Remarks. (1) Many other solutions of our system have been constructed in [4] for the case when $k = l = 3$, corresponding to the Grassmannian $G_3(\mathbf{C}^6)$. All these solutions are of the form $\Phi(\gamma(\beta; I); v)$ for some basis A and some A-admissible base I.

(2) Many nondegenerate strata of the Grassmannian $G_k(\mathbf{C}^{k+l})$ are represented by open subsets in $\mathbf{C}^J$ for some $J \subset [1,k] \times [1,l]$ (for example, in the case of $G_3(\mathbf{C}^6)$ only one stratum, up to the action of the Weyl group, is not of such a form). Hypergeometric functions on such strata satisfy the system of the form (1), (2) for some subgroup H_J and some lattice L^J (see the proof of Theorem 1). It is clear that all lattices L^J are simple, so that hypergeometric functions on these strata can be constructed with the use of Proposition 2 and Theorem 3. For example, if $k = l = p$ and J consists of $2p$ points $(1,1), (2,2), \ldots, (p,p), (1,2), (2,3), \ldots, (p-1,p), (p,1)$ then the action of H_J on $\mathbf{C}^J$ is isomorphic to the one from the examaple in Sect. 3, so that the hypergeometric functions constructed in the above example correspond to a special stratum in $G_p(\mathbf{C}^{2p})$.

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10.

(with A.N. Varchenko)

On Heaviside functions of configuration of hyperplanes

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§1. Introduction

The ring of integer-valued functions defined on M, the complement of a finite union of hyperplanes in a real affine space, whose values on the components are constants, has been considered. The ring is denoted by P; it contains distinguished multiplicative generators, namely, Heaviside functions of hyperplanes defined in the following manner: for a given hyperplane, fix a function which is equal to 1 on one side of the hyperplane and 0 on the other. Each element of the ring is a polynomial in Heaviside functions. A filtration of the ring by degrees of polynomials, denoted by $\{P^k\}$, $k \geqq 0$, introduces some properties into P which are close to those of the cohomology ring of the complement $M_{\mathbf{C}}$ to a union of complexified hyperplanes in a complexified affine space.

The ring $H^*(M_{\mathbf{C}})$ has been described by V.I. Arnold [1], E. Brieskorn [2], and P. Orlik and L. Solomon [3]. Orlik and Solomon have attracted attention to the fact that the dimension of the space $H^*(M_{\mathbf{C}})$ is equal to the number of components of the set M. The present paper proposes an explanation of this fact based on comparison of the rings P and H^*. P is a commutative ring endowed with an increasing filtration $\{P^k\}$, H^* is an anticommutative ring endowed with a graduation $\{H^k\}$. We formulate the properties of the ring P by referring to similar known properties of the ring H^*.

The rings P and H^* can possibly be included in a one-parameter family of rings which have independent meanings in themselves.

This paper is related to investigations of general hypergeometric functions [4–10] and deals with the geometrical aspects of the theory.

The authors express their gratitude to V.I. Arnold for useful discussions.

1. Definition. Consider a finite set of linear functions $\{f_i\}$, $i \in I$, on an n-dimensional affine space V over the field $\mathbf{R}$. Denote by S the union of hyperplanes $A_i = \{v \in V \mid f_i(v) = 0\}$, $i = I$. We call the couple S and $\{f_i\}$ a configuration of hyperplanes. Consider the ring $P(S, \mathbf{Z})$ of integer-valued functions on $M = V \setminus S$, which are constant on every connected component. We consider, in the ring P, the multiplicative generators, i.e., the Heaviside functions x_i, $i \in I$, determined by the conditions: $x_i(v) = 1$ if $f_i(v) > 0$ and $x_i(v) = 0$ if $f_i(v) < 0$. Every function $x \in P(S, \mathbf{Z})$ is written as a polynomial, with integer coefficients, in $\{x_i\}, i \in I$. The minimum degree of polynomials in $\{x_i\}$ representing x is called the degree of the function $x \in P(S, \mathbf{Z})$.

Define an increasing filtration

$$O \subset P^0 \subset P^1 \subset \cdots \subset P,$$

where P^k is a subspace of functions which can be represented by polynomials of degree not exceeding k. In particular, the P^0 are the constant functions. Obviously, $P^k P^l \subset P^{k+l}$. We call $\{P^k\}$ the degree filtration.

Example. Consider a configuration of lines on a plane, $\{A_i\}$, $i \in I$. The degree filtration in the ring P consists of three terms: the constant functions P^0, the linear combinations of Heaviside functions P^1 and $P^2 = P$. Suppose that no three lines intersect at one point. A basis over $\mathbf{Z}$ in the ring consists of the constant function which is equal to 1, the Heaviside functions and all the monomials $x_i x_j$ with intersecting lines A_i and A_j. The dimension of P^0 is equal to 1. The dimension of P^1/P^0 is equal to the number of lines. The dimension of P^2/P^1 is equal to the number of points of intersection of lines.

We say that if the alternating sum of four of the values of the function $x \in P$ on four components of the complement approaching the point of intersection of two lines is equal to zero, then the function has a zero index at the point. A function $x \in P$ has degree $\leqq 1$ if and only if it has zero index at every point of intersection.

2. Properties of the ring P

Theorem 1. *$P^n = P$, that is, any piecewise constant function on the complement to the union of hyperplanes in an n-dimensional affine space is a polynomial of degree not exceeding n in Heaviside functions of hyperplanes.*

Let V_{C} be the complexification of the space V, $A_{i,\mathrm{C}}$ the complexification of a hyperplane A_i, $i \in I$, S_{C} the union of hyperplanes $\{A_{i,\mathrm{C}}\}$, $i \in I$, and $M_{\mathrm{C}} = V_{\mathrm{C}} \backslash S_{\mathrm{C}}$. Theorem 1 is an analogue of the statement: $H^k(M_{\mathrm{C}}) = 0$ for $k > n$.

The configuration S on V naturally induces on every affine subspace $U \subset V$ a new configuration denoted by S_U. S_U consists of hyperplanes $\{A \cap U \mid A \in S, U \not\subset A\}$ determined by linear functions $\{f_i|_U\}, i \in I$.

If an affine subspace U is not contained in S then a natural homomorphism $j_U: P(S) \to P(S_U)$ is defined so as to restrict functions of $P(S)$ on $U \backslash S_U$.

Any non-empty intersection F of hyperplanes of a configuration is called an edge. The Codimension of an edge is denoted by $r(F)$. In particular, hyperplanes are the edges of codimension 1. The set of all edges is denoted by $\mathscr{L}$.

A d-dimensional affine subspace $U \subset V$ is called a generally positioned space if U is transversre with respect to all the edges and crosses all the edges whose codimensions do not exceed d.

Theorem 2. *If $U \subset V$ is a generally positioned subspace, then the homomorphism j_U restricted to $P^k(S)$ defines an isomorphism between $P^k(S)$ and $P^k(S_U)$, for $k \leqq d$.*

This theorem is an analogue of Brieskorn's theorem [2]: if $U_{\mathrm{C}} \subset V_{\mathrm{C}}$ is a

sufficiently general d-dimensional subspace, then $H^k(M_{\mathrm{C}}) \to H^k(M_{\mathrm{C}} \cap U_{\mathrm{C}})$ is an isomorphism for $k \leqq d$.

Let F be an edge of a configuration and $I^F \subset I$ the set of all the indices i such that $F \subset A_i$. We denote by S^F the configuration composed of hyperplanes $\{A_i\}$, $i \in I^F$, that is the hyperplanes containing F. We say that S^F is a localization of the configuration S at the edge F. Consider the ring $P(S^F, \mathbf{Z})$ of the configuration S^F. A natural inclusion $P(S^F, \mathbf{Z}) \to P(S, \mathbf{Z})$ exists, defined by restricting the functions of $P(S^F, \mathbf{Z})$ to M. Its image is the subring generated by functions $\{x_i\}$, $i \in I^F$. The inclusion pressrves the degree filtration.

Theorem 3. *The natural mapping*

$$\bigoplus_{\substack{F \in \mathscr{L} \\ r(F)=k}} P^k(S^F)/P^{k-1}(S^F) \to P^k(S)/P^{k-1}(S)$$

is an isomorphism for every $k > 0$.

Corollary 1. *If the configuration hyperplanes have only normal intersections, then $\dim_{\mathbf{Z}} P^k(S)/P^{k-1}(S)$ is equal to the number of k-codimensional edges.*

Let $M_{\mathrm{C}}^F = V_{\mathrm{C}} \backslash \bigcup_{i \in I^F} A_{i,\mathrm{C}}$. According to Brieskorn [2], the natural mapping $\bigoplus_{F \in \mathscr{L}, r(F)=k} H^k(M_{\mathrm{C}}^F, \mathbf{Z}) \to H^k(M_{\mathrm{C}}, \mathbf{Z})$ is an isomorphism for every $k > 0$. Theorem 3 is an analogue to Brieskorn's theorem.

