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The Existence of a Short Sequence of Admissible Pivots to an Optimal Basis in LP and LCP

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We say an LP (linear programming) is fully nondegenerate if both the primal and the dual problems are nondegenerate. In this paper, we prove the existence of a sequence of $|B^*\setminus B|$ admissible pivot from any basis B (not necessarily feasible) to the unique optimal basis B^*, if the given LP has an optimal solution and is fully nondegenerate. Here admissible pivots are those pivots (satisfying certain sign conditions) that exist if the current LP dictionary is not terminal, i.e., neither optimal, inconsistent nor dual inconsistent. A natural extension of the result to LCPs (linear complementarity problems) with sufficient matrices is given. The existence itself does not yield a strongly polynomial pivot algorithm for LPs but provides us with a good motivation to study the class of admissible pivot methods for LPs, as opposed to the narrower class of simplex methods for which the shortest sequence of pivots is not known to be polynomially bounded. © 1997 IFORS. Published by Elsevier Science Ltd.

Kew words: Pivot algorithsm, simplex methods, linear programming.

1. INTRODUCTION

Consider a standard form of the linear programming (LP) problem:

$$\text{maximize } c^T x \text{ subject to } Ax \leq b,$$

where x is a variable d-vector, c and b are given rational d-vector and m-vector, and A is given rational $m \times d$-matrix. The simplex method has been studied extensively by many researchers since it was invented in 1947 by Dantzig (1948, 1991) but yet it resists the resolution of the long-standing open problem:

Question 1: Is there any refinement of the simplex method which terminates in a number of pivots that is bounded by a polynomial in m and d?

Here we use the simplex method as a generic term for the family of methods which start with a feasible basis and use a pivot operation selected with the usual sign checking and ratio test. In particular the simplex method always preserves the feasibility and does not decrease the associated objective value. A *refinement* of the simplex method is a simplex algorithm which further restricts the choice of pivots with extra conditions. The simplex method is known to be extremely practical, but no one has been able to present such a refinement of the simplex method. In fact, many well known refinements, such as the simplex algorithms with the largest coefficient rule and the maximum improvement rule, are known to require an exponential number of pivots for a certain class of LPs.

A polynomial refinement of the simplex method, if it exists, will be a *strongly polynomial* algorithm, that is, the number of basic arithmetic operations required for solving any LP is bounded by m and d. Two polynomial algorithms for LPs, the ellipsoid method (Khachian, 1979) and the interior-point method (Karmarkar, 1984) are not strongly polynomial since in both methods the number of basic iterations to solve an LP depends on the size of binary encoding of the input matrix.

For the simplex method, the prospect of discovering a polynomial refinement is, unfortunately, quite obscure, since a much simpler existence question:

Question 2: Is there a short sequence of simplex pivots to an optimal basis from any given feasible basis?

appears to be already too hard. Here, the word 'short' means 'bounded by a polynomial in m and d'. It should be noted that for the first question to have the affirmative answer, this simpler question must

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have the affirmative answer. And the affirmative answer to this simpler question would imply the affirmative answer for the famous (and hard) diameter question for convex polytopes:

Question 3: Is the combinatorial diameter of a d-polytope with m-facets bounded by a polynomial in d and m?

Although some interesting results on the diameter of polytopes have been obtained recently (e.g., Kalai, 1991), it is fair to say that we are still quite far from a satisfactory answer to this 'simplest' question.

The present paper is to provide a good motivation to study a class of pivot algorithms that are more general than the simplex algorithm. The key notion is the admissible pivot as opposed to the simplex pivot. Admissible pivots are defined by certain sign conditions on the associated LP dictionary, and in particular they exist if the dictionary is not terminal, i.e., neither optimal, inconsistent nor dual inconsistent. See Section 2.2 for the formal definition and Fig. 2 for illustrative diagrams. One can consider both the simplex method and the dual simplex method to be admissible pivot methods. Also, Bland's recursive pivot method (Bland, 1977), Edmonds–Fukuda method (Fukuda, 1982; Björner *et al.*, 1993) and the criss-cross method (Terlaky, 1987; Wang, 1987) are admissible pivot methods.

A main theorem of this paper states that there exists a very short sequence of admissible pivots from any basis (not necessarily feasible) to an optimal basis, provided that the LP has an optimal solution and both the primal and the dual problems are nondegenerate. We shall present a natural extension of this result to the setting of linear complementarity problems (LCP).

In the next section, we give formal definitions and present the main results. We interpret the main results for general (possibly degenerate) problems with lexicographic perturbation scheme in Section 3.

2. MAIN RESULTS

Our notations are not very different from those employed in standard LP textbooks such as Chvatal (1983) and Fletcher (1987), but for completeness we start with some basic notations and definitions.

2.1. Matrix notation

First we present some basic notations and definitions for linear systems.

For finite sets M and E, an $M \times E$ *matrix* is an array of doubly indexed numbers or variables

$$A = (a_{ij}: i \in M, j \in E)$$

where each member of M is called a *row index*, each member of E is called a *column index* and each a_{ij} is called a *component*. For $R \subseteq M$ and $S \subseteq E$, the $R \times S$ matrix $(a_{rs}: r \in R, s \in S)$ is called a submatrix of A, and will be denoted by A_{RS}. We use simplified notations like, A_R for A_{RE}, $A_{\cdot S}$ for A_{MS}, A_i for $A_{\{i\}E}$, and $A_{\cdot j}$ for $A_{M\{j\}}$. Also, for a positive integer m, we use expressions such as $m \times E$ *matrix* to mean an $M \times E$ matrix with the usual index set $M = \{1, 2, \ldots, m\}$. Thus, our matrix notation is simply an extension of the usual matrix notation, and this extension enables us to describe pivot algorithms in a simplest and rigorous way.

