HYPERGEOMETRIC FUNCTIONS AND TORAL MANIFOLDS

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INTRODUCTION

This paper is related to a series of papers devoted to the general theory of hypergeometric functions but it can be read independently. According to [1], one can define general hypergeometric functions as functions (and more precisely, sections of a line bundle) on a Grassman manifold satisfying a holonomic system of linear differential equations. In [2] a more general system of equations related to a finite-dimensional representation of a complex torus $(\mathbf{C}^*)^n$ was introduced. The present paper is devoted to the detailed study of this system. The basic results were announced in [3].

Suppose given an $n \times N$-matrix of integers χ_{ij}, $1 \le i \le n$, $1 \le j \le N$, which have the following properties:

a) the columns $\chi_j = (\chi_{1j}, \ldots, \chi_{nj})$ of the matrix $\|\chi_{ij}\|$ generate a lattice $\mathbf{Z}^n$;

b) there exist integers $c_1, \ldots, c_n$, such that for all $j \in [1, N]$

$$\sum_{1\le i\le n} c_i \chi_{ij} = 1. \tag{0.1}$$

Condition b) means that all the vectors $\chi_j \in \mathbf{Z}^n$ lie on a primitive affine hyperplane.

One can also consider the matrix $\|\chi_{ij}\|$ from another point of view. We consider the complex torus $H = (\mathbf{C}^*)^n = \{(t_1, \ldots, t_n) \in \mathbf{C}^n\colon t_i \neq 0 \ \forall i\}$. Then the vector χ_j can be considered as a character of the torus H, namely $\chi_j(t_1, \ldots, t_n) = t_1^{\chi_{1j}} \ldots t_1^{\chi_{nj}}$. Here the lattice of characters of the torus H, which will be denoted by $\check{H}$, is identified with $\mathbf{Z}^n$. We consider the N-dimensional vector space $V = \mathbf{C}^N$ with coordinates $(v_1, \ldots, v_N)$. A collection of characters $\{\chi_j \in \check{H}\}$ defines an action of H on V:

$$t(v_1, \ldots, v_N) = (\chi_1(t)v_1, \ldots, \chi_N(t)v_N), \quad t \in H.$$

Condition a) means that the homomorphism $H \to GL(V)$ obtained is an imbedding and condition b) means that its image contains the subgroup of scalar multiples of the identity operator in GL(V). Namely, this subgroup is the image of the one-parameter subgroup of H consisting of elements $(\lambda^{c_1}, \ldots, \lambda^{c_n})$, $\lambda \in \mathbf{C}^*$.

We shall use additive notation to denote the group operation in the group $\check{H}$ (corresponding to multiplication of characters).

Let $\mathfrak{h} = \mathbf{C}^n$ be the Lie algebra of the torus H. We denote by $X_i (1 \le i \le n)$ the natural basis of $\mathfrak{h}$. A linear functional $\beta\colon \mathfrak{h} \to \mathbf{C}$ is defined by a collection $(\beta_1, \ldots, \beta_n)$, where $\beta_i = \beta(X_i)$. In what follows we shall assume that such a functional is fixed. We consider the differential operators $Z_i = L_{X_i} - \beta_i$ on $V (i = 1, \ldots, n)$. Here L_{X_i} is the operator of the Lie derivative along the vector field on V defined by $X_i \in \mathfrak{h}$. Explicitly,

$$Z_i = \Big(\sum_{1\le j\le N} \chi_{ij} v_j \cdot \frac{\partial}{\partial v_j}\Big) - \beta_i.$$

We introduce the lattice of relations among characters. By definition this is the set $L = \{(a_1, \ldots, a_N) \in \mathbf{Z}^N\colon \sum_j a_j \chi_{ij} = 0 \ \forall i \in [1, n]\}$. For any $a = (a_1, \ldots, a_N) \in L$ we consider the differential operator

$$\Box_a = \prod_{a_j > 0} (\partial/\partial v_j)^{a_j} - \prod_{a_j < 0} (\partial/\partial v_j)^{-a_j}$$

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on functions $\Phi(v_1, \ldots, v_N)$.

Definition 1. The system of hypergeometric type connected with a collection of characters $\{\chi_1, \ldots, \chi_N\}$ and a collection of exponents $(\beta_1, \ldots, \beta_n)$ has the form

$$Z_i\Phi(v_1, \ldots, v_N) = 0 \quad (1 \leqslant i \leqslant n); \quad \square_a\Phi = 0 \quad (a \in L). \tag{0.2}$$

Formally the number of equations in this system is infinite since the lattice L contains infinitely many elements. Actually, one can manage with a finite set of equations of the form $\square_a\Phi = 0$.

We formulate the fundamental properties of the system (0.2).

THEOREM 1 [2]. The hypergeometric system (0.2) is holonomic. In particular, its solutions form a locally constant sheaf of finite rank outside a hypersurface in V.

We denote by P the polyhedron in $\mathbf{R}^n = \check{H} \otimes \mathbf{R}$, which is the convex hull of 0 and all χ_j, $j \in [1, N]$. In addition, let Q be the convex hull of the χ_j. Thus, P is a pyramid with vertex 0 and base Q. We introduce a volume form Vol on $\mathbf{R}^n$, with respect to which the volume of the unit cube is equal to n!. Then the volume of any polyhedron with vertices in $\mathbf{Z}^n \subset \mathbf{R}^n$ will be integral.

THEOREM 2. The number of linearly independent solutions of the system (0.2) at a general point is equal to the volume of the polyhedron P.

This theorem will be proved in Sec. 2.

The proof of Theorem 2 is based on the study of the D-module corresponding to the hypergeometric system (0.2). The number of linearly independent solutions of a system at a general point is the multiplicity of occurrence of the zero section of the cotangent bundle T*V in the characteristic cycle of the system (cf. [4]). We calculate the characteristic cycle completely in terms of the volumes of suitable Newton polyhedra.

In Sec. 1 we construct the number of solutions needed explicitly as power series of hypergeometric type whose coefficients are products of Γ-factors.

In the concluding Sec. 3 we give a number of examples showing that many classical hypergeometric series of several variables (Horn, Appel, Lauricelli series, etc.) occur in our scheme. We also show that similarly to the Grassmanian, each Hermitian symmetric space naturally generates a system of type (0.2).

Generally, several approaches to the theory of hypergeometric functions are possible — in terms of differential equations, hypergeometric series, Euler integrals [5], Barnes integrals. In this paper we consider only the first two approaches and do not concern ourselves with integral representations. We only note that among the Euler type integrals associated with systems of the form (0.2) there are the integrals $\int \Pi P_i(t_1, \ldots, t_n)^{\alpha_i} t_1^{\beta_1} \ldots t_n^{\beta_n} dt_1 \ldots dt_n$, where P_i are polynomials, i.e., practically all integrals which arise in quantum field theory. A separate paper will be devoted to these integrals.

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1. Γ-Series

In this section we concern ourselves with the construction of solutions of the system (0.2) as power series. We shall sometimes denote the expression $v_1^{b_1} \ldots v_N^{b_N}$ by v^b.

1.1 Γ-Series and Convergence Conditions. Suppose given a vector $\gamma = (\gamma_1, \ldots, \gamma_N) \in \mathbf{C}^N$. We consider the formal series

$$\Phi_\gamma(v) = \sum_{a \in L} v^{\gamma+a} \Big/ \prod_{1 \leqslant j \leqslant N} \Gamma(\gamma_j + a_j + 1),$$

where Γ is the Euler gamma-function.

LEMMA 1. The series $\Phi_\gamma(v)$ formally satisfies the system (0.2) with collection of exponents $(\beta_1, \ldots, \beta_n)$, where $\beta_i = \sum_j \chi_{ij}\gamma_j$. □

If the collection of exponents β is fixed, then the possible γ run through the affine plane $\Pi(\beta) = \{(\gamma_1, \ldots, \gamma_N) : \sum_j \chi_{ij}\gamma_j = \beta_i \ \forall i \in [1, n]\}$. It is parallel to the linear space $L_C = L \otimes C$.

We shall call a subset $I \subset [1, N]$ a base if the vectors χ_j $(j \in I)$ form a basis for the space $\mathbf{R}^n = \check{H} \otimes \mathbf{R}$. Thus, the set $[1, N]$ acquires the structure of a matroid.

For each $J \subset [1, N]$ we denote by $\mathbf{C}^J$ the coordinate subspace $\{v \in \mathbf{C}^N\colon v_j = 0 \ \forall j \notin J\}$ in $V = \mathbf{C}^N$.

LEMMA 2. The subset $I \subset [1, N]$ is a base if and onlv if the linear functions v_j $(j \in \bar{I} = [1, N] \setminus I)$ form a system of coordinates on $\Pi(\beta)$ i.e., the projection $\Pi(\beta) \to \mathbf{C}^{\bar{I}}$ is bijective.

