Докл. Акад. Наук СССР
Том 307 (1989), № 4

Soviet Math. Dokl.
Vol. 40 (1990), No. 1

ALGEBRAIC AND TOPOLOGICAL INVARIANTS OF ORIENTED MATROIDS

UDC 517.58–517.88

I. M. GEL'FAND AND G. L. RYBNIKOV

In a number of questions of geometry and algebra, and recently in the theory of generalized hypergeometric functions (see [1] and [2]) an important role is played by the concept of a hyperplane configuration in l-dimensional real or complex space. The concept of matroid (or oriented matroid in the real case) describes the combinatorial structure of a configuration. For the concept of matroid, its geometric aspect, and generalizations, see [3]–[7].

In this note we extend to arbitrary oriented matroids a number of important constructions and theorems contained in [8] and [9] on real hyperplane configurations. We hope to return to other essential concepts (imaginary cones, double loops and the like [2], [10]) and analyze them from the general point of view of matroids.

1. Matroids and Orlik-Solomon rings.

DEFINITION. A pair $M_0 = (E_0, \mathfrak{C}_0)$, where E_0 is a finite set and $\mathfrak{C}_0$ is a set of nonempty subsets of E_0, is called a *matroid* if the following axioms hold:

(I) If $R_1, R_2 \in \mathfrak{C}_0$ and $R_1 \subseteq R_2$, then $R_1 = R_2$.

(II) If $R_1, R_2 \in \mathfrak{C}_0$, $R_1 \neq R_2$, and $a \in R_1 \cap R_2$, then there exists $T \in \mathfrak{C}_0$ such that $T \subseteq (R_1 \cup R_2)\setminus\{a\}$.

In this context the elements in $\mathfrak{C}_0$ are called *cycles*.

EXAMPLE. Let V be a finite-dimensional linear space over an arbitrary field. A *configuration* S is a finite family $(s_i)_{i\in I}$ of hyperplanes in V going through 0. Let $\mathfrak{C}_0(S)$ be the set of all subsets $R \subseteq I$ such that the intersection of the hyperplanes of $(s_i)_{i\in R}$ is not normal (i.e. codim $\bigcap_{i\in R} s_i < |R|$), but such that an arbitrary proper subfamily intersects normally. It is not hard to check that $M_0(S) = (I, \mathfrak{C}_0(S))$ is a matroid.

Let $M_0 = (E_0, \mathfrak{C}_0)$ be an arbitrary matroid. A set $A \subseteq E_0$ is called *independent* if it does not contain cycles. It is not hard to show that for an arbitrary $B \subseteq E_0$ all maximal independent subsets of B have the same number of elements. This number is called the *rank* of B and is denoted $r_{M_0}(B)$. The *rank of the matroid* M_0 is $r(M_0) = r_{M_0}(E_0)$.

DEFINITION. Let $M_0 = (E_0, \mathfrak{C}_0)$ be a matroid. $\mathscr{A}(M_0)$ the **Z**-algebra given by the generators e_a, $a \in E_0$, and the relations

$$e_a^2 = e_a e_b + e_b e_a = 0, \qquad a, b \in E_0,$$

$$\sum_{i=1}^{k} (-1)^i e_{a_1} \cdots e_{a_{i-1}} e_{a_{i+1}} \cdots e_{a_k} = 0, \qquad \{a_1, \ldots, a_k\} \in \mathfrak{C}_0.$$

We call $\mathscr{A}(M_0)$ the *Orlik-Solomon ring* of M_0.

REMARK. Our definition is equivalent to the one proposed in [9], with the proviso that we consider an algebra not over **C**, but over **Z**.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 33A35.

EXAMPLE. Let $V = \mathbf{C}^l$, and let $S = (s_i)_{i\in I}$ be a hyperplane configuration in V. We set $s = \bigcup s_i$. We select for each hyperplane s_i in S a linear function f_i defining it. Brieskorn [12] proved that the cohomology ring $H^*(V\setminus s, \mathbf{Z})$ is isomorphic to the subring of the ring of regular differential forms on $V\setminus s$ generated by the forms $df_i/2\pi\sqrt{-1}f_i$ and 1. It follows from results in [11] that this subring is isomorphic to $\mathscr{A}(M(S))$, where $df_i/2\pi\sqrt{-1}f_i$ corresponds to e_i. Thus $H^*(V\setminus s, \mathbf{Z}) \simeq \mathscr{A}(M(S))$.