Let E be a finite set, m be a positive integer, and let A be a real $m \times E$ matrix of *rank* $(A) = m$. Consider the system of linear equalities:

$$Ax = \mathbf{0}, \tag{1}$$

where x is an unknown vector in $\mathbb{R}^E$. The system can be alternately written as

$$\Sigma_{j \in E} a_{ij} x_j = 0 \qquad \forall i \in M. \tag{2}$$

A subset B of E is said to be *basis* if $|B| = m$ and the square submatrix $A_{\cdot B}$ is nonsingular. The complement $E \backslash B$ of a basis B is called a *nonbasis* and denoted by N. When B is basis, we denote by $(A_{\cdot B})^{-1}$ the $B \times m$ matrix T such that the product $T \cdot A_{\cdot B}$ is the $B \times B$ identity matrix $I^{(B)}$, i.e., T is the left inverse. Since we consider matrix rows and columns to be indexed by sets, for two matrices to be multipliable, the column index set of the first matrix must be equal to the row index set of the second matrix.

Let B be any basis. The *dictionary* $D = D(B)$ of the basis B is the $B \times N$ matrix $-(A_{\cdot B})^{-1}A_{\cdot N}$. It is important to note that the dictionary represents the system of linear equations, equivalent to (1), given by

$$x_B = Dx_N, \tag{3}$$

which can be also written as

$$x_i = \Sigma_{j \in N} d_{ij} x_j \qquad \forall i \in B. \tag{4}$$

Here d_{ij}'s denote the components of D. For any nonbasic element $g \in N$, the unique solution $\bar{x}$ to the system (1) with $\bar{x}_g = 1$ and $\bar{x}_j = 0$ for all $j \in N - g$ is called the *basic solution* (associated with B and g), and denoted by $x(B,g)$. Once the dictionary is given, the basic solution can be easily read: $x(B,g)_B = D_{\cdot g}$.

For those who are not familiar with the notion of dictionary [first introduced in Chvatal (1983)], we note that the LP tableau matrix (see, e.g., Fletcher, 1987) $T = T(B)$ is related to the dictionary $D = D(B)$ by $T = [I^{(B)} - D]$.

For any $r \in B$ and $s \in N$ with $d_{rs} \neq 0$, the set $B - r + s$ is again a basis, and the replacement of B by $B - r + s$ is called the *pivot operation on* (r, s). Here, $-r$ and $+s$ are a short form of single-element deletion $\backslash\{r\}$ and addition $\cup\{s\}$, respectively.

We define two vector subspaces of $\mathbb{R}^E$ associated with the matrix A:

$$N(A) = \{x \in \mathbb{R}^E: \quad Ax = \mathbf{0}\} \tag{5}$$
$$R(A) = \{A^t\lambda: \quad \lambda \in \mathbb{R}^m\}, \tag{6}$$

where A^t is the transpose of A, $N(A)$ is the *null space* of A and $R(A)$ is the *row space* of A. When a basis B for $Ax = \mathbf{0}$ is given, these two spaces are given symmetrically as

$$x \in N(A) \Leftrightarrow x_B = Dx_N \tag{7}$$
$$y \in R(A) \Leftrightarrow y_N = -D^t y_B. \tag{8}$$

The basic solution of the dual system $y_N = -D^t y_B$ associated with a dual basis N and $f \in B$ is denoted by $y(N,f)$. Note that $y(N,f)_N = (-D^t)_{\cdot f} = (-D_f)^t$, i.e., the dual basic vector is represented in the fth row of D.

2.2. Linear programming case

Let E be a finite set, let f and g be two distinguished elements of E, called the *objective* and the *infinity* element, respectively. Given an $m \times E$ matrix A, a linear programming problem is to find x to

$$\begin{aligned} \text{maximize} \quad & x_f \\ \text{such that} \quad & x \in N(A) \\ & x_j \geq 0 \qquad \forall j \in E\backslash\{f, g\} \\ & x_g = 1. \end{aligned} \tag{LP}$$

The dual LP is then defined as to find y to

$$\begin{aligned} \text{maximize} \quad & y_g \\ \text{such that} \quad & y \in R(A) \\ & y_j \geq 0 \qquad \forall j \in E\backslash\{f, g\} \\ & y_f = 1. \end{aligned} \tag{DP}$$

This definition of LP is not quite usual, but one can easily transform any standard form of LP to our form.

An *LP basis* is a basis B of the system $Ax = \mathbf{0}$ with $f \in B$ and $g \in N$. Let B be any LP basis and let $D = D(B)$ be the associated dictionary. A basis B is said to be *feasible* if $d_{ig} \geq 0$ for all $i \in B - f$, and *dual feasible* if $d_{fj} \leq 0$ for all $j \in N - g$. A basis B is *optimal* if it is both feasible and dual feasible. A basis B is *inconsistant* if there exists a basic index $r \in B - f$ such that $d_{rg} < 0$ and $d_{rj} \leq 0$ for all $j \in N - g$. Similarly, a basis B is *dual inconsistent* if there exists a nonbasic index $s \in N - g$ such that

$d_{fs} > 0$ and $d_{is} \geq 0$ for all $i \in B - g$. A basis B is called *terminal* if it is either optimal, inconsistent or dual inconsistent. We often identify a basis with its dictionary, and may say 'feasible dictionary' instead of 'feasible basis' etc. See Fig. 1.

