EXTREME POINTS AND ADJACENCY RELATIONSHIP IN THE FLOW POLYTOPE

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ABSTRACT - Extreme flows, that is extreme points of the feasible set for network flow problems, play a fundamental role in most optimization problems. The adiacency relation between extreme flows is investigated, and a theorem is stated, which, for any extreme flow on a given network, defines a one-to-one correspondence between the set of its neighboring extreme flows and a set of cycles.

1. Introduction.

Let $G(N, A, b)$ be a capacitated network, with $N=\{1, 2, \dots, n\}$ the set of nodes, $A=\{a_1, a_2, \dots, a_m\}$ the set of arcs and b the m vector of capacities; let node 1 and node n denote the source and the sink respectively. The set of feasible flows, $x \in R^m$, of given value v is the polytope defined by:

$$\begin{aligned} Ex &= e \\ x &\leq b \\ x &\geq 0 \end{aligned} \tag{1.1}$$

where E, a $(n-1)\,x\,m$ matrix, is the node arc incidence matrix with the last row dropped, e is a $(n-1)$ vector with v as first component and 0 elsewhere. Such a polytope will be referred to as the Flow Polytope, and its extreme points will be called Extreme Flows.

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A variety of real life problems leads to the minimization of a function $f(x)$ over the Flow Polytope. In most cases $f(x)$ is a quasi concave function, which implies that there exists at least an extreme flow on which $f(x)$ attains its minimum. Hence the set of extreme flows plays a major role in such minimization problems.

A characterization of such set can be found in Zangwill (1968) for the uncapacitated case, in Roy (1970) and Florian et al. (1971) for the capacitated one. An analysis of corrispondences between concepts from Flow Network Theory and from Convex Polytope Theory can be found in Dantzig (1963), Johnson (1966), Hartmann (1972) and Maier (1973). In the present paper our main concern is to characterize the set of extreme flows from the point of view of the adjacency relation. The relationship between pairs of adjacent extreme flows and cycles in the network is investigated. Such a relationship is formally stated in Theorem 3.2.

An algorithm, which finds al the vertices adjacent to a given vertex of the flow polytope, is presently being implemented and tested. A description of such an algorithm together with the results of a wide numerical experimentation, will be contained in a forthcoming companion paper.

2. Basic definitions.

Given a flow x on the network $G=(N, A, b)$, we partition the set of arcs A in the following two subsets:

$$A_1(x)=\{a_j \in A:\ 0<x_j<b_j\}$$

$$A_2(x)=A-A_1(x).$$

The arcs of $A_1(x)$ will be called floating arcs, while the arcs of $A_2(x)$ will be called blocked arcs.

If $\tilde{A}\subseteq A$, $G(\tilde{A})$ will denote the graph induced by $\tilde{A}$, that is the graph with $\tilde{A}$ as the set of arcs and $\tilde{N}=\{i$: either $(i,j)\in\tilde{A}$, or $(j,i)\in\tilde{A}\}$ as the set of nodes. We will call chain a sequence $\gamma=(i_1, i_2, \ldots, i_r)$ of nodes of G, such that, for $k=1,2,\ldots,r-1$, either $(i_k, i_{k+1})\in A$, or $(i_{k+1}, i_k)\in A$. If $i_1=i_r$, the sequence will be called a cycle.

Given a chain (cycle) γ let us define:

$$A^+(\gamma)=\{a_j \in A:\ a_j=(i_k, i_{k+1}) \text{ for some } k,\ 1\leq k\leq r-1\}$$

$$A^-(\gamma)=\{a_j \in A:\ a_j=(i_{k+1}, i_k) \text{ for some } k,\ 1\leq k\leq r-1\}$$

$$A(\gamma)=A^+(\gamma)$$

The arcs of $A^-(\gamma)$ will be called forward
be called reverse arcs of γ.

A chain (cycle) γ for which $A^-(\gamma)=$
To each chain (cycle) γ an m vector μ

$$\mu_i(\gamma)=\begin{cases} 1 & \text{if} \\ 0 & \text{if} \\ -1 & \text{if} \end{cases}$$

Given the sequence γ, let $-\gamma$ denot
Clearly $\mu(-\gamma)=-\mu(\gamma)$.

