ALGEBRAIC AND COMBINATORIAL ASPECTS OF THE GENERAL THEORY OF HYPERGEOMETRIC FUNCTIONS

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INTRODUCTION

1. This article is a part of a program of study of the general hypergeometric functions introduced in [6]. Basically, we set forth here the algebraic and combinatorial aspects of the theory; other results on general hypergeometric functions are given in [7, 9]. We also cite some applications, including some applications to the study of the continuous analogue of the partition functions of Kostant [21] introduced in [4].

General hypergeometric functions are essentially functions on a Grassmannian. A Grassmannian is a manifold of k-dimensional subspaces in an n-dimensional vector space V over the field **R** or **C**. At the present time, these manifolds play an important role in a large number of problems which are closely related to each other (see [8, 10], for example; we note also that the well-known twistor program of Penrouz [12] is based on a study of Gassmannians of two-dimensional planes in C^4). The Grassmannian of k-dimensional subspaces in V will be denoted by $G_k(V)$.

Although the Grassmannian $G_k(V)$ is a homogeneous space with respect to the group GL(V), we will first be interested in the action of a maximal torus H in the group GL(V) on it. The choice of a maximal torus fixes a basis $e_1,\ldots,e_n$ in V, i.e., it allows us to identify V with the coordinate space $\mathbf{R}^n$ or $\mathbf{C}^n$; moreover, in the complex case, this torus is the group of all diagonal matrices, and, in the real case, the group of diagonal matrices with positive elements. The orbits of H on $G_k(\mathbf{C}^n)$ are toroidal manifolds; in the real case these orbits reduce to interesting manifolds, so-called Grassman simplexes [19] (see also [13, 20]). In this article we will concern ourselves only with real Grassmannians, which makes it possible for us to give a well-rounded treatment of the combinatorial and geometric aspects. However, a complete understanding of the situation is impossible without entering the complex domain; the complex theory will be treated in a subsequent article.

2. Definition of a General Hypergeometric Function. The existence of the torus H in $GL(\mathbf{R}^n)$ allows us to distinguish the class of homogeneous functions $\pi(x)$ in $\mathbf{R}^n$. For each set $\alpha = (\alpha_1,\ldots,\alpha_n)$ of complex numbers with sum $n-k$, we consider the homogeneous function $\pi_\alpha(x)=\prod_{1\leqslant i\leqslant n}(x_i)_+^{\alpha_i-1}$; here $x=(x_1,\ldots,x_n)\in\mathbf{R}^n$, and $(x_i)_+^{\alpha_i-1}=x_i^{\alpha_i-1}$ for $x_i>0$ and $(x_i)_+^{\alpha_i-1}=0$ for $x_i\leqslant 0$.

Now let $\zeta\in G_k(\mathbf{R}^n)$ be a k-dimensional subspace in $\mathbf{R}^n$, and let $S(\zeta)$ be a "sphere" in ζ (by a "sphere" in ζ we understand a factor space $\zeta\setminus 0$ with respect to the action of the multiplicative group $\mathbf{R}_+\setminus 0$, or an arbitrary smooth surface in ζ which intersects every ray from 0 in one point). Let ω be a nonzero exterior k-form on ζ. The general hypergeometric function $\Phi(\alpha;\zeta,\omega)$ is defined by

$$\Phi(\alpha;\zeta,\omega)=\int_{S(\zeta)}\pi_\alpha(x)\tilde{\omega}(x). \tag{1}$$

Here $\tilde{\omega}(x)$ is a $(k-1)$-form on $\zeta\setminus 0$ induced by ω in a natural way: if $t_1,\ldots,t_k$ are coordinates in ζ and $\omega=dt_1\wedge\ldots\wedge dt_k$, then $\tilde{\omega}=\sum_i(-1)^{i-1}t_i dt_1\wedge\ldots dt_{i-1}\wedge dt_{i+1}\wedge\ldots dt_k$. Since $\sum_i\alpha_i=n-k$, the form $\pi_\alpha(x)\tilde{\omega}(x)$ has degree of homogeneity 0, which means it decreases on the "sphere" $S(\zeta)$; in addition, $S(\zeta)$ is furnished with a natural orientation which is induced in a natural way by ω. An expresssion for $\Phi(\alpha;\zeta,\omega)$ in the local coordinates on the Grassmannian is given in Sec. 1.

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Since the integral in (1) diverges for some α and ζ, the definition of $\Phi(\alpha; \zeta, \omega)$ requires some revision for these values; the precise definition, using analytic continuation with respect to α and the method of partitionings, is given in [6]. We note that, along with $\Phi(\alpha; \zeta, \omega)$, the function $\Phi(\alpha, \varepsilon; \zeta, \omega)$ is also defined for all sets $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, where $\varepsilon_i = \pm 1$; we have the relation $\Phi(\alpha, \varepsilon; \zeta, \omega) = \Phi(\alpha; \varepsilon\zeta, \varepsilon\omega)$, where the pair $(\varepsilon\zeta, \varepsilon\omega)$ is obtained by acting on (ζ, ω) with the diagonal matrix $\varepsilon = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n)$. A holonomic system of differential equations which all of the functions $\Phi(\alpha, \varepsilon; \zeta, \omega)$ satisfy is constructed in [7].

Every function $\Phi(\alpha, \varepsilon; \zeta, \omega)$ satisfies the intrinsic homogeneity condition with respect to the action of the group H: for $\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in H$ we have that $\Phi(\alpha, \varepsilon; \lambda\zeta, \lambda\omega) = \pi_\alpha(\lambda)\Phi(\alpha, \varepsilon; \zeta, \omega)$. Thus if the value of this function is known at any point ζ whatever of the Grassmannian, then it is known on the entire orbit of ζ under the action of H, i.e., it is actually a function on the space of orbits $G_k(\mathbf{R}^n)/H$.

3. Decomposition of a Grassmannian into Strata. In view of the fact that we have identified a maximal torus of H, the coordinate subspaces $\mathbf{R}^{i_1 \ldots i_m} = \mathbf{R}^J$ play an important role in $\mathbf{R}^n$. We say that two points ζ and ζ' of the Grassmannian $G_k(\mathbf{R}^n)$ are equivalent if $\dim \times (\zeta \cap \mathbf{R}^J) = \dim(\zeta' \cap \mathbf{R}^J)$ for all $\mathbf{R}^J$. The equivalence classes are called strata in $G_k(\mathbf{R}^n)$. There exists a unique open stratum called the general stratum; it consists of subspaces $\zeta \in G_k(\mathbf{R}^n)$, which are in general position with respect to all coordinate subspaces.

It is possible to define strata in many ways: as the intersection of Schubert cells related to various orderings of the basis vectors in $\mathbf{R}^n$; by means of Plücker coordinates; in terms of mappings of moments [19, 13, 20]; and from the point of view of combinatorial geometry (see below). The relationships among these definitions are studied in an article of Segranov and Gel'fand, and they are carried over to "Grassmannians" and flag manifolds for all semisimple groups.

As shown in [6], the restriction of every function $\Phi(\alpha, \varepsilon; \zeta, \omega)$ to any stratum in $G_k(\mathbf{R}^n)$ is real analytic. In addition, on a general stratum the number of linearly independent ones among them is $\leqslant \binom{n-2}{k-1}$ (see [7]); the homological interpretation of this number is given in the present article. By considering the restrictions of general hypergeometric functions to various strata, it is possible to obtain many classical hypergeometric functions of one and several variables (see [6]; this question is treated in greater detail in [9]). Thus the Gauss hypergeometric function $F(a, b; c; z)$ turns out to be related to the general stratum in $G_2(\mathbf{R}^4)$ (see [6] and Sec. 2 of the present article).

4. In this and subsequent articles we study the analytic behavior of the restrictions of the general hypergeometric function to various strata in $G_k(\mathbf{R}^n)$ [henceforth, we will speak only of the function $\Phi(\alpha; \zeta, \omega)$, but everything we say carries over with obvious modifications to all functions $\Phi(\alpha, \varepsilon; \zeta, \omega)$]. Since we are interested here in the algebraic and combinatorial aspects of the theory, we consider only special integer sets of the indices α called polynomial sets (see Sec. 4). In this case $\Phi(\alpha; \delta, \omega)$ is actually piecewise rational. The case of general indices will be treated in a subsequent article.

The restriction of $\Phi(\alpha; \zeta, \omega)$ to a stratum Γ in $G_k(\mathbf{R}^n)$ has singularities on the boundaries of Γ; therefore it behaves differently on different connected components of Γ. The connected components of Γ are called cells. The description of cells in the general case is very difficult and can even be an unsolvable problem. It turns out, however, that in the study of $\Phi(\alpha; \zeta, \omega)$, not individual cells, but certain unions of them, which we call large cells, are essential. Large cells on the stratum Γ are parametrized by the $(n-k+1)$-dimensional coordinate subspaces $\mathbf{R}^J$ in $\mathbf{R}^n$ in general position with respect to Γ, i.e., such that $\dim(\zeta \cap \mathbf{R}^J) = 1$ for $\zeta \in \Gamma$; the corresponding large cell $\Gamma(J)$ consists of subspaces $\zeta \in \Gamma$ such that the direction vector of the line $\zeta \cap \mathbf{R}^J$ has positive coordinates.

As we shall see in Sec. 4, for polynomial α the function $\Phi(\alpha; \zeta, \omega)$ admits an expansion of the form

$$\Phi(\alpha; \zeta, \omega) = \sum_J \Phi_J(\alpha; \zeta, \omega), \qquad (2)$$

on Γ, where every function $\Phi_J(\alpha; \zeta, \omega)$ is rational in the large cell $\Gamma(J)$ and equal to 0 outside of $\Gamma(J)$. However this expansion is not unique. A central result of this article consists in the choice of a "basis in the space of large cells," i.e., a system B of large cells such that the expansion of the form (2), where the sum extends over all $J \in B$, exists and is unique; we call such a system B fundamental.