Throughout, we assume that an LP basis is given. Finding one LP basis (or proving that none exists) can be easily done by Gaussian elimination. The *goal* of a pivot algorithm is to find a terminal basis by a sequence of pivot operations starting with a given LP basis.

For $r \in B - f$ and $s \in N - g$ with $d_{rs} \neq 0$, a pivot on (r, s) is said to be *admissible* if either (I) $d_{rg} < 0$ and $d_{rs} > 0$ or (II) $d_{fs} > 0$ and $d_{rs} < 0$. See Fig. 2.

Admissible pivots are very natural elementary operations to be used in any pivot algorithms by the following reasons.

Proposition 1. Let B be any LP basis. Then the following statements hold.

(a) If B is not terminal, then there exists an admissible pivot at B.

(b) If the LP has an optimal solution, then there exists an admissible pivot at B if and only if B is not optimal.

Proof. The statement (a) is clear with the definitions. To prove (b), assume that the LP has an optimal solution. This together the LP duality theorem implies that neither an inconsistent nor a dual inconsistent basis exists. Thus if a basis B is not optimal, there exists an admissible pivot. The other implication is trivial.

We define an *admissible pivot method* for LP as a pivot method that only uses admissible pivots. It is not difficult to verify the following.

Remark 2. The primal simplex method, the dual simplex method, Bland's recursive pivot method (Bland, 1977), Edmonds–Fukuda method (Fukuda, 1982; Björner *et al.*, 1993) and the criss-cross method (Terlaky, 1987;Wong, 1987) are all admissible pivot methods. More precisely, the primal simplex method and Edmonds–Fukuda method use admissible pivots of type II only, the dual simplex method uses admissible pivots of type I only, and both Bland's recursive method and the criss-cross method use admissible pivots of type I and type II.

Now a simple fundamental question arises.

Question 4: What is the length of shortest sequence of admissible pivots from a given LP basis B to any fixed optimal basis B^*?

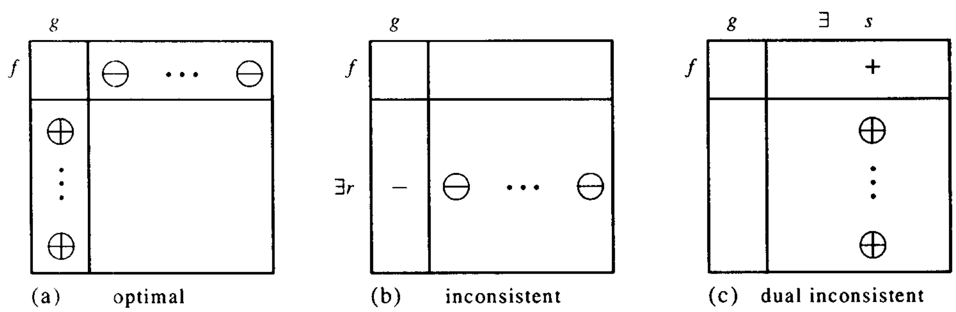


Fig. 1. Three types of terminal dictionaries for LP.

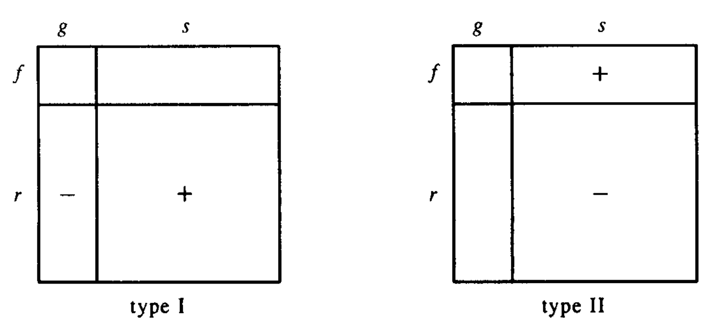


Fig. 2. Two types of admissible pivots for LP.

It turns out that this question is not a very good one, since there is a rather disappointing answer. In some cases, no such sequence exists and thus the shortest length is undefined, as shown in the following example.

Example 3. Consider the LP illustrated in Fig. 3. In this example, $E = \{f, g, 1, 2, 3, 4\}$, and the set of feasible solutions is the shaded triangle region. There are two optimal bases $B^* = \{f, 1, 4\}$ and $B^{**} = \{f, 1, 3\}$. At the dictionary for the basis $B = \{f, 3, 4\}$, there is only one admissible pivot, that is at (4, 1). Clearly by this pivot, one reaches the basis B^{**}. Thus, the basis B^* is not reachable from B by any admissible pivots.

The question becomes valid once we assume nondegeneracy. We say that an LP is *nondegenerate* if $d_{ig} \neq 0$ for all basic variable $i \in B - f$, for each LP basis B and its dictionary $D = [d_{ij}]$. An LP is said to be *dual-nondegenerate* if the dual problem is nondegenerate, or equivalently $d_{fj} \neq 0$ for all nonbasic variable $j \in N - g$. Finally, and LP is *fully nondegenerate* if both the primal and the dual problems are nondegenerate. One can easily prove the following.

Lemma 4. Every fully nondegenerate LP has at most one optimal basis.

Our main theorem for LP is the following which is a consequence of the uniqueness lemma above.

