INEQUALITIES BETWEEN PROJECTION FUNCTIONS OF CONVEX BODIES

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Abstract. We obtain injectivity results for certain integral transforms on Grassmann manifolds. These are used to solve the intermediate cases of a problem of Shephard and to give other applications to projection functions of convex bodies of revolution.

1. Introduction. A convex body in n-dimensional Euclidean space $\mathbb{E}^n$ is a compact convex subset with nonempty interior. It is said to be centrally symmetric if it is a translate of its reflection in the origin. The starting point of these investigations can be found in the work of Shephard [1964] who asked the following question:

If K and L are centrally symmetric convex bodies in $\mathbb{E}^n$, is there the implication

$$\mathrm{vol}_{n-1}(K \mid u^{\perp}) \geq \mathrm{vol}_{n-1}(L \mid u^{\perp}), \quad \forall u \in S^{n-1} \quad \Longrightarrow \quad \mathrm{vol}_n(K) \geq \mathrm{vol}_n(L)?$$

Here, $K \mid u^{\perp}$ is the orthogonal projection of K onto the subspace of $\mathbb{E}^n$ orthogonal to the unit vector $u \in S^{n-1}$. The question was answered independently by Petty [1967] and Schneider [1967]. They both showed that the answer is affirmative if it is further assumed that K is a zonoid. *Zonoids* are limits of vector sums of line segments. Both authors also showed that this further assumption cannot be suppressed. In addition, Schneider showed that the implication is not true if L is any nonzonoid whose boundary is C^{n+2} and has positive curvature. Schneider's results are further refined by Ball [1991], and by Goodey and Zhang [1996]. Interesting variations of the Shephard problem are considered by Chakerian and Lutwak [1992 and 1996]. The generalization of the Shephard problem for lower dimensional projections of convex bodies has been open since Petty and Schneider's work, see Question 4.2.1 of Gardner [1995]. It states:

If K and L are centrally symmetric convex bodies in $\mathbb{E}^n$, and $i \in \{1, \ldots, n-1\}$, is there the implication

$$\mathrm{vol}_i(K \mid E) \geq \mathrm{vol}_i(L \mid E) \quad \forall E \in \mathcal{L}_i^n \quad \Longrightarrow \quad \mathrm{vol}_n(K) \geq \mathrm{vol}_n(L)?$$

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Here $\mathcal{L}_i^n$ denotes the Grassmann manifold of all i-dimensional subspaces of $\mathbb{E}^n$. The counterpart of this problem for cross sections of convex bodies is considered by Bourgain and Zhang [1996], and by Zhang [1996].

A key idea for the case of hyperplane projections is the use of the spherical cosine transform. This arises naturally since zonoids (centred at the origin) are precisely those bodies whose support function is the cosine transform of a positive measure on S^{n-1}. The injectivity of the spherical cosine transform as well as Minkowski's existence theorem for surface area measures play important roles in the solution of the Shephard problem for hyperplane projections. For the intermediate cases $1 < i < n-1$, the corresponding cosine transform on a Grassmann manifold is not injective and there is no known analogue of Minkowski's result. To overcome these problems, we will use techniques from harmonic analysis to examine the kernel of the cosine transform on a Grassmann manifold. We will also use techniques from integral geometry. In particular, we use appropriate Radon transforms on Grassmann manifolds. We will show that, despite the lack of injectivity of these transforms, by concentrating on bodies of revolution, we can find results analogous to those of Petty and Schneider. To be more precise, we will find a class of bodies which yields an affirmative answer to the generalized Shephard problem. This class is bigger than the class of zonoids. We will prove that, corresponding to any centrally symmetric convex body of revolution outside this class, whose boundary is sufficiently smooth and has positive curvature, there are counterexamples to the generalized Shephard problem. It will be shown that examples of such bodies of revolution are provided by deformations of double cones in $\mathbb{E}^n$. The combination of these results provides a solution to the generalized Shephard problem.

In Section 2, we carry out the harmonic analysis which provides the necessary information about the kernels of the cosine and Radon transforms. Our solution of the generalization of Shephard's problem is the topic of Section 3. In the final section, we give some further applications of our techniques to bodies of revolution. In particular, we obtain information about the space spanned by the intermediate projection functions of bodies of revolution.