An m vector x such that $Ex=e$ (E
flow) if there exists a chain (cycle) γ s

$$x_i \neq 0 \Leftrightarrow a_i$$

If the chain (cycle) is a path (circuit) an
a path flow (circuit flow).

It is well known that there is a one-
bases of E and the set of spanning trees
call T_B the relative spanning tree ([3]).

3. Adjacent extreme flows.

We give first a Theorem which is

THEOREM 3.1. A flow x is an extre
by $A_1(x)$ does not have any cycle.

From Theorem 3.1. it follows the
case, that is:

COROLLARY 3.1. A flow x is an ex
if and only if it is a path flow.

(3) For notations and concepts concer
treme points, adjacent extreme points, etc.)
other standard Linear Programming textbo

$$A(\gamma) = A^{+}(\gamma) \cup A^{-}(\gamma).$$

The arcs of $A^{-}(\gamma)$ will be called forward arcs of γ, while the arcs of $A^{-}(\gamma)$ will be called reverse arcs of γ.

A chain (cycle) γ for which $A^{-}(\gamma) = \Phi$ will be said to be a path (circuit).

To each chain (cycle) γ an m vector $\mu(\gamma)$ will be associated with components

$$\mu_i(\gamma) = \begin{cases} 1 & \text{if } a_i \in A^{+}(\gamma) \\ 0 & \text{if } a_i \notin A(\gamma) \\ -1 & \text{if } a_i \in A^{-}(\gamma). \end{cases}$$

Given the sequence γ, let $-\gamma$ denote the sequence $(i_r, i_{r-1}, \dots, i_1)$.

Clearly $\mu(-\gamma) = -\mu(\gamma)$.

An m vector x such that $Ex = e$ $(Ex = 0)$ will be called chain flow (cycle flow) if there exists a chain (cycle) γ such that

$$x_i \neq 0 \Leftrightarrow a_i \in A(\gamma).$$

If the chain (cycle) is a path (circuit) and $x \geq 0$, the flow will be referred to as a path flow (circuit flow).

It is well known that there is a one-to-one correspondence between the set of bases of E and the set of spanning trees of G. Given a basis E_B of E, we will call T_B the relative spanning tree [3].

3. Adjacent extreme flows.

We give first a Theorem which is reported in [2] and [6].

THEOREM 3.1. A flow x is an extreme flow if and only if the graph induced by $A_1(x)$ does not have any cycle.

From Theorem 3.1. it follows the result given in [7] for the uncapacitated case, that is:

COROLLARY 3.1. A flow x is an extreme flow for an uncapacitated network if and only if it is a path flow.

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(3) For notations and concepts concerning polyhedral sets (bases, basic solutions, extreme points, adjacent extreme points, etc.) the reader may refer either to [1] or to any

In order to characterize the extreme flows adjacent to a given extreme flow x^0, we need some intermediate results.

Let us consider a basis E_B of E and the relative spanning tree T_B.

We can assume without any loss of generality, that E_B contains the last $n-1$ columns of E, that is T_B is the graph induced by arcs from a_{m-n+2} to a_m.

Let us cally γ^0 the chain from node 1 to node n in T_B, and γ^i the cycle obtained by addiction of a_i to T_B, with a_i as a forward arc, for $i=1,2,\ldots,m-n+1$.

LEMMA 3.1. For any t_i, $i=1,2,\ldots,m-n+1$, the vector:

$$x=v\,\mu(\gamma^0)+\sum_{i=1}^{m-n+1} t_i\,\mu(\gamma^i)$$

is a solution for $Ex=e$.