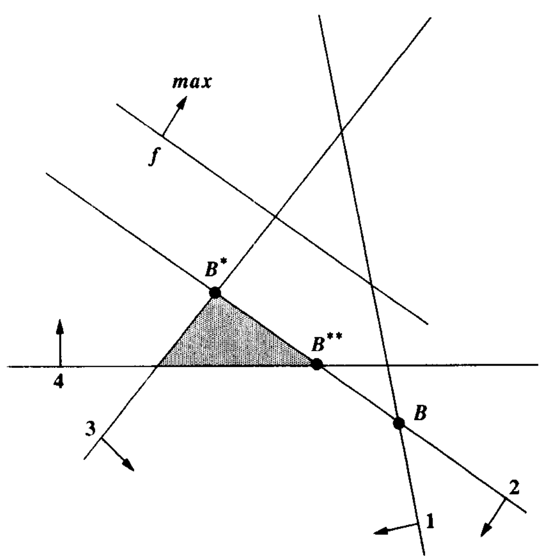
Theorem 5. If an LP has an optimal solution and is fully nondegenerate, then there exists a sequence of $|B^* \backslash B|$ admissible pivots from any LP basis B to the unique optimal basis B^*.

One might suspect that this theorem is a disguised form of a simple well-known lemma in linear algebra: any two bases B and B' of a matrix A can be connected by a sequence of $|B' \backslash B|$ pivot operations. As it was illustrated by Example 3, the validity of our theorem critically depends on the full nondegeneracy assumption, which has no counterpart in this lemma. Therefore we do not see any easy way to relate these two statements with similar appearances.

Instead of proving this theorem directly, we shall present and prove a more general theorem in the LCP setting below.

2.3. *LCP case*

Let E_{2n} denote a finite set with $2n$ elements with a *complementarity map* $\bar{\ }$, i.e., for every $i \in E_{2n}$, its complement $\bar{i}$ is an element of E_{2n} different from i and $\bar{\bar{i}} = i$. Also we denote by E_{2n+1} a set with $2n + 1$ elements of form $E_{2n} + g$, where g is a distinguished element, called the infinity element. A subset B of E_{2n+1} is said to be *complementary* if $\{i, \bar{i}\} \not\subseteq B$ for all $i \in E_{2n}$.



		g	1	2
	f		0	−
B	3	+	−	+
	4	−	+	−

		g	3	2
	f		0	−
B^*	1	+	−	+
	4	+	−	−

Fig. 3. B^* unreachable from B by admissible pivots.

Given positive integer n and an $n \times E_{2n+1}$ matrix A, a linear complementarity problem (LCP) is to find $x \in E_{2n+1}$ satisfying

$$\begin{aligned} & x \in N(A) \\ & x \geq \mathbf{0} \\ & x_g = 1 \\ & x_i = 0 \text{ or } x_{\bar{i}} = 0 \qquad \forall i \in E_{2n}. \end{aligned} \tag{LCP}$$

By replacing $N(A)$ with $R(A)$ we obtain the *dual LCP problem*, i.e., it is to find $y \in E_{2n+1}$ satisfying

$$\begin{aligned} & y \in R(A) \\ & y \geq \mathbf{0} \\ & y_g = 1 \\ & y_i = 0 \text{ or } y_{\bar{i}} = 0 \qquad \forall i \in E_{2n}. \end{aligned} \tag{DCP}$$

The last condition of LCP is known to be the *complementarity*. A vector x is said to be (*weakly*) *feasible* for an LCP if it satisfies all the conditions (except for the complementarity). An LCP is said to be (*weakly*) *feasible* if it admits a (weakly) feasible solution. Note that a weakly feasible LCP may not be feasible, although for the special case of LCP's that we consider below, the weak feasibility implies the feasibility.

An LCP basis is a complementary basis B of the system $Ax = \mathbf{0}$ with $g \in N$ (i.e., $g \notin B$). Let B be any LCP basis and let $D = D(B)$ be the associated dictionary. The basis B is said to be *feasible* if $d_{ig} \geq 0$ for all $i \in B$. The basis is *inconsistent* if there exists a basic index $r \in B$ such that $d_{rg} < 0$ and $d_{rj} \leq 0$ for all $j \in N - g$. The basis is called *terminal* if it is either feasible or inconsistent. See Fig. 4.

When B is feasible, the basic solution $x(B, g)$ solves the primal problem LCP. When B is inconsistent, LCP admits no solution, and if in addition $d_{r\bar{r}} = 0$ holds, then the dual solution $y(N, r)$ solves the dual DCP. Note that for most of the well-solved classes of LCPs, the diagonal entries $d_{i\bar{i}}$ always stay nonnegative and thus $d_{r\bar{r}} = 0$ holds for the evidence row r in any inconsistent dictionary.

Throughout we assume that an LCP basis is given. Finding one LCP basis (or proving that none exists) can be polynomially solved by Edmond's matroid intersection algorithm (Edmonds, 1979). The *goal* of a pivot algorithm for LCP is to find a terminal basis by a sequence of pivot operations, starting from a given LCP basis.

A vector x in $\mathbb{R}^{E_{2n+1}}$ is called *strictly sign-preserving* (s.s.p.) if

$$x_i \cdot x_{\bar{i}} \geq 0 \text{ for all } i \in E_{2n} \text{ and} \tag{9}$$
$$x_j \cdot x_{\bar{j}} > 0 \text{ for some } j \in E_{2n}, \tag{10}$$

and called *strictly sign-reversing* (s.s.r.) if

$$x_i \cdot x_{\bar{i}} \leq 0 \text{ for all } i \in E_{2n} \text{ and} \tag{11}$$
$$x_j \cdot x_{\bar{j}} < 0 \text{ for some } j \in E_{2n}. \tag{12}$$

An LCP is said to be *column sufficient* if $N(A)$ contains no s.s.r. vector z with $z_g = 0$, *row sufficient* if $R(A)$ contains no s.s.p. vector w, and *sufficient* if it is both row and column sufficient. Note that the sufficiency requires nothing on the gth column of A.

