It follows that

$$1 - d_n \le \frac{|C| - |U|}{|C|} < \frac{3\sqrt{3}\, ckt^2 + 2\sqrt{3} - \pi}{\dfrac{9\sqrt{3}}{2}\, k^2}.$$

Let p be a positive number less than 1/8. Letting n tend to infinity and choosing l so that $n^p < l \le m^{1/2-p}$ we have $\lim_{n\to\infty} n/k^2 = 3$ and $\lim_{n\to\infty} k/l^2 = 0$. Therefore multiplying the last inequality by n and going over to the limiting value as $n \to \infty$ we obtain inequality (4).

It is conjectured that in (4) the sign of equality holds. We formulate the following sharper

Conjecture 3. *Let $D(u)$ be the density of the densest packing of congruent copies of a domain u. Then*

$$\sup_{u \in \mathcal{S}_n} D(u) \sim 1 - \frac{4\sqrt{3} - 2\pi}{3\sqrt{3}\, n}.$$

References

1. C. A. Rogers, The closest packing of convex two-dimensional domains, *Acta Math.* 86 (1951), 309–321.
2. L. Fejes Tóth, Some packing and covering theorems, *Acta Sci. Math.* (Szeged) 12/A (1950), 62–67.
3. L. Fejes Tóth, On the densest packing of convex discs, *Mathematika* 30 (1983), 1–3.
4. L. Fejes Tóth, Densest packing of translates of a domain, *Acta Math. Acad. Sci. Hungar.* 45 (1984), 437–440.
5. L. Fejes Tóth, Über einen geometrischen Satz, *Math. Z.* 46 (1940), 83–85.
6. J. M. Wills, Research Problem 35, *Period. Math. Hungar.* 14 (1983), 312–314.

Received December 4, 1985.

Random Polytopes in the d-Dimensional Cube

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Abstract. Let C^d be the set of vertices of a d-dimensional cube, $C^d = \{(x_1, \ldots, x_d): x_i = \pm 1\}$. Let us choose a random n-element subset $A(n)$ of C^d. Here we prove that $\mathrm{Prob}(\text{the origin belongs to the conv } A(2d + x\sqrt{2d})) = \Phi(x) + o(1)$ if x is fixed and $d \to \infty$. That is, for an arbitrary $\varepsilon > 0$ the convex hull of more than $(2 + \varepsilon)d$ vertices almost always contains 0 while the convex hull of less than $(2 - \varepsilon)d$ points almost always avoids it.

1. Convex Hull of Subsets of Vertices

Let C^d denote the set of vertices of a d-dimensional cube, $C^d \subset \mathbf{R}^d$, $C^d = \{(x_1, \ldots, x_d): x_i = \pm 1\}$. Let $A(n)$ be a random n-element subset of C^d. We have $\binom{2^d}{n}$ possibilities for $A(n)$, hence $\mathrm{Prob}(A(n)$ has property $\pi) = (\#$ of n-tuples of C^d with property $\pi) / \binom{2^d}{n}$. The threshold function of the property π is n_d if for every $\varepsilon > 0$ we have

$$\mathrm{Prob}(A((1+\varepsilon)n_d) \text{ has property } \pi) \to 1 \quad \text{whenever} \quad d \to \infty$$

and

$$\mathrm{Prob}(A((1-\varepsilon)n_d) \text{ has property } \pi) \to 0 \quad \text{whenever} \quad d \to \infty.$$

J. Mycielski posed the following problem: How large should we choose n so that conv $A(n)$, the convex hull of $A(n)$, contains almost surely the origin 0 [2]? P. Erdős conjectured that the threshold function of this property is $O(d)$ [2]. This was proved by Komlós [5] in 1980.

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Theorem 1.1 [5]. $1.3d < n_d < 4.4d$, i.e.,

$$\lim_{d\to\infty} \text{Prob}(0 \in \text{conv } A(1.3d)) = 0$$

and

$$\lim_{d\to\infty} \text{Prob}(0 \in \text{conv } A(4.4d)) = 1.$$

Here we improve this result. Denote $1/\sqrt{2\pi}\int_{-\infty}^{c} e^{-x^2/2}\,dx$ by $\Phi(c)$, as usual.