Corollary 2. $\dim_{\mathbf{Z}} P^k(S, \mathbf{Z})/P^{k-1}(S, \mathbf{Z}) = \dim_{\mathbf{Z}} H^k(M_{\mathrm{C}}, \mathbf{Z})$ *for $k \geqq 0$.*

This corollary can easily be deduced by induction with respect to the dimension of the containing space, from an observation by Orlik and Solomon: $\dim_{\mathbf{Z}} H^*(M_{\mathrm{C}}) = \dim_{\mathbf{Z}} C_n(S)$, from Theorem 3, and from Brieskorn's theorem.

Remarks: 1. In particular, the corollary implies that $\sum_{k \geqq 0} (-1)^k \dim_{\mathbf{Z}} P^k(S)/P^{k-1}(S)$ is equal to the number of bounded components of M (cf. the combinatorial formulae for $\dim H^k(M_{\mathrm{C}})$ and the number of bounded components [3, 13]). 2. If k, i are positive numbers, then the unique decomposition of k exists:

$$k = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j},$$

where $n_i > n_{i-1} > \cdots > n_j \geqq j \geqq 1$. Following [14], we define

$$k^{\langle i \rangle} = \binom{n_i + 1}{i+1} + \binom{n_{i-1} + 1}{i} + \cdots + \binom{n_j + 1}{j+1}, \quad O^{\langle i \rangle} = 0.$$

An integer vector $(k_0, k_1, \ldots, k_d)$ is called an M-vector if $k_0 = 1$ and $0 \leqq k_{i+1} \leqq k_i^{\langle i \rangle}$ for $1 \leqq i \leqq d-1$. It follows from [15] that the sequence of numbers $\dim_{\mathbf{Z}} P^k(S)/P^{k-1}(S)$, $k \geqq 0$, is the M-vector.

We call a monomial $x_{i_1} \cdots x_{i_k} \in P$ an admissible monomial if $df_{i_1} \wedge \cdots \wedge df_{i_k} \neq 0$.

A k-dimensional simplicial cone multiplied by an $(n-k)$-dimensional affine subspace is a support of an admissible monomial.

Corollary 3. *The set of admissible monomials generates P as a module over $\mathbf{Z}$.*

3. Dual degree filtration. Consider the ring $P(S, \mathbf{Z})$ of a configuration of hyperplanes in an n-dimensional affine space as a linear space over $\mathbf{Z}$. We define on the dual space P^* a decreasing filtration

$$O \subset P_n^* \subset P_{n-1}^* \subset \cdots \subset P_0^* = P^*$$

by making use of the condition $P_k^* = \mathrm{Ann}\, P^{k-1}$. We call $\{P_k^*\}$ a degree filtration.

We present here another construction of the filtration. We point out a finite set of vectors of P^* which are called flag cochains. Every flag cochain has a degree. Then P_k^* coincides with the linear hull of all the flag cochains of degree not less than k. Now we proceed to the construction itself.

The connected components of M are called the regions. The regions are the n-dimensional polyhedra (not necessarily bounded). Open facets of any dimensions of these polyhedrons are called the facets of the configuration. In particular, n-dimensional facets are regions. Zero-dimensional facets are called vertices.

Let $F_{n-k} \subset F_{n-k+1} \subset \cdots \subset F_n = V$ be a sequence of edges of a configuration S, the dimension of F_j being equal to j, and let the coorientation of the edge F_j in the edge F_{j+1} be given. We call this the flag of the edges of degree k and we denote it by F. Let Δ be a $(n-k)$-dimensional facet of the configuration lying in F_{n-k}. The flag F, together with the facet Δ, is called a distinguished flag.

2^k regions are related to a distinguished flag, with Δ being included in the closures of every region. Indeed, there are exactly 2^k regions that can be reached, first by a small move in any direction along F_{n-k+1}, then by a still smaller move in any direction along F_{n-k+2}, etc. until a move is made from F_{n-1} in any direction. To any such region corresponds an ordered sequence α of length k, which consists of pluses and minuses, $+$ or $-$, occupying the position j, depending on whether the motion into the region was along or against the coorientation of $F_{n-k-j-1}$ in F_{n-k-j}. The region with the index α is denoted Δ_α (see Fig. 1).

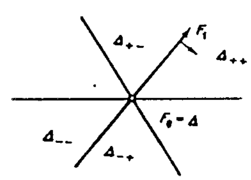

Fig. 1.

A vector $\psi_{F,\Delta} \in P^*$ is defined as follows: for any $x \in P$,

$$\psi_{F,\Delta}(x) = \sum_\alpha (-1)^{\varepsilon(\alpha)} x(\Delta_\alpha),$$

where $\varepsilon(\alpha)$ is the number of minuses in the sequence α, is called a flag cochain of the distinguished flag. k is called the degree of the flag cochain.

Theorem 4. *The linear hull of flag cochains of degree not less than k coincides with* $\mathrm{Ann}\, P^{k-1}$.

Theorem 4 can be used to determine the degree of a given function (cf. the example in Sect. 1.1).

4. Relations between heaviside functions. Here we suppose that V is a linear n-dimensional space, the $\{f_i\}$, $i \in I$, are linear functions on V and that all the hyperplanes $\{A_i\}$ pass through the origin.

Let $\alpha_1 f_{j_1} + \cdots + \alpha_s f_{j_s} = 0$ be a linear relation; the J_+ are the numbers of all the linear functions with positive factors in the relation, and the J_- are the numbers of those with negative factors.

Theorem 5. *The Heaviside functions of the configuration $\{f_i\}$, $i \in I$, satisfy the relations:*

(1) $x_i^2 - x_i = 0$, $i \in I$,

and

(2) $\prod_{j \in J_+} x_j \prod_{k \in J_-} (x_k - 1) - \prod_{j \in J_+} (x_j - 1) \prod_{k \in J_-} x_k = 0$

for any linear dependence $\alpha_1 f_{j_1} + \cdots + \alpha_s f_{j_s} = 0$.

If all the coefficients in the linear relation differ from zero, then there is a polynomial in (2) of degree $s-1$ having precisely s monomials of degree $s-1$. Equation (2) is an even analogue of the Orlik–Solomon relation [3] for differential forms, namely, consider differential forms $w_i = df_i/2\pi\sqrt{-1} f_i$. If $f_{j_1}, \ldots, f_{j_s}$ are linearly dependent, then

$$\sum_{l=1}^{s} (-1)^{l-1} w_{j_1} \wedge \cdots \hat{w}_{j_l} \cdots \wedge w_{j_s} = 0.$$

Theorem 6. *Equations (1) and (2) determine P. More precisely, if ϑ is an ideal in the ring of polynomials $\mathbf{Z}[X_i; i \in I]$ generated by the left-hand sides of the relations listed in Theorem 5, then the natural homomorphism $\mathbf{Z}[X_i; i \in I]/\vartheta \to P$ is an isomorphism.*

Theorem 6 is an even analogue of an Orlik-Solomon theorem [3], which describes the ring $H^*(M_{\mathbf{C}}, \mathbf{Z})$, namely, consider the external algebra of the vector space whose basis is e_i, $i \in I$. Consider its ideal generated by the elements $\sum_{l=1}^{s} (-1)^{l-1} e_{j_1} \cdots \hat{e}_{j_l} \cdots e_{j_s}$ where $(f_{j_1}, \ldots, f_{j_s}) \subset \{f_i\}$, $i \in I$, is an arbitrary subset of linearly dependent elements. Then the factor algebra of the external algebra with respect to the ideal is naturally isomorphic to $H^*(M_C, \mathbf{Z})$. Under the isomorphism, elements e_i are transformed into the cohomology classes of forms w_i. Consider in a Euclidean space with the coordinates $\{x_i\}$, $i \in I$, a unit cube whose boundaries are the hyperplanes $x_i = 0, x_i = 1, i \in I$. A subset of vertices of the cube is related to the configuration $\{f_i\}$: for every component M_0 of the set M, we mark the vertex of the cube at which $x_i = 1$ if $f_i(M_0) > 0$, and we mark the vertex at which $x_i = 0$ if $f_i(M_0) < 0$. Let X be the set of all marked vertices of the cube. Obviously, the ring of integer-valued functions on X is isomorphic to P. Theorem 6 yields the system of equations defining X as a subset of the Euclidean space: if, in Euclidean space, the system of equations cited in Theorem 5 is considered, the set of its solutions coincides with X.

We point out a basis over $\mathbf{Z}$ in the ring P. A subset $J = (j_1, \ldots, j_k) \subset I$ is called a circuit if covectors $f_{j_1}, \ldots, f_{j_k}$ are linearly dependent, but this does not hold for any proper subset of J.

