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ON THE GENERAL DEFINITION OF A MATROID AND A GREEDOID

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In this note we propose the concept of a (W, P)-matroid, which includes in a natural way the usual concept of a matroid, as well as a large class of greedoids [6], [7]. We were led to formulate this concept while studying certain geometric and analytic questions connected with the general theory of hypergeometric functions [1] and strata in compact homogeneous spaces G/P, on which acts a maximal torus [2]–[4]. It seems to use that it undoubtedly has an independent interest in combinatorics.

Let W be a Coxeter group, i.e. a group with a set of generators R, subject to the relations $r^2 = 1$ for all $r \in R$ and $(r_1r_2)^{m(r_1,r_2)} = 1$ for all $r_1, r_2 \in R$, where $m(r_1, r_2) \in N \cup \infty$ [5].

Suppose that $w \in W$. The minimal number of factors in a decomposition $w = r_1 \cdots r_l$, where $r_i \in R$, is called the *length* of the element w and is denoted by $l(w)$. The Bruhat partial order in W is defined as follows: $w_1 \leqslant w_2$ if there exist $s_1, s_2 \in W$ such that $w_2 = s_1w_1s_2$ and $l(w_2) = l(s_1) + l(w_1) + l(s_2)$.

With each $w \in W$ we associate a new order in W as follows: $w_1 \leqslant^w w_2$ if $w^{-1}w_1 \leqslant w^{-1}w_2$. It is clear that the Bruhat order coincides with $\leqslant^1$.

Let L be an arbitrary subset of the group W. An element $s \in L$ is called *w-minimal in* L if for all $u \in L$ we have $s \leqslant^w u$. We shall say that the subset $L \subseteq W$ *satisfies the minimality condition* if for each $w \in W$ there exists a w-minimal element in L.

DEFINITION 1. A *flag W-matroid* is a pair (W, L), where W is a Coxeter group and L is a subset of W satisfying the minimality condition.

The set L is called the *base set of a flag W-matroid*.

Let P be an arbitrary subset of R. Then W_P denotes the subgroup of W generated by P. Subgroups of the form W_P are called *parabolic subgroups* of W. Let W^P denote the set of left cosets W/W_P. If $\alpha \in W^P$, then the left coset α, regarded as a subset of W, satisfies the minimality condition. For any coset $\alpha \in W^P$ let α_w be a w-minimal element in α. Introduce a partial order $\leqslant^w$ on the set W^P by letting $\alpha \leqslant^w \beta$ if $\alpha_w \leqslant^w \beta_w$. Then the minimality condition makes sense for any subset $L \subseteq W^P$.

DEFINITION 2. A (W, P)*-matroid* is a triple (W, P, L), where W is a Coxeter group, $P \subset R$ is a subset of the generators, and L is a subset of W^P satisfying the minimality condition. The set L is called the *base set of the* (W, P)*-matroid*.

A flag W-matroid (W, L) is called W_P*-invariant* if $W_P \cdot L = L$.

PROPOSITION 1. *Let W be a Coxeter group and $P \subset R$. The natural projection of W on W^P realizes a one-to-one correspondence between the set of W_P-invariant flag W-matroids and the set of (W, P)-matroids.*

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EXAMPLE 1. Let $W = S_n$ be the group of permutations of $I_n = \{1, \dots, n\}$, R the set of transpositions $(i, i+1)$ for $i = 1, \dots, n$, and $P = R \setminus \{(k, k+1)\}$. Then $W_P = \{w \in W \mid w(I_k) = I_k\}$. We show that in this case, the definition of a (W, P)-matroid coincides with the definition of an ordinary matroid for a finite set.

The set W^P of left cosets of the group $W = S_n$ modulo the subgroup P can be naturally identified with the set $B_k(I_n)$ of all k-element subsets of I_n. With this identification, the Bruhat order in $B_k(I_n)$ takes the following form: for any $A, B \in B_k(I_n)$, with $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_k\}$, where $a_1 < \cdots < a_k$ and $b_1 < \cdots < b_k$, we have $A \leqslant B$ if $a_i \leqslant b_i$ for all $i = 1, \dots, k$. An arbitrary element $w \in W$ gives a new linear order in I_n: $w(1) < \cdots < w(n)$, which defines the order $\leqslant^w$ in $B_k(I_n)$.