The proof is obvious.

For each base I we shall denote by $\Pi_Z(\beta, I)$ the set of those $\gamma \in \Pi(\beta)$, for which $\gamma_j \in \mathbf{Z}$ for $j \notin I$. By virtue of Lemma 2, this is a lattice on the affine plane $\Pi(\beta)$.

We shall call the series $\Phi_\gamma(v)$ for $\gamma \in \Pi_Z(\beta, I)$ (multidimensional) Γ-series associated with the base I. We show that each such series is the product of a monomial and an integral power series in the auxiliary variables $x_1, \ldots, x_{N-n}$, which is convergent for small $|x_k|$.

First of all it is clear that $\Phi_\gamma(v) = \Phi_{\gamma+a}(v)$ for $a \in L$. Thus, the number of different Γ-series associated with the base I is equal to the number of orbits of L under the action of translations by $\Pi_Z(\beta, I)$. It is easy to see that one has

LEMMA 3. $|\Pi_Z(\beta, I)/L| = |\det(\chi_{ij})_{1\leq i\leq n,\, j\in I}|$.

Let $A = \{a^{(1)}, \ldots, a^{(N-n)}\}$ be a Z-basis for the free Abelian group L. We set $\Pi_Z^A(\beta, I) = \{(\gamma_1, \ldots, \gamma_N) \in \Pi_Z(\beta, I)\colon$ for $j \notin I$ $\gamma_j = \sum_{1\leq k\leq N-n} \lambda_k a_j^{(k)}$, where $0 \leq \lambda_k < 1\}$. Clearly the set $\Pi_Z^A(\beta, I)$ is a system of representatives in $\Pi_Z(\beta, I)/L$.

We shall say that the base I and the basis $A \subset L$ are compatible if any vector $a = (a_1, \ldots, a_N) \in L_R$, for which $a_j \geq 0$ for $j \notin I$, can be represented in the form $a = \sum_{1\leq k\leq N-n} \lambda_k a^{(k)}$, where all $\lambda_k \geq 0$. It is clear that any base I has an infinite set of Z-bases A which are compatible with it.

With the Z-basis A we associate the auxiliary variables $x_k = v^{a(k)}$, $1 \leq k \leq N - n$.

Proposition 1. If A is compatible with the base I, then each series $\Phi_\gamma(v)$ for $\gamma \in \Pi_Z^A(\beta, I)$ has the form $\Phi_\gamma(v) = v^\gamma \sum_{m_1, \ldots, m_{N-n}\geq 0} c_m x^m$, where the power series $\sum_m c_m x^m$ converges for sufficiently small $|x_k|$.

Proof. By definition the series $\Phi_\gamma(v)$ has the form $v^\gamma \sum_{(m_1, \ldots, m_{N-n})\in \mathbf{Z}^{N-n}} c_m x^m$, where $c_m = \left(\prod_j \Gamma\left(\gamma_j + \sum_k m_k a_j^{(k)} + 1\right)\right)^{-1}$. Let $\gamma \in \Pi_Z^A(\beta, I)$ so all $(\gamma_j + \sum_k m_k a_j^{(k)} + 1)$ for $j \notin I$ are integral. Hence only those coefficients c_m for which $\gamma_j + \sum_k m_k a_j^{(k)} \geq 0$ for $j \notin I$ are nonzero. Let $\gamma_j = \sum_k \lambda_k a_j^{(k)}$, so it follows from the conditions for the compatibility of A with I that $m_k + \lambda_k \geq 0$ for all k. But by hypothesis, $0 \leq \lambda_k < 1$, and the numbers m_k are integral so that one can only get a nonzero coefficient c_m when all $m_k \geq 0$. The convergence of the series $\sum_m c_m x^m$ for sufficiently small $|x_k|$ can be established with the help of standard exstimates. □

It follows from the proposition proved that the series $\Phi_\gamma(v)$ for $\gamma \in \Pi_Z(\beta, I)$ have common nonempty domain of convergence. We describe it in more invariant terms. We introduce the "logarithmic" space $\mathbf{R}^N$ with coordinates $w_1, \ldots, w_N$. For each base $I \subset [1, N]$ and point $w \in \mathbf{R}^N$ we define a linear function $\varphi_{I,w}$ on $\check{H} \otimes \mathbf{R} = \mathbf{R}^N$, which on χ_j, $j \in I$, assumes the values w_j. We define the cone $C(I) \subset \mathbf{R}^N$, consisting of those w for which the inequalities $\varphi_{I,w}(\chi_j) \leqslant w_j \ \forall j \notin I$ hold.

Proposition 2. Let $I \subset [1, N]$ be a base, $\gamma \in \Pi_Z(\beta, I)$. The series $\Phi_\gamma(v)$ converges for those $v \in \mathbf{C}^N$, for which $(-\ln|v_1|, \ldots, -\ln|v_N|) \in \mathbf{R}^N$ lies in a sufficiently far translation of the cone C(I) inside itself.

Proof. For each $k \in \bar{I}$ we denote by $b^{(k)} \in L_Q$ the vector for which $b_j^{(k)} = \delta_{jk}$ for $j \in \bar{I}$; these vectors are well-defined by Lemma 2 and form a basis for L_Q. We set $y_k = v^{b(k)}$, so the series $\Phi_\gamma(v)$ can be represented as the product of a monomial and an integral power series in the variables y_k, which converges for small $|y_k|$ (analogously to the proof of Proposition 1). Listing the conditions for the smallness of $|y_k|$ in terms of the coordinates v_j we get what is needed.

1.2. Triangulations of the Newton Polyhedron and a Basis for the Space of Solutions. We recall that P denotes the polyhedron in $\mathbf{R}^n$, which is the convex hull of the χ_j $(j \in [1, N])$, and zero (cf. Introduction). For a base $I \subset [1, N]$ we denote by $\Delta(I)$ the simplex in $\mathbf{R}^n$, which is the convex hull of χ_j $(j \in I)$ and 0. Speaking loosely we shall sometimes identify the base I and the simplex $\Delta(I)$.

Definition 1. A collection of bases T is called a triangulation of the polyhedron P, if $\bigcup_{I \in T} \Delta(I) = P$ and for any $I_1, I_2 \in T$ the simplices $\Delta(I_1)$ and $\Delta(I_2)$ intersect in a common face (possibly empty).

For a triangulation T we shall denote by C(T) the cone in the logarithmic space $\mathbf{R}^N$, which is the intersection of all the $C(I)$, $I \in T$. By Vert(T) we shall denote the set of vertices of the triangulation T, i.e., $\bigcup_{I \in T} I$.

Let $w \in \mathbf{R}^N$ be a point of the logarithmic space. With it we associate a piecewise-linear function $\varphi_{T,w}: P \to \mathbf{R}$, which is characterized by the following properties:

a) $\varphi_{T,w}$ is continuous and for each base $I \in T$ the restriction $\varphi_{T,w}|_{\Delta(I)}$ is a linear function;

b) for $j \in \mathrm{Vert}(T)$ $\varphi_{T,w}(\chi_j) = w_j$.

Proposition 3. The cone $C(T) \subset \mathbf{R}^N$ consists of those w for which the function $\varphi_{T,w}$ is convex and for $j \notin \mathrm{Vert}(T)$ one has $\varphi_{T,w}(\chi_j) \leqslant w_j$.

The proof is obvious.

Definition 2. A triangulation T of the Newton polyhedron P is called regular if the cone $C(T) \in \mathbf{R}^N$ contains interior points. In other words, T is regular if there exist strictly convex T-piecewise-linear functions on P.

Proposition 4. For any collection of vectors $\{\chi_1, \ldots, \chi_N\}$ satisfying (0.1), there exists a regular triangulation of the Newton polyhedron P.

Proof. We shall construct a triangulation together with a strictly convex piecewise-linear function. Suppose given a collection of real numbers $a_1, \ldots, a_N$. In the space $\mathbf{R}^{N+1} = (\check{H} \otimes \mathbf{R}) \times \mathbf{R}$ we consider the convex hull of the union of the vertical half-lines $[(\chi_j, a_j), (\chi_j, +\infty)]$ and $[(0, 0), (0, +\infty)]$. Under projection to $\check{H} \otimes \mathbf{R}$ the nonvertical faces of this polyhedron give a partition of the polyhedron P into pyramids with vertex at 0 (possibly not simplicial), and the numbers a_j define a convex piecewise-linear function with respect to this partition.

We shall now alter the collection (a_j) stepwise. Let us assume that among the pyramids obtained there is at least one which is not a simplex. Let χ_j be a vertex of it with respect to which this pyramid is not a cone. We decrease the value of a_j slightly. Then the pyramid with which we are concerned separates into cones with vertex χ_j. Continuing in this way we get a partition into simplices together with a strictly convex piecewise-linear function. □

Definition 3. Let T be a triangulation of the Newton polyhedron P. The collection of exponents $\beta = (\beta_1, \ldots, \beta_n)$ is said to be T-nonresonance if the sets $\Pi_Z(\beta, I)$ for $I \in T$ are pairwise disjoint.