Fix a linear ordering in I. A subset $J \subset I$ is called a broken circuit if there exists an index $j_0 \in I$ such that $(j_0, j_1, \ldots, j_k)$ is a circuit, j_0 being less than any element of J.

We make a monomial $x_{j_1} \cdots x_{j_k} \in P$ correspond to any subset $I \subset J$. $1 \in P$ will correspond to the empty set.

Theorem 7. *The system of all the monomials corresponding to subsets of I which do not contain broken circuits forms a basis over $\mathbf{Z}$ in P. Moreover, the system of all distinguished monomials of degree not exceeding k is a basis in $P^k, k \geqq 0$.*

Theorem 7 is an analogue of Theorem II.1 of [7], which in turn goes back to [11]. By Theorem II.1, the system of differential forms $w_{j_1} \wedge \cdots \wedge w_{j_k}$ for all the subsets $J \subset I$ not containing broken circuits is a basis in $H^*(M_C, \mathbf{Z})$. See also the theorem in [12].

6. A comparison with cohomologies. Here we define a non-canonical linear mapping $\pi_k: P^k \to H^k(M_C, \mathbf{Z})$ whose kernel coincides with P^{k-1}. π_k is determined by a choice of coorientations of all the edges of codimension k.

Fix coorientations of all the k-codimensional edges. Determine the image of a monomial $x = x_{i_1} \cdots x_{i_l}$, $l \leqq k$. Set $\pi_k(x) = 0$ when $l < k$ or when $l = k$ and $df_{i_1} \wedge \cdots \wedge df_{i_k} = 0$. If $l = k$ and $df_{i_1} \wedge \cdots \wedge df_{i_k} \neq 0$ set $\pi_k(x) = \pm[w_{i_1} \wedge \cdots \wedge w_{i_k}]$, where the plus sign is chosen when the fixed coorientation of an edge $f_{i_1} = \cdots = f_{i_k} = 0$ coincides with the coorientation induced by the form $df_{i_1} \wedge \cdots \wedge df_{i_k}$, and the minus sign is chosen otherwise.

Theorem 8. *π_k can be continued so as to become a well-defined linear mapping $P^k \to H^k(M_C, \mathbf{Z})$ whose kernel coincides with P^{k-1}.*

Define π_k in a geometrical way. Set $\pi_k(x) = 0$ if $l < k$, or if $l = k$ and $df_{i_1} \wedge \cdots \wedge df_{i_k} = 0$. If $l = k$ and $df_{i_1} \wedge \cdots \wedge df_{i_k} \neq 0$ then set $\pi_k(x)$ equal to the following linear function on $H_k(M_C, \mathbf{Z})$: the index of intersection of classes of $H_k(M_C, \mathbf{Z})$ with a non-compact $(2n-k)$-dimensional cycle $\{v \in M_C \mid f_{i_1}(v) > 0, \ldots,$

$f_{i_k}(v) > 0\}$ whose orientation we define with the help of the complex orientation of $\{v \in M_C \mid f_{i_1}(v) = 0, \ldots, f_{i_k}(v) = 0\}$ and of the previously fixed coorientation of the edge $\{v \in V \mid f_{i_1}(v) = 0, \ldots, f_{i_k}(v) = 0\}$ in V.

7. The ring of functions which are constant on facets. Let a configuration S of hyperplanes in an n-dimensional real affine space V be given. Consider the ring $Q(S, \mathbf{Z})$ of integer-valued functions on V which are constant on every facet of the configuration. Consider, in the ring Q, the multiplicative generators which are the Heaviside functions x_i, X_i, $i \in I$, given by the relations: $x_i(v) = 1$ if $f_i(v) > 0$, $x_i(v) = 0$ if $f_i(v) \leqq 0$, and $X_i(v) = 1$ if $f_i(v) = 0$, $X_i(v) = 0$ if $f_i(v) \neq 0$. In other words, $\{x_i\}$ are the functions defined by the conditions of Sect. 1.1 and $\{X_i\}$ are the characteristic functions of hyperplanes of the configuration. Every function $x \in Q(S, \mathbf{Z})$ is written in the form of a polynomial in $\{x_i, X_i\}$, $i \in I$, with integer coefficients. We call the minimum degree of polynomials in $\{x_i, X_i\}$ representing x the degree of the function x. Define a degree filtration

$$O \subset Q^0 \subset Q^1 \subset \cdots \subset Q$$

where Q^k is the subspace of functions which can be presented as polynomials of degree not exceeding k.

The properties of the ring Q and its filtration are analogous to the properties of the ring P. We have discussed in detail in this paper the analogues of Theorems 1–4 for Q. It is not difficult to produce the analogues of Theorems 5–8.

8. Chains. An integer linear combination of facets is called the integer chain of the configuration. An integer-valued linear function on a linear space of chains is called an integer cochain. The functions of $Q(S, \mathbf{Z})$ are in 1–1 correspondence to the chains of the configuration $x \in Q \mapsto \sum x(\Delta)\Delta$ with the summation running over all the facets Δ of the configuration. The functions of $P(S, \mathbf{Z})$ are in 1–1 correspondence to n-dimensional chains of the configuration. This is a linear correspondence. The present paper uses the geometrical language of chains and cochains.

Denote by $C_k(S)$ the space linear over $\mathbf{Z}$ of integer k-dimensional chains, by $C^k(S)$ the space linear over $\mathbf{Z}$ of integer k-dimensional cochains, and by $C_k^{\mathrm{comp}}(S) \subset C_k(S)$ the subspace of integer linear combinations of bounded k-dimensional facets.

Key issues in the present paper are dimensional and degree filtrations in the space of chains $C_*(S) = \bigoplus_{k=0}^n C_k(S)$ defined below.

Set $D_k(S) = C_0(S) \oplus C_1(S) \oplus \cdots \oplus C_k(S)$, $k \geqq 0$.

Then

$$O \subset D_0(S) \subset D_1(S) \subset \cdots \subset D_n(S) = C_*(S).$$

This filtration is called the dimensional filtration.

We say that a configuration S_1 is included in a configuration S_2 if the union of hyperplanes of the first configuration is contained in the union of hyperplanes of the second configuration. Any facet of the configuration S_1, regarded as a set, is represented as a sum of the facets of the configuration S_2. Thus, a natural inclusion $C_*(S_1) \subsetneq C_*(S_2)$ of the chains is defined which preserves the degree filtration.

Example. Let V be one-dimensional, S_1 a point a, and S_2 two points $a<b$. Then the facet $\{v \in V \mid a<v\}$ of the configuration S_1 is the sum of the three facets of configuration S_2: $\{v \in V \mid a<v<b\} + \{b\} + \{v \in V \mid b<v\}$.

We call a configuration which has all its hyperplanes passing through one point and intersecting normally an elementary configuration.

Let S_1 be an elementary configuration consisting of k planes. Clearly, $k \leqq n$. Suppose that S_1 is included in S_2. Images of the facets of the configuration S_1 in $C_*(S_2)$ will be called the elementary chains of degree k of the configuration S_2. The space V itself will be called the elementary chain of degree 0.

Examples. A hyperplane and the open subspace bounded by it are elementary chains of degree 1. A vertex of a configuration is an elementary chain of degree n.

Define a subspace $W_k \subset C_*(S)$ as the linear hull of elementary chains of degree $\leqq k$, $0 \leqq k \leqq n$. Then

$$O \subset W_0 \subset W_1 \subset \cdots \subset W_n \subset C_*(S).$$

This filtration will be called the degree filtration.

Set $gr_l D = D_l / D_{l-1}$, $W_k gr_l D = (W_k \cap D_l + D_{l-1})/D_l$, $gr_k W gr_l D = W_k gr_l D / W_{k-1} gr_l D$. $gr_l D$ is canonically isomorphic to $C_l(S)$, $\{W_k gr_l D\}$, $k \geqq 0$, is the degree filtration induced on $gr_l D$, and $\{gr_k W gr_l D\}$, $k \geqq 0$, its factor-spaces.

It is rather easy to see that Theorems 1–8 are statements on degree filtration induced on $gr_n D$. It is not difficult to produce generalizations of Theorems 5–8, while the generalizations of Theorems 1–4 are in Sect. 4.

Remark. Theorems 1–4 are proved in Sect. 4, Theorems 5–8 in Sect. 5.

The appendix to this paper (Sect. 6) contains a multidimensional generalization of the theorem on decomposition of a rational function into simple fractions, the idea behind which is linked to geometric constructions of the present paper.

§2. Chains of a configuration

This section contains some combinatorial information, preparatory to proving the theorems formulated above.