For the case of a general stratum Γ as B, it is possible to take the system of all large cells Γ(J) such that $\mathbf{R}^J$ contains a certain fixed two-dimensional coordinate subspace in $\mathbf{R}^n$. Thus the number of functions $\Phi_J(\alpha; \zeta, \omega)$ in the expansion (2) is $\binom{n-2}{k-1}$ in this case.

An expansion of the form (2) also has meaning for general indices α. Every function $\Phi_J(\alpha; \zeta, \omega)$ is equal to 0 outside of Γ(J) as before, and "has unique analytic behavior" in Γ(J); this means that its analytic continuations into the complex domain from different cells in Γ(J) are consolidated into a single branch of an analytic function. If the stratum Γ is general, we therefore get $\binom{n-2}{k-1}$ linearly independent solutions of the holonomic system of equations in [7]; taking into account an estimate in [7], we get that these functions form a basis in the space of solutions. Explicit expressions for the Φ_J can be given in terms of integral representations in the complex domain; this will be done in a subsequent article.

A precise statement of the result concerning the expansion (2) is given in Sec. 4 (Theorem 4.2). Roughly speaking, this result is as follows: we construct a group of homological origin with respect to the stratum Γ, and we define a class of special bases in it; to each basis in this class there corresponds an expansion of $\Phi(\alpha; \zeta, \omega)$ on Γ in a sum (2) whose terms are parametrized by elements of this basis.

The above homology group can be defined in several different ways topologically (see [2, 15]), algebraically, and geometrically (see Sec. 3). It is remarkable that, making use of results of Orlik and Solomon [22], it is possible to give a purely combinatorial definition in terms of so-called combinatorial geometry (or the theory of matroids). One of the conclusions of the present article is the fact that this theory, which has been developed in the last 30 years by Whitney and by Birkhoff and MacLane and which has received a new impetus thanks to the work of Rota and his school (see [1, 17, 23]), is a natural combinatorial basis for the theory of general hypergeometric functions.

5. Theorem 4.2 is closely related to a result concerning the expansion of rational functions which is of interest in itself (Theorem 5.2); this result gives a multidimensional generalization of the expansion of a rational function in simple fractions. Let ξ be an l-dimensional vector space (over **R** or **C**) in which there is given a finite family $\mathcal{F} = (f_i)_{i \in I}$ of nonzero linear forms. For each set $\alpha = (\alpha_i)_{i \in I}$ of nonnegative integers, we let F_α denote a rational function $(\prod_{i \in I} f_i^{\alpha_i})^{-1}$ on ξ. The problem solved in Theorem 5.2 consists in the construction of (linear) basis in the family of all functions F_α. The rank of a subset $J \subset I$ is defined to be the rank of the family of linear forms $(f_j)_{j \in J}$. We will assume that I has rank l, and we will consider functions F_α only for sets $\alpha = (\alpha_i)_{i \in I}$, such that $\operatorname{supp} \alpha = \{i \in I\colon \alpha_i \neq 0\}$ has rank l (the general case can easily be reduced to this one).

We denote the vector space $H = H(\xi, \mathcal{F})$ generated by the functions $F_J = \prod_{j \in J} f_j^{-1}$, where J runs through all l-subsets of rank I in l.

We call a system B consisting of l-subsets in I of rank l *fundamental* if the functions F_J for $J \in B$ form a basis in the space H. Theorem 5.2 asserts that, for every fundamental system B, the functions $F_{\alpha'}$ with $\operatorname{supp}\alpha' \in B$ form a basis in the space spanned by all F_α.

There is also an explicit method for constructing fundamental systems in arbitrary linearly ordered sets I (Theorem 3.1), which makes Theorem 5.2 more effective.

We note that (for a suitable choice of ξ and $\mathcal{F}$) the space $H(\xi, \mathcal{F})$ is one of the realizations of the homology group with which we were concerned in the preceding section; indeed, the special bases mentioned there are bases of functions F_J, where J runs through any fundamental system. It is interesting that the concept of a fundamental system also admits a purely combinatorial definition; the method of constructing them given in Theorem 3.1 in a combinatorial situation is due essentially to Bjorner [14] (more precisely, it is obtained by adapting a result of Bjorner to a construction of Orlik and Solomon [22]).

6. Another application of Theorem 4.2 relates to the following beautiful geometry problem. In $\mathbf{R}^n$ we consider the family of k-dimensional affine planes ζ + f which are parallel to a given subspace $\zeta \in G_k(\mathbf{R}^n)$. We consider the polyhedra obtained by taking the intersection of these planes with the positive octant $\mathbf{R}^n_+$. The problem consists in studying the volume of such

a polyhedron as a function of f; we denote this function by $\Psi(f)$. This function has a complicated piecewise polynomial behavior. In Sec. 5 we show that it can be obtained as a restriction of the general hypergeometric function $\Phi(\alpha; \zeta, \omega)$ (for special values of α) to some submanifold. Applying Theorem 4.2, we get that the behavior of $\Psi(f)$ is governed by the same homology group as above, and that there is a class of explicit expressions for it corresponding to the various choices of a basis in this homology group.

It turns out that the continuous analogue of Kostant's partition function [21] which is introduced in [4] and which plays an important role in representation theory can also be defined as a function of the form $\Psi(f)$ for some special choice of $\mathbf{R}^n$ and of ζ; this means that all of what has been described above is also applicable to it. For systems of roots of type A_l the continuous analogue of Kostant's function is calculated in [11]; this result is included in our general scheme.

A very interesting question is whether it is possible to apply the various methods described here to a study of Kostant's function itself. In geometric terms, it revolves around the study of the "discrete analogue" of $\Psi(f)$ obtained by replacing the volume of the polyhedron by the number of integer points in it. This question will be taken up in another article, where explicit expressions will be obtained for various systems of roots.

7. This article is organized in the following way. The necessary definitions and notation are brought together in Sec. 1. In Sec. 2, we establish useful functional relations for general hypergeometric functions, which we call Gauss relations (for the Gauss hypergeometric function they reduce to the classical Gauss relations [3]). In Sec. 3 we study a homology group which plays a central role in the article, and we obtain a number of realizations of it. In Sec. 4, we obtain the main theorem, Theorem 4.2 and in Sec. 5 we bring together some of its applications.

The material from combinatorial geometry which we need is developed in the Appendix. We describe there the construction of Orlik and Solomon [22] and Bjorner's theorem [14] in forms which are convenient for our purposes. So that this article can be read independently, we give a new proof of this theorem.

1. DEFINITIONS AND NOTATION

Instead of $\mathbf{R}^n$, it will be convenient to consider the vector space $\mathbf{R}^I$ with a preferred basis $(e_i)_{i\in I}$, indexed by the finite set I. For each $J \subset I$ we let $\mathbf{R}^J$ denote the subspace of $\mathbf{R}^I$ spanned by the vectors $(e_j)_{j\in J}$, and we let $\mathbf{R}^J_+$ be the (open) positive octant in $\mathbf{R}^J$, i.e., $\mathbf{R}^J_+ = \{\sum_{j\in J} x_j e_j : x_j > 0\}$. We denote the number of elements in the finite set J by $|J|$; if $|J| = m$, then we say that J is an m-set.

We denote by $G_k(\mathbf{R}^I)$ the Grassmannian of k-dimensional vector subspaces in $\mathbf{R}^I$. The codimension of the subspaces, i.e., the number $|I| - k$, will always be denoted by l.

Let $\zeta \in G_k(\mathbf{R}^I)$. We denote by $\zeta^\perp$ the subspace of linear forms on $\mathbf{R}^I$ whose restrictions to ζ are equal to 0. We put $L = L(\zeta) = \mathbf{R}^I/\zeta$. We denote the projection $\mathbf{R}^I \to L$ by q, and we put $f_i = q(e_i) \in L$ for $i \in I$. We identify L with the dual space of $\zeta^\perp$; in particular, the f_i will be regarded as linear forms on $\zeta^\perp$. The family $(f_i)_{i\in I}$ of vectors in L will be denoted by $\mathcal{F} = \mathcal{F}(\zeta)$.

We consider the pregeometry of rank l on I corresponding to $\mathcal{F}(\zeta)$ (see the Appendix); the rank function of this pregeometry is given by $r(J) = \mathrm{rk}\,(f_j)_{j\in J} = \dim(\mathbf{R}^J/\zeta \cap \mathbf{R}^J)$. It is clear that two points ζ and ζ' lie on a single stratum Γ in $G_k(\mathbf{R}^I)$ if and only if the pregeometries on I corresponding to them coincide. We will use the terminology of the Appendix relative to this pregeometry, adding, if necessary, a designation to ζ (or Γ). Thus a subset J of I is independent for ζ if $\zeta \cap \mathbf{R}^J = 0$, and is a basis (of the pregeometry) for ζ if $\zeta \oplus \mathbf{R}^J = \mathbf{R}^I$. The set of all bases of the pregeometry for ζ is denoted by $B(\zeta)$. We note that the list of Γ defined in [6] consists of the k-subsets of I which are complements of subsets of $B(\zeta)$.