For example, it is sufficient that β_i and 1 be linearly independent over $\mathbf{Q}$.

Definition 4. Let T be a triangulation of the polyhedron P, $A = \{a^{(1)}, \ldots, a^{(N-n)}\}$ be a $\mathbf{Z}$-basis for L. We shall say that A and T are compatible if A is compatible with I for all $I \in T$.

Proposition 5. Let T be a regular triangulation of the polyhedron P. Then there exist infinitely many $\mathbf{Z}$-bases of L compatible with T.

Proof. Let $I \subset [1, N]$ be a base. We denote by K(I) the cone in L_R consisting of those $(a_1, \ldots, a_N)$ for which $a_j \geq 0$ for $j \notin I$. The basis A is compatible with I if the cone generated by A contains K(I). We consider the projection $p: (\mathbf{R}^N)^* \to L^*_{\mathbf{R}}$, dual to the natural imbedding $L_{\mathbf{R}} \to \mathbf{R}^N$. We identify $(\mathbf{R}^N)^*$ with the logarithmic space $\mathbf{R}^N$. Then the cone $C(I) \subset (\mathbf{R}^N)^*$ is the preimage under the projection p of the cone $\check{K}(I)$ dual to K(I). If the triangulation is regular, i.e., $\bigcap_{I \in T} C(I)$ has nonempty interior, then $\bigcup_{I \in T} K(I)$ is contained in an open half-space and consequently in a closed convex cone K not containing any lines. Hence we can choose a sufficiently "obtuse" $\mathbf{Z}$-basis of L such that the cone it generates contains K. □

We set $\Pi_Z^A(\beta, T) = \bigcup_{I \in T} \Pi_Z^A(\beta, I)$.

THEOREM 3. Let T be a regular triangulation of the polyhedron P, and $A = \{a^{(1)}, \ldots, a^{(N-n)}\}$ be a $\mathbf{Z}$-basis of L compatible with T. Then each series $\Phi_\gamma(v)$, $\gamma \in \Pi_Z^A(\beta, T)$ is equal to the product of a monomial v^γ by an integral power series in the variables $x_k = v^{a(k)}$, which converges for sufficiently small $|x_k|$. If the collection β is T-nonresonance, then these series are linearly independent and the number of them is equal to the volume of P.

The proof is almost completely contained in the preceding arguments. One can verify the linear independence of the functions Φ^γ not in the space of analytic functions but in the space of formal power series. But in this space the linear independence is obvious since in Φ^γ there occur only monomials of the form v^δ, where $\delta \in \gamma + L$. By virtue of the T-nonresonance condition, in different series there cannot be identical monomials, from which the linear independence follows.

Finally, the assertion that the number of series is equal to Vol P follows immediately from Lemma 3 above since $|\det(\chi_{ij})_{1 \le i \le n, j \in I}| = \mathrm{Vol}\,\Delta(I)$.

Remarks. a) By virtue of Proposition 2 all the series $\Phi_\gamma(v)$, $\gamma \in \Pi_Z^A(\beta, T)$ converge on the set of those $(v_1, \ldots, v_N)$ for which the vector $(-\ln|v_1|, \ldots, -\ln|v_N|)$ lands in a translation of the cone C(T) inside itself by a sufficiently large vector.

b) By Theorem 2 from the Introduction, the Γ-series constructed form a basis for the space of solutions of the system (0.2).

2. Hypergeometric D-Module

2.1. Definition of the Module $\mathcal{M}_\beta$ and Formulation of the Theorems. In this section we study the hypergeometric system (0.2) in detail. For this it is very helpful to use the language of D-modules [4].

We shall denote the space C^N with coordinates $(v_1, \ldots, v_N)$ on which the hypergeometric functions are defined by V. Let D_V be the sheaf of rings of linear differential operators of finite order on V with holomorphic coefficients. The operators Z_i and $\Box_a$ are global sections of D_V. With the hypergeometric system (0.2) there is associated the sheaf of left D_V-modules (or for simplicity the D_V-module) $\mathcal{M} = \mathcal{M}_\beta = D_V / (\Sigma D_V Z_i + \Sigma D_V \Box_a)$.

The most important invariant of a D_X-module $\mathcal{N}$ on the manifold X is its characteristic manifold $SS(\mathcal{N}) \subset T^*X$ and its characteristic cycle $\widehat{SS}(\mathcal{N})$, which is a linear combination of the irreducible components of $SS(\mathcal{N})$ with the multiplicities which arise naturally [4]. In our case $SS(\mathcal{M}_\beta)$ is the set of zeros of the highest symbols of operators from the left ideal generated by Z_i and $\Box_a$. The D_X-module $\mathcal{N}$ is called holonomic if $\dim SS(\mathcal{N}) = \dim X$.

We consider the space V^* dual to V with coordinates $\xi_1, \ldots, \xi_N$ dual to $v_1, \ldots, v_N$. For each $a \in L$ we consider the polynomial $\check{\Box}_a(\xi) = \prod_{a_j>0} \xi_j^{a_j} - \prod_{a_j<0} \xi_j^{-a_j}$, which is the symbol of the differential operator $\Box_a$. By S we denote the algebraic submanifold of V^* defined by the equations $\check{\Box}_a(\xi) = 0\ \forall a \in L$. For each face Γ of the polyhedron Q (possibly $\Gamma = \varnothing$) we consider the submanifold $S(\Gamma) \subset S$ defined by the conditions $\xi_j = 0$ for $\chi_j \notin \Gamma$.

Proposition 1. a) The manifold S is the closure of the orbit of a torus H in V^* passing through the point (1, ..., 1). This orbit is a principal homogeneous space over H.

b) The orbits of H on S are in bijective correspondence with the faces of the polyhedron Q. The closure of the orbit corresponding to the face Γ is S(Γ).

Proof. a) Let Σ be the subsemigroup of $\check{H}$, generated by χ_j. It follows from the definition of S that the ring $\mathbf{C}[S]$ of regular functions on S is isomorphic to the semigroup algebra $\mathbf{C}[\Sigma]$ of the semigroup Σ. Hence dim S = n and S is irreducible. It is clear that the orbit of the point (1, ..., 1) is contained in S. Since the characters χ_j generate $\check{H}$ it is a principal homogeneous space over H. Hence, its closure coincides with S.

b) These assertions follow from standard facts of the theory of toral manifolds [6].

By virtue of (0.1) the manifold S and all the manifolds S(Γ) are conical, i.e., invariant with respect to the action of homotheties in V^*. For a conical submanifold $Y \subset V^*$ there is defined the dual conical manifold $\check{Y} \subset V$. It is characterized by the fact that its projectivization $\mathbf{P}(\check{Y}) \subset \mathbf{P}(V)$ is the manifold projectively dual to $\mathbf{P}(Y) \subset \mathbf{P}(V^*)$ (cf. [7]). Let $V(\Gamma) \subset V$ be the conical manifold dual to $S(\Gamma) \subset V^*$.

THEOREM 4. The characteristic manifold of the D_V-module $\mathcal{M}_\beta$ coincides with the union of the conormal bundles $T^*_{V(\Gamma)}V$ to $V(\Gamma)$, where Γ runs through all faces of the polyhedron Q.

It follows in particular from this theorem that $\mathcal{M}$ is holonomic (Theorem 1 of Introduction) since $\dim T^*_{V(\Gamma)}V = \dim V$ for all Γ. In order to find the characteristic cycle $\widetilde{SS}(\mathcal{M}_\beta)$, we must find the numbers c_Γ equal to the multiplicities of occurrences of $T^*_{V(\Gamma)}V$ in $\widetilde{SS}(\mathcal{M}_\beta)$.

We consider the quotient lattice $\check{H}/\mathbf{Z}\Gamma$, where $\mathbf{Z}\Gamma \subset \check{H}$ is the sublattice generated by characters from Γ. We define two polyhedra $P(\Gamma)$ and $Q(\Gamma)$ in the real space $(\check{H}/\mathbf{Z}\Gamma)\otimes\mathbf{R}$. $P(\Gamma)$ is, by definition, the convex hull of the images of all $\chi_j (1 \le j \le N)$ (and of zero for $\Gamma = \varnothing$), and $Q(\Gamma) \subset P(\Gamma)$ is the convex hull of the images of only those χ_j which do not lie in Γ. The integral structure on the real space $(\check{H}/\mathbf{Z}\Gamma)\otimes\mathbf{R}$ (i.e., the definition of the integral sublattice $\check{H}/\mathbf{Z}\Gamma$ in it) defines a volume form Vol in it with respect to which the volume of the elementary simplex with vertices on the lattice is equal to 1.