1. Cones and angles. A configuration having at least one vertex is called a regular configuration. If a configuration has a non-empty intersection of all its hyperplanes, it is called a central configuration. Any chains of a regular central configuration will be called a linear cone. Any chain of a non-regular central configuration will be called an angle. The form of the angle is the direct product

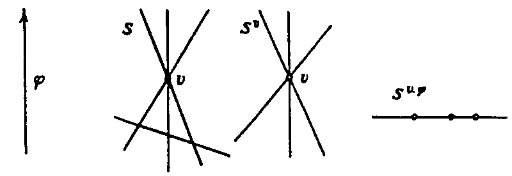

Fig. 2.

of a line and a chain of the configuration induced on a generally situated hyperplane by the given configuration.

Any chain of the configuration S^v considered as a chain in S is called a cone with vertex v of the configuration S. Any chain of the configuration S^F considered as a chain in S is called an angle with edge F of the configuration S. We recall that S^v and S^F are localizations of the configuration in the edges v and F.

2. Linear functions and configurations. A facet of a configuration is said to be bounded from above in relation to a linear function φ defined on V if the facet lies in an appropriate half-space $\varphi \leqq$ const. The set of all facets bounded from above is called a skeleton of the configuration S with respect to φ, and is written S_φ. The space of integer linear combinations of the facets of S_φ is denoted by $C_*(S_\varphi)$.

We call an affine localization $S^{v,\varphi}$ of the configuration S with respect to a point $v \in V$ and a linear function φ, a configuration cut out from the configuration S^v on a level hyperplane $\{x \in V \mid \varphi(x) = \varphi(v) - 1\}$ (see Fig. 2). Let Γ be a bounded facet of the configuration $S^{v,\varphi}$. Consider a cone with vertex v and guide Γ, the vertex v being deleted from it. This set is denoted by $K(\Gamma, v)$. The mapping $p: \Gamma \mapsto K(\Gamma, v)$ gives a monomorphism of the space $C_*^{\text{comp}}(S^{v,\varphi})$ into the subspace of chains of the configuration S^v bounded from above.

A linear function on V is called a function in general position with respect to the vertex v of the configuration S if it is not constant on edges of positive dimension passing through v, and it is called a function in general position with respect to the configuration S if it is not constant on all the edges with positive dimension and has pairwise different values on vertices.

Lemma 1. *Let φ be a generally positioned function with respect to the vertex v. Then p sets an isomorphism $C_k^{\text{comp}}(S^{v,\varphi}) \simeq C_{k+1}(S_\varphi^v)$.*

The proof is obvious.

3. Loose configurations. We call a configuration a loose configuration if every one of its hyperplanes intersects with every one of its edges of positive dimension. We list its obvious properties.

Lemma 2. 1. *Let S be a loose configuration and U a hyperplane which transversally crosses all its edges. Then S_U is a loose configuration.*

2. *Let S be any configuration and φ a function generally positioned with respect to the vertex v. Then $S^{v,\varphi}$ is a loose configuration.*

3. *Let S be a loose configuration and F its edge. Then S_F is a loose configuration.*

4. *Let S be a loose configuration and φ a linear function vanishing on the hyperplane $A \in S$. Then φ is generally positioned with respect to all the vertices of the configuration S which do not lie in A.*

4. Distinguished substar (Cf. [9]). Let $f_1, \ldots, f_N$ be an ordered set of linear functions on the n-dimensional affine space V and S the configuration determined by the functions. Define a linear order of vertices of the configuration S: $v < w$ if for a certain k, $(f_j(v))^2 = (f_j(w))^2$ if $j < k$, $(f_k(v))^2 < (f_k(w))^2$. The maximal vertex of a bounded facet will be said to be distinguished. The set of all bounded facets with a common distinguished vertex is called a distinguished substar of the vertex. The dimension of a distinguished substar is called the multiplicity of the corresponding vertex. We describe a distinguished substar of a vertex of a loose configuration.

Let v be a vertex of the configuration S. For any facet Γ^v of the configuration S^v there exists a unique facet Γ of the configuration S whose germ in v coincides with the germ of the facet Γ^v in v. The facets Γ^v, Γ will be said to be mutually induced in v.

Lemma 3. *Let v be a vertex of the loose ordered configuration $f_1(v) > 0$. Then an edge Γ belongs to a distinguished substar of a vertex v if and only if Γ is induced from the facet of the configuration S^v bounded from above with respect to f_1.*

The proof is obvious.

Similarly, let $v \in A_1 \cap A_2 \cap \cdots \cap A_{k-1}$, $f_k(v) > 0$. Consider a configuration $\tilde{S}$ cut out from S by $A_1 \cap \cdots \cap A_{k-1}$.

Lemma 4. *The facet Γ belongs to a distinguished substar of the vertex v if and only if Γ is induced from a facet of the configuration $\tilde{S}^v$ bounded from above with respect to f_k.*

Corollary. *A loose regular configuration has exactly one vertex of zero multiplicity.*

5. Euler characteristic of some chains. Let Γ be a facet of the configuration S, Γ^* a cochain which is equal to 1 on Γ and 0 on other facets, and $d(\Gamma)$ the dimension of a facet. The cochain $\chi = \sum(-1)^{d(\Gamma)}\Gamma^*$, summation being carried out over all the facets, will be called the Euler characteristic cochain.

Let Δ be a chain which is equal to the sum of all bounded facets of the configuration S. Let F be an edge of the configuration S, Γ a non-closed facet of the configuration S^F, and Δ_Γ the chain which is equal to the sum of all bounded facets of the configuration S which lie in Γ.

Theorem 9. *Let S be a regular loose configuration. Then $\chi(\Delta) = 1$, $\chi(\Delta_\Gamma) = 0$.*

The proof is obtained by induction on the dimension of the configuration. If $\dim V = 1$ Theorem 9 is obvious. Suppose that the theorem is proved for loose configurations in a space of dimension not exceeding $n-1$. Prove it for dimension n, the case $d(\Gamma) = n$ being sufficient.

Enumerate the hyperplanes of S so that the first hyperplanes are those containing $(n-1)$-dimensional facets of the polyhedron Γ.

Prove that $\chi(\Delta) = 1$. Let $\Delta(v)$ be a distinguished substar of the vertex v. We have $\Delta = \sum_v \Delta(v)$. For only one vertex v with multiplicity 0, we have $v = \Delta(v)$. Prove that $\chi(\Delta(v)) = 0$ if the multiplicity of the vertex is positive. Indeed, in this case, by virtue of Lemmas 1–4, k-dimensional facets of the distinguished substar correspond 1-1 to $(k-1)$-dimensional bounded facets of a suitable loose configuration in a space whose dimension is less than n. By assumption in the induction, it follows that the Euler characteristic of the distinguished substar is 0.

Prove that $\chi(\Delta_\Gamma) = 0$. Denote by $\Delta_\Gamma(v)$ the sum of the facets of the distinguished substar which fall within Γ. We have $\Delta_\Gamma = \sum_v \Delta_\Gamma(v)$, summation being carried out over all vertices which are in the closure of the set Γ. If a vertex v belongs to the interior of the set Γ, its multiplicity is positive, its distinguished substar coincides with $\Delta_\Gamma(v)$ and, as proved above, $\chi(\Delta_\Gamma(v)) = 0$. Prove that $\chi(\Delta_\Gamma(v)) = 0$ if v belongs to the boundary of the set Γ and $\Delta_\Gamma(v)$ is non-empty. In this case v does not belong to the first of the hyperplanes and $\Delta_\Gamma(v)$ consists of positive-dimensional facets. Let, to be certain, $f_1(v) > 0$. By Lemmas 3 and 1, the facets of $\Delta_\Gamma(v)$ correspond 1-1 to bounded facets of the affine localization S^{v,f_1} on the hyperplane $H = \{x \in V \mid f_1(x) = f_1(v) - 1\}$ which are in a set $\Gamma(v)$ determined by v and Γ. Describe $\Gamma(v)$ and prove that Theorem 1 is applicable to it. Thus Theorem 1 for $\dim V = n$ will be proved.

Let a facet Γ be defined by the inequalities $f_{i_1} > 0, f_{i_2} > 0, \ldots, f_{i_k} > 0$ in a small neighbourhood of the point v. The hyperplanes $A_{i_1}, \ldots, A_{i_k} \in S^F$ and v belong to $A = A_{i_1} \cap \cdots \cap A_{i_k}$. The conditions $f_{i_1} > 0, \ldots, f_{i_k} > 0$ define a subset $\Gamma(v)$ of H. The set A contains F lying in A_1. Thus $\Gamma(v)$ is bounded by hyperplanes whose intersection is non-empty, and, therefore, $\Gamma(v)$ is a union of non-closed facets of the configuration $(S^{v,\varphi})^{A \cap H}$. For $\Gamma(v)$ in H, Theorem 1 implies that $\chi(\Delta_\Gamma(v)) = 0$.