Theorem 1.2. *Let c be a real number. Then*

$$\lim_{d\to\infty} \text{Prob}(0 \in \text{conv } A(2d + c\sqrt{2d})) = \Phi(c).$$

Corollary 1.3. $n_d = 2d$.

Our results are strongly related to an old theorem of Wendel [7] (see Theorem 4.1). More results and an extensive literature about random polytopes can be found in a recent paper of Buchta and Müller [1]. See also Mycielski [6].

2. Lemmas

P. Erdős conjectured that a random ±1 matrix is almost always regular. This was proved by Komlós [4] in 1967:

Lemma 2.1 [4]. *Let M be a $d\times d$ random ±1 matrix (i.e., every entry a_{ij} is chosen independently and with probabilities $\text{Prob}(a_{ij}=1) = \text{Prob}(a_{ij}=-1) = \frac{1}{2}$). Then*

$$\text{Prob}(M \text{ is regular}) > 1 - O(1/\sqrt{d}).$$ (1)

Komlós conjectures that one can replace $O(1/\sqrt{d})$ by $O(1/(1+\varepsilon)^d)$ for some positive ε. The following lemma is a simple generalization of a result due to Harding [3]. Let $P = \{p_1, p_2, \ldots, p_n\} \subset \mathbf{R}^d$ be a point-set ($p_i = p_j$ is possible). The partition $P = U \cup V$ is induced by a hyperplane if there exists a hyperplane $H \subset \mathbf{R}^d$ such that $P \cap H = \emptyset$ and H splits every segment $[u, v]$ for $u \in U$, $v \in V$. Denote by $h(P)$ the number of such partitions of P, $h(d, n) = \max\{h(P): P \subset \mathbf{R}^d, |P| = n\}$. Harding proved that

$$h(d, n) = \binom{n-1}{d} + \cdots + \binom{n-1}{0}.$$

Denote by $a(k, P)$ the number of affine dependent k-tuples of P. The lower bound in the following lemma is an easy consequence of a theorem of Winder [8] (see also Zaslavsky [9]).

Lemma 2.2. *For every $P \subset \mathbf{R}^d$, $|P| = n$ we have*

$$h(d, n) - \sum_{2 \le k \le d+1} a(k, P) \le h(P) \le h(d, n) = \sum_{k \le d} \binom{n-1}{k}.$$ (2)

This lower bound is not the best possible, but it is sufficient for our purposes.

3. Proof of Theorem 1.2

The first observation is that we can neglect those $A(n)$'s which contain a pair of opposite vertices because the probability of this event tends to 0 when $d\to\infty$. In fact, we have

$$\text{Prob}(A(n) \cap (-A(n)) \ne \emptyset) \le \frac{2^d \binom{2^d}{n-2}}{\binom{2^d}{n}} < \frac{n^2}{2^d} = o(1) \quad \text{if} \quad n = O(d) \text{ and } d\to\infty.$$ (3)

The main idea in our argument is that we obtain a random n-set A in two steps. First we choose n pairs from the 2^{d-1} pairs of the form $\{x, -x\}$, $x \in C^d$. Then we choose an element from each pair. Let H be a hyperplane of $\mathbf{R}^d$ in general position with respect to C^d, $0 \notin H$. For $x \in C^d$ we denote the point $H \cap (0, x)$ by $\pi(x)$.

Lemma 3.1. *Let $x_1, \ldots, x_n \in C^d$, $\Pi = \{\pi(x_i): 1 \le i \le n\}$ and*

$$Y = \{\{y_1, \ldots, y_n\}: y_i = x_i \text{ or } -x_i\}.$$

Then # (*members of Y whose convex hull avoids 0*) = $2h(\Pi)$.

