UDC 513.6

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1. Let A be a finite set of Laurent monomials in the variables $x_1, \dots, x_n$, and let $\mathbf{C}^A$ be the space of complex linear combinations of monomials belonging to A. In [1] we constructed a polynomial $E_A(a)$ from the coefficients $(a_\omega)_{\omega\in A}$ of an undetermined polynomial from $\mathbf{C}^A$, which we call the *principal A-determinant.* It is important both for the study of A-discriminants [1] and independently. The search for an explicit form of the polynomial E_A is of interest from different points of view. Thus, for the case when A consists of the monomials $x_i y_j$, $i \in [1, k]$, $j \in [1, n-k]$, the polynomial E_A is the product of all the minors (of all possible orders) of the matrix $\|a_{ij}\|$. Note that this principal determinant describes the singularities of a hypergeometric system on the Grassmannian $G_k(\mathbf{C}^n)$ [2].

Since E_A is a polynomial, we can construct its Newton polyhedron $M(A) \subset \mathbf{R}^A$. It is "secondary" in relationship to a polyhedron Q_A that is constructed with respect to the same A. The central fact of this note is an unexpected one-to-one correspondence between the triangulations of Q_A and the vertices of $M(A)$.

2. Triangulations of a Newton polyhedron. As in [3], we shall represent the set A as a finite subset of the integer lattice $\mathbf{Z}^n$ that satisfies the following conditions:

1) A generates the group $\mathbf{Z}^n$.

2) There exists a homomorphism of groups $\lambda: \mathbf{Z}^n \to \mathbf{Z}$ such that $\lambda(\omega) = 1$ for all $\omega \in A$.

We denote by $\mathbf{C}^A$ the space of all functions $a: A \to \mathbf{C}$, i.e., the complex vector space with coordinates $(a_\omega)_{\omega\in A}$ (the notation $\mathbf{R}^A$, $\mathbf{Z}^A_+$, etc. has an analogous meaning).

Let $Q = Q_A \subset \mathbf{R}^n$ be the convex hull of the set A. This is an $(n-1)$-dimensional polyhedron in the affine hyperplane $\{u \in \mathbf{R}^n : \lambda(u) = 1\}$. We shall call it the *Newton polyhedron of the set A.*

DEFINITION 1. A *triangulation of Q* with vertices in A is the set T of $(n-1)$-dimensional simplices in Q that possesses the following properties:

1) The vertices of any simplex from T lie in A.

2) The intersection of any two simplices of T is either empty or is their common face.

3) $Q = \bigcup_{\sigma\in T} \sigma$.

We denote by T_0 the set of all vertices of simplices of the triangulation T.

Let T be a triangulation of Q with vertices in A. A function $g: Q \to \mathbf{R}$ is said to be *T-piecewise-linear* if it is continuous and its restriction to each simplex $\sigma \in T$ is a linear (inhomogeneous) function. Each function $\psi: A \to \mathbf{R}$ gives a unique T-piecewise-linear function $g_{\psi,T}: Q \to \mathbf{R}$ such that $g_{\psi,T}(\omega) = \psi(\omega)$ for all $\omega \in T_0$. We let $C(T) \subset \mathbf{R}^A$ denote the cone consisting of those ψ for which $g_{\psi,T}$ is a convex function and $\psi(\omega) \geq g_{\psi,T}(\omega)$ for $\omega \in A - T_0$.

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The triangulation T is said to be *regular* if the cone $C(T)$ has a nonempty interior in the space $\mathbf{R}^A$ (see [3]).

3. Vertices of the polyhedron $M(A)$. We introduce a volume form Vol on Q by setting the volume of an elementary $(n-1)$-dimensional simplex with vertices on the lattice $\{\omega \in \mathbf{Z}^n : \lambda(\omega) = 1\}$ equal to 1. For each triangulation T we define a function $\varphi_T : A \to \mathbf{Z}_+$ by setting $\varphi_T(\omega) = \sum_{\omega \in \sigma \in T} \mathrm{Vol}(\sigma)$.