For every l-subset $J \subset I$ we put $\Gamma^J = \{\zeta \in G_k(\mathbf{R}^I): \zeta \oplus \mathbf{R}^J = \mathbf{R}^I\}$. The set Γ^J is a coordinate neighborhood in $G_k(\mathbf{R}^I)$: the elements of Γ^J are parametrized by the real matrices $Z = (z_{ij})_{i\in I\setminus J,\, j\in J}$, and the subspace $\zeta(Z) \in \Gamma^J$ with basis $(e_i + \sum_{j\in J} z_{ij} e_j)_{i\in I\setminus J}$ corresponds to the matrix Z.

We denote by $\tilde{G}_k(\mathbf{R}^I)$ the set of pairs $\tilde{\zeta} = (\zeta, \omega)$, where $\zeta \in G_k(\mathbf{R}^I)$, and ω is a nonzero skew symmetric k-linear form on ζ. The projection $\tilde{\zeta} \to \zeta$ converts $\tilde{G}_k(\mathbf{R}^I)$ into a fiber bundle over $G_k(\mathbf{R}^I)$ with fiber $\mathbf{R}\setminus 0$; for each subset $\Gamma \subset G_k(\mathbf{R}^I)$ we will denote the preimage of Γ under this projection by $\tilde{\Gamma} \subset \tilde{G}_k(\mathbf{R}^I)$. In particular, the sets $\tilde{\Gamma}$, where Γ is a stratum in $G_k(\mathbf{R}^I)$, are called strata in $\tilde{G}_k(\mathbf{R}^I)$.

Suppose that $\tilde{\zeta} = (\zeta, \omega) \in \tilde{G}_k(\mathbf{R}^I)$, and let $\alpha = (\alpha_i)_{i\in I}$ be a set of complex numbers whose sum is l. The general hypergeometric function $\Phi(\alpha; \tilde{\zeta}) = \Phi(\alpha; \zeta, \omega)$ is defined in the Introduction [see (1)]. For fixed $\tilde{\zeta}$, the function $\Phi(\alpha; \tilde{\zeta})$ is univalent and meromorphic in α, and for fixed α it is a real analytic function of $\tilde{\zeta}$ if we restrict it to any stratum in $\tilde{G}_k(\mathbf{R}^I)$ (see [6]).

From our agreement concerning the choice of an orientation on the "sphere" $S(\zeta)$ it follows that $\Phi(\alpha; \zeta, \lambda\omega) = |\lambda| \Phi(\alpha; \zeta, \omega)$ for $0 \neq \lambda \in \mathbf{R}$. In particular, the function $\Phi(\alpha; \zeta, \omega)$ remains unchanged if we replace ω by $-\omega$. Making use of this, we will often give ω only to within its sign.

Example 1.1. Let $k = 1$. We choose a generator $\sum_{i\in I} b_i e_i$ of the line $\zeta \in G_1(\mathbf{R}^I)$ so that at least one of the coordinates b_i is positive. Then

$$\Phi(\alpha; \zeta, \omega) = \prod_{i\in I} (b_i)_+^{\alpha_i - 1} \left| \omega\left(\sum_i b_i e_i\right) \right|.$$

We write $\Phi(\alpha; \tilde{\zeta})$ in the local coordinates introduced above. Let J be an l-subset in I, and let $Z = (z_{ij})_{i\in I\setminus J,\, j\in J}$ be a real matrix. Let $\omega(Z)$ be a k-form on $\zeta(Z)$ taking on the values ± 1 on the set of vectors $(e_i + \sum_{j\in J} z_{ij} e_j)_{i\in I\setminus J}$. Let $I\setminus J = \{i_1, \ldots, i_k\}$. In this notation, (1) can be written

$$\Phi(\alpha; \zeta(Z), \omega(Z)) = \int_{S(\mathbf{R}^{I\setminus J})} \prod_{i\in I\setminus J} (x_i)_+^{\alpha_i - 1} \prod_{j\in J} \Big(\sum_{i\in I\setminus J} z_{ij} x_i\Big)_+^{\alpha_j - 1} \tilde{\omega}(x), \tag{3}$$

where

$$\tilde{\omega}(x) = \sum_{1\leq r\leq k} (-1)^{r-1} x_{i_r} dx_{i_1} \wedge \ldots \wedge dx_{i_{r-1}} \wedge dx_{i_{r+1}} \wedge \ldots \wedge dx_{i_k}. \tag{4}$$

For brevity, we will write $\Phi(\alpha; Z)$ instead of $\Phi(\alpha; \zeta(Z), \omega(Z))$.

If $\alpha = (\alpha_i)_{i\in I}$ and $\beta = (\beta_i)_{i\in I}$ are two sets of indices, then the set $(\alpha_i + \beta_i)_{i\in I}$ will be denoted by $\alpha + \beta$. For each $J \subset I$, we use 1_J to denote the set $(\iota_i)_{i\in I}$, where $\iota_j = 1$ for $j \in J$ and $\iota_i = 0$ for $i \notin J$. We will write 1_i instead of $1_{\{i\}}$. For example, $\alpha - 1_j$ denotes the set $(\alpha'_i)_{i\in I}$, where $\alpha'_i = \alpha_i - \delta_{ij}$. Finally, we put $|\alpha| = \sum_i \alpha_i$ and $\operatorname{Supp} \alpha = \{i \in I\colon \alpha_i \neq 0\}$.

2. GAUSS RELATIONS

THEOREM 2.1. Let $\tilde{\zeta} = (\zeta, \omega) \in \tilde{G}_k(\mathbf{R}^I)$.

a) Let $\beta = (\beta_i)_{i\in I}$ be a set of complex numbers with sum $l - 1$, and let $v = \sum_{i\in I} a_i x_i \in \zeta^\perp$ be a linear form which is identically 0 on ζ. Then

$$\sum_{i\in I} a_i \Phi(\beta + 1_i; \tilde{\zeta}) = 0. \tag{5}$$

b) Let $\gamma = (\gamma_i)_{i\in I}$ be a set of complex numbers with sum $l + 1$, and let $y = \sum_{i\in I} b_i e_i \in \zeta$. Then

$$\sum_{i\in I} b_i (\gamma_i - 1) \Phi(\gamma - 1_i; \tilde{\zeta}) = 0. \tag{6}$$

The relations (5) and (6) are called the Gauss relations for the general hypergeometric function on an arbitrary stratum (see example 2.1 below); altogether, they give $k + l$ I|* independent relations which relate the values of $\Phi(\alpha; \tilde{\zeta})$ for fixed $\tilde{\zeta}$ and "contiguous" indices α. The relations (5) follow directly from the definitions, and the relations (6) follow from (5) and a duality theorem in [9]. For a general stratum, it is convenient to construct the proof in local coordinates. We choose an l-subset $J \subset I$, and we consider the function $\Phi(\alpha; Z)$ defined in (3).

Proposition 2.1. a) For fixed $j \in J$, the function $\Phi(\alpha; Z)$, as a function of the column $(z_{ij})_{i\in I\setminus J}$, has degree of homogeneity $(\alpha_j - 1)$ (i.e., for $\lambda > 0$, if we make the replacement $z_{ij} \to \lambda z_{ij}$ for all $i \in I\setminus J$ in $\Phi(\alpha; Z)$, this multiplies the functions by $\lambda^{\alpha_j - 1}$).

*Something missing in the Russian original – Publisher.

b) For fixed $i \in I \setminus J$, the function $\Phi(\alpha; Z)$, as a function of the row $(z_{ij})_{j \in J}$, has degree of homogeneity $(-\alpha_i)$.

c) For all $i \in I \setminus J$ and all $j \in J$,

$$\partial \Phi(\alpha; Z)/\partial z_{ij} = (\alpha_j - 1)\, \Phi(\alpha + 1_i - 1_j; Z). \tag{7}$$

All of these assertions follow immediately from (3).

COROLLARY. a) For every $j \in J$,

$$\Phi(\alpha; Z) = \sum_{i \in I \setminus J} z_{ij} \Phi(\alpha + 1_i - 1_j; Z). \tag{8}$$

b) For every $i \in I \setminus J$,

$$\alpha_i \Phi(\alpha; Z) + \sum_{j \in J} (\alpha_j - 1) z_{ij} \Phi(\alpha + 1_i - 1_j; Z) = 0. \tag{9}$$

To prove (8), we write the homogeneity condition of $\Phi(\alpha; Z)$ with respect to the column $(z_{ij})_{i \in I \setminus J}$ in the Euler form

$$\sum_{i \in I \setminus J} z_{ij} \partial \Phi(\alpha; Z)/\partial z_{ij} = (\alpha_j - 1)\, \Phi(\alpha; Z).$$

Substituting this into (7), we get (8). The result in (9) is proved in exactly the same way.

Equations (8) and (9) are the Gauss relations for the general hypergeometric function on a general stratum. We note that (7) and (8) (in our terms, for a general stratum) have been proved by Aomoto [24, 25].

Example 2.1. Let $J = \{1, 2\}$, $I \setminus J = \{0, 3\}$, and let $k = 2$, so that we are dealing with the Grassmannian of two-dimensional subspaces of $\mathbf{R}^4$. By virtue of Proposition 2.1 (a), (b), to calculate $\Phi(\alpha; Z)$ we may assume that all of the matrix elements of Z, except the first, are fixed. We put $z_{31} = z_{32} = -1$, $z_{01} = 1$, $z_{02} = z$; by virtue of (3) we have for such a matrix Z

$$\Phi(\alpha; Z) = \int_0^1 x^{\alpha_1 - 1} (1 - x)^{\alpha_3 - 1} (z - x)_+^{\alpha_2 - 1}\, dx.$$

Hence it follows that

$$\left.\begin{aligned} &\Phi(\alpha; Z) = 0 \quad \text{for} \quad z \leqslant 0, \\ &\Phi(\alpha; Z) = \frac{\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\alpha_2 + \alpha_3)} z^{\alpha_1 + \alpha_2 - 1} F(1 - \alpha_1, \alpha_3; \alpha_2 + \alpha_3; z) \quad \text{for} \quad 0 \leqslant z \leqslant 1, \\ &\Phi(\alpha; Z) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_3)}{\Gamma(\alpha_1 + \alpha_3)} z^{\alpha_2 - 1} F(1 - \alpha_2, \alpha_3; \alpha_1 + \alpha_3; z^{-1}) \quad \text{for} \quad z \geqslant 1, \end{aligned}\right\} \tag{10}$$

where $F(a, b; c; z)$ is the classical Gauss hypergeometric function.