2. Oriented matroids and Varchenko-Gel'fand rings.

DEFINITION. A triple $M = (E, \mathfrak{C}, *)$, where E is a finite set $*: a \mapsto a^*$ is a fixed-point-free involution on E, and $\mathfrak{C}$ is a set of nonempty subsets of E, is called an *oriented matroid* if the following axioms hold:

(0) If $R \in \mathfrak{C}$, then $R^* = \{a^* | a \in R\} \in \mathfrak{C}$ and $R \cap R^* = \varnothing$.

(I) If $R_1, R_2 \in \mathfrak{C}$ and $R_2 \subseteq R_1 \cup R_1^*$, then either $R_2 = R_1$ or $R_2 = R_1^*$.

(II) If $R_1, R_2 \in \mathfrak{C}$, $a \in R_1 \cap R_2^*$, and $R_1 \neq R_2^*$, then there exists $T \in \mathfrak{C}$ such that $T \subseteq (R_1 \cup R_2)\setminus\{a, a^*\}$.

In this context the elements of $\mathfrak{C}$ are called *oriented cycles* [7].

EXAMPLE. Let V be a finite-dimensional space over an ordered field, and $S = (s_i)_{i\in I}$ a hyperplane configuration in V. We denote by $E(S)$ the set of all pairs (i, h), where $i \in I$ and h is one of the two closed half-spaces with boundary s_i. An involution $*$ on $E(S)$ is given by $(i, h)^* = (i, -h)$. We denote by $\mathfrak{C}(S)$ the set of all subsets $R \subseteq E(S)$ such that $\bigcup_{(i,h)\in R} h = V$, $R \cap R^* = \varnothing$, and R is minimal with these properties. Then $M(S) = (E(S), \mathfrak{C}(S), *)$ is an oriented matroid [7].

An oriented matroid $M = (E, \mathfrak{C}, *)$ gives rise to an ordinary matroid. Let $\overline{E}$ be the quotient set $E/*$. We denote by $\overline{B}$ the image of the set $B \subseteq E$ under the natural projection $E \to \overline{E}$. Let $\overline{\mathfrak{C}}$ be the set of subsets of $\overline{E}$ having the form $\overline{R}$, where $R \in \mathfrak{C}$. Obviously $\overline{M}(\overline{E}, \overline{\mathfrak{C}})$ is a matroid.

DEFINITION. Let $M = (E, \mathfrak{C}, *)$ be an oriented matroid, and $P(M)$ the commutative **Z**-algebra given by the generators x_a, $a \in E$, and the relations

$$x_a^2 = x_a, \qquad x_a + x_{a^*} = 1, \qquad a \in E,$$

$$\prod_{a\in R} x_a = 0, \qquad R \in \mathfrak{C}.$$

The ring $P(M)$ is called the *Varchenko-Gel'fand ring* of the oriented matroid M.

EXAMPLE. Let $V = \mathbf{R}^l$, $S = (s_i)_{i\in I}$ a hyperplane configuration in V, and $s = \bigcup s_i$. It follows from the results in [8] that the ring of integer-valued functions on $V\setminus s$ constant on connected components is isomorphic to $P(M(S))$, where $x_{(i,h)}$ corresponds to the characteristic function of $h\setminus s$.

3. Comparison of the rings $P(M)$ and $\mathscr{A}(\overline{M})$. Let $M = (E, \mathfrak{C}, *)$ be an oriented matroid, and $\overline{M} = (\overline{E}, \overline{\mathfrak{C}})$ the corresponding ordinary matroid. The ring $\mathscr{A} = \mathscr{A}(\overline{M})$ has a natural grading $\mathscr{A} = \bigoplus_{r\geq 0} \mathscr{A}^r$ where $\mathscr{A}^r$, consists of linear combinations of products of r generators e_a, $a \in \overline{E}$. The rin g $P = P(M)$ is equipped with the filtration $P^0 \subseteq P^1 \subseteq \cdots \subseteq P$, where P^r consists of polynomials in x_a, $a \in E$, of degree at most r.

We denote by $l(E)$ the set of all finite sequences $(a_1, \ldots, a_k)$, $a_i \in E$. From Theorem 1 in [13] we get

PROPOSITION. *There exists* $\varepsilon: L(E) \to \{0, 1, -1\}$ *such that*

a) $\varepsilon(a_1, \ldots, a_k) = \operatorname{sgn}\sigma \cdot \varepsilon(a_{\sigma(1)}, \ldots, a_{\sigma(k)})$ *for any permutation* $\sigma \in S_k$.

b) $\varepsilon(a_1^*, a_2, \ldots, a_k) = -\varepsilon(a_1, a_2, \ldots, a_k)$.

c) if $a_1, \dots, a_r$ are distinct and the set $\{a_1, \dots, a_r\}$ is independent in $\overline{M}$, then $\varepsilon(a_1, \dots, a_r) \neq 0$, and

d) for any $a_1, \dots, a_r \in E$, $R \in \mathfrak{C}$, and $b \in R$ such that $\varepsilon(b, a_1, \dots, a_r) = 1$ there exists $b' \in R$ such that $\varepsilon(b', a_1, \dots, a_r) = -1$.