THEOREM 5. The multiplicty c_Γ of occurrence of $T^*_{V(\Gamma)}$ in $\widetilde{SS}(\mathcal{M}_\beta)$ is equal to Vol P (Γ) − Vol Q (Γ). In particular the number of linearly independent solutions of the system (0.2) at a generic point is equal to the volume of the polyhedron $P = P(\varnothing)$.

2.2. Proof of Theorem 4. We consider the Fourier transform of the system (0.2), i.e., in the equations we replace v_j by $(-\partial/\partial\xi_j)$, $\partial/\partial v_j$ by ξ_j. Let $\check{\mathcal{M}}_\beta$ be the D_{V^*}-module corresponding to the Fourier transformed system. We denote by $\varphi\colon T^*V^* \to T^*V$ the natural isomorphism consisting of identifying both spaces with $V \times V^*$. By (0.1) the module $\mathcal{M}_\beta$ is monodromic in the sense of [8]. Hence it follows from the paper cited that $\widetilde{SS}(\mathcal{M}_\beta)$ and $SS(\mathcal{M}_\beta)$ can be obtained from $\widetilde{SS}(\check{\mathcal{M}}_\beta)$ and $SS(\check{\mathcal{M}}_\beta)$ by the action of the isomorphism φ.

The module $\check{\mathcal{M}}_\beta$ is a special case of the following construction. Given a smooth algebraic manifold Y with the action of algebraic group G, submanifold $W \subset Y$ which is invariant with respect to G and splits into a finite number of orbits W_α and $\beta\colon \mathfrak{g} \to \mathbf{C}$, a character of the Lie algebra of the group G, then one can define a D_Y-module $\mathcal{N}$ as follows.

Let $J_W \subset \mathcal{O}_Y$ be the sheaf of functions which are zero on W, $e_i \in \mathfrak{g}$ be a basis. We set $\mathcal{N} = D_Y/(D_Y J_W + \Sigma D_Y (L_{e_i} - \beta(e_i)))$, where L_{e_i} is the Lie derivative. In this situation one can assert that $\mathcal{N}$ is holonomic and $SS(\mathcal{N})^1 \subset \bigcup T^*_{W_\alpha}Y$ (for smooth W this is proved in [4, Theorem 5.2.12]; for arbitrary W the proof is analogous). It follows from this that $SS(\mathcal{M}_\beta) \subset \cup T^*_{\check{V}(\Gamma)}V$ if one considers that the conormal bundles to mutually dual conical submanifolds $K \subset V$, $\check{K} \subset V^*$ are carried into one another by the isomorphism $\varphi : T^*V^* \to T^*V$. The fact that there is precise equality will follow from Theorem 5.

2.3. Local Degree of S along S(Γ). Let Γ be a face of the polyhedron Q (possibly $\Gamma = \varnothing$). We define a toral manifold $Y(\Gamma)$. For this, in the lattice $\check{H}/\mathbf{Z}\Gamma$ we consider the subsemigroup $\Sigma(\Gamma)$ generated by the images of the characters χ_j. Let $\mathbf{C}[\Sigma(\Gamma)]$ be the semigroup algebra of this semigroup, i.e., the ring of Laurent polynomials with exponents from $\Sigma(\Gamma)$. We set $Y(\Gamma) = \operatorname{Spec}\mathbf{C}[\Sigma(\Gamma)]$.

LEMMA 1. A small neighborhood of a generic point $s \in S(\Gamma)$ in the manifold S is analytically isomorphic to the product of a ball in $\mathbf{C}^{\dim\Gamma}$ and a neighborhood of the point 0 in the manifold $Y(\Gamma)$.

The proof follows from familiar facts about toral manifolds. Thus, the structure of S along $S(\Gamma)$ is the same as the structure of $Y(\Gamma)$ near 0.

An important invariant of a singular point y on the manifold Y is its local degree, $\deg_y Y$. For Y imbedded in affine space it is defined [9] as the multiplicity of intersection at the point y of the manifold Y and a generic affine subspace of complementary dimension passing through y. Analogously one can define the local degree $\deg_\Lambda Y$ of the manifold Y along an irreducible submanifold Λ (cf. [9]). It depends only on Y and Λ.

Proposition 2. The local degree $\deg_{S(\Gamma)} S = \deg_0 Y(\Gamma)$ is equal to Vol $P(\Gamma)$ − Vol $Q(\Gamma)$.

Proof. We shall assume that among the characters $\chi_1, \ldots, \chi_N$ precisely $\chi_1, \ldots, \chi_m$ do note lie in the face Γ. The lattice $\check{H}/\mathbf{Z}\Gamma$ corresponds to the torus $H(\Gamma) = \operatorname{Spec}\mathbf{C}[\check{H}/\mathbf{Z}\Gamma]$ whose lattice of characters it is. Then the images of the characters $\chi_1, \ldots, \chi_m$ in $\check{H}/\mathbf{Z}\Gamma$ which we denote by χ_j, define an imbedding of the toral manifold $Y(\Gamma)$ in $\mathbf{C}^m$. Let $\overline{Y(\Gamma)} \subset \mathbf{P}^m$ be the closure of $Y(\Gamma)$ in the projective space. The degree of the manifold $\overline{Y(\Gamma)}$ (or $Y(\Gamma)$) is equal to the volume of the polyhedron $P(\Gamma)$ (cf. [10]). In other words, if a generic plane Π of complementary dimension is drawn through the point $0 \in \mathbf{C}^m$, then the sum of the multiplicities of all points of intersection (including 0) is equal to Vol P (Γ). On the other hand, the number of nonzero points of intersection is the number of solutions in the torus $H(\Gamma)$ of a system of equations with Newton polyhedron $Q(\Gamma)$. According to [10] it is equal to Vol $Q(\Gamma)$. Hence $\deg_0 Y(\Gamma) = \mathrm{Vol}\, P(\Gamma) - \mathrm{Vol}\, Q(\Gamma)$.

2.4. Proof of Theorem 5. To find the characteristic cycle we must define [4] a good filtration on the D_V-module $\mathcal{M}$. Let $p\colon D_V \to \mathcal{M} = D_V/(\Sigma D_V Z_i + \Sigma D_V \Box_a)$ be the canonical homomorphism, I be its kernel, $F.D_V$ be the filtration by order of differential operators. It induces good filtrations $F_q I = I \cap F_q D_V$ and $F_q\mathcal{M} = p(F_q D_V)$. Then if we denote the ring of symbols $\mathrm{gr}^F_. D_V = \mathcal{O}_V[\xi_1, \ldots, \xi_N]$ by A, then the A-module $\mathrm{gr}^F \mathcal{M}$ is the quotient module of A by the ideal consisting of highest symbols of all operators from the ideal I. The operators Z_i and $\Box_a$ lie in I. Let $\bar{Z}_i$ and $\bar{\Box}_a \in A$ be their highest symbols.

Proposition 3. The surjection $A/(\bar{Z}_i, \bar{\Box}_a) \to \mathrm{gr}^F_. \mathcal{M}$ is an isomorphism.

In other words, to calculate the characteristic cycle it is not necessary to consider the symbols of differential implications of the equations (0.2). The proof is based on the following lemma.

LEMMA 2. The following equations

$$[Z_i, \Box_a] = c_{ia}\Box_a, \quad [Z_i, Z_j] = [\Box_a, \Box_b] = 0$$

hold, where $c_{ia} \in \mathbf{C}$ are certain constants.

The proof is obvious.

We consider the ring $A/(\bar{\Box}_a)_{a\in L}$. Its spectrum is $V \times S$. The elements $\bar{Z}_1, \ldots, \bar{Z}_n$ form a regular sequence in this ring (i.e., each $\bar{Z}_i$ is not a divisor of 0 in $A/(\bar{\Box}_a, \bar{Z}_1, \ldots, \bar{Z}_{i-1})$). Hence $A/(\bar{\Box}_a, \bar{Z}_i)$ has a Koszul resolution $\bar{K}^\cdot$:

$$\ldots \to A/(\bar{\Box}_a)[\bar{e}_i \wedge \bar{e}_j] \xrightarrow{\bar{d}_{-1}} A/(\bar{\Box}_a)[\bar{e}_i] \xrightarrow{\bar{d}_0} A/(\bar{\Box}_a),$$

where $\bar{e}_1, \ldots, \bar{e}_n$ are anticommuting generators. The differential in the resolution carries $\bar{e}_i$ into $\bar{Z}_i$ and satisfies the Leibnitz rule.