If we make a suitable choice of a basis of four relations in the space of relations of the form (5) and (6), and if we transform them by making use of (10), we get the classical Gauss relations for F ([3], 2.8. (38), (42), (35), (43)); it is easy to verify that all 15 of the Gauss relations ([3], 2.8. (31)-(45)) are linear combinations of these four relations (applied to various sets of indices α_i).

3. THE SPACE H(ζ) AND ITS REALIZATIONS

We fix a stratum Γ in $G_k(\mathbf{R}^I)$ and a point $\zeta \in \Gamma$. We construct a finite-dimensional space $H(\zeta)$ and we describe a special class of bases in it; these concepts play a central role in what follows. We obtain several different realizations of $H(\zeta)$. If ζ contains any basis vector e_i, then we put $H(\zeta) = 0$; we note that the strata Γ with this property are said to be degenerate in the terminology of [6]. Thus in what follows we assume that this is not the case; in other words, all of the vectors f_i in the space $L = L(\zeta)$ are different from 0.

Algebraic Definition. For each $J \in B(\zeta)$ we let F_J denote the rational function $\prod_{j \in J} f_j^{-1}$ on $\zeta^\perp$ (see Sec. 1). By definition $H(\zeta)$ is the vector space generated by all such functions F_J.

Topological Definition. Let $\zeta_{\mathbf{C}}^\perp$ be the complexification $\zeta^\perp$, and suppose that $X = \zeta_{\mathbf{C}}^\perp \setminus \bigcup_{i \in I} f_i^\perp$ is obtained from $\zeta_{\mathbf{C}}^\perp$ by deleting the hyperplanes orthogonal to all of the f_i. The space

H(ζ) is isomorphic to the highest cohomology group $H^l(X)$ of X. More precisely, we have the following result.

Proposition 3.1 [2, 15]. The mapping $H(\zeta) \otimes \Lambda^l(L) \to H^l(X)$, which sends each element $F \otimes \omega$ into the class of cohomologies of the differential l-form $F\omega$ on X is an isomorphism.

Combinatorial Definition. We consider the pregeometry of rank l on I constructed with respect to ζ, i.e., corresponding to the family of vectors $\mathcal{F} = (f_i)_{i \in I}$ in L. Let $\mathcal{A}_l = \mathcal{A}_l(\zeta)$ be the space corresponding to this pregeometry under the construction of Orlik and Solomon (see the Appendix). Then H(ζ) and $\mathcal{A}_l(\zeta)$ are isomorphic. More precisely, we can construct a natural isomorphism between $\mathcal{A}_l(\zeta)$ and $H(\zeta) \otimes \bigwedge^l(L)$.

The requirement that $J \in B(\zeta)$ means that the restriction of the projection $q: \mathbf{R}^I \to L$ to $\mathbf{R}^J$ is an isomorphism of $\mathbf{R}^J$ with L; for each $J \in B(\zeta)$, we again denote by $q = q_{J,\zeta}$ the isomorphism $\Lambda^l(\mathbf{R}^J) \simeq \Lambda^l(L)$, introduced by this isomorphism. We recall (see Appendix) that there is a natural epimorphism $\varepsilon: \mathcal{C}_l \to \mathcal{A}_l$ with kernel $\partial\mathcal{C}_{l+1}$, where the space $\mathcal{C}_l$ is identified with $c \bigoplus_{J \in B(\zeta)} \Lambda^l(\mathbf{R}^J)$.

Proposition 3.2. The mapping $\mathcal{C}_l \to H(\zeta) \otimes \Lambda^l(L)$, which carries $\omega \in \Lambda^l(\mathbf{R}^J) \subset \mathcal{C}_l$ into $F_J \otimes q_{J,\zeta}(\omega)$, is an epimorphism, and its kernel coincides with $\partial\mathcal{C}_{l+1}$. Thus this mapping induces a natural isomorphism of $\mathcal{A}_l$ with $H(\zeta) \otimes \Lambda^l(L)$.

This result is essentially due to Orlik and Solomon [22]. Combining Propositions 3.1 and 3.2, we get an isomorphism between $\mathcal{A}_l$ and $H^l(X)$; in [22] it is shown that the algebra $\mathcal{A} = \bigoplus_{m \leq l} \mathcal{A}_m$ constructed in the Appendix is isomorphic in a natural way to the cohomology ring $H^*(X)$.

It is possible to restate Proposition 3.2 in a purely algebraic fashion in terms of H(ζ). Indeed, let $\hat{J}$ be an $(l + 1)$-subset in I having rank l (for ζ); this means that $\dim(\zeta \cap \mathbf{R}^{\hat{J}}) = 1$. Let $y = \sum_{j \in \hat{J}} b_j e_j$ be a nonzero vector in ζ; in other words, this means that $\sum_{j \in \hat{J}} b_j f_j = 0$. Dividing this last equality by $\prod_{j \in \hat{J}} f_j$, we get the following linear relation in H(ζ):

$$\sum_{j \in \hat{J}} b_j F_{\hat{J} \setminus j} = 0 \tag{11}$$

(it is easy to see that $b_j \neq 0$ if and only if $\hat{J} \setminus j \in B(\zeta)$).

Proposition 3.3. Every linear relation among the elements F_J $(J \in B(\zeta))$ in H(ζ) is a consequence of relations of the form (11).

This follows directly from Proposition 3.2.

We note a similarity between (11) and the Gauss relations (6). This similarity is not accidental; we will make the connection with the hypergeometric function later.

Geometric Definition. We assume that ζ satisfies the additional restriction that $\zeta \cap \mathbf{R}_+^I = \varnothing$ (it is easy to see that, for any stratum Γ, the set of such ζ is nonempty and open in Γ). In other words, this means that all vectors f_i lie in some semispace in L(ζ); if the f_i are regarded as linear forms on $\zeta^\perp$, then this means that there exists a point $x \in \zeta^\perp$, for which $f_i(x) > 0$ for all $i \in I$. For each subset $J \subset I$ of rank l (for ζ), we let C_J denote the open cone in L generated by the f_j for $j \in J$ (in other words, C_J is the image of $\mathbf{R}_+^J$ under the projection $q: \mathbf{R}^I \to L$). Let L^0 be the subset in L consisting of those vectors f which are in general position with respect to the system $\mathcal{F} = (f_i)_{i \in I}$ (i.e., f does not lie in any characteristic subspace in L spanned by a subsystem of $\mathcal{F}$). We put $C_J^0 = C_J \cap L^0$ and we let $\chi_J = \chi_{J,\zeta}$ denote the characteristic function of the set C_J^0. Let $H' = H'(\zeta)$ be the vector space of functions on L generated by the functions χ_J for $J \in B(\zeta)$. We choose a nonzero skew-symmetric l-linear form ω on L, and, for each $J = \{j_1, \ldots, j_l\} \in B(\zeta)$ we put $c_J(\omega) = |\omega(f_{j_1}, \ldots, f_{j_l})|$.

Proposition 3.4. The mapping which carries χ_J into $c_J(\omega)F_J$ for all $J \in B(\zeta)$, can be extended to an isomorphism of $H'(\zeta)$ and H(ζ).

Proof. We put $C_I^* = \{x \in \zeta^\perp : f(x) > 0 \text{ for } f \in C_I\}$; from the condition that $\zeta \cap \mathbf{R}_+^I = \varnothing$ is a nonempty open convex cone in $\zeta^\perp$. We consider the Laplace transformation which carries a function φ on C_I into the function $P\varphi$ on C_I^* given by

$$P\varphi(x) = \int_L \varphi(f) e^{-f(x)} \omega(f). \tag{12}$$

We can see immediately that $P\chi_J = c_J(\omega)F_J$ for all $J \in B(\zeta)$. In addition, it is easy to see that the restriction of P to $H'(\zeta)$ is injective; therefore it gives the required isomorphism.

We pass to the special basis in $H(\zeta)$. We call a subset $B \subset B(\zeta)$ a fundamental system for ζ if the functions F_J for $J \in B$ form a basis in $H(\zeta)$. By virtue of Proposition 3.2, this definition agrees with the combinatorial definition in the Appendix; according to Proposition 3.4, a system B is fundamental if and only if the function χ_J for $J \in B$ form a basis in $H'(\zeta)$. We describe a general method for constructing fundamental systems.

We say that a subset $I' \subset I$ is a circuit (for ζ) if the vectors f_i for $i \in I'$ are linearly dependent, but for any characteristic subset of I' this is not true. Now suppose that a linear ordering has been introduced on I; we call a subset of I an open circuit if it is obtained from some circuit in I by deleting the maximal element.

THEOREM 3.1. The system of all subsets in $B(\zeta)$ which do not contain open circuits (with respect to an arbitrary given linear ordering of I) is fundamental.

In the combinatorial situation, this result (and an even more general one) has been established by Bjorner [14] (Theorem II.1 in the Appendix). Theorem 3.1 follows from Theorem II.1 if we make use of Proposition 3.2.