We fix such a function ε.

THEOREM 1. *There exists an epimorphism of Abelian groups $F_\varepsilon^r : P^r \to \mathscr{A}^r$ such that $F_\varepsilon^r(x_{a_1} \cdots x_{a_r}) = \varepsilon(a_1, \dots, a_r) e_{a_1} \cdots e_{a_r}$ for any $a_1, \dots, a_r \in E$. The kernel of this epimorphism is P^{r-1}. Thus $\mathscr{A}^r \simeq P^r / P^{r-1}$.*

EXAMPLE. Let $S = (s_i)_{i \in I}$ be a hyperplane configuration in the space $V = \mathbf{R}^l$, and let $S^{\mathbf{C}} = (s_i^{\mathbf{C}})_{i \in I}$, where $s_i^{\mathbf{C}} \subset V^{\mathbf{C}} = \mathbf{C}^l$, be its complexification: let $s = \bigcup_{i \in I} s_i$ and $s^{\mathbf{C}} = \bigcup_{i \in I} s_i^{\mathbf{C}}$. Clearly

$$E(S)/* \simeq E_0, \ \overline{M(S)} \simeq M_0(S) = M_0(S^{\mathbf{C}}), \ \mathscr{A}(M_0(S^{\mathbf{C}})) \simeq H^*(V^{\mathbf{C}} \setminus s^{\mathbf{C}}, \mathbf{Z}).$$

Thus for the case of this configuration Theorem 1 asserts that

$$H^r(V^{\mathbf{C}} \setminus s^{\mathbf{C}}, \mathbf{Z}) \simeq P^r(M(S))/P^{r-1}(M(S)).$$

This is proved in [8].

4. Big Varcheko-Gel′fand ring and faces of an oriented matroid. Let $M = (E, \mathfrak{C}, *)$ be an oriented matroid, and $Q(M)$ the commutative **Z**-algebra given by the generators y_a, $a \in E$, and $z_{\bar{a}}$, $\bar{a} \in \overline{E}$, and the relations $y_a^2 = y_a$, $z_{\bar{a}}^2 = z_{\bar{a}}$, $y_a + y_{a^*} + z_{\bar{a}} = 1$, $y_a z_{\bar{a}} = y_a y_{a^*} = 0$, $a \in E$, and $\prod_{a \in R}(y_a + z_{\bar{a}}) = \prod_{a \in R}(y_{a^*} + z_{\bar{a}})$, $R \in \mathfrak{C}$.

DEFINITION. We call $Q(M)$ the *big Varchenko-Gel′fand ring* of M.

We remark that the ring $P(M)$ is isomorphic to the quotient of $Q(M)$ by the ideal generated by $z_{\bar{a}}$, $\bar{a} \in \overline{E}$.

DEFINITION. Ring-homomorphisms $P(M) \to \mathbf{Z}$ are called *faces* of the oriented matroid M. We say that the face γ_1 *adjoins* the face γ_2 if, for any $a \in E$, $\gamma_2(y_a) = 1$ implies $\gamma_1(y_a) = 1$.

It is clear that any face carries every generator of $Q(M)$ either to 1 or to 0.

EXAMPLE. Let S be a hyperplane configuration in $V = \mathbf{R}^l$. We say that two points of V lie in the same face of S if the segment joining them is contained entirely in every hyperplane s_i meeting it. It follows from the results in [8] that $Q(M(S))$ is isomorphic to the ring of integer-valued functions on V which are constant on each face of S. By this isomorphism $y_{(i,h)}$ corresponds to the characteristic function of the closed halfspace $h \setminus s_i$, and $z_{\overline{(i,h)}}$ to the characteristic function of the hyperplane s_i.

Let $M = (E, \mathfrak{C}, *)$ be an oriented matroid, and Γ_M its set of faces. We remark that any face $\gamma \in \Gamma_M$ carries exactly one generator from every set $\{y_a, z_{\bar{a}}, y_{a^*}\}$, $a \in E$, to 1. For each face γ we define a mapping $\tau_\gamma : E \to \{0, 1, -1\}$ in the following way

$$\tau_\gamma(a) = \begin{cases} 1, & \text{if } \gamma(y_a) = 1, \\ 0, & \text{if } \gamma(z_{\bar{a}}) = 1, \\ -1, & \text{if } \gamma(y_{a^*}) = 1. \end{cases}$$

It is clear that τ_γ uniquely determines γ.