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2. Cosine and Radon transforms on Grassmann manifolds. For $1 \le i \le n-1$, the *cosine transform* T_i^n: $L^2(\mathcal{L}_i^n) \to L^2(\mathcal{L}_i^n)$ is defined by

$$(T_i^n f)(X) = \int_{\mathcal{L}_i^n} |\langle X, Y\rangle| f(Y)\nu_i^n(dY) \qquad \text{for } f \in L^2(\mathcal{L}_i^n) \text{ and } X \in \mathcal{L}_i^n,$$

where $|\langle X, Y\rangle|$ is the absolute value of the determinant of the orthogonal projection of X onto Y, and ν_i^n is the rotation invariant probability measure on $\mathcal{L}_i^n$. For

$1 \leq i \neq j \leq n-1$ the *Radon transform* $R^n_{i,j}\colon L^2(\mathcal{L}^n_i) \to L^2(\mathcal{L}^n_j)$ is defined by

$$(R^n_{i,j}f)(X) = \int_{\mathcal{L}^n_i(X)} f(Y)\, \nu^X_i(dY),$$

where $\mathcal{L}^n_i(X)$ is the submanifold of $\mathcal{L}^n_i$ which comprises all $Y \in \mathcal{L}^n_i$ which contain (respectively, are contained in) X, and ν^X_i is the invariant probability measure on $\mathcal{L}^n_i(X)$. It is well-known that if $f, g \in L^2(\mathcal{L}^n_i)$ and if $h \in L^2(\mathcal{L}^n_j)$ then

$$\int_{\mathcal{L}^n_i} (T^n_i f)(X) g(X)\, \nu^n_i(dX) = \int_{\mathcal{L}^n_i} f(X)(T^n_i g)(X)\, \nu^n_i(dX)$$

and

$$\int_{\mathcal{L}^n_j} (R^n_{i,j}f)(Y) h(Y)\, \nu^n_j(dY) = \int_{\mathcal{L}^n_i} f(X)(R^n_{j,i}h)(X)\, \nu^n_i(dX).$$

Combining this with the observations

$$T^n_i\colon C^\infty(\mathcal{L}^n_i) \to C^\infty(\mathcal{L}^n_i) \qquad \text{and} \qquad R^n_{i,j}\colon C^\infty(\mathcal{L}^n_i) \to C^\infty(\mathcal{L}^n_j),$$

we see that both transforms can be extended to distributions δ on $\mathcal{L}^n_i$ by

$$(T^n_i \delta)(f) = \delta(T^n_i f) \qquad \text{and} \qquad (R^n_{i,j}\delta)(g) = \delta(R^n_{j,i} g),$$

for $f \in C^\infty(\mathcal{L}^n_i)$ and $g \in C^\infty(\mathcal{L}^n_j)$. We also note that, with this extension, T^n_i maps a measure μ on $\mathcal{L}^n_i$ to a continuous function $T^n_i \mu$ on $\mathcal{L}^n_i$ defined by

$$(T^n_i \mu)(X) = \int_{\mathcal{L}^n_i} |\langle X, Y\rangle|\, \mu(dY).$$

If f is a function on $\mathcal{L}^n_i$, we denote by $f^\perp$ the function on $\mathcal{L}^n_{n-i}$ defined by $f^\perp(E) = f(E^\perp)$. Clearly $f \in L^2(\mathcal{L}^n_i)$ if and only if $f^\perp \in L^2(\mathcal{L}^n_{n-i})$. It is also easy to see that, if $1 \leq i \neq j \leq n-1$, then

(2.1) $$\left(R^n_{i,j} f\right)^\perp = R^n_{n-i,n-j} f^\perp \qquad \text{and} \qquad \left(T^n_i f\right)^\perp = T^n_{n-i} f^\perp.$$

Furthermore, for $1 \leq i < j < k \leq n-1$, we have

(2.2) $$R^n_{i,k} = R^n_{j,k} R^n_{i,j} \qquad \text{and} \qquad R^n_{k,i} = R^n_{j,i} R^n_{k,j}.$$

Of course, (2.1) shows that these two statements are equivalent.