In particular, if ζ is a point of a general stratum in $G_k(\mathbf{R}^I)$, then $\dim H(\zeta) = \dim H'(\zeta) = \binom{|I|-1}{l-1}$, and we may take as a basis in $H(\zeta)$ ($H'(\zeta)$) the family of functions F_J (χ_J), where J runs through all l-subsets of I which contain a certain fixed element $i \in I$ (Appendix, Example 2).

4. FUNDAMENTAL SYSTEMS AND GENERAL HYPERGEOMETRIC FUNCTIONS

In this section we apply the concepts developed in Sec. 3 to a study of the general hypergeometric function. Again, let Γ be some fixed stratum in $G_k(\mathbf{R}^I)$. We consider the set $\hat{I} = I \cup \{0\}$, obtained by adjoining a distinguished point, denoted here by 0, to I. A technicality that arises here is that the space $H(\zeta)$ and the fundamental system constructed with reference to the stratum Γ are to be applied to a study of the general hypergeometric function, not on Γ, but on some stratum $\hat{\Gamma}^0$ in $G_{k+1}(\mathbf{R}^{\hat{I}})$. More precisely, we put $\hat{\Gamma} = \{\hat{\zeta} \in G_{k+1}(\mathbf{R}^{\hat{I}}): (\hat{\zeta} \cap \mathbf{R}^I) \in \Gamma\}$, and we let $p: \hat{\Gamma} \to \Gamma$ denote the projection given by $p(\hat{\zeta}) = \hat{\zeta} \cap \mathbf{R}^I$. It is not difficult to see that $\hat{\Gamma}$ is a fiber bundle over Γ whose fiber at each point $\zeta \in \Gamma$ is isomorphic in a natural way to the space $L(\zeta) = \mathbf{R}^I/\zeta$. Indeed, every subspace $\hat{\zeta} \in G_{k+1}(\mathbf{R}^{\hat{I}})$ such that $\hat{\zeta} \cap \mathbf{R}^I = \zeta$, is obtained by adjoining to ζ some vector of the form $e_0 + f$, where f is in $\mathbf{R}^I$ and is defined uniquely modulo ζ; thus it is possible to assume that $f \in L(\zeta)$, and the mapping $\hat{\zeta} \to f$ is the required isomorphism between $p^{-1}(\zeta)$ and $L(\zeta)$. Taking this isomorphism into account, we will write the elements of $\hat{\Gamma}$ as pairs (ζ, f), where $\zeta \in \Gamma$, and $f \in L = L(\zeta)$. It is easy to see that there is precisely one open stratum $\hat{\Gamma}^0 = \{(\zeta, f) \in \hat{\Gamma}: f \in L^0\}$ in $\hat{\Gamma}$ (we recall that an open subset $L^0 \subset L(\zeta)$ consists of vectors in general position with respect to the family of vectors $(f_i)_{i \in I}$; see Sec. 3).

We put $\hat{\Gamma}_+ = \{(\zeta, f) \in \hat{\Gamma}: \hat{\zeta} \cap \mathbf{R}_+^{\hat{I}} \neq \varnothing\}$, and we let $\hat{\alpha} = (\alpha_i)_{i \in \hat{I}}$ be a set of indices with $|\hat{\alpha}| = l = |I| - k$. We will be concerned with the restriction of the function $\Phi(\hat{\alpha}; \hat{\zeta}, \hat{\omega})$ to the open subset $\hat{\Gamma}_+^0 = \hat{\Gamma}^0 \cap \hat{\Gamma}_+$ of $\hat{\Gamma}^0$.

We recall that, for each subset $J \subset I$ of rank l (for ζ), we use C_J to denote the open convex cone in $L = L(\zeta)$ generated by the vectors f_j for $j \in J$. We put $\hat{\Gamma}_+(J) = \{(\zeta, f) \in \hat{\Gamma}_+: f \in C_J\}$ and $\hat{\Gamma}_+^0(J) = \hat{\Gamma}^0 \cap \hat{\Gamma}_+(J)$. It is clear that $\hat{\Gamma}_+(I)$ consists of those subspaces $\hat{\zeta} \in \hat{\Gamma}_+$, for which $\hat{\zeta} \cap \mathbf{R}_+^{\hat{I}} \neq \varnothing$; hence it follows that the restriction of $\Phi(\hat{\alpha}; \hat{\zeta}, \hat{\omega})$ to $\hat{\Gamma}_+^0$ is concentrated on the subset $\hat{\Gamma}_+^0(I)$. We note that if $|J| = l$, then $\hat{\Gamma}_+^0(J)$ is the intersection of $\hat{\Gamma}_+^0$ with the large cell in $\hat{\Gamma}^0$ corresponding to the coordinate subspace $\mathbf{R}^{J \cup \{0\}}$ (see the Introduction).

As we have already remarked, we will be concerned with special integer indices $\hat{\alpha}$. But first we discuss the situation for general $\hat{\alpha}$ in an informal way. Let B be an arbitrary fundamental system of l-subsets in I for the stratum Γ (see Sec. 3). We assert that the function $\Phi(\hat{\alpha}; \hat{\zeta}, \hat{\omega})$ on $\hat{\Gamma}_+^0$ can be expanded in a sum

$$\Phi(\hat{\alpha}; \hat{\zeta}, \hat{\omega}) = \sum_{J \in B} \Phi_J^{(B)}(\hat{\alpha}; \hat{\zeta}, \hat{\omega}), \qquad (*)$$

where $\Phi_J^{(B)}$ is equal to 0 for $\hat{\zeta} \notin \hat{\Gamma}_+^0(J)$ and "has unique analytic behavior" in $\hat{\Gamma}_+^0(J)$; this means that if we continue the restrictions of $\Phi_J^{(B)}$ to the various cells of $\hat{\Gamma}_+^0(J)$ analytically into the complex domain, we get a unique branch of an analytic function in some region U in the complexification $\hat{\Gamma}_{\mathbf{C}}^0$ of the stratum $\hat{\Gamma}^0$. If U is a sufficiently small region and E(U) is the (finite-dimensional) space of restrictions to U of solutions of the holonomic system of equations in [7], then it is possible to suggest that $\dim E(U) = \dim H(\zeta)$ (where ζ lies in the stratum Γ, covered by $\hat{\Gamma}^0$) and that it is possible to obtain a basis in E(U) by continuing $\Phi_J^{(B)}$ analytically into the complex domain. These questions will be treated in another article.

We pass to the precise statements. First of all, we introduce the normalization of the function $\Phi(\hat{\alpha}; \hat{\zeta}, \hat{\omega})$ which is suitable for continuing it analytically with respect to $\hat{\alpha}$. Let $\hat{\zeta} = (\zeta, f) \in \hat{\Gamma}_+^0$, and let ω be a nonzero k-form on ζ such that $\tilde{\zeta} = (\zeta, \omega) \in \tilde{\Gamma}$ (see Sec. 1). From ω we construct the (k + 1)-form $\hat{\omega}$ on $\hat{\zeta}$ given by $\hat{\omega}(y_1, \ldots, y_k, e_0 + \bar{f}) = \omega(y_1, \ldots, y_k)$, where $\{y_1, \ldots, y_k\}$ is a basis in ζ, and $\bar{f}$ is an arbitrary representative of the vector $f \in L = \mathbf{R}^I/\zeta$ in $\mathbf{R}^I$. We will write the point $(\hat{\zeta}, \hat{\omega}) \in \hat{\Gamma}_+^0$ in the form $(\tilde{\zeta}, f) = (\zeta, \omega, f)$, and the set of indices $\hat{\alpha}$ as a pair (α, α_0), where $\alpha = (\alpha_i)_{i \in I}$ is an arbitrary set of complex numbers, and $\alpha_0 = l - \sum_{i \in I} \alpha_i$. We put

$$\Psi(\alpha; \zeta, \omega, f) = \Phi(\hat{\alpha}; \zeta, \omega, f) / \prod_{i \in I} \Gamma(\alpha_i). \tag{13}$$

It will be important to clarify the behavior of $\Psi(\alpha; \zeta, \omega, f)$ as a function of α in a neighborhood of a point where some of the α_i can vanish. More precisely, let $\alpha^{(0)} = (\alpha_i^{(0)})_{i \in I}$ be a set such that $\operatorname{Supp} \alpha^{(0)} = J$, where J is some m-subset in I which has rank l for ζ; we denote the subset $(\alpha_j^{(0)})_{j \in J}$ by $\alpha_J^{(0)}$. We put $\zeta_J = \zeta \cap \mathbf{R}^J$; thus $\zeta_J \in G_{m-l}(\mathbf{R}^J)$. From the k-form ω on ζ we construct the $(m-1)$-form ω_J on ζ_J whose value at the vectors $y_1, \ldots, y_{m-l} \in \zeta_J$ is $\omega(y_1, \ldots, y_{m-l}, \bar{e}_{i_1}, \ldots, \bar{e}_{i_r})$, where $\{i_1, \ldots, i_r\} = I \setminus J$, and the $\bar{e}_i$ are vectors in ζ congruent to e_i modulo $\mathbf{R}^J$ (it is easy to see that such a form ω_J is uniquely determined up to its sign; see Sec. 1). We note that there is a natural isomorphism between $L(\zeta_J) = \mathbf{R}^J/\zeta_J$ and $L(\zeta) = \mathbf{R}^I/\zeta = (\mathbf{R}^J + \zeta)/\zeta$.

<u>Proposition 4.1.</u> Under the above assumptions, $\Psi(\alpha; \zeta, \omega, f)$ can be continued analytically with respect to α to the point $\alpha^{(0)}$, and its value at this point is $\Psi(\alpha_J^{(0)}; \zeta_J, \omega_J, f)$.