§3. Decomposition into cones

1. Theorem 10. *Let φ be a linear generally positioned function with respect to the configuration S and Δ the chain composed of facets bounded from above. Then Δ can be represented as the sum of cones of the configuration S bounded from above:*

(3) $\Delta = \sum_v K_v$ where the summation is over the vertices of the configuration. The decomposition (3) is unique. Every cone in (3) has dimension not greater than the dimension of the chain Δ.

Proof. The values of the function at the vertices determine the order of the vertices. In a neighbourhood of the greatest of the vertices, v, of the chain Δ, the chain has the form of a cone K_v of the configuration S^v bounded from above. Subtract the cone from Δ. We do the same with the rest of the chain. The uniqueness of the representation is obvious.

For a description of the cone K_v of the decomposition (3) in terms of the behaviour of the chain Δ in a neighbourhood of the vertex v, see Sect. 3.5.

Corollary. *The natural mapping*

$$\bigoplus_{F\in\mathcal{S}} C_*(S^F)\to C_*(S)$$

is an epimorphism.

Proof. It is easy to show, by making use of Theorem 10, that every facet of the configuration S is a sum of the cones (provided it has a vertex) or a sum of angles.

Denote by $I(S)$ the subspace of chains generated by the angles of the configuration. In other words, $I(S)$ is the image of the natural mapping

$$\bigoplus_{\substack{F\in\mathcal{S}\\ d(F)<n}} C_*(S^F)\to C_*(S).$$

The dimension filtration induces a filtration on $I: I_l(S)=I(S)\cap D_l(S)$, $l\geqq 0$. Denote by $CI(S)$ the factor-space $C_*(S)/I(S)$. The dimension filtration $CI_l(S)=(D_l+I)/I$ is defined on $CI(S)$.

Theorem 11. *Let φ be a linear function, and Δ the chain whose dimension does not exceed l, $l\geqq 0$. Then there exists a linear combination of angles $\sum\Gamma_\alpha$, with dimensions not greater than l such that the chain $\Delta-\sum\Gamma_\alpha$ is bounded from above with respect to φ. In other words, the natural mapping $C_l(S_\varphi)\to CI_l(S)$ is an epimorphism.*

Proof. Let a number t_0 be given, such that for any $t>t_0$, the level hyperplane of the level t of φ crosses transversally all the edges of the configuration S. Consider a configuration S_U cut out on the hyperplane $U=\{x\in V\,|\,\varphi(x)=t_0\}$.

Let Γ be a cone of the configuration S_U with vertex at v. Then there exists an angle $\tilde{\Gamma}$ of the configuration S such that $\Gamma=\tilde{\Gamma}\cap U$. The edge of $\tilde{\Gamma}$ is one-dimensional and passes through v. Similarly, let Γ be an angle of the configuration S_U. Then there exists an angle $\tilde{\Gamma}$ of the configuration S whose intersection with U is Γ. The intersection of an edge of the angle $\tilde{\Gamma}$ is an edge of the angle Γ.

Let Δ be the original chain and $\Delta\cap U$ its intersection with U. By the corollary of Theorem 10, $\Delta\cap U$ can be represented as a linear combination of angles and cones of the configuration S_U: $\Delta\cup U=\sum\Gamma_\alpha$. Let $\tilde{\Gamma}_\alpha$ be the angle of the configuration for which $\Gamma_\alpha=\tilde{\Gamma}_\alpha\cap U$. It is easy to see that the chain $\Delta-\sum\tilde{\Gamma}_\alpha$ is bounded from above.

2. Skew cochains. A cochain of the configuration is said to be localized at a given vertex if it is equal to zero on any facet which is not included in the star of the vertex. A cochain is said to be a skew cochain if it is equal to zero on any angle of the configuration.

We will produce a great store of skew local chains.

Let φ be a linear function on V. The cochain $\chi_\varphi=\sum_{\Gamma^*:\varphi(\Gamma)\leq 0}(-1)^{d(\Gamma)}\Gamma^*$ is called the cochain associated to φ. The value of the cochain on an arbitrary chain is equal to the Euler characteristics of the part of the chain which falls into the semispace $\varphi\leqq 0$. Any cochain associated to a linear function in general position is called an Euler cochain of configuration. The set of all Euler cochains is finite.

Theorem 12. *An Euler cochain is equal to zero on any angle.*

Proof. Let χ_φ be an Euler cochain and Δ an angle with an edge F. It suffices to investigate the case when Δ is a facet of dimension n of the configuration S^F.

Let Γ be a facet of the configuration S lying in $\Delta\cap\{\varphi\leqq 0\}$. Let the maximum of the function φ, which is bounded on the closure of the facet Γ be reached at the vertex v. Denote by $\Delta(v)$ the sum of all such facets. Then $\chi(\Delta)=\sum_v\chi(\Delta(v))$. Prove that $\chi(\Delta(v))=0$. If v belongs to the interior of the set Δ, then $\Delta(v)$ consists of all the facets of the star of the vertex v on which the maximum of the function φ is reached at v. These positive-dimensional facets correspond 1–1 to bounded facets of the affine localization $S^{v,\varphi}$. Now the equality $\chi(\Delta(v))=0$ follows from the first part of Theorem 9.

If v belongs to the boundary of the set Δ, then the equality similarly follows from the second part of Theorem 9.

3. Linear combinations of Euler cochains. Let φ be a generally positioned function and $t_1<t_2<\dots<t_N$ its critical values, i.e. its values at the vertices of the configuration S. Consider an Euler cochain $\chi_{\varphi-t}$. It does not change when $t\in[t_j,t_{j+1})$. $\chi_{\varphi-t}\equiv 0$ for $t<t_1$. If v is a vertex such that $\varphi(v)=t$, set $\chi_{\varphi,v}=\chi_{\varphi-t+\varepsilon}-\chi_{\varphi-t-\varepsilon}$ where ε is a small positive number. $\chi_{\varphi,v}$ is the cochain localized at v. More precisely, $\chi_{\varphi,v}=\sum(-1)^{d(\Gamma)}\Gamma^*$ with the summation taken over all the facets of the star of the vertex v such that the maximum of φ is reached at v. That is, all the facets going from v in the direction of decreasing φ are taken into the summation. The cochain $\chi_{\varphi,v}$ will be called the local Euler cochain centred at v.

Theorem 13. *Let u,v be two different vertices of the configuration S, K_u the cone of the configuration S with vertex at u, and $\chi_{\varphi,v}$ the local Euler cochain centred at v. Then $\chi_{\varphi,v}(K_u)=0$.*

Proof. The cone K_u in a neighbourhood of v looks like an angle. Thus, by Theorem 12, $\chi_{\varphi,v}(K_u)=0$.

4. There is a sufficient number of Euler cochains.

Theorem 14. *Let Δ be a non-zero cochain of the configuration S, bounded from above with respect to a function φ which is in general position. Then there exists a linear combination of Euler cochains whose value on Δ is not equal to zero.*

Corollaries. 1. *A non-zero chain which is bounded from above is not a linear combination of angles, that is the natural mapping $C_*(S_\varphi)\to CI(S)$ is an isomorphism.*

2. *Any skew cochain is a linear combination of Euler cochains. In other words, Euler cochains generate a space dual to $CI(S)$.*

Proof. This is proved by induction in the configuration's dimension. If $\dim V = 1$ the theorem is obvious. We prove an inductive step.

Decompose Δ into a sum of cones bounded from above with respect to $\varphi: \Delta = \sum_v K_v$. Choose a vertex v at which the cone does not equal zero. By Theorem 13, it suffices to prove the existence of a linear combination of local Euler cochains centred at v whose value on K_v differs from zero.

If the vertex v enters K_v with the coefficient λ, then $\chi_{-\varphi,v}(K_v) = \lambda$. If $\lambda \neq 0$, then the theorem is proved. Further, we assume that v enters K_v with zero coefficient.

Let $\chi_{\alpha,v}$ be a local Euler cochain. The isomorphism of Lemma 1 transforms it into a cochain ψ on bounded chains of the localization $S^{v,\varphi}$.

Lemma 5. *ψ is an Euler cochain of the configuration $S^{v,\varphi}$ associated to a linear function α–$\alpha(v)$ restricted to the space of affine localization.*

Proof. Lemma is obvious.

Turn back to the cone K_v. It induces a non-zero bounded chain K in $S^{v,\varphi}$. By inductive assumption, for the configuration $S^{v,\varphi}$ there exists a linear combination of its Euler cochains having non-zero value at K. This fact and Lemma 5 imply the theorem.

5. Characterization of the cones of decomposition (3). Let φ be a linear function in general position on S, Δ a chain bounded from above, and $\Delta = \sum_v K_v$ a decomposition into a sum of cones of the configuration S bounded from above with respect to φ.