Now we construct a complex $(K^\cdot, F)$ of filtered left D_V-modules such that $\mathrm{gr}^F_. K^\cdot = \bar{K}^\cdot$. We set $K^{-p} = (D_V/\Sigma D_V \Box_a)[e_{i_1} \wedge \ldots \wedge e_{i_p}]$ and we define the differential $d_{1-p}\colon K^{-p} \to K^{1-p}$ by

$$d_{1-p}\left((P \bmod \sum D_V \Box_a)\, e_{i_1} \wedge \cdots \wedge e_{i_p}\right) = \sum_{1\leqslant k\leqslant p} (-1)^{k-1} \left(PZ_{i_k} \bmod \sum D_V \Box_a\right) e_{i_1} \wedge \cdots \wedge \hat{e}_{i_k} \wedge \cdots \wedge e_{i_p}.$$

That this definition is proper follows from Lemma 2. In $K^\cdot$ we introduce the filtration F induced by the filtration by order of differential operators. The equality $\mathrm{gr}^F_. K^\cdot = \bar{K}^\cdot$ follows from the fact that the operators $\Box_a$ have constant coefficients and hence $\mathrm{gr}^F_. (D_V/\Sigma D_V \Box_a) = A/\Sigma A \bar{\Box}_a$. From the exactness of $\mathrm{gr}^F_. K^\cdot = \bar{K}^\cdot$ we get that $K^\cdot$ is a resolution of $\mathcal{M} = D_V/(\Sigma D_V \Box_a + D_V Z_i)$.

LEMMA 3. Differential $d_0 : K^{-1} \to K^0$ is strictly compatible with the filtration, i.e., $d_0(K^{-1}) \cap F_p K^0$ coincides with $d_0(F_p K^{-1})$.

The lemma follows purely formally from the exactness of the sequence $\mathrm{gr}^F_. K^{-2} \to \mathrm{gr}^F_. K^{-1} \to \mathrm{gr}^F_. K^0$ at the middle term.

From Lemma 3 we get that the sequence $\mathrm{gr}^F_. K^{-1} \to \mathrm{gr}^F_. K^0 \to \mathrm{gr}^F_. \mathcal{M} \to 0$ is exact. This also means that $\mathrm{gr}^F_. \mathcal{M} = A/(\bar{\Box}_a, \bar{Z}_i)$. Proposition 3 is proved.

Now we can finish the proof of Theorem 5. By definition the multiplicity c_Γ is equal to the multiplicity of the $A = \mathbf{C}[v, \xi]$-module $\mathrm{gr}\, M = A/(\bar{\Box}_a, \bar{Z}_i)$ along $T^*_{V(\Gamma)}V = T^*_{S(\Gamma)}V^*$ (cf. [4]). Let $s = (\xi_1^0, \ldots, \xi_N^0)$ be a generic point of $S(\Gamma)$. Any $v \in V$ can be considered as an element of $T^*_S V^*$. Here $v \in T^*_{S(\Gamma)}V^*$, if and only if the plane $\Pi_v = \{(\xi_1, \ldots, \xi_N)\colon \sum_{1\leqslant j\leqslant N} \chi_{ij}\xi_j v_j = 0\ \forall i \in [1, n]\}$ contains s and the whole tangent space at s to $S(\Gamma)$. Through s we draw an affine subspace Ξ transverse to $S(\Gamma)$. Then S cuts out of Ξ an affine manifold $Y'(\Gamma)$ which is locally analytically isomorphic to $Y(\Gamma)$ and Π_V cuts out an affine subspace of dimension complementary to $Y(\Gamma)$. The multipicity c_Γ is equal to the multiplicity of intersection at the point s of the manifolds $Y'(\Gamma)$ and $\Pi_V \cap \Xi$ for generic $v \in (T^*_{S(\Gamma)}V^*)_s$. This latter multiplicity is the local degree of $Y'(\Gamma)$. By Proposition 2 it is equal to Vol $P(\Gamma)$ − Vol $Q(\Gamma)$. Theorem 5 is proved.

3. Examples and Applications

In this section we apply the results of Sec. 1 to some classical hypergeometric series of one and several variables. In addition we introduce a new class of hypergeometric series connected with Hermitian symmetric spaces.

In each of the examples considered below we give the following data: (a) the collection of characters $\chi_1, \ldots, \chi_N \in \check{H} \simeq \mathbf{Z}^n$ and the lattice of relations $L \subset \mathbf{Z}^N$; (b) the higher order equations in the hypergeometric system (0.2); (c) the polyhedra P and Q and Vol P, i.e., the number of linearly independent solutions of the system (0.2); (d) the regular triangulation T (one or several) of the polyhedron P; (e) a Z-basis $A = \{a^{(1)}, \ldots, a^{(N-n)}\}$ of L compatible with T, and a collection of variables $x_k = v^{a(k)}$; (f) series $\Phi^\gamma(v)$ for $\gamma \in \Pi_{\mathbf{Z}}^{A}(\beta, T)$ which form a basis for the space of solutions.

We note that as L any primitive sublattice of $\mathbf{Z}^N$ such that $\Sigma a_j = 0$ for $a = (a_1, \ldots, a_N) \in L$ can arise. Here the collection of characters $\chi_1, \ldots, \chi_N$ can be determined uniquely up to isomorphism from L: as χ_j one can take the image of the standard basis vector $e_j \in \mathbf{Z}^N$ in the quotient lattice $\mathbf{Z}^N/L \simeq \mathbf{Z}^n$. Sometimes it is helpful for us to use this realization.

3.1. Series in One Variable. In this point we analyze the case n = N − 1, i.e., rk L = N − n = 1.

(a) $L = \mathbf{Z}a$ for some primitive vector $a \in \mathbf{Z}^N$ with coordinate sum 0. Without loss of generality one can assume that $a = (a_1, \ldots, a_r, -b_1, \ldots, -b_s, 0, \ldots, 0)$, where all a_j, $b_k > 0$, $\Sigma a_j = \Sigma b_k = p$.

(b) There is one higher order equation:

$$\Big[\prod_{1\leq j\leq r} (\partial/\partial v_j)^{a_j}\Big]\Phi(v) = \Big[\prod_{1\leq k\leq s} (\partial/\partial v_{r+k})^{b_k}\Big]\Phi(v).$$

(c) The polyhedron Q is the convex hull of N vectors $\chi_1, \ldots, \chi_N$ on an affine (N − 2)-dimensional plane related by the unique relation $\sum_{1\leq j\leq r} a_j\chi_j = \sum_{1\leq k\leq s} b_k\chi_{r+k}$. It is easy to see that in the normalization we use Vol P = p.

(d) For j = 1, ..., N we set $I_j = [1, N] \setminus \{j\}$. Clearly I_j is a base if and only if $1 \leq j \leq r + s$. There are two triangulations of the polyhedron P: $T_1 = \{I_j\colon 1 \leq j \leq r\}$, $T_2 = \{I_{r+k}\colon 1 \leq k \leq s\}$. It is easy to see that both of them are regular.

(e) By definition the basis $A_1 = \{a\}$ in L is compatible with the triangulation T_1, and the basis $A_2 = \{-a\}$ with the triangulation T_2.

(f) Let β be a T_1-nonresonance collection of exponents. We fix $\gamma^0 = (c_1, \ldots, c_r, -d_1, \ldots, -d_s, e_1, \ldots, e_{N-r-s}) \in \Pi(\beta)$. By definition, for each j = 1, ..., r the set $\Pi_{\mathbf{Z}}^{A_1}(\beta, I_j)$ consists of a_j vectors $\gamma_{j,\nu} = \gamma^0 + \frac{\nu - c_j}{a_j} a$ $(\nu = 0, 1, \ldots, a_j - 1)$, so that $\Pi_{\mathbf{Z}}^{A_1}(\beta, T_1)$ consists of p vectors $\gamma_{j,\nu}$ $(1 \leq j \leq r;\ 0 \leq \nu \leq a_j - 1)$. The series $\Phi_{\gamma_{j,\nu}}(v)$ is proportional to

$$v^{\gamma_{j,\nu}} \cdot \sum_{m\geq 0} \Big(\prod_{1\leq l\leq r} \Gamma\Big(ma_l + c_l + \frac{\nu - c_j}{a_j} a_l + 1\Big) \prod_{1\leq k\leq s} \Gamma\Big(-mb_k - d_k - \frac{\nu - c_j}{a_j} b_k + 1\Big)\Big)^{-1} \cdot v^{am}.$$