The proof of this proposition will be given in another article.

<u>COROLLARY.</u> We suppose that the set $J = \operatorname{Supp} \alpha$ consists of l elements. Let $I \setminus J = \{i_1, \ldots, i_k\}$, and suppose that the vectors $\bar{e}_i \in \zeta$ for $i \in I \setminus J$ are as above. Then

$$\Psi(\alpha; \zeta, \omega, f) = |\omega(\bar{e}_{i_1}, \ldots, \bar{e}_{i_k})| \cdot \prod_{j \in J} (x_j)_+^{\alpha_j - 1} / \Gamma(\alpha_j), \tag{14}$$

where the x_j are the coordinates in the expansion of f with respect to the basis $(f_j)_{j \in J}$ of L. In particular, $\Psi(\alpha; \zeta, \omega, f)$ is concentrated on $\hat{\Gamma}_+^0(J)$, and can be continued analytically to $\hat{\Gamma}_+(J)$.

This follows directly from Proposition 4.1 and Example 1 of Sec. 1.

We call a set $\alpha = (\alpha_i)_{i \in I}$ *polynomial* for ζ (or for the stratum Γ) if all of the α_i are nonnegative integers and $\operatorname{Supp} \alpha$ has rank l for ζ. The following theorem gives a sharpening of the expansion (*) for polynomial α.

<u>THEOREM 4.2.</u> Let the set $\alpha = (\alpha_i)_{i \in I}$ be polynomial for Γ, and let $B \subset B(\zeta)$ be some fundamental system of l-subsets in I. Then the restriction of the function $\Psi(\alpha; \zeta, \omega, f)$ to Γ_+^0 can be written as a linear combination of the form

$$\Psi(\alpha; \zeta, \omega, f) = \sum_{\alpha'} c_{\alpha\alpha'}(\zeta, \omega) \Psi(\alpha'; \zeta, \omega, f),$$

where α' runs through the polynomial sets for Γ such that $|\alpha'| = |\alpha|$ and $\operatorname{Supp} \alpha' \in B$, and the coefficients $c_{\alpha\alpha'}(\zeta, \omega)$ do not depend on f but are analytic functions of $(\zeta, \omega) \in \tilde{\Gamma}$. The expansion with these properties is uniquely determined.

We note that every function $\Psi(\alpha'; \zeta, \omega, f)$ in the theorem is given by (14); hence it follows, in particular, that $\Psi(\alpha; \zeta, \omega, f)$ is piecewise polynomial with respect to f of degree $|\alpha| - l$. If we collect the terms corresponding to those sets α' with the same support, $\operatorname{Supp} \alpha'$, we get an expansion of the form (*).

Main Steps in the Proof of Theorem 4.2. 1. Let Γ' be a general stratum in $G_k(\mathbf{R}^I)$. We adjoin to B certain l-subsets which are dependent for Γ in order to obtain a system B' which is fundamental for Γ' (see Proposition II.1). We show that Theorem 4.2 holds for Γ' and the fundamental system B', and that, in addition, we have the following refinement: all of the coefficients $c_{\alpha\alpha'}(\zeta, \omega)$ can be continued analytically from $\tilde{\Gamma}'$ to $\tilde{\Gamma}$, and this continuation is equal to 0 if $\operatorname{Supp}\alpha'$ is dependent for Γ. It is clear that the existence of the expansion in Theorem 4.2 for arbitrary Γ follows from this refinement.

2. We fix the system B'. We call a polynomial set of indices α good if the refinement in Step 1 holds for $\Psi(\alpha;\ \zeta,\ \omega,\ f)$. We must show that every set α is good.

LEMMA 1. If the set α is good and $i \in \operatorname{Supp}\alpha$, then $\alpha + \mathbf{1}_i$ is also good.

LEMMA 2. Suppose that $\hat{J} \subset I$ and that $|\hat{J}| > l$. We assume that all of the sets α' such that $\operatorname{Supp}\alpha'$ is obtained from $\hat{J}$ by deleting some elements are good. Then $\mathbf{1}_{\hat{J}}$ is good.

Lemma 1 can be deduced from (7) in Sec. 2, and Lemma 2 from the Gauss relations (6) (or (9)).

3. By virtue of Lemmas 1 and 2, it suffices to verify the assertion that all of the α are good for $\alpha = \mathbf{1}_J$, where $J \in B(\zeta)$. But by virtue of (14) the function $\Psi(\mathbf{1}_J;\ \zeta,\ \omega,\ f)$, regarded as a function of f for fixed ζ and ω, is proportional to the function χ_J introduced in Sec. 3. Therefore the fact that $\mathbf{1}_J$ is good follows from the results in Sec. 3. This proves the existence of the expansion in Theorem 4.2. The uniqueness requires a separate argument, which we do not give here.

We postpone a detailed proof of Theorem 4.2 to a later article, where we will also consider the case of general indices.

5. AN APPLICATION OF THEOREM 4.2

We regard the function $\Psi(\alpha;\ \zeta,\ \omega,\ f)$ defined by (13) as a function of $f \in L = L(\zeta)$ for fixed ζ and ω. For brevity, we will write $\Psi(\alpha;\ f)$ instead of $\Psi(\alpha;\ \zeta,\ \omega,\ f)$. We note that, with respect to the family $\mathcal{F} = (f_i)_{i \in I}$ of rank l in L, the subspace $\zeta \in G_k(\mathbf{R}^I)$ is restored as the kernel of the projection $q\colon \mathbf{R}^I \to L$ which carries e_i into f_i. As before, we will assume that all of the f_i are different from 0 and lie in some open halfspace in L. We will be interested only in the case where all of the indices α_i are positive integers. In this case, $\Psi(\alpha;\ f)$ admits a completely "elementary" definition.

For each vector $f \in L$ we put $\Delta(f) = q^{-1}(f) \cap \mathbf{R}^I_+$, which is a bounded (open) polyhedron in the k-dimensional affine plane $q^{-1}(f)$ in $\mathbf{R}^I$; it is obvious that $\Delta(f) \neq \varnothing$ if and only if $f \in C_I$, where C_I is the open convex cone in L generated by all of the f_i. We denote by ω the k-form on $q^{-1}(f)$ obtained from the form ω on $\zeta = q^{-1}(0)$ by parallel translation, as well as its restriction to $\Delta(f)$.

Proposition 5.1. Using the above notation, we can write $\Psi(\alpha;\ f)$ in the form

$$\Psi(\alpha; f) = \Big(\prod_{i \in I} \Gamma(\alpha_i)\Big)^{-1} \int_{\Delta(f)} \Big(\prod_{i \in I} x_i^{\alpha_i - 1}\Big)\, \omega(x). \tag{15}$$

This follows directly from the definition.

In particular, the function $\Psi(\mathbf{1}_I;\ f)$ is simply the volume of the polyhedron $\Delta(f)$ with respect to the form ω.

THEOREM 5.1. Let $\alpha = (\alpha_i)_{i \in I}$ be a set of positive integral indices. Let B be some fundamental system of l-subsets in I for ζ. Then for each $J \in B$ there exists a unique polynomial $\Psi_J^{(B)}(\alpha;\ f)$ on the space L of degree $|\alpha| - l$ such that for $f \in L^0$ we have the expansion

$$\Psi(\alpha; f) = \sum_{J \in B} \Psi_J^{(B)}(\alpha; f)\, \chi_J(f)$$

(for the definitions of L^0 and χ_J, see Sec. 3).

This theorem follows immediately from Theorem 4.2.

A theorem on the expansion of rational functions is another interesting application of Theorem 4.2.

From the form ω on ζ, we construct the l-form ω_L on L in the following way: for $v_1, \ldots, v_l \in L$ we put $\omega_L(v_1, \ldots, v_l) = \omega_I(y_1, \ldots, y_k, \bar{v}_1, \ldots, \bar{v}_l) / \omega(y_1, \ldots, y_k)$, where $y_1, \ldots, y_k$ is some basis in ζ, where $\bar{v}_i \in q^{-1}(v_i)$ for $i = 1,\ldots,l$, and where ω_I is a $(k + l)$-form on $\mathbf{R}^I$ which takes on the values ± 1 on the set $(e_i)_{i\in I}$ (it is obvious that the form ω_L is defined uniquely up to its sign). As in Sec. 3, we identify vectors in L with linear forms on $\zeta^\perp$. Let P be the Laplace transformation relative to ω_L [see (12) in Sec. 3].

Proposition 5.2. We have that $P\Psi(\alpha; f) = \prod_{i\in I} f_i^{-\alpha_i}$.

The proof follows directly from the definitions.

For each polynomial set α, we define the rational function F_α on $\zeta^\perp$ by putting $F_\alpha = \prod_{i\in I} f_i^{-\alpha_i}$.

THEOREM 5.2. Let B be a fundamental system of l-subsets for ζ. Every rational function F_α can be represented in the form of a linear combination $\sum_{\alpha'} c_{\alpha\alpha'} F_{\alpha'}$ with constant coefficients $c_{\alpha\alpha'}$, where α' runs through a set such that $|\alpha'| = |\alpha|$ and $\operatorname{Supp}\alpha' \in B$. This representation is unique.

This follows immediately from Theorem 4.2 and Proposition 5.2.

The expansions in Theorem 5.1 and 5.2 are closely related to each other.

