# 1.

(with M. L. Tsetlin)

# Finite-dimensional representations of the group of unimodular matrices

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1. Let $K_n$ be a Lie algebra of all matrices of order $n$, i.e. the set of all $n \times n$ matrices $e, f, \ldots$ for which the operations of addition and multiplication by scalars are defined in the usual way and the commutator is defined by the formula $[f, g] = fg - gf$.

We say that a finite-dimensional representation of the Lie algebra $K_n$ is given if for each $f \in K_n$ there is a matrix $F$ of order $N$ defined in such a way that $f \to F$ and $g \to G$ implies that $\lambda f + \mu g \to \lambda F + \mu G$ and $[f, g] \to [F, G]$. The aim of this paper is to find all irreducible representations of $K_n$. A system of invariants which uniquely define a representation has been given by Cartan. He has also shown that it is feasible to construct for each such system of invariants (so-called highest weight of the representation) the representation which corresponds to that system (i.e., to the given highest weight). However, one could not say that the problem of describing each representation effectively had been solved[1].

In this paper we give explicit formulas which effectively define the representation.

Denote by $e_{ik}$ the matrix of order $n$ which has 1 at the intersection of the $i$-th row and $k$-th column and zeros in all other places. Denote by $E_{ik}$ the matrix of order $N$ which, under our representation, corresponds to the element $e_{ik} \in K_n$. Since each matrix in $K_n$ is a linear combination of $e_{ik}$, the set of $E_{ik}$ uniquely defines the representation. Thus the task of finding all irreducible representations of $K_n$ can be reformulated in a very simple way: *find $n^2$ matrices $E_{ik}$ $(i, k = 1, \ldots n)$ of order $N$ satisfying the following conditions:*

$$[E_{ik}, E_{kl}] = E_{il} \quad (i \neq l), \qquad [E_{ik}, E_{ki}] = E_{ii} - E_{kk},$$
$$[E_{i_1 k_1}, E_{i_2 k_2}] = 0 \qquad \text{for } k_1 \neq i_2 \text{ and } i_1 \neq k_2.$$

The system of matrices $E_{ik}$ is required to be irreducible (which means that there is no subspace invariant for all $E_{ik}$).

The effective enumeration of all representations is now achieved by actual explicit description of linear transformations (matrices) $E_{ik}$.

2. Let us now proceed to the enumeration of all representations. For the sake of clarity we first consider the formulas for $n = 2$ and $n = 3$.

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[1] To see that the existing construction of irreducible representations is far from perfect, it is sufficient to say that the dimensions of irreducible representations have not been computed from their constructions but have been obtained much later from completely different considerations.

The case $n=2$ is trivial and has been well known for a long time. The answer here is: each representation is defined by two integers $m_1$ and $m_2$ ($m_1 \geqq m_2$). The operators $E_{ik}$ ($i,k=1,2$) act in the space $R$ and one can choose such a basis consisting of vectors $\xi_q$ (where $q$ is an integer such that $m_1 \geqq q \geqq m_2$) in $R$ in which the representation is given by the formulas

$$\begin{aligned} &E_{11}\xi_q = q\xi_q, \qquad E_{22}\xi_q = (m_1+m_2-q)\xi_q, \\ &E_{12}\xi_q = \sqrt{(m_1-q)(q-m_2+1)}\,\xi_{q+1}, \\ &E_{21}\xi_q = \sqrt{(m_1-q+1)(q-m_2)}\,\xi_{q-1}. \end{aligned} \tag{1}$$