Let $E_A(a) = E_{A,\mathbf{Z}^n}(a)$ be the principal A-determinant, i.e., the determinant of the logarithmic Cayley-Koszul complex associated to the set A and the lattice $\mathbf{Z}^n$; this is a polynomial function with integer coefficients in $\mathbf{C}^A$, defined up to sign (see [1]). Let $M(A) \subset \mathbf{R}^A$ be the convex hull of the set of functions $\varphi : A \to \mathbf{Z}_+$ for which the monomial $\prod_{\omega \in A} a_\omega^{\varphi(\omega)}$ is included in $E_A(a)$ with a nonzero coefficient.

THEOREM 1. a) *The vertices of the polyhedron* $M(A)$ *are exactly the functions* φ_T *corresponding to all the possible regular triangulations* T *of the polyhedron* Q *with vertices on* A (*in particular, all the* φ_T *are distinct*).

b) *The coefficient of the monomial* $\prod_{\omega \in A} a_\omega^{\varphi_T(\omega)}$ *in* $E_A(a)$ *is equal to*

$$\epsilon(T) \prod_{\sigma \in T} \mathrm{Vol}(\sigma)^{\mathrm{Vol}(\sigma)},$$

where $\epsilon(T) = \pm 1$.

The signs $\epsilon(T)$ will be described in §6.

4. Support cones at the vertices of $M(A)$ **and the asymptotic behavior of** $E_A(a)$. We identify the space $\mathbf{R}^A$ with its dual by means of the pairing $(\varphi, \psi) = \sum_{\omega \in A} \varphi(\omega)\psi(\omega)$. If φ_0 is a vertex of $M(A)$, we call the cone of linear forms ψ on $\mathbf{R}^A$ such that $(\varphi, \psi) \geq (\varphi_0, \psi)$ for all $\varphi \in M(A)$ the *support cone to* $M(A) \subset \mathbf{R}^A$ at φ_0.

THEOREM 2. *The support cone to the polyhedron* $M(A)$ *at the vertex* φ_T *corresponding to a regular triangulation* T *coincides with the cone* $C(T)$ *defined above.*

Let $U(T)$ be the domain in $(\mathbf{C}^*)^A$ consisting of the $a = (a_\omega)_{\omega \in A}$ for which the vector $(-\ln|a_\omega|)_{\omega \in A} \in \mathbf{R}^A$ lies in the cone $C(T)$.

COROLLARY 1. *As* $a \to \infty$ *in* $U(T)$ *the limit of the fraction*

$$E_A(a) \Big/ \left[\epsilon(T) \cdot \prod_{\sigma \in T} \mathrm{Vol}(\sigma)^{\mathrm{Vol}(\sigma)} \cdot \prod_{\omega \in A} a_\omega^{\varphi_T(\omega)}\right]$$

is equal to 1.

REMARK. In [3] and [6] for each regular triangulation T we constructed a basis of the space of solutions of the hypergeometric system, consisting of series whose common domain of convergence is a "neighborhood of infinity" in $U(T)$.

5. Edges of the polyhedron $M(A)$ **and rearrangements of triangulations.** In what follows we denote by $\langle Y \rangle$ the convex hull of the set $Y \subset \mathbf{R}^n$.

A *cycle* in A is a minimal linearly dependent subset $Z \subset A$ (the term comes from the theory of matroids). As shown in [4], for every cycle $Z \subset A$ the polyhedron $\langle Z \rangle$ has exactly two triangulations, $T_+(Z)$, and $T_-(Z)$, with vertices in Z. Namely, the set Z is partitioned in a unique way into two subsets Z_+ and Z_- (up to replacing Z_+ by Z_-) such that $\langle Z_+ \rangle$ and $\langle Z_- \rangle$ intersect in their common interior point. The triangulation $T_+(Z)$ consists of the simplices $\langle Z - \{\omega\} \rangle$, $\omega \in Z_-$, and $T_{-}(Z)$ consists of the simplices $\langle Z - \{\omega\}\rangle$, $\omega \in Z_{+}$. Thus, each simplex of maximal dimension with vertices in Z is included in exactly one triangulation of the polyhedron $\langle Z\rangle$.

DEFINITION 2. Let T be a triangulation of the polyhedron Q with vertices in A, and let $Z \subset A$ be a cycle. We say that the *triangulation T is supported on Z* if the following conditions hold:

1) Inside $\langle Z\rangle$ there are no vertices of T besides the elements of Z itself.

2) The polyhedron $\langle Z\rangle$ is the union of faces of simplices from T.