THEOREM 1. *Suppose that $L \subset B_k(I_n) = W^P$. A triple (W, P, L) is a (W, P)-matroid if and only if L is a base set of some matroid of rank k over the set I_n* [6].

EXAMPLE 2. We shall determine how the concept of a greedoid with a finite alphabet [7] is related to the concept of a (W, P_l)-matroid in the case when $W = S_n$ and

$$P_l = \{(l+1, l+2), \dots, (n-1, n)\}.$$

Suppose that we are given a certain set S (the alphabet). We only consider the case $|S| < \infty$ and identify S with I_n. A *word* is defined to be an arbitrary sequence of letters from S. The *length* of a word α is denoted by $|\alpha|$. An arbitrary collection $\mathfrak{L}$ of words is called a *language*. The language $\mathfrak{L}$ is called a *greedoid* if the following conditions are satisfied:

1) $\varnothing \in \mathfrak{L}$.
2) No word $\alpha \in \mathfrak{L}$ contains repeating letters.
3) For any $\alpha \in \mathfrak{L}$, $\alpha = \beta\gamma$ implies that $\beta \in \mathfrak{L}$.
4) If $\alpha, \beta \in \mathfrak{L}$ and $|\alpha| < |\beta|$, then there exists an $x \in \beta$ such that $\alpha x \in \mathfrak{L}$.

Axiom 4 implies that all maximal words of a greedoid $\mathfrak{L}$ have the same length, which is called the *rank of the greedoid* $\mathfrak{L}$. The maximal words of a greedoid $\mathfrak{L}$ are called *bases*.

Let $\mathfrak{L}$ be a greedoid of rank l with an alphabet I_n. Then the set L of its bases can be naturally identified with a certain subset of the set W^{P_l} of left cosets. On the other hand, to every subset $L \subset W^{P_l}$ one can associate a language $\hat{L}$, consisting of all possible initial segments of words from L.

THEOREM 2. a) *Let L be a base set of a (W, P_l)-matroid. Then $\hat{L}$ is a greedoid.*

b) *Let $\mathfrak{L}$ be a greedoid of rank l with an alphabet I_n such that $i \in \mathfrak{L}$ for any $i \in I_n$. Then the base set L of $\mathfrak{L}$ is a base set for some (W, P_l)-matroid.*

REMARK. For any set $P = R \setminus \{(k_1, k_1+1), \dots, (k_m, k_m+1)\}$, with $k_m \leqslant l$, a (W, P_l)-matroid is W_P-invariant if the corresponding greedoid contains together with each word all words obtained by permuting letters 1 to k_1, k_1+1 to k_2, etc. Thus all matroids connected with the Coxeter group S_n are greedoids.

EXAMPLE 3. Let $J_n = \{1, \dots, n, 1^*, \dots, n^*\}$ and suppose that an involution $*$ is defined on J_n such that $(i)^* = i^*$ and $(i^*)^* = i$. A subset $A \subseteq J_n$ is called *isotropic* if $A \cap A^* = \varnothing$. The set of all isotropic subsets of J_n is denoted by $R(J_n)$ and the set of all k-element isotropic subsets by $R_k(J_n)$.

Let $L \subseteq R_k(J_n)$. The pair (J_n, L) is called a *symplectic matroid of rank k* if the following conditions are satisfied:

1) For any $A, B \in L$ and $a \in A \setminus B$ there exists a $b \notin B$ such that either $(A \setminus \{a\}) \cup \{b\} \in L$, or $(A \setminus \{a, b^*\}) \cup \{a^*, b\} \in L$.

2) For any A, $B \in L$ and $b \in B \setminus (A \cup A^*)$ there exists an $a \in A$ such that $(A \setminus \{a\}) \cup \{b\} \in L$.

The set L is called the *base set of the matroid* (J_n, L).