PROOF: Since $v\,\mu(\gamma^0)$ is a chain flow and $t_i\,\mu(\gamma^i)$ a cycle flow for any $i=1,2,\ldots,m-n+1$, it is

$$Ex=E(v\,\mu(\gamma^0))+\sum_{i=1}^{m-n+1} E(t_i\,\mu(\gamma^i))=e$$

and the lemma is so proven. □

Let us write

$$\mu(\gamma^i)=\begin{bmatrix}\varepsilon^i\\ \mu_B(\gamma^i)\end{bmatrix},\qquad e=\varepsilon^1 v,\qquad E=[E_N, E_B]$$

where ε^i, for $i\neq 0$, is the i^{th} unit vector, $\varepsilon^0=0$ and $\mu_B(\gamma^i)$ is the portion of $\mu(\gamma^i)$ corresponding to the last $n-1$ arcs, that is to the arcs of T_B.

LEMMA 3.2. Let $E_N^{(k)}$, $k=1,2,\ldots,m-n+1$, denote the k^{th} column of E_N, then

$$\mu_B(\gamma^k)=-E_B^{-1}E_N^{(k)}$$

$$\mu_B(\gamma^0)=E_B^{-1}\varepsilon^1.$$

PROOF: For any t_i it is:

$$E(v\,\mu(\gamma^0)+\sum_{i=1}^{m-n+1} t_i\,\mu(\gamma^i))=e$$

$$E_N(\sum_{i=1}^{m-n+1} t_i\,\varepsilon^i)+E_B(v\,\mu_B(\gamma^0)+\sum_{i=1}^{m-n+1} t_i\,\mu_B(\gamma^i))=\varepsilon^1 v$$

$$v\,\mu_B(\gamma^0)+\sum_{i=1}^{m-n+1} t_i\,\mu_B(\gamma^i)=v\,E_B^{-1}\varepsilon^1-\sum_{i=1}^{m-n+1} t_i\,(E_B^{-1}E_N^{(i)}).$$

Hence:

$$\mu_B(\gamma^0)=E_B^{-1}\varepsilon^1$$

$$\mu_B(\gamma^k)=-E_B^{-1}E_N^{(k)},\qquad k=1,2,\dots,m-n+1. \;\square$$

Let us now rewrite constraints (1.1) as

(3.1)
$$D_1x+D_2y=d$$
$$x\geq 0,\quad y\geq 0$$

where y is the vector of slack variables, and

$$D_1=\begin{bmatrix}E\\ I_m\end{bmatrix},\qquad D_2=\begin{bmatrix}0\\ I_m\end{bmatrix},\qquad d=\begin{bmatrix}e\\ b\end{bmatrix}$$

with I_m the unit matrix of order m.

A basis for (3.1) is:

$$D_B=\begin{bmatrix}E_B & C' & 0\\ I_{n-1} & 0 & I_{n-1}\\ 0 & I_{m-n+1} & 0\end{bmatrix}$$

where E_B is a basis for E and $C'=[E_{N1},0]$ with E_{N1} a submatrix of E_N. The non basic part of $[D_1,D_2]$ is:

$$D_N=\begin{bmatrix}C''\\ 0\\ I_{m-n+1}\end{bmatrix}$$

where $C''=[0,E_{N2}]$, with E_{N2} denoting the non basic part of E. By properly reordering the variables, (3.1) can be written as:

(3.2)
$$D_N\begin{bmatrix}y_N\\ x_N\end{bmatrix}+D_B\begin{bmatrix}x_B\\ y_B\end{bmatrix}=d$$

$$x_B,\,y_B,\,x_N,\,y_N\geq 0.$$

It is:

$$D_B^{-1} = \begin{bmatrix} E_B^{-1} & 0 & -E_B^{-1} C' \\ 0 & 0 & I_{m-n+1} \\ -E_B^{-1} & I_{n-1} & E_B^{-1} C' \end{bmatrix}$$

$$D_B^{-1} D_N = \begin{bmatrix} E_B^{-1} & (C''-C') \\ I_{m-n+1} & \\ -E_B^{-1} & (C''-C') \end{bmatrix}$$

$$C''-C' = [-E_{N1}, E_{N2}].$$

Let now assume without any loss of generality that E_{N1} contains columns from 1 to r of E, and E_{N2} columns from $r+1$ to $m-n+1$. We can now write (3.1) as:

$$(3.3) \quad \begin{bmatrix} x_B \\ y_B \end{bmatrix} = \begin{bmatrix} x^0_B \\ y^0_B \end{bmatrix} - \sum_{k=1}^{r} \begin{bmatrix} -E_B^{-1} E_N^{(k)} \\ \varepsilon^k \\ E_B^{-1} E_N^{(k)} \end{bmatrix} y_k - \sum_{k=r+1}^{m-n+1} \begin{bmatrix} E_B^{-1} E_N^{(k)} \\ \varepsilon^k \\ -E_B^{-1} E_N^{(k)} \end{bmatrix} x_k$$

$$x_B, y_B, x_k, y_k \geq 0$$

where $E_N^{(k)}$ is the k^{th} column of E_N, x_k and y_k are flow and slack relative to arc k respectively, and:

$$\begin{bmatrix} x^0_B \\ y^0_B \end{bmatrix} = D_B^{-1} d.$$

From Lemma 3.2 and from (3.3), remembering that x_B is a $(n-1+r)$ vector with components from $m-n+2$ to m of x in the first $n-1$ positions and with components from 1 to r of x in the last r positions, it follows that any feasible flow x can be written as:

$$(3.4) \quad x_i = \begin{cases} x_i^0 - \theta_i & i=1, \dots, r \\ \theta_i & i=r+1, \dots, m-n+1 \\ x_i^0 - \sum_{k=1}^{r} \mu_i(\gamma^k)\, \theta_k + \sum_{k=r+1}^{m-n+1} \mu_i(\gamma^k)\, \theta_k & i=m-n+2, \dots, m \end{cases}$$

where θ_i, $i=1, 2, \dots, m-n+1$, are nonnegative parameters. In a more compact form (3.4) can be written as:

$$x = x^0 - \sum_{k=1}^{r} \mu(\gamma^k)\theta_k + \sum_{k=r+1}^{m-n+1} \mu(\gamma^k)\theta_k. \tag{3.5}$$

It is easy to see that an extreme flow adjacent to x^0 can be obtained by giving value zero to all the parameters θ_k but one, say θ_h, which will be given a value:

$$\bar{\theta}_h = \begin{cases} \delta(-\gamma^k) & \text{if} \quad 1 \leq h \leq r \\ (\delta(\gamma^k) & \text{if} \quad r+1 \leq h \leq m-n+1 \end{cases}$$

We can state now the following theorem which characterize the adjacency relation on the Flow Polytope.

THEOREM 3.2. Let x^0 be an extreme feasible flow and $\bar{\gamma}$ a cycle with $\delta(\bar{\gamma}) > 0$. Then $x^1 = x^0 + \mu(\bar{\gamma})\,\delta(\bar{\gamma})$ is an extreme flow adjacent to x^0 if and only if the graph induced by $A(\bar{\gamma}) \cup A_1(x^0)$ does not have cycles distinct from $\pm\bar{\gamma}$.

PROOF: $\Rightarrow$ We prove first the necessity. Let $x^1 = x^0 + \mu(\bar{\gamma})\,\delta(\bar{\gamma})$ be an extreme flow adjacency to x^0. By hypothesis there exists a basis corresponding to x^0, with T as associated spanning tree, and an $a_k \notin T$, such that:
either

$$x^1 = x^0 - \mu(\gamma^k)\,\delta(-\gamma^k)$$

or

$$x^1 = x^0 + \mu(\gamma^k)\,\delta(\gamma^k)$$

where γ^k is the cycle obtained by addiction of a_k to T. From the definition of x^1 it follows that:
either

$$\mu(\bar{\gamma})\,\delta(\bar{\gamma}) = -\mu(\gamma^k)\,\delta(-\gamma^k)$$

or

$$\mu(\bar{\gamma})\,\delta(\bar{\gamma}) = \mu(\gamma^k)\,\delta(\gamma^k).$$

Since the vector $\mu(\cdot)$ has components $-1, 0, 1$ and by definition $\delta(\cdot)$ is a non negative scalar, it follows that:
either

$$\delta(\bar{\gamma}) = \delta(-\gamma^k), \quad \text{and} \quad \mu(\bar{\gamma}) = -\mu(\gamma^k)$$

or

$$\delta(\bar{\gamma}) = \delta(\gamma^k), \quad \text{and} \quad \mu(\bar{\gamma}) = \mu(\gamma^k).$$

Then the cycle $\bar{\gamma}$, except for the orientation is equal to cycle γ^k and is the only cycle of the graph induced by $A(\bar{\gamma}) \cup A_1(x^0)$.