We will use the cosine transform T^n_i to define a special class of convex bodies, which includes the zonoids. Our interest in the Radon transform $R^n_{i,j}$

stems largely from its occurrence in integral geometry. In particular, the Cauchy-Kubota formulas (see Schneider and Weil [1992], for example) imply that if $1 \le i < j \le n-1$, then

$$\alpha^{0,j}_{i,j-i}\big(R^n_{i,j} V_i(K \mid \cdot)\big)(F) = V_i(K \mid F) \qquad \text{for all } F \in \mathcal{L}^n_j. \tag{2.3}$$

Here, for $i = 1, 2, \cdots, n$, we denote by $V_i(K)$ the ith intrinsic volume of a convex body K, and

$$\alpha^{i,j}_{k,l} = \frac{i!\kappa_i j!\kappa_j}{k!\kappa_k l!\kappa_l},$$

where κ_i denotes the i-dimensional volume of the unit i-dimensional ball B_i. We recall that intrinsic volumes can be defined by

$$V_i(K) = \alpha^{n,0}_{i,n-i} \int_{\mathcal{L}^n_i} \mathrm{vol}_i(K \mid E)\, \nu^n_i(dE),$$

and that, in case $\dim K = i$, we have $V_i(K) = \mathrm{vol}_i(K)$.

For $X \in \mathcal{L}^n_i$, we denote by δ_X the probability measure on $\mathcal{L}^n_i$ which is concentrated on X. Clearly

$$T^n_i \delta_X = |\langle X, \cdot\rangle| \qquad \text{and} \qquad R^n_{i,j}\delta_X = \nu^X_j. \tag{2.4}$$

In the case $i < j$, the Cauchy-Kubota formulas (2.3) give

$$R^n_{i,j}|\langle X, \cdot\rangle| = \alpha^{i,j-i}_{0,j} V_i(C_X \mid \cdot) \tag{2.5}$$

and, for $E \in \mathcal{L}^d_j$

$$\begin{aligned}(T^n_j \nu^X_j)(E) &= \big(R^n_{j,i}|\langle E, \cdot\rangle|\big)(X) = \big(R^n_{n-j,n-i}|\langle E^\perp, \cdot\rangle|\big)(X^\perp) \\ &= \alpha^{n-j,j-i}_{0,n-i} V_{n-j}(C_{E^\perp} \mid X^\perp) = \alpha^{n-j,j-i}_{0,n-i} V_j(C_X \mid E),\end{aligned} \tag{2.6}$$

where C_X denotes the unit cube in X. It follows from (2.4), (2.5) and (2.6) that

$$R^n_{i,j} T^n_i \delta_X = \alpha^{i,n-i}_{j,n-j} T^n_j R^n_{i,j} \delta_X$$

for each $X \in \mathcal{L}^n_i$. Combining this with (2.1) and using the fact that all measures can be approximated by sums of atomic measures, we have, for all $1 \le i \ne j \le n-1$,

$$R^n_{i,j} T^n_i = \alpha^{i,n-i}_{j,n-j} T^n_j R^n_{i,j}. \tag{2.7}$$

Both the cosine and Radon transforms are continuous, linear and intertwine the group action of $SO(n)$ and so we will examine them using harmonic analysis on $\mathcal{L}_i^n$. First we will bring together some results concerning the irreducible invariant subspaces of $L^2(\mathcal{L}_i^n)$. These results can be found in the books of Boerner [1963] and Helgason [1984], for example. In view of the above observations regarding orthogonality, it is convenient for the purposes of harmonic analysis to identify the functions f and $f^\perp$. Thus $\mathcal{L}_i^n = \mathcal{L}_{n-i}^n$ and so we may assume that $2i \leq n$. The fundamental result is the fact that $L^2(\mathcal{L}_i^n)$ is an orthogonal sum of invariant irreducible subspaces, namely

$$L^2(\mathcal{L}_i^n) = \bigoplus_\delta H_\delta^{n,i} \qquad \text{for } 1 \leq i \leq \left[\frac{n}{2}\right], \tag{2.8}$$

where the summation is over equivalence classes δ of irreducible representations of $SO(n)$; see Helgason [1984, p. 391]. Two representations π_1 on V_1 and π_2 on V_2 are said to be equivalent if they admit an intertwining operator A: $V_1 \to V_2$, that is, if A is linear and $A\pi_1 = \pi_2 A$.