Let us now write the formulas for $n=3$. It turns out that in that case one can also write formulas similar to formulas (1) in the case $n=2$. Each irreducible representation for $n=3$ is defined by three integers $m_1, m_2, m_3$ ($m_1 \geqq m_2 \geqq m_3$). Vectors of the basis in the representation space $R$ (where the operators $E_{ik}$ act) are now conveniently numbered by triples $p_1, p_2, q$ (and not by the sole integer $q$ as in (1)). We shall denote vectors of that basis in the following way: $\begin{pmatrix} p_1 & & p_2 \\ & q & \end{pmatrix}$. Here $p_1, p_2, q$ are any integers which satisfy the inequalities

$$m_3 \leqq p_2 \leqq m_2 \leqq p_1 \leqq m_1$$

and

$$p_2 \leqq q \leqq p_1.$$

The representation is now defined by the following formulas:

$$\begin{aligned}
&E_{11}\begin{pmatrix} p_1 & & p_2 \\ & q & \end{pmatrix} = q\begin{pmatrix} p_1 & & p_2 \\ & q & \end{pmatrix}, \\
&E_{22}\begin{pmatrix} p_1 & & p_2 \\ & q & \end{pmatrix} = (p_1+p_2-q)\begin{pmatrix} p_1 & & p_2 \\ & q & \end{pmatrix}, \\
&E_{12}\begin{pmatrix} p_1 & & p_2 \\ & q & \end{pmatrix} = \sqrt{(p_1-q)(q-p_2+1)}\begin{pmatrix} p_1 & & p_2 \\ & q+1 & \end{pmatrix}, \\
&E_{21}\begin{pmatrix} p_1 & & p_2 \\ & q & \end{pmatrix} = \sqrt{(p_1-q+1)(q-p_2)}\begin{pmatrix} p_1 & & p_2 \\ & q-1 & \end{pmatrix}, \\
&E_{23}\begin{pmatrix} p_1 & & p_2 \\ & q & \end{pmatrix} = \sqrt{\frac{(m_1-p_1)(m_2-p_1-1)(m_3-p_1-2)(p_1-q+1)}{(p_1-p_2+2)(p_1-p_2+1)}}\begin{pmatrix} p_1+1 & & p_2 \\ & q & \end{pmatrix} \\
&\qquad + \sqrt{\frac{(m_1-p_2+1)(m_2-p_2)(m_3-p_2-1)(p_2-q)}{(p_1-p_2+1)(p_1-p_2)}}\begin{pmatrix} p_1 & & p_2+1 \\ & q & \end{pmatrix}, \\
&E_{32}\begin{pmatrix} p_1 & & p_2 \\ & q & \end{pmatrix} = \sqrt{\frac{(m_1-p_1+1)(m_2-p_1)(m_3-p_1-1)(p_1-q)}{(p_1-p_2+1)(p_1-p_2)}}\begin{pmatrix} p_1-1 & & p_2 \\ & q & \end{pmatrix} \\
&\qquad + \sqrt{\frac{(m_1-p_2+2)(m_2-p_2+1)(m_3-p_2)(p_2-q)}{(p_1-p_2+2)(p_1-p_2+1)}}\begin{pmatrix} p_1 & & p_2-1 \\ & q & \end{pmatrix}, \\
&E_{33}\begin{pmatrix} p_1 & & p_2 \\ & q & \end{pmatrix} = (m_1+m_2+m_3-p_1-p_2)\begin{pmatrix} p_1 & & p_2 \\ & q & \end{pmatrix}.
\end{aligned} \tag{2}$$

We see that the formulas for $E_{11}, E_{22}, E_{21}, E_{12}$ coincide with the formulas (1) for $n = 2$. Due to lack of space, we do not write formulas for $E_{13}$ and $E_{31}$. They can be obtained from the relations $E_{13} = [E_{12}, E_{23}]$, $E_{31} = [E_{32}, E_{21}]$.

3. Now we write explicit formulas for irreducible representations for arbitrary $n$. We have, therefore, to define linear transformations $E_{ik}$ $(i, k = 1, \dots, n)$ which act in some linear space $R$. Each representation is defined by $n$ integers $m_1, m_2, \dots, m_n$ $(m_1 \geqq m_2 \geqq \dots \geqq m_n)$. These numbers coincide with the highest weight as defined by Cartan.