$\Leftarrow$ Let us now assume that the graph induced by $A(\bar{\gamma}) \cup A_1(x^0)$ has $\bar{\gamma}$ as its only cycle.

We prove first that $x_1 = x^0 + \mu(\bar{\gamma}) \cdot \delta(\bar{\gamma})$ is an extreme flow. By definition of $\delta(\bar{\gamma})$ there exists an arc $a_h \in A(\bar{\gamma})$ such that either $x_h{}^1 = 0$ or $x_h{}^1 = b_h$. Then $a_h \notin A_1(x^1)$ and the graph induced by $A_1(x^1)$ does not contain $\bar{\gamma}$. Since $A_1(x^1) \subset A(\bar{\gamma}) \cup A_1(x^0)$, and $\bar{\gamma}$ is the unique cycle of the graph induced by $A(\bar{\gamma}) \cup A_1(x^0)$, the graph induced by $A_1(x^1)$ does not have any cycle, and, by Theorem 3.1, x^1 is an extreme flow.

Let us now prove that x^1 is adjacent to x^0. Let $a_k \in A(\bar{\gamma})$ be an arc such that either $x_k{}^0 = b_k$ or $x_k{}^0 = 0$, and T be a spanning tree associated to the basic solution x^0, which contains all the arcs of $A(\bar{\gamma})$ but a_k (4). It is always possible to find such a spanning tree, by adding to the graph induced by $A(\bar{\gamma}) \cup A_1(x^0) - \{a_k\}$, which, by hypotesis, does not contain sycles, a proper, may be empty, subset of $A_2(x^0)$.

Let us consider the extreme flow adjacent to x^0 defined by:

$$\bar{x} = \begin{cases} x^0 + \mu(\gamma^k)\,\delta(\gamma^k), & \text{if } x_k{}^0 = 0 \\ x^0 - \mu(\gamma^k)\,\delta(-\gamma^k), & \text{if } x_k{}^0 - b_k \end{cases}$$

where γ^k is defined as usual, after T and a_k. From the definition of γ^k, being $\delta(\bar{\gamma}) > 0$, it follows that:

$$\bar{\gamma} = \gamma^k, \qquad \text{if } x_k{}^0 = 0$$

$$\bar{\gamma} = -\gamma^k, \qquad \text{if } x_k{}^0 = b_k$$

hence:

$$x^1 = x^0 + \mu(\bar{\gamma})\ \delta(\bar{\gamma}) = \begin{cases} x^0 + \mu(\gamma^k)\ \delta(\gamma^k), & \text{if } x_k{}^0 = 0 \\ x^0 - \mu(\gamma^k)\ \delta(-\gamma^k), & \text{if } x_k{}^0 = b_k \end{cases}$$

from which it follows that $x^1 = \bar{x}$, and the proof is completed. □

The results of Theorem 3.2 can be restated for the uncapacitated case as follows:

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(4) Notice that, because of degeneracy, many distinct bases and hence many distinct spanning trees may correspond to the same basic solution.

COROLLARY 3.2. Let x^1 and x^2 be two path flow on an uncapacitated network G, γ^1 and γ^2 be the corresponding paths. Then x^1 and x^2 are adjacent extreme flows if and only if the graph induced by $A(\gamma^1) \cup A(\gamma^2)$ contains, except for the orientation, only one cycle.

COROLLARY 3.3. In an uncapacitated network any two path flows, such that their paths have in common only the origin and the destination, are adjacent extreme flows.

4. Numerical example.

Let us consider the network of fig. 1, on which each arc has been assigned a pair of values (c_{ij}, x^0_{ij}), the capacity and flow respectively.