We recall that the generalized hypergeometric series ${}_pF_{p-1}$ is defined by the formula

$${}_pF_{p-1}\left(\begin{matrix}\alpha_1, \ldots, \alpha_p\\ \beta_1, \ldots, \beta_{p-1}\end{matrix}; x\right) = \sum_{m\geq 0} \frac{(\alpha_1)_m \cdots (\alpha_p)_m}{(\beta_1)_m \cdots (\beta_{p-1})_m} \frac{x^m}{m!},$$

where $(\alpha)_m = \Gamma(\alpha + m)/\Gamma(\alpha) = \alpha(\alpha + 1)\ldots(\alpha + m - 1)$ (cf. [11]). Using the obvious formulas

$$\left.\begin{aligned} &\Gamma(ma + \alpha) = \Gamma(\alpha)\cdot(\alpha)_{ma}, \quad \Gamma(-ma - \alpha + 1) = (-1)^{ma}\frac{\Gamma(-\alpha + 1)}{(\alpha)_{ma}},\\ &(\alpha)_{ma} = a^{ma}\Big(\frac{\alpha}{a}\Big)_m \Big(\frac{\alpha + 1}{a}\Big)_m \cdots \Big(\frac{\alpha + a - 1}{a}\Big)_m, \end{aligned}\right\} \tag{3.1}$$

it is easy to see that the series $\Phi_{\gamma_{j,\nu}}(v)$ is proportional to

$$v^{\gamma_{j,\nu}}{}_pF_{p-1}\left(\begin{matrix}\alpha_1, \ldots, \alpha_p\\ \beta_1, \ldots, \beta_{p-1}\end{matrix}; x\right),$$

where $x = (-1)^p v^a \prod_k b_k^{b_k} \Big/ \prod_j a_j^{a_j}$, and the parameters $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_{p-1}$ are linear functions of $c_1, \ldots, c_r, d_1, \ldots, d_s$. We shall not write them out explicitly in general due to the cumbersomeness of the formulas; we consider only the case p = 2 in which one gets the classical Gauss function.

3.1.1. Let $N = 4$, $n = 3$, $L = \mathbf{Z}\cdot a$, $a = (1, 1, -1, -1)$. The following functions form a basis in the space of solutions of the system (0.2):

$$v_2^{c_2-c_1} v_3^{c_1-d_1} v_4^{c_1-d_2} {}_2F_1\left(\begin{matrix} d_1-c_1,\ d_2-c_1 \\ c_2-c_1+1 \end{matrix};\ \frac{v_1v_2}{v_3v_4}\right)$$

and

$$v_1^{c_1-c_2} v_3^{c_2-d_1} v_4^{c_2-d_2} {}_2F_1\left(\begin{matrix} d_1-c_2,\ d_2-c_2 \\ c_1-c_2+1 \end{matrix};\ \frac{v_1v_2}{v_3v_4}\right).$$

3.1.2. Let $N = 3$, $n = 2$, $L = \mathbf{Z}\cdot a$, $a = (2, -1, -1)$. The following functions form a basis in the space of solutions of the system (0.2):

$$v_2^{\frac{c_1}{2}-d_1} v_3^{\frac{c_1}{2}-d_2} {}_2F_1\left(\begin{matrix} d_1-c_1/2,\ d_2-c_1/2 \\ 1/2 \end{matrix};\ \frac{v_1^2}{4v_2v_3}\right)$$

and

$$\left(\frac{v_1^2}{v_2v_3}\right)^{1/2} v_2^{\frac{c_1}{2}-d_1} v_3^{\frac{c_1}{2}-d_2} {}_2F_1\left(\begin{matrix} d_1-(c_1-1)/2,\ d_2-(c_1-1)/2 \\ 3/2 \end{matrix};\ \frac{v_1^2}{4v_2v_3}\right).$$

We note that the Gauss functions which arise in this case with special values of the exponents satisfy quadratic transformations (cf. [11]).

<u>3.2. Horn Series</u>. As is known [11], there are 14 complete hypergeometric Horn series in two variables: F_1, F_2, F_3, F_4, G_1, G_2, G_3, H_1, ..., H_7. It turns out that there are only 8 essentially different systems (0.2) corresponding to these series so that some of them really represent one and the same analytic function, i.e., can be obtained from one another by a monomial change of variables and analytic continuation. In each of the cases n = dim H is the number of parameters on which the corresponding series depends and N = n + 2. Here are these 8 groups:

$$\begin{aligned}
&\dim H = 2\colon \{G_3\};\\
&\dim H = 3\colon \{G_1, H_3, H_6\},\ \{H_5\};\\
&\dim H = 4\colon \{F_1, G_2\},\ \{F_4\},\ \{H_1\},\ \{H_4, H_7\};\\
&\dim H = 5\colon \{F_2, F_3, H_2\}.
\end{aligned}$$

As an example we consider the system associated with the series G_1, H_3 and H_6; the other cases can be analyzed analogously. We recall that

$$G_1(\alpha, \beta, \gamma; x, y) = \sum_{m,\, n\geqslant 0} \frac{(\alpha)_{m+n}(\beta)_{n-m}(\gamma)_{m-n}}{m!\, n!} x^m y^n,$$

$$H_3(\alpha, \beta, \gamma; x, y) = \sum_{m,\, n\geqslant 0} \frac{(\alpha)_{2m+n}(\beta)_n}{(\gamma)_{m+n} m!\, n!} x^m y^n,$$

$$H_6(\alpha, \beta, \gamma; x, y) = \sum_{m,\, n\geqslant 0} \frac{(\alpha)_{2m-n}(\beta)_{n-m}(\gamma)_n}{m!\, n!} x^m y^n.$$

(a) Let $N = 5$, $n = 3$, $\chi_1 = (1, 0, 0)$, $\chi_2 = (0, 1, 0)$, $\chi_3 = (0, 0, 1)$, $\chi_4 = (1, -1, 1)$, $\chi_5 = (1, 1, -1) \in \mathbf{Z}^3$. The lattice L of relations among the characters χ_j is generated by the two relations

$$\chi_2 + \chi_4 = \chi_1 + \chi_3,\quad \chi_3 + \chi_5 = \chi_1 + \chi_2;$$

in other words, the vectors $a^{(1)} = (-1, 1, -1, 1, 0)$ and $a^{(2)} = (-1, -1, 1, 0, 1)$ form a basis for L.

(b) The higher order equations corresponding to $a^{(1)}$ and $a^{(2)}$ have the form

$$\frac{\partial^2\Phi}{\partial v_1 \partial v_3} = \frac{\partial^2\Phi}{\partial v_2 \partial v_4},\quad \frac{\partial^2\Phi}{\partial v_1 \partial v_2} = \frac{\partial^2\Phi}{\partial v_3 \partial v_5}.$$

(c) All the χ_j lie in the plane $x_1 + x_2 + x_3 = 1$ in $\mathbf{R}^3$. In Fig. 1 the polygon Q in this plane is shown.

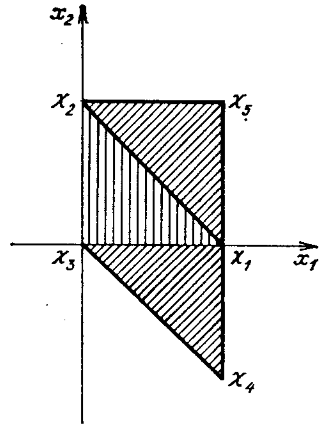

Fig. 1

Clearly Vol P = 3.

(d) The regular triangulation T = {{1, 2, 3}, {1, 3, 4}, {1, 2, 5}} is pictured in Fig. 1.

(e) It is easy to see that the basis $A = \{a^{(1)}, a^{(2)}\}$ of L (cf. (a)) is compatible with the triangulation T. The corresponding variables are: $x = v^{a^{(1)}} = v_2v_4/v_1v_3$, $y = v^{a^{(2)}} = v_3v_5/v_1v_2$.

(f) We fix the vector $\gamma^0 = (\beta_1, \beta_2, \beta_3, 0, 0) \in \Pi(\beta)$. Clearly $\gamma^0 \in \Pi_{\mathbf{Z}}^{A}(\beta, \{1, 2, 3\})$. The series $\Phi_{\gamma^0}(v)$ is proportional to

$$\Phi_1(v) = v_1^{\beta_1} v_2^{\beta_2} v_3^{\beta_3} G_1(-\beta_1, -\beta_2, -\beta_3;\ -x, -y).$$

Further, $\gamma^0 - \beta_3 a^{(2)} = (\beta_1 + \beta_3, \beta_2 + \beta_3, 0, 0, -\beta_3) \in \Pi_{\mathbf{Z}}^{A}(\beta, \{1, 2, 5\})$ and the corresponding series $\Phi_{\gamma^0 - \beta_3 a^{(2)}}(v)$ is proportional to

$$\Phi_2(v) = v_1^{\beta_1+\beta_3} v_2^{\beta_2+\beta_3} v_5^{-\beta_3} H_3(-\beta_1 - \beta_3, -\beta_2 - \beta_3, 1 - \beta_3;\ xy, y).$$

Finally, to the vector $\gamma^0 - \beta_2 a^{(1)} = (\beta_1 + \beta_2, 0, \beta_2 + \beta_3, -\beta_2, 0) \in \Pi_{\mathbf{Z}}^{A}(\beta, \{1, 3, 4\})$ corresponds a series proportional to

$$\Phi_3(v) = v_1^{\beta_1+\beta_2} v_3^{\beta_2+\beta_3} v_4^{-\beta_2} H_3(-\beta_1 - \beta_2, -\beta_2 - \beta_3, 1 - \beta_2;\ xy, x).$$

Thus, the functions $\Phi_1(v)$, $\Phi_2(v)$, $\Phi_3(v)$ form a basis in the space of solutions of the system (0.2) in the domain $0 < |x|, |y| < \varepsilon$ for sufficiently small $\varepsilon > 0$.