Proposition 5.3. Let $F_\alpha = \sum_{\alpha'} c_{\alpha\alpha'} F_{\alpha'}$ be the expansion in Theorem 5.2. Then the polynomial $\Psi_J^{(B)}(\alpha; f)$ in Theorem 5.1 is given by

$$\Psi_J^{(B)}(\alpha; f) = c_J^{-1} \sum_{\alpha'} c_{\alpha\alpha'} \prod_{j\in J} x_j^{\alpha'_j - 1} / \Gamma(\alpha'_j),$$

where $c_J = |\omega_L((f_j)_{j\in J})|$, α' runs through the sets such that $|\alpha'| = |\alpha|$ and $\operatorname{Supp}\alpha' = J$, and the x_j are the coordinates in the expansion of f with respect to the basis $(f_j)_{j\in J}$.

This follows immediately from Proposition 5.2 and (14) in Sec. 4.

Remarks. a) It is possible to derive an algorithm for calculating the $\Psi_J^{(B)}(\alpha; f)$ from the proof of Theorem 4.2; another method is to apply Proposition 5.3. Since we have a general method for constructing fundamental systems (Theorem 3.1), it is possible in principle to obtain an explicit formula for $\Psi(\alpha; f)$. It is clear that, in concrete examples, obtaining such a formula may not be a simple matter.

b) Proposition 5.3 is closely related to the duality theorem for general hypergeometric functions obtained in [9].

We turn to Kostant's partition function. Let $\mathcal{F} = (f_i)_{i\in I}$ be the positive roots of some system R of roots in L; we normalize the form ω so that the l-form ω_L on L corresponding to it takes on the values ± 1 on the set of simple roots in R. We recall that, by definition, Kostant's function $K_R(f)$ is the number of representations of f in the form $\sum_{i\in I} m_i f_i$, where all of the m_i are nonnegative integers [21]. Using the terminology introduced above, we can reformulate this definition thus: $K_R(f)$ is the number of integer points in the closure of the polyhedron $\Delta(f)$. We define the *continuous analog* of $K_R(f)$ as the volume of the polyhedron $\Delta(f)$; it is easy to verify that this definition is equivalent to that given in [4]. Thus, in the notation introduced above, the continuous analog of Kostant's function is $\Psi(\mathbf{1}_I; f)$. All of the results obtained above are applicable to this function. In particular, Theorem 5.1 shows that to each choice of a fundamental system B for the pregeometry on I given by $\mathcal{F}$, there is related an expansion $\Psi(\mathbf{1}_I; f) = \sum_{J\in B} \Psi_J^{(B)}(\mathbf{1}_I; f)\chi_J(f)$, where each function $\Psi_J^{(B)}(\mathbf{1}_I; f)\,\chi_J(f)$ is concentrated in the cone C_J and coincides with the restriction of the polynomial $\Psi_J^{(B)}(\mathbf{1}_I; f)$ in it; the degree of these polynomials in the case at hand is equal to the number of positive roots in the system R minus its rank.

We give a number of examples of choices of fundamental systems in this situation. The number of elements of B is given by the following proposition.

Proposition 5.4. The dimension of the space $\mathcal{A}_I$, corresponding to the pregeometry with respect to a family of positive roots of some system of roots R of rank l is equal to product $m_1 \ldots m_l$, where the numbers m_i are the indices of R (see [5]).

In view of Propositions 3.1 and 3.2, this result is proved in [2, 15] (see also [16]).

Examples. 1. Let R be of type A_l. In this case, the set I consists of pairs (i, j), where $1 \leqslant i < j \leqslant l + 1$, and the family $\mathcal{F}$ consists of vectors $\varepsilon_i - \varepsilon_j$, where $\{\varepsilon_1, \ldots, \varepsilon_{l+1}\}$ is the standard basis in $\mathbf{R}^{l+1}$. We choose the lexicographic order on I: $(i, j) < (i', j')$ if $i < i'$ or if $i = i'$ and $j < j'$. It is not difficult to verify that a subset J of I does not contain open circuits (with respect to this order) if and only if, for each $i = 1, \ldots, l$, there is not more than one vector of the form $\varepsilon_i - \varepsilon_j$. According to Theorem 3.1, the system B of all such l-subsets is fundamental. Hence $|B| = l!$, which agrees with Proposition 5.4.

2. Let R be of type A_l as before. Let w be a permutation of the set $\{1, \ldots, l + 1\}$ such that $w(l + 1) = l + 1$. For each $r = 1, \ldots, l$, we put $i(w, r) = \min(w(r), w(r + 1))$ and $j(w, r) = \max(w(r), w(r + 1))$; we define the l-subset $J_w \subset I$, by putting $J_w = \{(i(w, r), j(w, r)) : r = 1, \ldots, l\}$. It is not difficult to verify that the system of all subsets J_w is fundamental; we note that it is not obtained by means of the construction in Theorem 3.1. It is possible to interpret a result in Lidskii [11] as an explicit calculation of the polynomials $\Psi_J^{(B)}(\mathbf{1}_I; f)$ for this fundamental system.

3. Let R be of type B_l. The set I consists of the symbols $\{i, (i, j)^+, (i, j)^- : 1 \leqslant i < j \leqslant l\}$, and the family $\mathcal{F}$ consists of the vectors $f_i = \varepsilon_i$, $f_{(i,j)^\pm} = \varepsilon_i \pm \varepsilon_j$. We denote the subset of I consisting of the elements i, $(i, j)^+$, and $(i, j)^-$ for fixed i and all possible $j > i$ by I_i. We introduce a certain linear ordering on I such that if $i < i'$, the elements of I_i precede those of $I_{i'}$. It is not difficult to verify that the l-subset J of I does not contain open circuits relative to such an order if and only if J intersects every subset I_i in precisely one element. By virtue of Theorem 3.1, the system B of such l-subsets is fundamental. It is clear that $|B| = \prod_{1 \leqslant i \leqslant l} |I_i| = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2l - 1)$, which again agrees with Proposition 5.4.

4. For type C_l, the set I and the pregeometry on it induced by the family of positive roots is the same as for B_l. In particular, the system B constructed in Example 3 is also fundamental for C_l.

5. Suppose that R is of type G_2. In this case, $\mathcal{F}$ consists of 6 vectors in $\mathbf{R}^2$ in general position. By virtue of Example 2 of the Appendix, the system B of all 2-subsets in I containing an arbitrary fixed element is fundamental. In particular, $|B| = 5$.

Other examples, as well as explicit expressions for $\Psi(\mathbf{1}_I; f)$ related to various choices of fundamental systems, will be considered in a separate article.

APPENDIX. COMBINATORIAL GEOMETRY

The concept of a (combinatorial) pregeometry (or matroid) admits many equivalent definitions ([1, 17, 23]). For our purposes, a definition in terms of rank functions is convenient.

Definition. Let I be a finite set. We say that a pregeometry is given on I if there is defined on the set of all subsets of I an integer-valued function r which satisfies the following conditions:

(i) $0 \leqslant r(J) \leqslant |J|$ for all $J \subset I$;

(ii) $r(J_1) \leqslant r(J_2)$ for $J_1 \subset J_2$;

(iii) $r(J_1 \cap J_2) + r(J_1 \cup J_2) \leqslant r(J_1) + r(J_2)$ for all $J_1, J_2 \subset I$.

The number r(J) is called the rank of J, and r(I) the rank of the pregeometry.

Example. Let L be a vector space over some field. Then to every family of vectors $(f_i)_{i \in I}$ in L there corresponds a pregeometry on I: the rank r(J) is defined to be the dimension of the vector space spanned by the vectors f_j for $j \in J$. Pregeometries of this type are called *linear*. This example makes the terminology introduced below seem natural.

Let I be a pregeometry defined on I with rank function r and rank $r(I) = l$. A subset $J \subset I$ is said to be *independent* if $r(J) = |J|$ and *dependent* if $r(J) < |J|$. Maximal independent subsets in I are called *bases of the pregeometry*; it is well known that every basis has rank l. Minimal dependent subsets of I are called *circuits* (this terminology originates in graph theory).

With every pregeometry on I of rank l is associated a graded commutative superalgebra $\mathcal{A} = \bigoplus_{0 \leqslant m \leqslant l} \mathcal{A}_m$. Indeed, let $\mathcal{E} = \bigoplus_{0 \leqslant m \leqslant |I|} \mathcal{E}_m$ be the Grassman algebra generated by the elements

$(e_i)_{i\in I}$ (over the field $\mathbf{R}$, for definiteness). For each $J \subset I$ we put $[J] = \Lambda^{|J|}(\mathbf{R}^J)$ for brevity, so that $\mathscr{E}_m = \bigoplus_{|J|=m} [J]$. Let $\partial : \mathscr{E} \to \mathscr{E}$ be the (super)differentiation of $\mathscr{E}$ which carries all e_i into 1. We put $\mathscr{I}^0 = \bigoplus [J]$, where the sum is taken over all *dependent* $J \subset I$, and we let $\mathscr{I} = \mathscr{I}^0 + \partial\mathscr{I}^0$; it is easy to see that $\mathscr{I}^0$ and $\mathscr{I}$ are graded ideals in $\mathscr{E}$, and $\mathscr{I}^0_m = \mathscr{I}_m = \mathscr{E}_m$ for $m > l$. We define the algebra $\mathscr{A} = \bigoplus_{0\leqslant m\leqslant l} \mathscr{A}_m$ as the factor algebra $\mathscr{E}/\mathscr{I}$.

We describe a construction of special bases in $\mathscr{A}$. Let m be an integer between 0 and l, and let B be some system of independent m-subsets in I. We call a system *fundamental* if the restriction of the natural projection $\mathscr{E}_m \to \mathscr{A}_m$ to the subspace $\bigoplus_{J\in B} [J]$ gives its isomorphism with $\mathscr{A}_m$ (in other words, if e_J is a generator of the space [J], then the images in $\mathscr{A}_m$ of the elements e_J for $J \in B$ constitute a basis in $\mathscr{A}_m$). In particular, $|B| = \dim \mathscr{A}_m$ is an invariant of the pregeometry.