Theorem 15. 1. *For any local Euler cochain $\chi_{\alpha,v}$ centred at v*

$$\chi_{\alpha,v}(\Delta) = \chi_{\alpha,v}(K_v).$$

2. *Let K be a cone of the configuration S bounded from above with respect to φ. Suppose that, for any local Euler cochain $\chi_{\alpha,v}$ centred at v, $\chi_{\alpha,v}(\Delta) = \chi_{\alpha,v}(K)$. Then $K = K_v$.*

The first part of the theorem is a corollary of Theorem 13, the second is a corollary of Theorem 14.

6. Combinatorial connection (Corollaries of Theorem 14). Let φ_1, φ_2 be linear functions in general position. The following two statements result from Corollary 1.

Corollary 3. *$C_*(S_{\varphi_1})$ and $C_*(S_{\varphi_2})$ are canonically isomorphic to each other.*

Let S be a central regular configuration with vertex v and let φ_1, φ_2 be linear functions in general position.

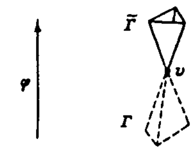

Fig. 3.

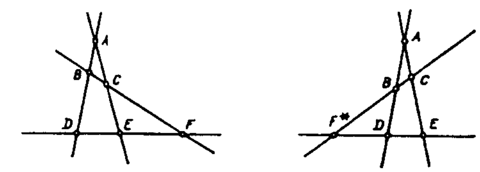

Fig. 4.

Corollary 4. *$C_*^{\mathrm{comp}}(S^{v,\varphi_1})$, $C_*^{\mathrm{comp}}(S^{v,\varphi_2})$ are canonically isomorphic to each other.*

Example 1. Let S be a regular central configuration and φ a function in general position. We point out an isomorphism $\pi: C_*(S_\varphi) \to C_*(S_{-\varphi})$ consisting in the fact that the cone going down with respect to φ is transformed into a cone going upwards, by adding angles.

Theorem 16 (*cf. Theorem 7 of* [9]). *Let $\Gamma \in C_*(S_\varphi)$ be a k-dimensional facet. Then $\pi(\Gamma) = (-1)^k \tilde{\Gamma}$, where $\tilde{\Gamma}$ is the closure of the reflection in the vertex v of the facet Γ (see Fig. 3).*

Example 2. Consider a configuration in $\mathbf{R}^3$ consisting of four planes passing through zero, and being in a general position. If φ is a function in general position, then $S^{v,\varphi}$ is a configuration on a plane consisting of four generally positioned lines. For suitable φ_1, φ_2, the configurations are given by Fig. 4. We point out the isomorphism of bounded chains: $F \mapsto -F^*$, $CF \mapsto -CF^* - C$, $EF \mapsto -EF^* - E$, $CEF \mapsto -CEF^* - CE$, where CF, CF^*, EF, EF^* are open intervals and CEF, CEF^* are the open triangles. The rest of the facets are transformed into the facets with the same names: $A \mapsto A$, etc.

4. Dimensional and degree filtrations

1. Decomposition into simplexes.

Theorem 17. *If S is a loose configuration then there exists a basis of $C_*^{\mathrm{comp}}(S)$ consisting of open simplexes of different dimensions. The basis has the following property: if there is a chain whose dimension does not exceed l, the dimensions of simplexes of its decomposition in terms of the basis elements are not greater than l.*

Proof is by induction on the dimension of a space. When $\dim V = 0$, the point is the only facet. Let $\dim V = n > 0$. Order the hyperplanes of the configuration. Let A be the hyperplane with minimal number. By inductive assumption, there is a simplicial basis of $C_*^{\text{comp}}(S_A)$. Complete it to a basis of $C_*^{\text{comp}}(S)$. Let v be a vertex outside A. Include in the basis the zero-dimensional chain v. Let φ be a linear function such that $\varphi(A) = 0$, $\varphi(v) > 0$. Consider an affine localization $S^{v,\varphi}$. Its hyperplanes form an ordered set. Using induction, we choose a simplicial basis of $C_*^{\text{comp}}(S^{v,\varphi})$. We make a simplex $K(\Gamma)$ correspond to each of its elements; this simplex is a cone with vertex v, guide Γ, base lying on A, with v and the bottom base not being included in $K(\Gamma)$. It is easy to see that the basis of $C_*^{\text{comp}}(S_A)$, together with the simplexes constructed for all vertices outside A, composes a basis of $C_*^{\text{comp}}(S)$ which has the property formulated in the theorem (see the proof of Theorem 10).

Theorem 18. *For any configuration of hyperplanes, S, in $C_*(S)$ there exists a basis consisting of elementary chains. The basis has the property that a chain whose dimension does not exceed l can be decomposed into a sum of elementary chains with dimensions not greater than l.*

The proof is by induction on the dimension of the configuration and follows easily from Theorem 17 (cf. Theorems 10, 11, and Lemma 1).

2. Properties of dimensional and weight filtrations in chains.

1. $W_n = C_*(S)$.
2. D_l is a linear hull of elementary chains whose dimensions do not exceed l, for $l \geqq 0$.
3. If $S_1 \subset S_2$ and $i: C_*(S_1) \to C_*(S_2)$ is the natural inclusion, then $i(W_k(S_1)) \subset W_k(S_2)$, $i(D_l(S_1)) \subset D_l(S_2)$ for $k, l \geqq 0$.
4. $W_k(S)$ coincides with the image under natural mapping:

$$\bigoplus_{\substack{F \in \mathscr{S} \\ r(F) \leq k}} C_*(S^F) \to C_*(S), \tag{4}$$

where S^F is the localization of the configuration at the edge F and $r(F)$ is the codimension of the edge.

5. If $U \subset V$ is a subspace and $j_U: C_*(S) \to C_*(S_U)$ is the natural epimorphism, then $j_U(W_k(S)) = W_k(S_U)$.

The third property is a corollary of the definition of filtrations. The first and second properties are corollaries of Theorem 18.

We prove Property 4. Obviously, $W_k(S)$ is contained in the image of mapping (4). By Property 1, $C_*(S^F) = W_{r(F)}(S^F)$. Thus, $W_k(S)$ coincides with the image of the mapping (4).

Property 4 can serve as a definition of a degree filtration.

We prove Property 5. Each elementary chain of degree k of $C_*(S_U)$ is an image of an elementary chain of degree k of $C_*(S)$. Thus $j_U(W_k(S)) \supset W_k(S_U)$. An image of an elementary chain of degree k is a facet of a configuration on U consisting of no more than k hyperplanes. Property 4 implies that the image belongs to $W_k(S_U)$.

6. If $\Delta \in W_k \cap D_l$ then Δ can be represented as a linear combination of elementary chains whose degrees do not exceed k and dimensions do not exceed l.

The proof is by induction with respect to the dimension of the configuration. For $n = 0$, the theorem is valid for any k, l. Let $n > 0$. Let φ be a linear function in general position with respect to the configuration S. Let t_0 be a number such that, for any $t > t_0$, the hyperplane of constant level t of φ crosses all the edges transversally. Consider the configuration S_U cut out on the hyperplane $U = \{x \in V \mid \varphi(x) = t_0\}$. Then the dimension of the chain $\Delta \cap U$ is $\leqq l - 1$.

Let $k < n$. Then $\Delta \cap U \in W_k(S_U) \cap D_{l-1}(S_U)$. By the inductive assumptions, $\Delta \cap U = \sum a_m \Delta_m$, where $a_m \in \mathbf{Z}$ and $\{\Delta_m\} \subset W_k(S_U) \cap D_{l-1}(S_U)$ are elementary chains. For every elementary chain Δ_m there exists an elementary chain $\tilde{\Delta}_m \in W_k(S) \cap D_l(S)$ such that $\tilde{\Delta}_m \cap U = \Delta_m$. The chain $\tilde{\Delta} = \Delta - \sum a_m \tilde{\Delta}_m$ has no intersection with U. By the choice of U, the chain $\tilde{\Delta}$ is bounded from above. By Theorems 12 and 14, $\tilde{\Delta} = 0$. Property 6 is proved. For $k = n$, the property follows from Theorem 18.