We note that the series $\Phi_\gamma(v)$, corresponding to the vector $\gamma = \gamma^0 + \beta_2 a^{(2)} = (\beta_2 - \beta_2, 0, \beta_2 + \beta_3, 0, \beta_2)$, is proportional to

$$\Phi_4(v) = v_1^{\beta_1-\beta_2} v_3^{\beta_2+\beta_3} v_5^{\beta_2} H_6(\beta_2 - \beta_1, -\beta_2, -\beta_2 - \beta_3;\ -xy, y^{-1}),$$

from which it is clear that the series H_6 can be obtained from the series G_1 and H_3 with the help of a change of variables and analytic continuation.

3.3. Systems Connected with Hermitian Symmetric Spaces. Let X = G/P be a compact homogeneous space of a connected, semisimple, complex Lie group G, let $\mathfrak{g} \supset \mathfrak{p}$ be the Lie algebras of the groups G and P, H be a Cartan subgroup of G, and $\mathfrak{n}$ be a supplementary nilpotent subalgebra to $\mathfrak{p}$ in $\mathfrak{g}$. It follows easily from the classification of Hermitian symmetric spaces that the following proposition holds.

Proposition 1. The following conditions are equivalent:

(1) The collection of characters $\chi_\alpha \in \check{H}$ which are roots for the adjoint action of H on $\mathfrak{n}$ satisfy conditions (0.1) from the Introduction.

(2) The algebra $\mathfrak{n}$ is Abelian.

(3) X is a Hermitian symmetric space.

Thus, with each Hermitian symmetric space X there is associated a system (0.2) on the function $\Phi(v)$, $v \in V = \mathfrak{n}$. We note that the space V can be identified naturally with an open Schubert cell in X.

If $X = G_k(\mathbf{C}^{k+l})$ is the Grassmanian (cf. Example 3.3.1 below), then the construction indicated leads to the system of equations from [1] rewritten in local coordinates. Hence, the solutions of systems (0.2) associated with the remaining Hermitian symmetric spaces are natural analogs of the general hypergeometric functions from [1]. Here we shall consider only the classical Hermitian symmetric spaces.

3.3.1. X is the Grassmanian $G_k(\mathbf{C}^{k+l})$.

(a) $N = k\ell$, $n = k + \ell - 1$. The space V consists of $k \times \ell$-matrices $(v_{ij})_{1\le i\le k, 1\le j\le \ell}$, and the action of the complex torus H on V is generated by dilatations of the rows and columns of the matrix (v_{ij}). The lattice $\check{H} \simeq \mathbf{Z}^{k+l-1}$ can be realized as the sublattice $\{(c_1, \ldots, c_k, d_1, \ldots, d_l) \in \mathbf{Z}^k \times \mathbf{Z}^l\colon \ \Sigma c_i = \Sigma d_j\}$ in $\mathbf{Z}^k \times \mathbf{Z}^l$, and the character $\chi_{ij} \in \check{H}$ is equal to $e_i + e_j'$, where $e_1, \ldots, e_k$ is the standard basis in $\mathbf{Z}^k$, and $e_1', \ldots, e_\ell'$ is the standard basis in $\mathbf{Z}^l$. The lattice L is generated by the relations $\chi_{ij} + \chi_{i'j'} = \chi_{ij'} + \chi_{i'j}$ for all $i, i' \in [1, k]$ $j, j' \in [1, l]$.

(b) The higher order equations in the system (0.2) have the form $\frac{\partial^2\Phi}{\partial v_{ij}\partial v_{i'j'}} = \frac{\partial^2\Phi}{\partial v_{ij'}\partial v_{i'j}}$.

(c) The polyhedron Q is the direct product of simplices $\Delta^{k-1} \times \Delta^{l-1}$, $\operatorname{Vol} P = \binom{k+l-2}{k-1}$.

(d) A regular triangulation T of the polyhedron P with $\binom{k+l-2}{k-1}$ simplices of volume 1 is familiar in combinatorial topology [12, Chapter II]. Bases $I \in T$ are indexed by all sequenes $\Omega = (j_1, j_2, \ldots, j_{k+1})$ of natural numbers such that $1 = j_1 \ge j_2 \ge \ldots \ge j_k \ge j_{k+1} = \ell$; the corresponding base I_Ω is $\{(i, j)\colon 1 \le i \le k,\ j_{i+1} \le j \le j_i\}$.

(e) It is easy to see that the basis $A = \{a^{(i,j)} = e_{ij} - e_{i,j+1} - e_{i+1,j} + e_{i+1,j+1} (1 \le i \le k-1,\ 1 \le j \le \ell - 1)\}$ for L is compatible with the triangulation T. The corresponding variables are: $x_{ij} = v_{ij}v_{i+1,j+1}/v_{i+1,j}v_{i,j+1}$.

(f) For simplicity we give explicit formulas for the series $\Phi_\gamma(v)$ for the case $k = 2$ only. In this case T consists of ℓ bases $I_j (1 \le j = j_2 \le \ell)$. We introduce the notation

$$F_{D,j}(\alpha, \beta_1, \ldots, \beta_{l-1}, \gamma; x_1, \ldots, x_{l-1}) = \sum_{m_1, \ldots, m_{l-1} \geqslant 0} \frac{(\alpha)_{-m_1-\ldots-m_{j-1}+m_j+\ldots+m_{l-1}}(\beta_1)_{m_1}\cdots(\beta_{l-1})_{m_{l-1}}}{(\gamma)_{-m_1-\ldots-m_{j-1}+m_j+\ldots+m_{l-1}}} \cdot \frac{x_1^{m_1}\ldots x_{l-1}^{m_{l-1}}}{m_1!\ldots m_{l-1}!};$$

in particular, $F_{D,1} = F_D$ is the Lauricelli function (cf., e.g., [13]). Then the series $\Phi_\gamma(v)$ is proportional to

$$v^\gamma F_{D,j}(-\gamma_{2j}, -\gamma_{21}, \ldots, -\gamma_{2,j-1}, -\gamma_{1,j+1}, \ldots, -\gamma_{1,l}, \gamma_{1j} + 1;$$
$$x_1 x_2 \ldots x_{j-1}, x_2 \ldots x_{j-1}, \ldots, x_{j-1}, x_j, x_j x_{j+1}, \ldots, x_j x_{j+1} \ldots x_{l-1}).$$

We note that the system (0.2) for the function F_D was found in [13]. For $\ell = 2$ one obtains the Gauss function again; for $\ell = 3$ the series $F_{D,1}$ and $F_{D,3}$ are the Appel series F_1, and the series $F_{D,2}$ is the Horn series G_2 [11].

3.3.2. X is a connected component of the manifold of n-dimensional isotropic subspaces of the 2n-dimensional orthogonal space.

(a) $N = n(n-1)/2$, $V = \Lambda^2\mathbf{C}^n = \{(v_{ij}), 1 \le j \le n\}$. The lattice $\check{H} \simeq \mathbf{Z}^n$ can be realized as the sublattice of vectors in $\mathbf{Z}^n$ with even coordinate sum, and the character $\chi_{ij} \in \check{H}$ is equal to $e_i + e_j$. The lattice L is generated by the relations $\chi_{ij} + \chi_{k\ell} = \chi_{ik} + \chi_{j\ell} = \chi_{i\ell} + \chi_{jk}$ $(1 \le i < j < k < \ell \le n)$.

(b) The higher order equations in the system (0.2) are:

$$\frac{\partial^2\Phi}{\partial v_{ij}\partial v_{kl}} = \frac{\partial^2\Phi}{\partial v_{ik}\partial v_{jl}} = \frac{\partial^2\Phi}{\partial v_{il}\partial v_{jk}} \qquad (1 \leqslant i < j < k < l \leqslant n).$$

(c) The polyhedron Q is the hypersimplex $\Delta(1, n-1)$ (cf. [14]), $\operatorname{Vol} P = 2^{n-1} - n$.