The following is a fundamental theorem, known as the LCP duality theorem (Fukuda and Terlaky, 1992).

Theorem 6. For any sufficient LCP, exactly one of the following statements holds:

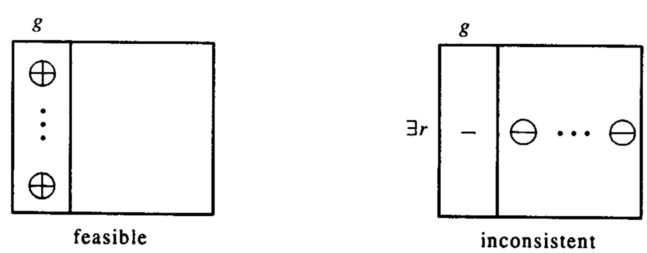


Fig. 4. Two types of terminal dictionaries for LCP.

(a) LCP has a feasible solution;
(b) DCP has a feasible solution.

Note that the original form of Theorem 6 is slightly more general. By the orthogonality of the two spaces $R(A)$ and $N(A)$, it is clear that both LCP and DCP cannot be weakly feasible simultaneously. Thus if a sufficient LCP is weakly feasible, Theorem 6 implies LCP is feasible.

There are several known special cases of sufficient LCP's. For example, when A is an $n \times (2n+1)$ matrix of form $[I^{(n)} - Q\ b]$ for some positive semidefinite matrix Q, with i and $i+n$ being complement, the associated LCP is sufficient. It is well known and easy to verify that any LP can be reduced to an LCP of this type where Q is a special positive semidefinite matrix of form $\begin{bmatrix} \mathbf{0} & C \\ -C^t & \mathbf{0} \end{bmatrix}$. When an LP dictionary $D = D(B)$ is given, C can be written explicitly as $D_{(B-f)(N-g)}$ for the objective element f and the infinity element g of the LP.

It should be remarked that our sufficiency of LCP coincides with the original notion of sufficiency of a square matrix Q in [5], when A is of form $[I^{(n)} - Q\ b]$.

A simple way to prove the duality theorem is by a finite algorithm. In Fukuda and Terlaky (1992), a pivot algorithm, called the criss-cross method, is given. This algorithm is a natural extension of the criss-cross method (Terlaky, 1987; Wong, 1987) for LP, and uses two types of admissible transformations that replace any LCP basis with another LCP basis with pivot operation(s). The following proposition shows some sign structure of row sufficient LCP dictionaries, that is exploited in several pivot algorithms including the principal pivot method (Cottle *et al.*, 1992) and the criss-cross method.

Proposition 7 (Cottle *et al.*, 1989, 1992; Fukuda and Terlaky, 1992). For a row sufficient LCP, and for the dictionary D of any LCP basis B, the following statements hold.

(a) The 'diagonal' entry $d_{r\bar{r}}$ is nonnegative for $r \in B$.
(b) If $r, s \in B$, $d_{r\bar{r}} = 0$ and $d_{r\bar{s}} > 0$ then $d_{s\bar{r}} < 0$.

See Fig. 5 illustrating the proposition above.

For $r \in B$ with $d_{r\bar{r}} \neq 0$, a pivot on $(r, \bar{r})$ is an admissible transformation of type I, if $d_{rg} < 0$ and

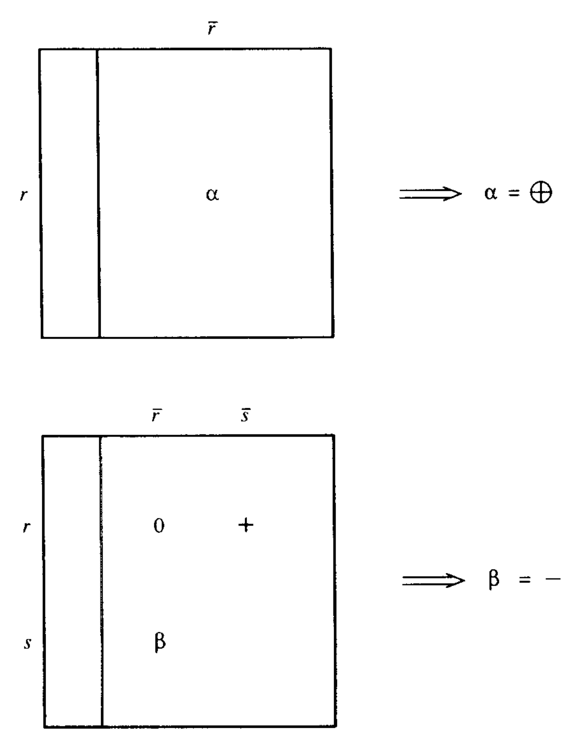


Fig. 5. Sign pattern of LCP dictionary.

$d_{r\bar{r}} > 0$. For $r, s \in B$ with $d_{r\bar{r}} = 0$, a two successive pivots on $(r, \bar{s})$ and $(s, \bar{r})$ is an admissible transformation of type II, if $d_{rg} < 0$, $d_{r\bar{s}} > 0$ and $d_{s\bar{r}} < 0$. See Fig. 6.