Let W be a group of permutations of the set J_n, commuting with the involution $*$, i.e. the Weyl group of the Lie algebras sp$(2n)$ and $o(2n+1)$. Then R consists of permutations $r_i = (i, i+1)\cdot(i^*, (i+1)^*)$, for $i = 1, \dots, n-1$, and the transposition $r_n = (n, n^*)$. Let $P = R \setminus \{r_k\}$. Then $W_P = \{w \in W \mid w(I_k) = I_k\}$, and the set W^P can naturally be identified with $R_k(J_n)$. The linear order in the set J_n given by $1 < \cdots < n < n^* < \cdots < 1^*$ induces a partial order on $R_k(J_n)$ as in Example 1. If $R_k(J_n)$ is identified with W^P, this order coincides with the Bruhat order. In an analogous way to Example 1, with every $w \in W$ its own order in $R_k(J_n)$ is associated.

THEOREM 3. *Let $L \subseteq R_k(J_n) = W^P$. A triple (W, P, L) is a (W, P)-matroid if and only if L is a base set of some symplectic matroid of rank k.*

One way of defining a matroid $M(I_n)$ consists in giving a rank function $r\colon B(I_n) \to \mathbf{Z}$; if L is the set of bases of a matroid $M(I_n)$, then the rank function of this matroid has the form $r(A) = \max_{B \in L}|A \cap B|$ [6]. It turns out that a symplectic matroid admits an analogous definition. The role of the Boolean algebra $B(I_n)$ is then played by the family of isotropic subsets $R(J_n)$, which becomes a lattice after a maximal element $\mathbf{1}$ is adjoined to it. We associate to each symplectic matroid (J_n, L) a rank function $r\colon R(J_n) \to \mathbf{Z}$ by letting $r(A) = \max_{B \in L}|A \cap B|$.

A symplectic matroid (J_n, L) is called *loop-free* if for each $a \in J_n$ there exists an $A \in L$ such that $a \in A$.

THEOREM 4. a) *Let (J_n, L) be a loop-free symplectic matroid. Then its rank function satisfies the following conditions:*

1) $0 < r(A) \leqslant |A|$ *for any* $A \in R(J_n) \setminus \varnothing$.

2) $r(A) \leqslant r(B)$ *if* $A \subseteq B$.

3) $r(A) + r(B) \geqslant r(A \cap B) + r(A \cup B)$ *for any* $A, B \in R(J_n)$ *such that* $A \cup B \in R(J_n)$.

b) *Conversely, suppose that $r\colon R(J_n) \to \mathbf{Z}$ satisfies conditions* 1)–3). *Then r is a rank function of some loop-free symplectic matroid* (J_n, L).

REMARK. The lattice $R(J_n) \cup \mathbf{1}$ is dual to the face lattice of an n-dimensional cube. This lattice may be defined axiomatically [8]. Apparently other W-matroids are related to other interesting nondistributive lattices.

EXAMPLE 4. A symplectic matroid of rank k with base set L is called *orthogonal* if for each $A \in L$ and $a \in A$ such that $(A \setminus \{a\}) \cup \{a^*\} \in L$ there exists $b \notin A$ for which $(A \setminus \{a\}) \cup \{b\}$, $(A \setminus \{a\}) \cup \{b^*\} \in L$. The same arguments as in Example 3 show that the concept of an orthogonal matroid is equivalent to the concept of a (W, P)-matroid, where W is the Weyl group of the Lie algebra $o(2n)$ and W_P is a maximal parabolic subgroup.

Let Σ be a root system in $\mathbf{R}^n$, equipped with a nondegenerate inner product $(\cdot,\cdot)$ [5], $\sigma_1, \dots, \sigma_n$ a system of simple roots and W the Weyl group of Σ, i.e. the subgroup of motions in $\mathbf{R}^n$ generated by reflections with respect to the simple roots $\sigma_1, \dots, \sigma_n$. As we know, W is a Coxeter group, so the concept of a (W, P)-matroid makes sense.

Let $P \subseteq \{\sigma_1, \dots, \sigma_n\}$, and let W_P be the corresponding parabolic subgroup. Consider the point $\omega_P \in \mathbf{R}^n$, defined by the conditions

$$\frac{(\omega_P, \sigma_i)}{(\sigma_i, \sigma_i)} = \begin{cases} 1 & \text{for } \sigma_i \notin P, \\ 0 & \text{for } \sigma_i \in P. \end{cases}$$

Since the group W_P is the stabilizer of the point ω_P, we have a map $\mu\colon W^P \to \mathbf{R}^n$ which sends a left coset $w \cdot W_P$ to the point $w(\omega_P)$. Call it the *moment map*. To any subset $L \subseteq W^P$ we associate the polytope Δ_L, equal to the convex hull of the points $\mu(L)$.