THEOREM 2. *Let $\tau : E \to \{0, 1, -1\}$ be a mapping such that $\tau(a^*) = -\tau(a)$ for any $a \in E$. Then the following statements are equivalent:*

1) $\tau = \tau_\gamma$ *for some $\gamma \in \Gamma_M$.*

2) *For any $R \in \mathfrak{C}$ the set $\tau(R) = \{\tau(a) | a \in R\}$ is either equal to $\{0\}$ or else contains $\{1, -1\}$.*

REMARK. The properties 1) and 2) are also equivalent to $\tau^{-1}(1)$ being a cell in the sense of [7].

DEFINITION. A *chamber* is any face $\omega \in \Gamma_M$ such that $\omega(z_{\bar{a}}) = 0$ for any $a \in E$. The set of chambers of M is denoted by Ω_M.

It is easy to see that chambers are in bijective correspondence with the ring homomorphisms $P(M) \to \mathbf{Z}$.

THEOREM 3. *$Q(M)$ is isomorphic to the ring of integer-valued functions on Γ_M. $P(M)$ is isomorphic to the ring of integer-valued functions on Ω_M.*

5. Salvetti complex of an oriented matroid. Let Φ_M be the set of functions $\varphi : E \to \{1, -1, \sqrt{-1}, -\sqrt{-1}\}$ satisfying:

a) $\varphi(a^*) = -\varphi(a)$, and for every $a \in E$

b) for any $R \in \mathfrak{C}$, either $\varphi(R)$ is equal to $\{\sqrt{-1}, -\sqrt{-1}\}$ or else it contains $\{1, -1\}$.

We define mappings π_0 and π_1 from $\{1, -1, \sqrt{-1}, -\sqrt{-1}\}$ to $\{0, 1, -1\}$ in the following way: $\pi_j(\pm 1) = \pm 1$, $j = 0, 1$, $\pi_0(\pm\sqrt{-1}) = 0$, and $\pi_1(\pm\sqrt{-1}) = \pm 1$. It is easy to see that if $\varphi \in \Phi_M$, then $\pi_0 \varphi = \tau_\gamma$ and $\pi_1 \varphi = \tau_\omega$ for some $\gamma \in \Gamma_M$ and $\omega \in \Omega_M$. In addition, for any pair (γ, ω), where $\gamma \in \Gamma_M$, $\omega \in \Omega_M$, and ω adjoins γ it is possible to define a unique $\varphi \in \Phi_M$ such that $\tau_\gamma = \pi_0 \varphi$ and $\tau_\omega = \pi_1 \varphi$.

REMARK. It is also possible to identify the elements in Φ_M with the homomorphisms of a ring into $\mathbf{Z}$ in a way similar to the definition of Γ_M and Ω_M.

We introduce a partial order relation in Φ_M be setting $\varphi_1 \geq \varphi_2$ if, for any $a \in E$, $\varphi_1(a) = 1$ implies $\varphi_2(a) = 1$ and $\varphi_2(a) = \sqrt{-1}$ implies $\varphi_1(a) = \sqrt{-1}$. We consider the nerve of the ordered set Φ_M. This is the simplicial complex $K(\Phi_M)$ whose vertices are the elements of Φ_M, and whose simplices are its linearly ordered subsets. Let $X(M)$ be a geometric realization of $K(\Phi_M)$. We denote by X_φ the union of the interiors of the simplices having $\varphi \in \Phi_M$ as a maximal vertex, and by $\overline{X}_\varphi$ the closure of X_φ.

THEOREM 4. *Let $\varphi \in \Phi_M$ and $d = r_{\overline{M}}(\varphi^{-1}(\{\sqrt{-1}, -\sqrt{-1}\}))$. There is a homeomorphism $\xi_\varphi : B^d \to \overline{X}_\varphi$ with $\xi_\varphi(\dot{B}^d) = X_\varphi$ and $\xi_\varphi(S^{d-1}) = \bigcup_{\varphi' < \varphi} X_{\varphi'}$, where B^d is the d-dimensional closed sphere, $\dot{B}^d$ its interior, and S^{d-1} its boundary. Thus $(X(M), (\xi_\varphi)_{\varphi \in \Phi_M})$ is a finite regular CW-complex* [15].

DEFINITION. The CW-complex $(X(M), (\xi_\varphi)\varphi \in \Phi_M)$ is called the *Salvetti complex* of the oriented matroid M.