7. $W_k \cap D_l = 0$ for $k + l < n$.
8. Let $U \subset V$ be a d-dimensional subspace in general position with respect to the configuration S. Then the natural homomorphism $j_U: C_*(S) \to C_*(S_U)$ reduces the dimension of any chain by $n - d$ and, for any $0 \leqq k, l \leqq d$, sets an isomorphism

$$W_k(S) \cap D_{l+n-d}(S) \to W_k(S_U) \cap D_l(S_U).$$

Proof. It suffices to consider the case of U being a hyperplane. The fact of the generality of the hyperplane U obviously implies $D_{l+1}(S) \to D_l(S_U)$, $W_k(S) \to W_k(S_U)$. We prove the absence of the kernel of $j_U|_{W_{n-1}}$. Indeed, if $\Delta \in W_{n-1}(S)$ belongs to the kernel of the homomorphism j_U, then $\Delta = \Delta_+ + \Delta_-$, with $\Delta_\pm$ lying at different sides of the hyperplane U. It follows from Theorems 10–13 and 14 that $\Delta_+ = \Delta_- = 0$. Thus, j_U sets an inclusion $W_k(S) \cap D_{l+1}(S) \subsetneq W_k(S_U) \cap D_l(S_U)$, for $k \leqq n-1$. If $\Delta \in W_k(S_U) \cap D_l(S_U)$ then, by Property 6, there exists $\tilde{\Delta} \in W_k(S) \cap D_{l+1}(S)$ such that $j_U(\tilde{\Delta}) = \Delta$. Property 8 is proved.

9. Under the conditions of Property 8, j_U sets an isomorphism $W_k gr_{l+n-d} D(S) \to W_k gr_l D(S_U)$ for any $0 \leqq k, l \leqq d$.

Property 9 is a corollary of Property 8, since $W_k gr_m D = W_k \cap D_m / D_{m-1} \cap W_k$.

10. For any $k \geqq 0$, the natural mapping

$$\bigoplus_{\substack{F \in \mathscr{S} \\ r(F) = k}} W_k(S^F)/W_{k-1}(S^F) \to W_k(S)/W_{k-1}(S)$$

is an isomorphism.

Proof. Let φ be a linear function in general position. Then, by Theorems 10–14, $C_*(S_\varphi) = \bigoplus_{\substack{F\in\mathcal{F}\\ r(F)=n}} C_*(S_\varphi^F)$. According to Corollary 1 of Theorem 14, this equality implies Property 10, for $k=n$. The case of arbitrary k can be reduced to the case just considered, with the help of Property 8. The same argument proves Property 11.

11. For any $k, l \geqq 0$, set $D_l gr_k W = D_l \cap W_k / D_l \cap W_{k-1}$. Then the natural mapping

$$\bigoplus_{\substack{F\in\mathcal{F}\\ r(F)=k}} D_l gr_k W(S^F) \to D_l gr_k W(S)$$

is an isomorphism.

12. A chain from $W_k(S)$ is uniquely determined by its general k-dimensional cross-section. More precisely, if $U \subset V$ is a subspace in general position with respect to S, $\dim U = k$, and $c \in W_k(S)$ is the chain with $c \cap U = 0$, then $c = 0$.

Property 12 follows from Property 8.

13. Let $U_1, U_2 \subset V$ be subspaces in general positions with respect to the configuration S, $\dim U_1 = \dim U_2$. Then $C_*(S_{U_1})$ is canonically isomorphic to $C_*(S_{U_2})$. The isomorphism is defined by Property 8.
The isomorphism will be called the combinatorial connection. In [9] the combinatorial connection is defined for a similar situation.

3. The ring $P(S, \mathbf{Z})$ defined in the introduction. Properties 1, 9 and 10 yield Theorems 1–3.

4. The filtration dual to degree filtration. Define a degree filtration in $C^*(S)$ by making use of the condition $W^k = \operatorname{Ann} W_{k-1}$. We have

$$O \subset W^n \subset W^{n-1} \subset \cdots \subset W^0 = C^*(S).$$

We give another construction of the filtration. Let $X(S) \subset C^*(S)$ be a linear hull of Euler cochains. Let $U_1, \ldots, U_{n-1}$, $U_n = V$ be affine subspaces whose dimensions are, respectively, $1, \ldots, n-1, n$. Suppose that the subspaces are in general positions with respect to the configuration S. Let $j_k: C^*(S_{U_k}) \to C^*(S)$ be the natural monomorphism.

Theorem 19. *For any* $k \geqq 0$,

$$W^k(S) = X(S) + j_{n-1}(X(S_{U_{n-1}})) + \cdots + j_k(X(S_{U_k})).$$

Theorem 19 follows from Corollary 2 of Theorem 14 and from Property 8 of Sect. 4.2.

5. Flag cochains.

Lemma 7. *An arbitrary flag cochain of degree n is a linear combination of Euler cochains.*

The proof is by induction with respect to the dimension of the configuration. In carrying out this proof, one has to descend to an affine localization of a zero-dimensional edge of a flag and to use Lemma 5.

Lemma 8. *Flag cochains of degree n generate the space which is dual to* $C_*(S)/(C_{n-1}(S) + W_{n-1}(C_*(S)))$.

The proof is by induction with respect to the dimension of the space. Let φ be a function in general position. By Theorems 10–14, it suffices to prove that the flag cochains of degree n generate $C^n(S_\varphi)$. Moreover, it suffices to investigate the case of S being a regular central configuration. We denote its vertex by v. Pass to the affine localization $S^{v,\varphi}$. Then $C_n(S_\varphi) \simeq C_{n-1}^{\mathrm{comp}}(S^{v,\varphi})$. Flag cochains of degree n on S transform into flag cochains of degree $n-1$ on $S^{v,\varphi}$. By the inductive assumption, for a non-zero chain from $C_{n-1}^{\mathrm{comp}}(S^{v,\varphi})$, there exists a flag cochain of degree $n-1$ which does not vanish on the former. The lemma is proved.

Proof of Theorem 4. Lemma 8 coincides with the statement of Theorem 4 for $k=n$. The case of arbitrary k follows from Lemma 8 and from Property 9 of Sect. 4.2.

§5. Comparison to cohomologies and relations

1. Comparison to cohomologies. Let S be a configuration of hyperplanes in a real n-dimensional affine space V.
Every oriented n-dimensional facet $\Delta \in C_n(S)$ determines a cohomology class $[\Delta] \in H^n(M_C, \mathbf{Z})$ which is equal to the index of intersection with a non-compact cycle Δ (we suppose M_C to be complex-oriented).
Fix an orientation on V and thus on every n-dimensional facet. Define a linear mapping $\pi: C_n(S) \to H^n(M_C, \mathbf{Z})$ by establishing correspondence between linear combinations of facets, $\sum a_\alpha \Delta_\alpha$, and classes $\sum a_\alpha [\Delta_\alpha]$.
In order to describe the kernel of the mapping π, we consider the natural isomorphism $i: C_n(S) \to gr_n D$.

Theorem 20. $\ker \pi = i^{-1}(W_{n-1} gr_n D)$.

Proof. Relate a class of homologies in $H_n(M_C, \mathbf{Z})$ of the torus defined below to each flag $F = \{F_0 \subset F_1 \subset \cdots \subset F_n\}$, namely, let $\varepsilon_1, \ldots, \varepsilon_n > 0$. Consider a torus $T(\varepsilon) = \{z \in \mathbf{C}^n \mid |z_j| = \varepsilon_j\}$ in $\mathbf{C}^n$. Fix its orientation. Affinely map $\mathbf{R}^n$ with coordinates $z_1, \ldots, z_n$ onto V so that the standard flag $\{z_1 = \cdots = z_n = 0\} \subset \{z_2 = \cdots = z_n = 0\} \subset \cdots \subset \{Z_n = 0\} \subset \mathbf{R}^n$ be mapped onto the flag F. Consider a complexification $\mathbf{C}^n \to V_C$ of the mapping. Torus $T(\varepsilon)$ for $0 < \varepsilon_1 \ll \varepsilon_2 \ll \cdots \ll \varepsilon_n \ll 1$ is mapped into M_C and defines a homology class not depending on ε. Denote it by T_F.

Lemma 9. *Consider a cochain on $C_n(S)$ which is equal to the index of intersection with T_F. Then the cochain is equal to the flag cochain ψ_F up to a multiplication by ± 1.*

The proof is obvious.

The linear hull of the classes T_F obtained for different flags F of degree n will be denoted by L. By Lemmas 8 and 9, $\dim_Z L = \dim_Z C_n(S)/i^{-1}(W_{n-1} gr_n D)$. By Corollary 2 of Theorem 3, $\dim_Z H_n(M_C, Z) = \dim_Z C_n(S)/i^{-1}(W_{n-1} gr_n D)$. As a corollary, we obtain Theorem 20 as well as the statement $L = H_n(M_C, Z)$.

Example. Consider a configuration of coordinate hyperplanes $A_j = \{z_j = 0\}$, $j = 1, \ldots, n$, in $\mathbf{R}^n$. By making use of the form $dz_1 \wedge \cdots \wedge dz_n$, set an orientation in $\mathbf{R}^n$. M consists of 2^n octants. The space $C_n(S)/i^{-1}(W_{n-1} gr_n D) = gr_n W gr_n D$ is one-dimensional and is generated by a positive octant $\{z_1 > 0, \ldots, z_n > 0\}$. Under the mapping π, the positive octant is transformed into a cohomology class of the form $(-1)^{n(n-1)/2} w_1 \wedge \cdots \wedge w_n$ where $w_j = dz_j / 2\pi\sqrt{-1} z_j$.