DEFINITION 3. The polytope Δ_L, where L is some subset of W^P, is called a *hypersimplex* if all of its edges are parallel to vectors in Σ.

THEOREM 5. *(W, P, L) is a (W, P)-matroid if and only if Δ_L is a hypersimplex.*

REMARK. The order $\leqslant^w$ on the set of vertices of a hypersimplex can be defined geometrically. Let C_w be the convex cone in $\mathbf{R}^n$ consisting of vectors $y = \sum_1^n m_i w(\sigma_i)$ such that $m_i \geqslant 0$ for $i = 1, \dots, n$. Note that $\Delta_W = \bigcap_{w \in W}(\mu(w) - C_w)$. Define in $\mathbf{R}^n$ a partial order $\leqslant^w$, letting $x \leqslant^w y$ if $y - x \in C_w$. The restriction of this order to the set $\mu(W^P)$ of vertices of the hypersimplex Δ_{W^P} coincides with the order $\leqslant^w$ on W^P.

EXAMPLE 1. Consider the polytope Δ_W corresponding to a maximal flag matroid. There exists a bijection between the set of its faces and the set $\bigcup_{P \subseteq R} W^P$, with the dimension of the face corresponding to a left coset $w \cdot W_P$ equal to $|P|$. The polytope Δ_W is called a *Coxeter complex*.

For $W = S_3$, Δ_W is a regular hexagon, while for $W = S_4$, Δ_W is a semiregular polyhedron in $\mathbf{R}^3$ with 24 vertices, 8 hexagonal faces, and 6 square faces.

EXAMPLE 2. Let $M(I_n)$ be a matroid with base set L. To each base $B \in L$ we associate a point δ_B with coordinates $(\delta_B)_i = 0$ if $i \notin B$, and 1 if $i \in B$. Then the hypersimplex Δ_L is the convex hull of the points δ_B.

Let $M_k(I_n)$ be a free matroid of rank k, i.e. $L = B_k(I_n)$. The corresponding hypersimplex $\Delta_{n,k} \subset \mathbf{R}^n$ is given by the constraints $\sum_1^n x_i = k$, $0 \leqslant x_i \leqslant 1$, $i \leqslant n$. The polytope $\Delta_{n,k}$ was considered in [2] and [9]. Note that $\Delta_{n,k}$ is the convex hull of centers of k-dimensional faces of a regular $(n-1)$-dimensional simplex.

Consider the matroid $MF(I_7)$, given by the configuration of all seven points of a projective plane over the field F_2 with 2 elements. Bases of the matroid $MF(I_7)$ are triples of points in general position. Then the corresponding hypersimplex Δ_F in $\mathbf{R}^7$ is given by the conditions $\sum_1^7 x_i = 3$, $0 \leqslant x_i \leqslant 1$, $x_i + x_j + x_k \leqslant 2$, $(i, j, k) \notin L$, and has the symmetry group PGL$(3, F_2)$. It has 28 vertices, 126 edges, and 245 two-dimensional, 238 three-dimensional, 112 four-dimensional, and 21 five-dimensional faces. Analogously, one can construct a series of polytopes with PGL(n, F_q) as a symmetry group.

EXAMPLE 3. Let (J_n, L) be a symplectic matroid of rank k. Then the hypersimplex Δ_L has as its vertices certain vertices of the cube $E_n = \{x \in \mathbf{R}^n \mid |x_i| \leqslant 1\}$ and as edges the edges of E_n or diagonals of its two-dimensional faces. The symplectic matroid constructed from Δ_L will be orthogonal if and only if all edges of the polytope Δ_L are diagonals of two-dimensional faces of the cube.