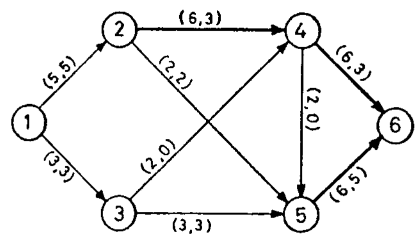


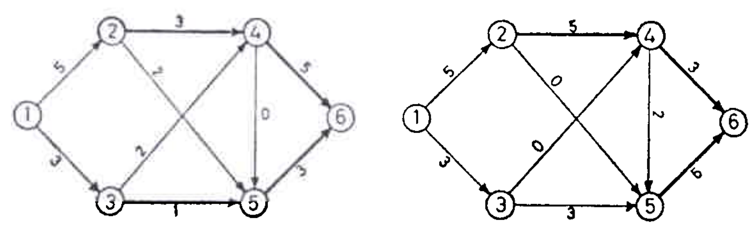
fig. 1

Since the graph induced by the set of floating arcs (thick arcs in figure) is a tree, the flow $x^0 = (x^0_{ij})$ is an extreme flow.

Let us consider the cycles $\gamma' = (3, 4, 6, 5, 3)$ and $\gamma'' = (2, 4, 5, 2)$ with $\delta(\gamma') = \delta(\gamma'') = 2$.

Let us call x' and x'' the flows obtained by superimposing to x^0 the cycle flows $2\mu(\gamma')$ and $2\mu(\gamma'')$ respectively.

In fig. 2 *a* and fig. 2 *b* flows x' and x'', with the relative set of floating arcs, are indicated.



(a) (b)

fig. 2

Both x' and x'' are extreme flows since the graphs induced by $A_1(x')$ and by $A_1(x'')$ are trees.

The graphs induced by $A(\gamma') \cup A_1(x^0)$ and by $A(\gamma'') \cup A_1(x^0)$ are given in fig. 3 *a* and in fig. 3 *b* respectively.

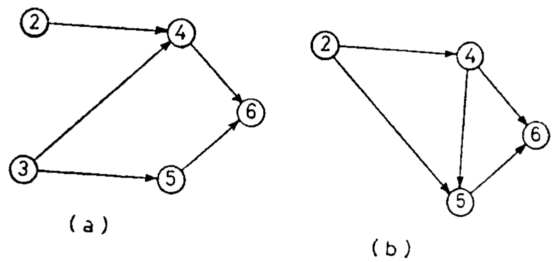


fig. 3

Clearly the former graph does not contain any cycle but $\pm\gamma$, while the latter does; then by theorem 3.2 x' is an extreme flow adjacent to x^0, while x'' is not. This can be checked by considering the tableaux of the equality constraints of (3.1) for our problem (table I).

In table I, for each of the basic solutions x^0, x' and x'', a row is added which contains a *'B'* in correspondence of the columns which, because of positivity of the value, have to be in basis, and *'N'* in correspondence of the columns which cannot stay in the basis, because of linear dependence. The empty entries are effect of the degeneracy.

It is easy to check that it is possible to find a basis corresponding to x^0 and a basis corresponding to x' which differ for only one column. The same is not true for x^0 and for x''; in fact the columns relative to x_{25} and to y_{46} are in basis for x^0 and are out of basis for any basis correspondent to x''.

	x_{12}	x_{13}	x_{24}	x_{25}	x_{34}	x_{35}	x_{45}	x_{46}	x_{56}	y_{12}	y_{13}	y_{24}	y_{25}	y_{34}	y_{35}	y_{45}	y_{46}	y_{56}
	1	1																
	—1		1	1														
		—1			1	1												
			—1		—1		1	1										
				—1		—1	—1		1									
	1									1								
		1									1							
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				1									1					
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						1									1			
							1									1		
								1									1	
									1									1
(x^0)	*B*	*B*	*B*	*B*		*B*	*N*	*B*	*B*			*B*	*N*	*B*		*B*	*B*	*B*
(x')	*B*	*B*	*B*	*B*	*B*	*B*	*N*	*B*	*B*			*B*	*N*	*N*	*B*	*B*	*B*	*B*
(x'')	*B*	*B*	*B*	*N*		*B*	*B*	*B*	*B*			*B*	*B*	*B*		*N*	*B*	*B*

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