Proposition 8. For any sufficient LCP, and for any LCP basis B, the following statements hold.

(a) If B is not terminal, then an admissible transformation exists at B.
(b) If the LCP is weakly feasible, then an admissible transformation exists at B if and only if B is not feasible.

Proof. To prove (a), assume that B is not terminal. Since it is not feasible, there exists a basic index $r \in B$ such that $d_{rg} < 0$. If $d_{r\bar{r}} > 0$, we have an admissible transformation of type *I*. Suppose $d_{r\bar{r}} = 0$. Since B is not inconsistent, there exists a basic index s such that $d_{r\bar{s}} > 0$. Clearly $s \neq r$, and we have $d_{s\bar{r}} < 0$ by Proposition 7(b). This implies that there exists an admissible transformation of type II. This completes the proof for (a).

The statement (b) is a direct consequence of (a).

An LCP is said to be *nondegenerate* if $d_{ig} \neq 0$ for all $i \in B$, for any LCP basis B and its dictionary $D = D(B)$. The following is basic. (For a stronger theorem, see Theorem 3.5.8 in Cottle *et al.*, 1992)

Proposition 9. Every column sufficient nondegenerate LCP has at most one feasible basis.

Proof. Suppose an LCP is column sufficient and nondegenerate, and assume it has two distinct feasible bases B and B'. Let z be $x(B, g) - x(B', g)$, the difference of the two associated feasible solutions to the LCP. Clearly, $z_g = 0$ and $z_i \cdot z_{\bar{i}} \leq 0$ for all $i \in E_{2n}$. Since the LCP is nondegenerate and the two bases are distinct, there exists at least one index $j \in E_{2n}$ such that $z_j \cdot z_{\bar{j}} < 0$. These imply that z is an s.s.r. vector in $N(A)$, contradicting the column sufficiency of the LCP.

Now we can state and prove the main theorem in this section, a generalization of the LP theorem, Theorem 5.

Theorem 10. If an LCP is sufficient, nondegenerate and weakly feasible, then there exists a sequence of admissible transformations involving exactly $|B^* \backslash B|$ pivots from any LCP basis B to the unique feasible LCP basis B^*.

Proof. Supose that the LCP is sufficient, nondegenerate and weakly feasible. Let B^* be the unique feasible basis and let B be any LCP basis different from B^*. Set $B_0 = B \backslash B^*$, $B_1 = B \cap B^*$, $N_0 = \overline{B_0}$ and $N_1 = \overline{B_1}$. Consider the submatrix $D' = D_{B_0(N_0+g)}$ of the dictionary $D = D(B)$. The matrix D' represents a smaller LCP problem LCP′ with variables $B_0 \cup (N_0 + g)$, see Fig. 7. Now we prove the important claim:

Claim: LCP′ is nondegenerate, feasible and sufficient.

Proof for the claim: Since each dictionary of the subproblem can be considered as a subdictionary of the original LCP, the nondegeneracy of LCP implies the same for LCP′. Also, the feasibility of LCP′ is implied by the existence of the feasible basis N_0. To prove the sufficiency, let $A' = [I^{(B_0)} \; - D']$. Note that the linear subspace associated with the subproblem LCP′ is $N(A')$. Now remark the followings.

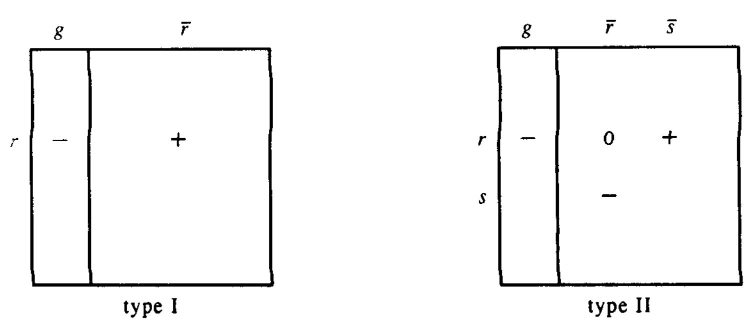


Fig. 6. Two types of admissible transformations for LCP.

(1) Every vector $z' \in N(A')$ is extendable to a unique vector $z \in N(A)$ by setting the remaining components with $z_{N_1} = \mathbf{0}$ and $z_{B_1} = D_{B_1 N_0} z'_{N_0}$.
(2) Every vector $w' \in R(A')$ is extendable to a unique vector $w \in R(A)$ by setting the remaining components with $w_{B_1} = \mathbf{0}$ and $w_{N_1} = -(D_{B_0 N_1})^t w'_{B_0}$.

These two facts imply that any s.s.r. vector $z' \in N(A')$ with $z'_g = 0$ induces an s.s.r. vector $z \in N(A)$ with $z_g = 0$, and any s.s.p. vector $w' \in R(A')$ induces an s.s.p. vector $w \in R(A)$. Therefore, the sufficiency of LCP implies the sufficiency of LCP′. This completes the proof of the claim.

It follows from Proposition 9 and this claim, LCP′ has a unique feasible basis, namely, N_0. This means that the basis B_0 can neither be feasible nor inconsistent. This together with Proposition 8 implies that there exists an admissible transformation in D'. This transformation makes the resulting basis closer to the feasible basis B^* by the number of pivots (either one or two). Hence, by applying the same argument, we can arrive at B^* by a sequence of admissible transformations involving exactly $|B^* \backslash B|$ pivots.