EXAMPLE 4. The moment map μ can be defined for any finite Coxeter group W, since W is generated by reflections with respect to a finite set of vectors in $\mathbf{R}^n$ [5]. Let $W = H_3$, i.e. W is generated by generators r_1, r_2, r_3 and the relations $(r_1r_2)^5 = (r_2r_3)^3 = (r_1r_3)^2 = 1$. For maximal parabolic subgroups $W_P \subseteq W$, we have the following hypersimplexes: Δ_{W^P} is a dodecahedron if $P = \{r_3, r_2\}$, and an icosahedron if $P = \{r_1, r_2\}$, whereas if $P = \{r_1, r_3\}$, then Δ_{W^P} is the convex hull of midpoints of edges of an icosahedron.

Let (W, P, L) be some (W, P)-matroid. A subgroup $\overline{W} \subseteq W$ conjugate to some parabolic subgroup of W is called a *separator* of the (W, P)-matroid if $L \subset \overline{W} \cdot \alpha$ for some $\alpha \in L$.

PROPOSITION 2. *Let $\overline{W}$ be a separator of the (W, P)-matroid (W, P, L), $L \subseteq \overline{W}sW_P = \overline{W} \cap wW_Pw^{-1}$, and $\overline{L} = \{\alpha w^{-1} \cap \overline{W} \mid \alpha \in L\}$. Then $(\overline{W}, \overline{P}, \overline{L})$ is a $(\overline{W}, \overline{P})$-matroid.*

A (W, P)-matroid is called *nondegenerate* if it does not have a separator other than W.

PROPOSITION 3. *Let W be a Weyl group, (W, P, L) a (W, P)-matroid, and Δ_L a corresponding hypersimplex. Then (W, P, L) is nondegenerate if and only if the dimension of the hypersimplex Δ_L is equal to the number of generators of the group W.*

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ON SUMMABILITY OF EXPANSIONS IN
OF SELFADJOINT OPERA

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Let $A \geqslant 0$ be a selfadjoint operator acting with discret
let $\{e_k\}_1^\infty$ be an orthonormal basis of its eigenvectors, an
corresponding eigenvalues, arranged in increasing order a
many times as its multiplicity. It is assumed that $\sum_{k:\lambda_k \neq 0}$
note we consider a linear topological space $\Phi' \supset \mathfrak{H}$ in wh
$\mathbf{C}^1$ being the complex plane) converges to an element of
describe in terms of the growth of Fourier coefficients
generalized elements in Φ' connected with A. Also, we
investigate the summability of Fourier series expansions w
in Banach spaces in Φ'.

1. Let

$$\Phi_m = \left\{ f \in \mathfrak{H} \mid f = \sum_{k=1}^{m} c_k e_k \quad \forall c_k \in \mathbf{C}^1 \right\},$$

(obviously, Φ is dense in $\mathfrak{H}$ and invariant under A),
continuous antilinear functionals on Φ, with weak conve
$\Leftrightarrow (\langle F^n, f\rangle \to \langle F, f\rangle,\ n \to \infty\ \forall f \in \Phi$ (the symbol over
which the convergence is being considered, and $\langle F, f\rangle$ is
an element f). The correspondence $\mathfrak{H} \ni f \to F_f \in \Phi'$: $\langle F$
the inner product in $\mathfrak{H}$) determines an imbedding $\mathfrak{H} \subset$
imbeddings are dense and continuous. Elements in Φ' are
Let s be the space of all numerical sequences $\{s_k\}_1^\infty$
convergence. The isomorphism $\mathcal{J}$: $F \to \{F_k = \langle F, e_k\rangle\}_1^\infty$
the set of finitary sequences (only finitely many nonzero
Here the operation $\{f_k\}_1^\infty \to \{\lambda_k f_k\}_1^\infty$ corresponds to
extended to a continuous operator $\hat{A}$: $\hat{A}F = J^{-1}\{\lambda_k F_k\}_1^\infty$.
The series $\sum_1^\infty F_k e_k$, where $F_k = \langle F, e_k\rangle$, is called the
$F \in \Phi'$, and the numbers F_k are called its Fourier coeffici
the Fourier series of any generalized element F converge
series $\sum_1^\infty c_k e_k$ converges in Φ' to some $F \in \Phi'$, and $c_k =$
as the space of formal series of the form $\sum_1^\infty F_k e_k$.

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