Proof. If the convex hull of $\{y_1, \ldots, y_n\}$ avoids 0 then there exists a hyperplane H_0 through 0 such that it separates $\{y_1, \ldots, y_n\}$ from $\{-y_1, \ldots, -y_n\}$. Hence $H_0 \cap H$ induces a partition of Π. Moreover, the converse is also true, every induced partition yields two members of Y. □

Using Lemma 2.2 for $n = 2d + c\sqrt{2d}$ we get

$$\text{Prob}(0 \notin \text{conv } A(n)) = \frac{1}{2^n\binom{2^d}{n}} \sum 2h(\Pi) \le \frac{h(d, n)}{2^{n-1}}$$

$$= \frac{1}{2^{n-1}} \sum_{i < n/2 - (c\sqrt{n})/2} \binom{n-1}{i} = \Phi(-c) + O\left(\frac{1}{n}\right).$$

(Actually, we have calculated $\text{Prob}(0 \notin \text{conv } A(n)|A\cap(-A) = \emptyset)$ and used (3).) Moreover, (3) implies

$$\text{Prob}(0 \notin \text{conv } A(n)) \ge \frac{h(d, n)}{2^{n-1}} - \frac{1}{2^{n-1}\binom{2^d}{n}} \sum \sum_{1 \le k \le d} a(k, \Pi).$$ (4)

Here Komlós' theorem (i.e., Lemma 2.1) implies

$$\frac{1}{2^{d-1}} \sum a(k, \Pi) \leq O\left(\frac{1}{\sqrt{d}}\right)\binom{n}{d}.$$

Similarly

$$\frac{1}{2^{d-1}}\frac{1}{n}\sum a(k,\Pi)$$

$$= \binom{n}{k}\text{Prob}(k \text{ random } \pm 1 \text{ sequences of length } d \text{ are linearly dependent})$$

$$\leq \binom{n}{k}O\left(\frac{1}{\sqrt{d}}\right).$$

Hence (4) gives

$$\text{Prob}(0 \notin \text{conv } A(n)) \geq \frac{h(d, n)}{2^{n-1}} - \frac{1}{2^{n-1}}\sum_{i<d}\binom{n}{i}O\left(\frac{1}{\sqrt{d}}\right) = \Phi(-c) + O\left(\frac{1}{n}\right) - O\left(\frac{1}{\sqrt{d}}\right).$$

4. Final Remarks

If Komlós' conjecture is true, then the method given above yields, for $a = O(d)$,

$$\text{Prob}(0 \notin \text{conv } A(n)) = \frac{h(n, d)}{2^{n-1}} + O(d^2/(1+\varepsilon)^d).$$

Of course, this method can be used in all cases when the underlying set of the points T is symmetric, and $\text{Prob}(x_1, \ldots, x_d, T \text{ are linearly dependent}) = 1 - o(1)$, for example,

Theorem 4.1 (Wendel [7]).   Let B^d denote the d-dimensional ball. Let us choose an n-element set P randomly. Then

$$\text{Prob}(0 \notin \text{conv } P) = h(d, n)/2^{n-1}.$$

Actually, Wendel used a similar inductional method (but he did not need our lemmas except the equality $h(P) = h(d, n)$ for affine independent n-sets $P \subset \mathbb{R}^d$). Finally, it is easy to see that in our case ($n \sim 2d, d \to \infty$) we have

$$\text{Prob}(0 \in \text{int conv } P) - \text{Prob}(0 \in \text{conv } P) \to 0.$$

References

1. C. Buchta and J. Müller, Random polytopes in a ball, J. Appl. Probab. 21 (1984), 753–762.
2. P. Erdős and J. Mycielski, private communication.
3. E. F. Harding, The number of partitions of a set of n points in k dimensions induced by hyperplanes, Proc. Edinburgh Math. Soc. (2) 15 (1966/67), 285–289.
4. J. Komlós, On the determinant of (0,1)-matrices, Studia Sci. Math. Hungar. 2 (1967), 7–21.
5. J. Komlós, unpublished.
6. J. Mycielski, On random convex hulls, Colloq. Math, to appear.
7. J. G. Wendel, A problem in geometric probability, Math. Scand. 11 (1962), 109–111.
8. R. O. Winder, Partitions of N-space by hyperplanes, SIAM J. Appl. Math. 14 (1966), 811–818.
9. T. Zaslavsky, Facing up to Arrangements: Face-count Formulas for Partitions of Space by Hyperplanes, Memoir 154, American Mathematical Society, Providence, RI, 1975.