3) If $\langle I\rangle$ and $\langle I'\rangle$ are maximal (i.e., $\dim\langle Z\rangle$-dimensional) simplices with vertices in Z, imbedded in the same triangulation of the polyhedron $\langle Z\rangle$, and $J \subset A - Z$ is a subset such that $\langle I \cup J\rangle \in T$, then $\langle I' \cup J\rangle \in T$ as well.

In case $\dim\langle Z\rangle = \dim Q$, condition 3) follows from 1) and 2).

Let T be a triangulation supported on a cycle Z, and suppose it induces a triangulation $T_{+}(Z)$ (say) on $\langle Z\rangle$. We denote by $s_Z(T)$ a new triangulation of Q obtained from T by removing all simplices of the form $\langle I \cup J\rangle$ with $\langle I\rangle \in T_{+}(Z)$ and adding in their place simplices of the form $\langle I' \cup J\rangle$ with $\langle I'\rangle \in T_{-}(Z)$ (the fact that $s_Z(T)$ is actually a triangulation follows from conditions 1)–3) and definition 5). We say that $s_Z(T)$ is obtained from T by a rearrangement along the cycle Z. It is clear that $s_Z(T)$ is also supported on Z and $s_Z(s_Z(T)) = T$.

THEOREM 3. *Let T and T' be two regular triangulations of Q with vertices in A. Then the vertices φ_T and $\varphi_{T'}$ of the polyhedron $M(A)$ are joined by an edge if and only if T is obtained from T' by a rearrangement along some cycle $Z \subset A$.*

6. Calculation of the signs $\epsilon(T)$. Let φ_T and $\varphi_{T'}$ be two neighboring vertices of $M(A)$; by Theorem 6, $T' = s_Z(T)$ for some cycle $Z \subset A$. A subset $J \subset A - Z$ is said to be *separating for T and T'* if there exists a simplex of maximal dimension $\langle I\rangle \subset \langle Z\rangle$ such that $I \cup J$ is the set of vertices of a simplex (of maximal dimension) included in the triangulation T. If $\dim\langle Z\rangle = n - 1 = \dim Q$, then $\varnothing$ is the unique separating subset. We set

$$p(T, T') = \sum_J (\mathrm{Vol}\langle Z \cup J\rangle + [\mathbf{Z}^n : \Xi(Z \cup J)]),$$

where the sum is taken over all separating subsets J for T and T', and $\Xi(Z \cup J)$ is the subgroup of $\mathbf{Z}^n$ generated by $Z \cup J$.

THEOREM 4. *If T and T' are two regular triangulations obtained from each other by a rearrangement along some cycle, then the signs $\epsilon(T)$ and $\epsilon(T')$ from Theorem 1 are connected by the relation $\epsilon(T)\epsilon(T') = (-1)^{p(T,T')}$.*

7. **Remarks.** a) The lattice of faces of the polyhedron $M(A)$ also admits a combinatorial-geometric description: the faces of $M(A)$ correspond to some polyhedral subdivisions of the polyhedron Q.

b) Let $D_A(a)$ be the regular A-determinant, i.e., the determinant of the regular Cayley-Koszul complex associated with A (see [1]), and let $M_D(A) \subset \mathbf{R}^A$ be the polyhedron constructed from $D_A(a)$ in the same way as $M(A)$ is constructed from $E_A(a)$. For $D_A(a)$ and $M_D(A)$ one can prove analogues of all the results presented above. The vertices of $M_D(A)$ also correspond to regular triangulations of the polyhedron Q_A; however, in contrast to the case of $M(A)$ different triangulations can correspond to the same vertex of $M_D(A)$.

c) In the case when A is the set of vertices of a convex polygon on the plane, the polyhedron $M(A)$ coincides with the polyhedron introduced by Stasheff in his study of the homotopy associativity of H-spaces [5], [7].

d) In the example of §1, when $E_A(a)$ is the product of all minors of the $k \times (n-k)$ matrix $a = \|a_{ij}\|$, the polyhedron Q is the product of simplices $\Delta^{k-1} \times \Delta^{n-k-1}$, and A the set of all its vertices. Examples of regular triangulations of Q are given in [3]. Even in this case, the problem of finding all regular triangulations of Q and a more explicit combinatorial description of $M(A)$ is a very interesting problem of combinatorial geometry.

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