Each vector of the basis in $R$ is defined by the set $\alpha$ of numbers $m_{pq}$, $p \leqq q$, $q = 1, \dots, n-1$ arranged in the following pattern:

$$\alpha = \begin{pmatrix} m_{1,n-1} \quad m_{2,n-1} \quad \cdots \quad m_{n-1,n-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ m_{13} \quad m_{23} \quad m_{33} \\ m_{12} \quad m_{22} \\ m_{11} \end{pmatrix}. \tag{3}$$

The numbers $m_{pq}$ are arbitrary integers which satisfy the inequalities $m_{p,q+1} \geqq m_{pq} \geqq m_{p+1,q+1}$. The numbers $m_1, \dots, m_i, \dots, m_n$ which define the representation are denoted here by $m_{in}$. The vector of the basis defined by the pattern $\alpha$ shall be written simply as $(\alpha)$.

Let $m_{pq} - p = l_{pq}$ and

$$a^j_{k-1,k} = \left[ (-1)^{k-1} \frac{\prod_{i=1}^{k} (l_{i,k} - l_{j,k-1}) \prod_{i=1}^{k-2} (l_{i,k-2} - l_{j,k-1} - 1)}{\prod_{i \neq j} (l_{i,k-1} - l_{j,k-1})(l_{i,k-1} - l_{j,k-1} - 1)} \right]^{1/2} \tag{4}$$

Denote now by $\alpha^i_{k-1,k}$ the pattern obtained from the pattern $\alpha$ by replacing $m_{i,k-1}$ with $m_{i,k-1} + 1$. Then the operator $E_{k-1,k}$ acts on the vector $(\alpha)$ in the following way

$$\begin{aligned} E_{k-1,k}(\alpha) &= \sum_j a^j_{k-1,k}(\alpha^j_{k-1,k}), \\ E_{kk}(\alpha) &= \left( \sum_{i=1}^{k} m_{ik} - \sum_{i=1}^{k-1} m_{i,k-1} \right)(\alpha), \\ E_{k,k-1}(\alpha) &= \sum_j b^j_{k,k-1}(\bar{\alpha}^j_{k,k-1}). \end{aligned} \tag{5}$$

Here $\bar{\alpha}^j_{k,k-1}$ denotes the pattern obtained from $\alpha$ by replacing $m_{j,k-1}$ with $m_{j,k-1} - 1$ and $b^j_{k,k-1}$ is defined by the formula

$$b^j_{k,k-1} = \left[ (-1)^{k-1} \frac{\prod_{i=1}^{k} (l_{i,k} - l_{j,k-1} + 1) \prod_{i=1}^{k-2} (l_{i,k-2} - l_{j,k-1})}{\prod_{i \neq j} (l_{i,k-1} - l_{j,k-1} + 1)(l_{i,k-1} - l_{j,k-1})} \right]^{1/2}. \tag{4'}$$

Formulas (5) completely define the representation since any operator $E_{pq}$ can be obtained by commuting operators of the form (5).

4. We shall, nevertheless, write out the formulas for $E_{pq}$. Let $p < q$. Denote by $\alpha_{pq}^{i_p \ldots i_{q-1}}$ the pattern obtained from $\alpha$ by increasing each of the indices $m_{i_p p}, \ldots, m_{i_{q-1} q-1}$ by 1 while leaving all other indices unchanged.

Then

$$E_{pq}(\alpha) = \sum_{(i_p, \ldots, i_{q-1})} a_{pq}^{i_p, \ldots, i_{q-1}} (\alpha_{pq}^{i_p, \ldots, i_{q-1}}),$$

where

$$a_{pq}^{i_p, \ldots, i_{q-1}} = \pm \frac{\prod_{k=p}^{q-1} a_{k,k+1}^{i_k}}{\prod_{k=p}^{q-2} [(l_{i_k k} - l_{i_{k+1} k+1})(l_{i_k k} - l_{i_{k+1} k+1} - 1)]^{1/2}}.$$

The sign is determined by the number of inversions in the sequence $i_p, \ldots, i_{q-1}$. The action of $E_{pq}$ for $p > q$ is defined in a similar way. In applications of representation theory it is usually required only to know those $(\alpha)$ which can be obtained from that given under the action of $E_{ik}$. The coefficients $a_{pq}^{j \cdots}$ are needed much more rarely.

Note that if the basis $(\alpha)$ is assumed to be orthonormal, then $(E_{pq})^* = E_{qp}$.

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