THEOREM 5. *There is an isomorphism of graded rings $\mathscr{A}(\overline{M}) \simeq H^*(X(M), \mathbf{Z})$.*

EXAMPLE. Let $S = (s_i)_{i \in I}$ be a hyperplane configuration in $V = \mathbf{R}^l$, and $s^{\mathbf{C}}$ its complexification. Salvetti [9] constructed a CW-complex X_S imbedded in $V^{\mathbf{C}} \setminus s^{\mathbf{C}}$. We will present a description of X_S communicated to us by V. A. Vasil′ev. Let $i \in I$, and let f_i be a linear function on V defining the hyperplane s_i. We set

$$U_1 = \{v \in V^{\mathbf{C}} \setminus s^{\mathbf{C}} | \operatorname{Re} f_i(v) > 0\}, \quad U_{-1} = \{v \in V^{\mathbf{C}} \setminus s^{\mathbf{C}} | \operatorname{Re} f_i(v) < 0\},$$

$$U_{\sqrt{-1}} = \{v \in V^{\mathbf{C}} \setminus s^{\mathbf{C}} | \operatorname{Re} f_i(v) = 0, \operatorname{Im} f_i(v) > 0\}$$

$$U_{-\sqrt{-1}} = \{v \in V^{\mathbf{C}} \setminus s^{\mathbf{C}} | \operatorname{Re} f_i(v) = 0, \operatorname{Im} f_i(v) < 0\}.$$

The intersections of sets of this type over all $i \in I$ constitute a cellular complex $V^C \backslash S^C$ (cf. [2]). X_S is a CW-complex dual to this cellular complex. It is easy to see that it is homeomorphic to $X(M(S))$. It is shown in [9] that X_S turns out to be the deformation retract of $V^C \backslash S^C$. Hence $H^*(X_S, Z) \simeq H^*(V^C \backslash S^C, Z)$, i.e. Theorem 5 for the oriented matroid $M(S)$ follows from results of Brieskorn, Orlik and Solomon [12], [11].

The authors are grateful to V. A. Vasil′ev and A. V. Zelevinskii for very helpful discussions.

Moscow State University
Schmidt Institute of Earth Physics
Academy of Sciences of the USSR
Moscow

Received 5/MAY/1989

BIBLIOGRAPHY

1. I. M. Gel′fand, Dokl. Akad. Nauk SSSR **288** (1986), 14–18; English transl. in Soviet Math. Dokl. **33** (1986).

2. V. A. Vasil′ev, I. M. Gel′fand, and A. V. Zelevinskii, Funktsional. Anal. i Prilozhen. **21** (1987), no. 1, 23–38; English transl. in Functional Anal. Appl. **21** (1987).

3. D. J. A. Welsh, *Matroid theory*, Academic Press, 1976.

4. Pierre Cartier, Séminaire Bourbaki 1980/81, Lecture Notes in Math., vol. 901, Springer-Verlag, 1981, Exposé 561, pp. 1–22.

5. I. M. Gel′fand and V. V. Serganova, Uspekhi Mat. Nauk **42** (1987), no. 2 (254), 107–134; English transl. in Russian Math. Surveys **42** (1987).

6. Robert C. Bland and Michel Las Vergnas, J. Combinatorial Theory Ser. B**24** (1978), 94–123.

7. Jon Folkman and Jim Lawrence, J. Combinatorial Theory Ser. B**25** (1978), 199–236.

8. A. N. Varchenko and I. M. Gel′fand, Funktsional. Anal. i Prilozhen. **21** (1987), no. 4, 1–18; English transl. in Functional Anal. Appl. **21** (1987).

9. M. Salvetti, Invent. Math. **88** (1987), 603–618.

10. T. V. Alekseevskaya, I. M. Gel′fand, and A. V. Zelevinskii, Dokl. Akad. Nauk SSSR **297** (1987), 1289–1293; English transl. in Soviet Math. Dokl. **36** (1988).

11. Peter Orlik and Louis Solomon, Invent. Math. **56** (1980), 167–189.

12. Egbert Brieskorn, Séminaire Bourbaki 1971/72, Lecture Notes in Math., vol. 317, Springer-Verlag, 1973, Exposé 401, pp. 21–44.

13. Michel Las Vergnas, J. Combinatorial Theory Ser. B **25** (1978), 283–289.

14. P. J. Hilton and S. Wylie, *Homology theory: an introduction to algebraic topology*, Cambridge Univ. Press, 1960.

15. Albert T. Lundell and Stephen Weingram, *The topology of CW complexes*, Van Nostrand, Princeton, N. J., 1969.

Translated by J. KUPLINSKY