We choose an arbitrary linear ordering "<" on I. We say that a subset J' of I is an *open circuit* if J' is obtained from some circuit in I by deleting a maximal element; we call $J \subset I$ *proper* if J does not contain any open circuits. It is easy to see that all proper subsets are independent.

<u>THEOREM II.1.</u> For every linear ordering on I and for all $0 \leqslant m \leqslant l$, the system of all proper m-subsets in I is fundamental.

<u>Remarks.</u> The definition of a pregeometry (or matroid) is due to Whitney, MacLane, and Birkhoff (see the historical notes in [17]). The construction of the algebra $\mathscr{A}$ is due to Orlik and Solomon [22], and Theorem II.1 is essentially due to Bjorner [14], but it is described there in different terms. As given in [14], the concept of an open circuit and the combinatorial ideas at the foundation of Theorem II.1 originated with Whitney and Rota. A more general construction of fundamental sets is given in [14], but Theorem II.1 is sufficient for our purposes.

We give a new proof of Theorem II.1 which is independent of [14].

First of all, it is not difficult to show that the proof of the theorem for arbitrary m reduces to the case m = l.

Now let m = l, i.e., B is the family of all proper bases of our pregeometry.

1. Let e_J be a generator of the space [J]. We show that the images of the vectors e_J $(J \in B)$ under the projection $\mathscr{E}_l \to \mathscr{A}_l$ generate all of $\mathscr{A}_l$. In other words, we must show that the subspace $\bigoplus_{J\in B} [J] + \mathscr{I}^0_l + \partial\mathscr{E}_{l+1}$ coincides with the entire space $\mathscr{E}_l$, i.e., contains all of the e_{J^0}, where J^0 is an arbitrary l-subset in I. If J^0 is dependent or $J^0 \in B$, then there is nothing to prove, so we suppose that J^0 is a basis which is not proper. By definition, this means that there are a subset $J' \subset J^0$ and an element $i \in I$ larger than all of the elements of J' such that $J' \cup \{i\}$ is a circuit in I. It is clear that $i \notin J^0$; we put $\hat{J} = J^0 \cup \{i\}$. The element $\partial e_{\hat{J}}$ is a linear combination with nonzero coefficients of elements $e_{\hat{J}\setminus j}$ for $j \in \hat{J}$; in addition, for $j \in J^0 \setminus J'$, the set $\hat{J} \setminus j$ is dependent. Therefore the element e_{J^0} is congruent modulo $\mathscr{I}^0_l + \partial\mathscr{E}_{l+1}$ to a linear combination of elements of the form $e_{\hat{J}\setminus j}$ for $j \in J'$. But it is easy to see that there exists a linear order on the set of l-subsets of I with respect to which all sets $\hat{J} \setminus j$ for $j \in J'$ are less than $J^0 = \hat{J} \setminus i$. Applying induction on J with respect to this order, we get that $e_{\hat{J}\setminus j}$ for $j \in J'$ lies in $\bigoplus_{J\in B} [J] + \mathscr{I}^0_l + \partial\mathscr{E}_{l+1}$; this means that this is also true for e_{J^0}, which is what is required.

2. For each m with $0 \leqslant m \leqslant |I|$, we let $\mathscr{B}_m \subset \mathscr{E}_m$ denote the sum of subspaces of [J] with respect to all m-subsets J of rank < l. It is obvious that $\mathscr{B}_l = \mathscr{I}^0_l$, $\mathscr{B}_m = \mathscr{E}_m$ for m < l and $\partial(\mathscr{B}_m) \subset \mathscr{B}_{m-1}$ for all m. We put $\mathscr{C}_m = \mathscr{E}_m/\mathscr{B}_m$ and we denote the mapping $\mathscr{C}_m \to \mathscr{C}_{m-1}$ induced by ∂ also by ∂. Since $\mathscr{C}_l = \mathscr{E}_l/\mathscr{I}^0_l$, and $\mathscr{A}_l = \mathscr{E}_l/(\mathscr{I}^0_l + \partial\mathscr{E}_{l+1})$, there is a natural projection $\mathscr{C}_l \to \mathscr{A}_l$; we denote it by ε. Then the sequence

$$0 \to \mathscr{C}_{|I|} \xrightarrow{\partial} \mathscr{C}_{|I|-1} \xrightarrow{\partial} \dots \xrightarrow{\partial} \mathscr{C}_l \xrightarrow{\varepsilon} \mathscr{A}_l \to 0. \qquad (*)$$

arises. We assert that this sequence is exact.

The exactness of the terms $\mathscr{A}_l$ and $\mathscr{C}_l$ is obvious. From Folkman's theorem on homologies of geometric lattices [18] it follows easily that the complex $((\mathscr{B}_m), \partial)$ is acyclic everywhere except for $\mathcal{B}_{l-1}$ (see [16], no. 17); in addition, it is known that the complex $((\mathcal{E}_m), \partial)$ is acyclic. The exactness of (*) follows immediately from this with the help of the exact sequence of the pair.

3. It is clear that $\mathcal{C}_m = \mathcal{E}_m/\mathcal{B}_m$ can be identified, as a vector space, with $\oplus [J]$, where the sum is taken over all m-subsets $J \subset I$ of rank l. In particular, $\dim \mathcal{C}_m$ is the number of such subsets; we denote it by r_m. Taking the Euler–Poincaré characteristic of the exact sequence (*), we get that $\dim \mathcal{A}_l = \sum_{m \geq l} (-1)^m r_m$.

4. It remains to show that $|B| = \sum_{m \geq l} (-1)^m r_m$. Let E be the set of all subsets of rank l in I which do not belong to B. For $J \in E$, we put $p(J) = (-1)^{|J|}$. We must show that

$$|\{J \in E: p(J) = 1\}| = |\{J \in E: p(J) = -1\}|. \qquad (**)$$

For the proof of (**), it suffices to construct an involution $\sigma: E \to E$ such that $p(\sigma(J)) = -p(J)$ for $J \in E$.

Suppose that $J \in E$. By definition, J contains an open circuit, i.e., there exists a circuit J^0 with maximal element $i \in I$ such that $J^0 \setminus i \subset J$. We denote the maximal element i having this property by i(J). We define $\sigma(J)$ thus: if $i(J) \notin J$, then $\sigma(J) = J \cup \{i(J)\}$; but if $i(J) \in J$, then $\sigma(J) = J \setminus \{i(J)\}$. Obviously $\sigma(J) \in E$ and $p(\sigma(J)) = -p(J)$. In addition, it follows easily from the construction that $i(\sigma(J)) = i(J)$, from which it is clear that $\sigma(\sigma(J)) = J$. Thus σ is the required involution, which proves (**) and completes the proof of the theorem.

Examples. 1. We suppose that there is an element $i \in I$ such that $r(\{i\}) = 0$. In this case $\mathcal{A} = 0$. In fact, the empty set $\varnothing$ is an open circuit, so there are no proper subsets.

2. Suppose that the rank function is defined by $r(J) = \min(l, |J|)$, so all m-subsets are independent for $m \leqslant l$. In this case, $\mathcal{A}_m = \mathcal{E}_m$ for $m < l$. We can take all l-subsets containing some fixed element $i \in I$ as a fundamental system. In particular, $\dim \mathcal{A}_l = \binom{|I|-1}{l-1}$.

Other examples are given in Sec. 5 of the main text.

Proposition II.1. Suppose that some pregeometry of rank l is given, and suppose that B is a fundamental system of l-subsets in I relative to this pregeometry. Then B can be extended by means of certain independent l-subsets to a fundamental system of l-subsets relative to the pregeometry in Example 2.

This follows directly from the definition.

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LAGRANGIAN IMBEDDINGS OF SURFACES AND UNFOLDED WHITNEY UMBRELLA

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INTRODUCTION

A smooth map $i:C^2 \to (M^4, \omega)$ of a surface into a four-dimensional symplectic manifold will be called *isotropic*, if $i^*\omega = 0$. In this paper we study isotropic maps in general position.

One can show that a closed surface with odd Euler characteristic has no isotropic immersions in symplectic space $\mathbf{R}^4$. Hence an isotropic map of such a surface has singularities. In this paper we produce an isolated singularity of an isotropic map $\mathbf{R}^2 \to (\mathbf{R}^4, \omega)$, an *unfolded Whitney umbrella*, and we prove its stability. By a *Lagrangian immersion* we mean an isotropic map whose singularities are transverse self-intersections and unfolded Whitney umbrellas.

THEOREM. The Lagrangian immersions form a nonempty open set in the C^∞-topology of the space of isotropic maps of a closed surface into a four-dimensional symplectic manifold.

Conjecture. The set of Lagrangian immersions is dense in the space of isotropic maps.

For a Lagrangian immersion $C^2 \to (M^4, \omega)$ of a closed surface we prove the formula

$$C\cdot C = \chi(C) + 2\# + T$$

(in the nonorientable case the equality is modulo 2), where $C\cdot C$ is the self-intersection index of the fundamental cycle of the surface $H_2(M)$, $\chi(C)$ is its Euler characteristic, $\#$ is the number of self-intersection points counted with signs, T is the number of unfolded umbrellas. We show that a closed surface with $\chi \leqslant -2$ has a one-to-one (i.e., without self-intersection points) Lagrangian immersion in a standard symplectic space, and a nonorientable closed surface with Euler characteristic with a negative multiple of 4 even has a nonsingular Lagrangian imbedding in $\mathbf{R}^4$. We note that among orientable surfaces only the torus has a nonsingular Lagrangian imbedding.

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