ALGEBRA 250A (Prof. De Loera)
Graduate Proficiency Exam **Fall 2005**

These problems are intended to help you review material covered in the prelim. Some problems are harder and longer than what you will find in the exam. Try to write the solutions as you would in the exam: Write all details when solving every problem. Be organized and use the notation appropiately. Initially try to solve the problems without any assistance. These problems cover only the first half of MAT 250A. The rest will be covered later in September.

- **Group Theory**

1. Prove: A subgroup of a cyclic group is cyclic.
2. Let G be a cyclic group of order 20. Indicate each element that generates G. List the distinct subgroups of G and a diagram of inclusion among them.
3. Let σ be the permutation of S_9 given by $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 1 & 5 & 8 & 6 & 3 & 2 & 4 & 7 \end{bmatrix}$
   Determine the order of σ. Is it an even or odd permutation?
4. Prove or disprove:
   a) A finite group always has a non-trivial center.
   b) A finite group always has a subgroup of order k for each divisor k of the order of G.
   c) All groups of order 325 must be Abelian.
   d) A group of order 12 must contain a normal subgroup of order 2 or 4.
5. Let G be a finite p-group. Show that
   a) If $|G| = p$, then G is cyclic.
   b) Show that if G is non-trivial, then the center of G is non-trivial.
   c) Show that if $|G| = p^2$, then G is Abelian.
   d) Every finite p-group is solvable.
6. Show that $Z_m \times Z_n$ is isomorphic to Z_{nm} if and only if $gcd(n, m) = 1$.
7. Show that $Aut(Z_n)$ is always Abelian, but in general not cyclic. Show that if p is a prime, then $Aut(Z_p)$ is cyclic of order $p - 1$.
8. For a group G denote by G' the commutator subgroup. Show that G' is a normal subgroup of G. Show that G/G' is Abelian.
9. Let $n \geq 3$ and $G = S_n$, Show that $G' = A_n$.
10. Let G be a finite group. Let K, S be subgroups of G and suppose that K is contained in the normalizer of S in G. Show that KS is a subgroup of G.
11. Show that if $|G| = 24$, then G is not simple; that is, G contains a non-trivial proper normal subgroup.
12. Show that if $|G| = 100$, then G is not simple.
13. Let G be the group defined by generators a, b, c and relations

$$a^2 = b^2 = c^2 = (ab)^3 = (ac)^2 = (bc)^3 = id.$$

   a) Determine the order and the index of the subgroup $H =< a, b >$ of G.
   b) Show that G is isomorphic to the symmetric group S_4.

14. If $p > q$ are prime numbers, a group of order pq has at most one subgroup of order p.
15. Find subgroups H, K of the dihedral group D_4 such that they are H is normal in K, K is normal in D_4 but H is not normal in D_4.
16. True of False? Give either a proof or a counterexample:

    a) Every finite Abelian p-group is generated by its elements of maximal order

    b) Every short exact sequence of Abelian groups is split.

    c) All solvable groups are nilpotent.

    d) The group A_4 has no subgroup of order 6.

    e) If H is a normal subgroup of a finite group G, then G has a subgroup that is isomorphic to G/H.
17. Determine all irreducible complex representations of the dihedral group D_n.
18. Let B the subgroup of $GL(C, n)$ consisting of all upper triangular matrices. Show that B is a solvable group
19. Prove that every complex representation T of an Abelian group G has a one-dimensional T-invariant subspace.
20. Determine up to isomorphism all Abelian groups of order 64.
21. If N is a normal subgroup of order p (prime) of a group G of order p^n, then N is in the center of G.
22. How many elements of order seven are there in a simple group of order 168?
23. Find the Sylow 2-subgroups and Sylow 3-subgroups of S_3, S_4
24. Classify up to isomorphism all groups of order 30.
25. Any group of order p^2q is solvable.
26. Let G be the group of 2×2 non-singular matrices with entries in a field of 2 elements (operation is matrix multiplication). What is the order of G? What are the possible orders of subgroups of G?
27. If in the group G, $a^5 = 1$ and $aba^{-1} = b^2$ and b is not the identity, find the order of b.
28. Find the number of elements in the conjugacy class of $g = (123)(456)$ in S_6. Find the order of the centralizer of g.
29. Find the order of a Sylow 5-subgroup of $GL_2(Z_5)$. Find one such subgroup explicitly.
30. Let M be an Abelian group with generators a, b, c, d, e satisfying the relations:

    $a-7b+14d-21e = 0$, $5a-7b-2c+10d-15e = 0$, $3a-3b-2c+6d-9e = 0$, $a-b+2d-3e = 0$.

    Give the structure of M as a direct sum of cyclic groups. Give the invariant factors and the elementary divisors of M_t, the torsion subgroup of M. What is the rank of the torsion free part of M?

- **Ring Theory**

1. Show that every ideal in the integers is principal.
2. Consider Z_{85} as a ring. Which elements have multiplicative inverses? 24 is one of them, find its inverse.

3. Let R be the ring of Gaussian integers, i.e.
   $R = \{a + bi | a, b \in \mathbb{Z}, i^2 = -1\}$
   a) Show that the principal ideal (2+i) is a maximal ideal of R.
   b) What is the quotient ring $R/(2 + i)$?
4. Show that $f(x) = 3x^5 - 2x^3 + 4x^2 - 8x + 6$ and $g(x) = 3x^3 + 4x^2 - 5x + 1$ are irreducible in $\mathbb{Q}[x]$.
5. Prove that in a commutative ring with identity every maximal ideal M is a prime ideal.
6. Consider the homomorphism of polynomial rings

$$\phi : \mathbb{Q}[x, y, z] \to \mathbb{Q}[t], \; x \to t^2, y \to t^3, z \to t^5$$

   (where $\phi(q) = q$ for all rational numbers), let I be the kernel of ϕ.
   Show that I is a prime ideal, but it is neither maximal nor principal.
7. Let I be an ideal generated by square free monomials inside $\mathbb{Q}[x_1, x_2, \ldots, x_n]$. Show that I can be written as a finite intersection of ideals of the form $< x_{i_1}, \ldots, x_{i_d} >$.
8. Prove or disprove:
   a) Every subring of $\mathbb{Q}[x_1, x_2]$ is finitely generated.
   b) Every subring of $\mathbb{Q}[x_1, x_2]$ is a unique factorization domain.
   c) Every non-zero homomorphic image of a local ring is local.
   d) $\mathbb{Q}[x]$ is a PID and a UFD
   e) $\mathbb{Z}[x]$ is a PID and a UFD
   f) Let I be an ideal of R, a commutative ring with identity, then R/I is a field if and only if I is a maximal ideal.
   g) f irreducible in $\mathbb{Z}[x]$ implies that $\mathbb{Z}[x]/I$ is a field.
9. Find all two-sided ideals of the ring of 2×2 matrices over Z_5.
10. Prove that there does not exist any ideal I such that $(x^4 + x + 1) \subset I \subset K[x]$ where K is the field with 2 elements and the inclusions are proper.