3. LEXICOGRAPHIC PERTURBATION

The main theorems in the previous section, Theorem 5 and Theorem 10 assume nondegeneracy of associated linear systems. However, by using a well known perturbation scheme, we can state the theorems for general LPs and LCPs with modified notion of admissible pivots. The purpose of this section is to present these theorems by using the lexicographic (symbolic) perturbation scheme.

Let E be a finite set with a distinguished element g, let A be an $m \times E$ matrix, and consider the linear system $Ax = \mathbf{0}$. Let $F = \{f_1, f_2, \ldots, f_k\}$ be any ordered k-subset of E. Typically we use a basis of A excluding g as F, but it can be any subset for the following definition. For a positive number ε, define an *ε-perturbation $A'_{\cdot g}$ of the gth column $A_{\cdot g}$ with respect to F* as

$$A'_{\cdot g} = A_{\cdot g} - \varepsilon A_{\cdot f_1} - \varepsilon^2 A_{\cdot f_2} - \cdots - \varepsilon^k A_{\cdot f_k}. \tag{13}$$

Let B be any fixed basis of A with $g \in B$, let $D = D(B)$ be the dictionary, and let $\bar{A} = (A_{\cdot B})^{-1} A$. Thus we have

$$\begin{aligned} \bar{A}_{\cdot N} &= -D \\ \bar{A}_{\cdot B} &= I^{(B)}. \end{aligned} \tag{14}$$

We denote by $D' = D'(B)$ the dictionary for the perturbed system $A'x = \mathbf{0}$ where A' is exactly the same as A except its gth column being $A'_{\cdot g}$. We define the *composition* $\alpha \circ \beta$ of two signs $\alpha, \beta \in \{+, 0, -\}$ as

$$\alpha \circ \beta = \begin{cases} \alpha & \text{if } \alpha \neq 0, \\ \beta & \text{otherwise.} \end{cases}$$

Given the signs of entries in D, one can easily read the signs of entries in the gth column of D', for sufficiently small ε.

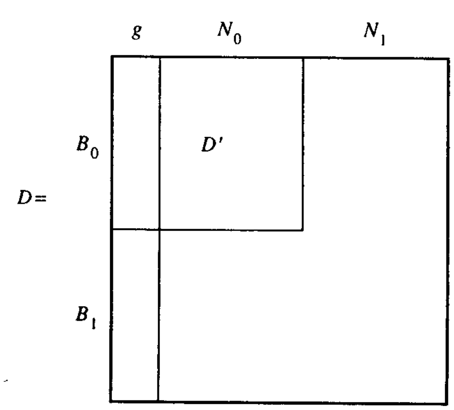


Fig. 7. A partition of the dictionary $D = D(B)$.

Proposition 11. For sufficiently small $\varepsilon > 0$ and for each $i \in B$, the sign of d'_{ig} is determined by

$$\text{sign}(d'_{ig}) = \text{sign}(d_{ig}) \circ \delta_{i1} \circ \delta_{i2} \circ \cdots \circ \delta_{ik}, \tag{15}$$

where, for each $j = 1, 2, \ldots, k$,

$$\delta_{ij} = \begin{cases} + & \text{if } f_j \in B, \\ -\,\text{sign}(d_{if_j}) & \text{if } f_j \in N. \end{cases} \tag{16}$$

Proof. This follows directly from the equations (13) and (14).

This proposition implies that any ε-perturbation for sufficiently small ε is combinatorially unique. We shall use the term *lexicographic perturbation* for any such ε-perturbation. The following is basic and important.

Proposition 12. If F is an ordered basis for $Ax = \mathbf{0}$ not containing g, the following statements hold for the associated lexicographic perturbation.

(a) For any basis B not containing g, the gth column of $D' = D'(B)$ has no zero components.
(b) If $x \in N(A)$ and $x_g = 1$, then there exists $x' \in N(A')$ such that $x'_g = 1$ and $x'_j \geq 0$ whenever $x_j \geq 0$.

Proof. The statement (a) comes from the fact that F is a basis and at least one of $\delta_{i1}, \delta_{i2}, \ldots, \delta_{im}$ (in Proposition 11) must be nonzero for each $i \in B$.

The statement (b) is immediate from (13).

3.1. Lexicographic perturbation scheme for LP's

Suppose an LP is given, and let B be any LP basis. Thus $f \in B$ and $g \in N$. The basis represents the two linear systems: $x_B = Dx_N$ and $y_N = -D^t y_B$. To ensure a perturbed LP to be nondegenerate, one can use $F := B - f$ with any linear ordering for the perturbation of gth column. To ensure a perturbed LP to be dual-nondegenerate, one can set $F^D := N - g$ with any linear ordering to perturb the fth column of the dual system $y_N = -D^t y_B$. These two perturbations are independent and thus can be performed at the same time. We call this perturbation scheme a *full lexicographic perturbation.*

We assume that a full lexicographic perturbation is applied to the LP, and for any LP basis B, denote by d'_{ig} and d'_{fj} the modified entries in the dictionary D', for $i \in B - f$ and $j \in N - g$. For $r \in B - f$ and $s \in N - g$ with $d_{rs} \neq 0$, a pivot on (r, s) is said to be *weakly admissible* if either (I) $d'_{rg} < 0$ and $d_{rs} > 0$ or (II) $d'_{fs} > 0$ and $d_{rs} < 0$. Thus a weakly admissible pivot is a pivot that would become admissible once the LP is replaced with the perturbed problem. In the original dictionary, the sign patterns of weakly admissible pivots are shown in Fig. 8. Now we can state Theorem 5 for the general LP.