Notice a useful corollary of Theorem 20. Let φ be a linear function in general position. Then the mapping π restricted on $C_n(S_\varphi)$ gives an isomorphism of n-dimensional chains of the configuration and the space $H^n(M_C, Z)$ which are bounded from above.

Proof of Theorem 8. For $k = n$, Theorem 8 is a corollary of Theorem 20, of the previous example and of Properties 10 and 11 of Sect. 4.2. The case $k < n$ follows from the case $k = n$, by making use of Property 9 of Sect. 4.2.

2. Relations.

Proof of Theorem 5. The statement is reduced to the case of the circuit $f_1, \ldots, f_{k-1}, f_k = -(f_1 + \cdots + f_{k-1})$ for which the relations $x_1 \cdots x_k = 0, (x_1 - 1) \cdots (x_k - 1) = 0$ hold.

Proof of Theorem 6. By Theorem 5, it suffices to show that $\dim \mathbf{Z}[x]/\vartheta = \dim H^*(M_C)$. The relations $x_i^2 = x_i$ annihilate the monomials which include at least one of variables raised to a power greater than 1. One has to prove that the work of the relation concerning the rest of the monomials can be deduced from (2). Under the isomorphism $P^s/P^{s-1} \to H^s(M_C)$, relation (2) is transformed into a homogeneous relation of degree $s - 1$:

$$w_{i_2} \wedge \cdots \wedge w_{i_s} - w_{i_1} \wedge w_{i_3} \wedge \cdots \wedge w_{i_s} + \cdots + (-1)^{s-1} w_{i_1} \wedge \cdots \wedge w_{i_{s-1}}. \tag{5}$$

According to [3], the external algebra spanned on $\{w_{i \in I}\}$ and factorized by making use of relations (5) for all the circuits is isomorphic to $H^*(M_C)$. Thus $\mathbf{Z}[x]/\vartheta$ has the needed dimension.

Theorem 7 is a corollary of Theorems 6 and 8 and of the Theorem based on [7, 11, 12]

§6. Appendix. Decomposition into elementary fractions

There exists an elementary analytical analogue of Theorem 18 on the decomposition into simplicial cones that is a generalization of the theorem on the decomposition of rational functions into elementary fractions.

1. Decomposition. Consider a rational function $R = P/Q$ on an n-dimensional linear space V, where P and Q are polynomials. Suppose that Q can be decomposed into a product of polynomials of degree 1: $Q = \prod_{i \in I} f_i^{k_i}$. Let $z_1, \ldots, z_n$ be the linear coordinates in V.

Theorem 21. *The function R can be represented in the form:*

$$R = \sum_{t, \alpha} (t_1)^{\alpha_1} \cdots (t_n)^{\alpha_n} A_{t,\alpha}, \tag{6}$$

where either $\alpha_j \geqq 0$ and $t_j = z_j$, or $\alpha_j < 0$ and t_j is a polynomial of degree 1 in $z_j, z_{j+1}, \ldots, z_n$, with 1 as the coefficient of z_j; the $A_{t,\alpha}$ are numbers. The representation is unique.

Proof. Restrict R to each of the lines parallel to the z_1-axis and decompose it into elementary fractions in z_1. We thus obtain a representation

$$R = \sum_{i \in I} \sum_{k_i \leq p < 0} f_i^p B_{p,i} + \sum_l z_1^l C_l,$$

where $B_{p,i}$ and C_l are rational functions in $z_2, \ldots, z_n$, whose denominators are products of polynomials whose degrees are equal to 1. This representation is unique. The functions $B_{p,i}$ and C_l can be treated similarly.

Example. $1/(z_1 - z_3)\,(z_1 - z_2)\,(z_1 - z_2 - z_3) = 1/(z_1 - z_3)\,(z_3 - z_2)\,(-z_2) + 1/(z_1 - z_2)\,(z_2 - z_3)\,(-z_3) + 1/(z_1 - z_2 - z_3) z_2 z_3 = 1/(z_1 - z_3)\,(z_3 - z_2)\,(-z_3) + 1/(z_1 - z_3)\,(-z_2) z_3 + 1/(z_1 - z_2)\,(z_2 - z_3)\,(-z_3) + 1/(z_1 - z_2 - z_3) z_2 z_3$.

Theorem 21 is close to Theorem 5.2 of [7].

Let all the linear functions $\{f_{i \in I}\}$ be homogeneous and the set $t_1, \ldots, t_n$ enter (6) with negative coefficients $\alpha_1, \ldots, \alpha_n$. We point out the relation between $t_1, \ldots, t_n$ and the covectors $\{f_{i \in I}\}$. t_1 is proportional to one of the $\{f_{i \in I}\}$. We project the rest of the covectors of $\{f_{i \in I}\}$ along t_1, onto a hyperplane in V^* which is orthogonal to the vector $(1, 0, \ldots, 0)$. Thus we obtain a set of covectors $\{g\}$. Then t_2 is proportional to one of those covectors. Project along t_2 the rest of the covectors of $\{g\}$, onto an $(n-2)$-dimensional plane orthogonal to the plane in V, spanned on $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$. We get a set of covectors $\{h\}$. Then t_3 is proportional to one of those, etc.

If the set of covectors $\{f_{i \in I}\}$ is closed with respect to the projection operations just described, then, for $\alpha_j < 0$, the polynomial t_j of (6) is proportional to one of $\{f_{i \in I}\}$. The following are examples of such families:

1. Type A. Q is the product of degrees of polynomials $z_j = z_k$, $j < k$.
2. Type B. Q is the product of degrees of polynomials $z_j, z_j \pm z_k, j < k$.

2. Application. Consider a linear mapping $h: \mathbf{R}^N \to \mathbf{R}^n$, under which the j-th basis vector is transformed into a vector h_j. Suppose that all the h_j lie in the half-space $x_1 > 0$, where $x_1, \ldots, x_n$ are the coordinates in $\mathbf{R}^n$. Define the function U on $\mathbf{R}^n$. $U(x)$ is the $(N-n)$-dimensional volume of the intersection of the fibre lying above x and of the positive octant in $\mathbf{R}^N$, where, as a form of volume, the ratio of, respectively, volume forms on $\mathbf{R}^N$ and $\mathbf{R}^n$ is taken. For details of the function see [7].

Theorem 22. *For the set* $\{h_{j \leq N}\}$ *in general position,*

$$U(x) = \sum_{\substack{1 \leq s_1, \ldots, s_{n-1} \leq N \\ s_i \neq s_j}} \frac{(h_{s_1}, \ldots, h_{s_{n-1}}, x)^{N-n} \chi_{s_1, \ldots, s_{n-1}}(x)}{(N-n)! \prod\limits_{\substack{s_n \notin \{s_1, \ldots, s_{n-1}\} \\ 1 \leq s_n \leq N}} (h_{s_1}, \ldots, h_{s_n})}, \tag{7}$$

where $(v_1, \ldots, v_n)$ *is the determinant whose columns are* $v_1, \ldots, v_n$ *and* $\chi_{s_1, \ldots, s_{n-1}}$ *is a characteristic function of the simplicial cone generated by the vectors* $h_{s_1}, h_{s_1, s_2}, \ldots, h_{s_1, \ldots, s_n}$, *where* $h_{s_1, \ldots, s_p}$, *for* $p > 1$, *is the projection of the vector* h_p *along* $\{h_1, \ldots, h_{p-1}\}$ *on the coordinate plane* $\{(0, \ldots, 0, 1_p, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\}$, *its sign being taken so that its p-th coordinate is positive.*

In particular, for $n = 2$, χ_j *is the characteristic function of the cone generated by the vectors* h_j, $(0, 1)$.

For arbitrary n and $N = n$, (7) assumes the form:

$$U(x) = \sum_{s \in S_n} \chi_{s_1, \ldots, s_{n-1}}(x) / (h_{s_1}, \ldots, h_{s_n}).$$

In this case, $U(x)$ is piecewise constant and is proportional to the characteristic function of the cone generated by the vectors $h_1, \ldots, h_n$.

Remark. If we drop χ from (7), the sum will be identically equal to zero. For example, for $n = 2$, $f_j = (1, a_j)$, we have

$$\sum_{i=1}^{N} \frac{(x_2 - a_i x_1)^{N-2}}{(a_1 - a_i) \cdots (a_{i-1} - a_i)(a_{i+1} - a_i) \cdots (a_N - a_i)} \equiv 0.$$

To prove the theorem, it suffices to consider the Laplace transformation of the function $U(x)$ (see [7]), to decompose the rational function thus obtained into elementary fractions and to perform the inverse Laplace transformation of the terms of the sum.

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