Reisner Ring
H:=$\{ m_\delta \mid \delta \in \Delta\}$ *where* $m_\delta \leq m_\tau$ ***iff*** $\delta \supset \tau$.

$$(\mathbf{H}_2) \quad m_\delta m_\tau := \begin{cases} 0 & \textit{if } \delta \cup \tau \notin \Delta \\ m_\delta m_\tau{}^{\dagger} \text{written as a standard monomial} & \textit{else} \end{cases}$$

† This condition was originally given in [E] as "$m_{\delta \cup \tau}$ else" which gives the wrong order if $\delta \cap \tau \neq \emptyset$.

The partial ordering, i.e. reverse inclusion, given to H, could seem somewhat odd, but the fact is that it is the most common way to order H in an ASL-structure, and so, it has been given a name, i.e. if $\deg h_1 \geq \deg h_2$ for all $h_1, h_2 \in$ H with $h_1 \leq h_2$ then the ASL-structures is said to be a "monotonely graded ASL" [B-H 1].

References.

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