Theorem 13. If an LP have an optimal solution, then under any full lexicographic perturbation scheme the perturbed LP has a unique optimal basis, and there exists a sequence of $|B^* \backslash B|$ weakly admissible pivots from any LP basis B to the basis B^* optimal for the perturbed problem.

Proof. Let an LP has an optimal solution, and thus both the primal LP and the dual problem DP are feasible. By Proposition 12(b), any full lexicographic perturbation preserves the feasibility of both and thus the perturbed LP must have an optimal solution by the strong duality theorem. Proposition 12(a) guarantees the uniqueness of an optimal basis of the perturbed problem. The remaining part follows from Theorem 5.

To interpret this theorem for Example 3, consider the perturbation of the dual problem with respect to $F^D = \{3, 2\}$ (in any ordering). Since the primal LP is nondegenerate, any primal perturbation has no effect. With this perturbation, an optimal basis $B^* = \{f, 1, 4\}$ becomes the unique optimal basis for the perturbed problem, because the dictionary entry d^*_{f3} would become minus after perturbation. On the other hand, for the basis $B = \{g, 3, 4\}$ the dictionary entry d_{f1} which is zero would become plus, and thus a pivot on the position (3, 1) is in fact a weakly admissible pivot. Clearly by making a pivot on this position, one obtains the optimal basis B^* in one pivot.

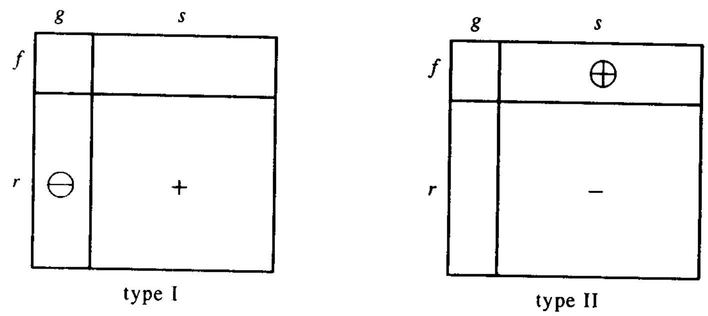


Fig. 8. Two types of weakly admissible pivots for LP.

3.2. *Lexicographic perturbation scheme for LCPs*

It is quite straightforward to make any LCP nondegenerate by a lexicographic perturbation. We can simply use any LCP basis B with any linear ordering as F. A *weakly admissible transformation* at an LCP basis B is a transformation that would become admissible once the gth column is interpreted as perturbed. Fig. 9 shows the sign patterns of admissible pivots.

Since the sufficiency of LCP has nothing to do with the gth column, it is preserved under lexicographic perturbation. Thus we have a general version of our LCP theorem for nondegenerate LCPs, Theorem 10.

Theorem 14. If an LCP is weakly feasible, then under any lexicographic perturbation scheme the perturbed LCP has a unique feasible basis, and there exists a sequence of weakly admissible transformations involving exactly $|B^*\backslash B|$ pivots from any LCP basis B to the basis B^* feasible for the perturbed problem.

Proof. Let an LCP be weakly feasible, By Proposition 12(b), any full lexicographic perturbation preserves the weak feasibility of the LCP, and thus the perturbed problem that is sufficient must have a feasible basis. Proposition 12(a) guarantees the uniqueness of a feasible basis. The remaining part follows from Theorem 10.

4. CONCLUDING REMARKS

We have shown that there is a very short sequence of admissible pivots from any LP basis to the unique optimal basis when an LP is fully nondegenerate. In fact, the length is the shortest possible. Also we extended this theorem to the larger class of sufficient LCP's. With the standard symbolic perturbation scheme, these results were interpreted in the general (degenerate) setting, using the notion of weakly admissible pivots.

The LP result has a strong contrast to the case of 'simplex' pivots as opposed to 'admissible' pivots, since even under the full nondegeneracy assumption, there is no easy way to analyze the length of a shortest sequence of simplex pivots from any basis to the optimal basis.

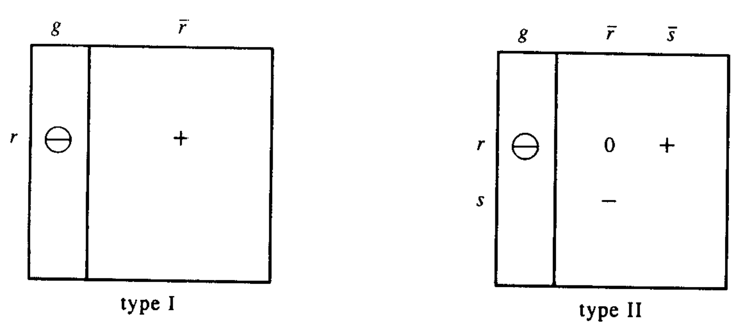


Fig. 9. Two types of weakly admissible transformations for LCP.

Of course, we have proved merely the existence of a short sequence of weakly admissible pivots for the general LP and any sufficient LCP. To design a polynomial admissible pivot algorithm is an ultimate goal. We hope that our existence results provide many researchers with a good incentive and some guidance to look for such a jewel.

Finally, it should be noted that all the results in the present paper can be naturally extended to a more general setting of oriented matroids, just like in Fukuda and Terlaky (1992) where the LCP duality theorems are proved in this setting.

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