(d) The regular triangulation T of 2^{n-1} simplices of volume 1 consists of the basis I_Ω where Ω runs through the sequences $\{1 = i_1 < i_2 < \ldots < i_r \le n\}$ with $r \ge 3$, and

$$I_\Omega = \{(i_p, i)\colon\ 1 \leqslant p \leqslant r-2,\ i_p < i \leqslant i_{p+1}\} \cup \{(i_{r-2}, i_r)\} \cup \{(i, i_r)\colon\ i_{r-1} \leqslant i < i_r\} \cup \{(i_r, i)\colon\ i_r < i \leqslant n\}.$$

In general we have not been able to produce a Z-basis A compatible with T.

3.3.3. X is a Lagrangian Grassmanian in 2n-dimensional sympletic space.

(a) $N = n(n+1)/2$, $V = S^2\mathbf{C}^n = \{(v_{ij}), 1 \leqslant i \leqslant j \leqslant n\}$. The lattice $\check{H}$ is the same as in point 3.3.2, $\chi_{ij} = e_i + e_j$ $(1 \le i \le j \le n)$. The lattice L is generated by the relations $\chi_{ij} + \chi_{k\ell} = \chi_{ik} + \chi_{j\ell}$ for all i, j, k, and ℓ; here one assumes that for $i > j$, $\chi_{ij} = \chi_{ji}$.

(b) The higher order equations in (0.2) have the same form as in point 3.3.1, where it is assumed that $v_{ij} = v_{ji}$. We note that in [3] the system is not given completely.

(c) The polyhedron Q is a simplex with vertices $\chi_{ii} = 2e_i$, Vol P = 2^{n-1}.

(d) As regular triangulation T one can take one base $I = \{(i, i): 1 \le i \le n\}$. Another regular triangulation can be obtained from the triangulation of the hypersimplex in point 3.3.2 by adding n simplices at vertices of Q.

(e) It is easy to verify that the Z-basis $A = \{a^{(ij)} = e_{ij} - e_{i,j-1} - e_{i+1,j} + e_{i+1,j-1}$ $(1 \le i < j \le n)\}$ is compatible with the triangulation T.

(f) We do not write down the series $\Phi_\gamma(v)$ in general due to their awkwardness. If n = 2 we arrive at the series of Example 3.1.2.

3.3.4. $X \subset \mathbf{P}^{2l+3}$ is an even-dimensional quadric.

(a) $N = 2l + 2$, $n = l + 2$, $V = \mathbf{C}^{2l+2} = \{(v_i): \ i = \pm 1, \ \pm 2, \ldots, \pm(l+1)\}$. The character $\chi_{\pm i} \in \check{H} = \mathbf{Z}^{l+2}$ $(1 \leqslant i \leqslant l+1)$ is equal to $e_{\ell+2} \pm e_i$. The lattice L is generated by the relations $\chi_1 + \chi_{-1} = \chi_2 + \chi_{-2} = \ldots = \chi_{\ell+1} + \chi_{-\ell-1}$.

(b) The higher order equations in (0.2) are:

$$\frac{\partial^2\Phi}{\partial v_1 \partial v_{-1}} = \frac{\partial^2\Phi}{\partial v_2 \partial v_{-2}} = \ldots = \frac{\partial^2\Phi}{\partial v_{l+1} \partial v_{-l-1}}.$$

(c) The polyhedron Q is an $(\ell + 1)$-dimensional "octahedron" with center $e_{\ell+2}$, Vol P = 2^ℓ.

(d) A regular triangulation T consists of 2^ℓ bases

$$I_{\varepsilon_1 \ldots \varepsilon_l} = \{\varepsilon_i \cdot i \ (1 \leqslant i \leqslant l), \ \pm(l+1)\}, \ \varepsilon_1, \ldots, \varepsilon_l = \pm 1.$$

(e) The basis $A = \{a^{(k)} = e_k + e_{-k} - e_{\ell+1} - e_{-\ell-1}$ $(1 \le k \le \ell)\}$ of L is compatible with the triangulation T. The corresponding variables are: $x_k = v^{a(k)} = v_k v_{-k}/v_{\ell+1} v_{-\ell-1}$.

(f) For $\gamma \in \Pi_Z^A(\beta, I_{\varepsilon_1 \ldots \varepsilon_l})$ the series $\Phi_\gamma(v)$ is proportional to

$$v^\gamma F_C(-\gamma_{l+1}, -\gamma_{-l-1}; \gamma_{\varepsilon_1 \cdot 1} + 1, \gamma_{\varepsilon_2 \cdot 2} + 1, \ldots, \gamma_{\varepsilon_l \cdot l} + 1; x_1, \ldots, x_l),$$

where

$$F_C(\alpha, \beta; \gamma_1, \ldots, \gamma_l; x_1, \ldots, x_l) = \sum_{m_1, \ldots, m_l \geqslant 0} \frac{(\alpha)_{m_1+\ldots+m_l} (\beta)_{m_1+\ldots+m_l}}{(\gamma_1)_{m_1} \cdots (\gamma_l)_{m_l}} \cdot \frac{x_1^{m_1} \ldots x_l^{m_l}}{m_1! \ldots m_l!}$$

is the Lauricelli function (cf., e.g., [13]). For $\ell = 2$ the series F_C becomes the Appel series F_4 ([11]).

3.3.5. $X \subset \mathbf{P}^{2l+2}$ is an odd-dimensional quadric.

(a) $N = 2l + 1$, $n = l + 1$, $V = \mathbf{C}^{2l+1} = \{(v_i), -l \leqslant i \leqslant l\}$, $\check{H} = \mathbf{Z}^{l+1}$. The characters are: $\chi_0 = e_{\ell+1}$, $\chi_{\pm i} = e_{\ell+1} \pm e_i (1 \le i \le \ell)$. The lattice L is generated by the relations $\chi_1 + \chi_{-1} = \chi_2 + \chi_{-2} = \ldots = \chi_\ell + \chi_{-\ell} = 2\chi_0$.

(b) The higher order equations in (0.2) are:

$$\frac{\partial^2\Phi}{\partial v_i \partial v_{-i}} = \frac{\partial^2\Phi}{\partial v_0^2} \qquad (1 \leqslant i \leqslant l).$$

(c) The polyhedron Q is the same as for a 2ℓ-dimensional quadric (point 3.3.4), Vol P = 2^ℓ (owing to a different normalization of the volume compared with point 3.3.4).

(d) A regular triangulation T_1 of $2^{\ell-1}$ simplices of volume 2 is constructed in point 3.3.4. Another regular triangulation T_2 (of 2^ℓ simplices of volume 1) consists of the bases $I_{\varepsilon_1 \ldots \varepsilon_\ell} = \{\varepsilon_i \cdot i \ (1 \le i \le \ell), 0\}$, $\varepsilon_1, \ldots, \varepsilon_\ell = \pm 1$.

(e) The basis $A = \{a^{(k)} = e_k + e_{-k} - 2e_0, 1 \le k \le \ell\}$ is compatible with the triangulation T_2. The corresponding variables are: $x_k = v^{a(k)} = v_k v_{-k}/v_0^2$.

(f) For $\gamma \in \Pi_Z^A(\beta, I_{\varepsilon_1 \dots \varepsilon_l})$ the series $\Phi_\gamma(v)$ is proportional to $v^\gamma F_C\left(-\frac{\gamma_0}{2}, \frac{1-\gamma_0}{2}; \gamma_{\varepsilon_1 \cdot 1} + 1, \dots, \gamma_{\varepsilon_\ell \cdot \ell} + 1; 4x_1, 4x_2, \dots, 4x_\ell\right)$, i.e., it is the Lauricelli series F_C with one linear relation between the parameters.

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COBORDISMS OF SYMPLECTIC AND CONTACT MANIFOLDS

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The classes of symplectic and contact cobordisms of symplectic and contact manifolds defined below form groups. In this paper we shall evaluate the group of symplectic cobordisms of two-dimensional manifolds: it is isomorphic to $\mathbf{Z} \oplus \mathbf{R}$. It will be proved that if the manifold is tightened by a film of Euler characteristic zero, then the standard contact structure on the spherization of its tangent bundle is cobordant to zero. It will also be shown that the image of the group of symplectic cobordisms of two-dimensional manifolds with integer symplectic structures under the homomorphism onto the compact contactization, defined below in Sec. 6, is zero.

We shall assume throughout that all manifolds are oriented.

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1. Definition 1. An odd-dimensional manifold with a closed 2-form whose kernel is one-dimensional at every point will be called 1-symplectic (or quasisymplectic).

Any such manifold (M, ω) with symplectic boundary $(\partial M, \omega/\partial M)$ symplectomorphic to $(B_0, \Omega_0) \sqcup (-B_1, \Omega_1)$ is called a symplectic cobordism betwen the symplectic oriented manifolds (B_0, Ω_0) and (B_1, Ω_1) (the orientations of B and Ω^n, $2n = \dim B$, may be different). Notation: $(B_0, \Omega_0) \sim (B_1, \Omega_1)$. Symplectic cobordism is an equivalence relation.

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