Equivalence classes of irreducible representations of $SO(n)$ are characterized by their highest weights; if $n = 2p$ is even, these are the integer p-tuples of the form $(m_1, \dots, m_p)$ with $m_1 \geq m_2 \geq \cdots \geq m_{p-1} \geq |m_p|$, whereas if $n = 2p+1$ is odd, they are again integer p-tuples but now those which satisfy $m_1 \geq m_2 \geq \cdots \geq m_p \geq 0$. So the δ appearing in (2.8) may be replaced by the highest weight $(m_1, \dots, m_p)$. The highest weights corresponding to the δ for which $H_\delta^{n,i}$ is nontrivial are precisely those p-tuples $(m_1, \dots, m_p)$ where $p = [\frac{n}{2}]$ which satisfy the following two conditions:

(a) $m_j = 0$ for all $j > i$,

(b) the integers $m_1, \dots, m_p$ are even;

see Strichartz [1975], Sugiura [1962] and Takeuchi [1973]. Consequently, (2.8) can be rewritten as

$$L^2(\mathcal{L}_i^n) = \bigoplus_{(2m_1,\dots,2m_i,0,\dots,0)} H^{n,i}_{(2m_1,\dots,2m_i,0,\dots,0)}. \tag{2.9}$$

In fact, the $H^{n,i}_{(2m_1,\dots,2m_i,0,\dots,0)}$ are the only nontrivial invariant irreducible subspaces of $L^2(\mathcal{L}_i^n)$. In particular, we have

$$L^2(\mathcal{L}_{n-1}^n) = L^2(\mathcal{L}_1^n) = \bigoplus_{m=0}^{\infty} H^{n,1}_{(2m,0,\dots,0)}. \tag{2.10}$$

Of course, the spaces $H^{n,1}_{(2m,0,\dots,0)}$ are precisely the spaces of spherical harmonics of degree $2m$ in dimension n. We also note that the nonzero constant functions in $L^2(\mathcal{L}_i^n)$ are members of $H^{n,i}_{(0,\dots,0)}$.

The intertwining property of T_i^n and $R_{i,j}^n$ shows that, for each highest weight

$$(2m_1, \ldots, 2m_i, 0, \ldots, 0),$$

the spaces $T_i^n H^{n,i}_{(2m_1,\ldots,2m_i,0,\ldots,0)}$ and $R_{i,j}^n H^{n,i}_{(2m_1,\ldots,2m_i,0,\ldots,0)}$ are invariant irreducible subspaces of the appropriate Hilbert space. So we deduce from (2.9) that these are either trivial or one of the subspaces $H^{n,i}_{(2m_1,\ldots,2m_i,0,\ldots,0)}$, $H^{n,j}_{(2m_1,\ldots,2m_j,0,\ldots,0)}$. In the latter two cases, we must have

$$(2m_1, \ldots, 2m_i, 0, \ldots, 0) = (2m_1, \ldots, 2m_j, 0, \ldots, 0)$$

since the intertwining property of the transforms shows that these weights correspond to equivalent representations. Furthermore, Schur's Lemma shows that, in case

$$R_{i,j}^n H^{n,i}_{(2m_1,\ldots,2m_i,0,\ldots,0)} = H^{n,j}_{(2m_1,\ldots,2m_j,0,\ldots,0)},$$

we can identify the subspaces $H^{n,i}_{(2m_1,\ldots,2m_i,0,\ldots,0)}$ and $H^{n,j}_{(2m_1,\ldots,2m_j,0,\ldots,0)}$, and then the operators T_i^n and $R_{i,j}^n$ act as multiples of the identity on $H^{n,i}_{(2m_1,\ldots,2m_i,0,\ldots,0)}$. We write

$$T_i^n f = \lambda^{n,i}_{(2m_1,\ldots,2m_i,0,\ldots,0)} f \qquad \text{and} \qquad R_{i,j}^n f = \nu^{n,i,j}_{(2m_1,\ldots,2m_i,0,\ldots,0)} f$$

if $f \in H^{n,i}_{(2m_1,\ldots,2m_i,0,\ldots,0)}$. Injectivity questions about T_i^n and $R_{i,j}^n$ can be reduced to questions as to which of these multipliers are zero.

The spherical cosine transform $T_1^n = T_{n-1}^n$ mentioned in the introduction is injective. That is

$$\lambda^{n,1}_{(2m,0,\ldots,0)} \neq 0 \qquad \text{for all } m = 0, 1, 2, \ldots.$$

For the generalized Shephard problem, we will be interested in T_i^n for $1 < i < n-1$. These transforms were shown not to be injective by Goodey and Howard [1990]. If $i < j$ the transform $R_{i,j}^n$ is injective (on C^∞ functions and on distributions) if and only if $i+j \leq n$, whereas if $i > j$, it is injective if and only if $i+j \geq n$; see, for example Grinberg [1986] and Gelfand, Graev and Roşu [1984].

In order to describe further injectivity properties of these intertwining transforms we introduce two subspaces of $L^2(\mathcal{L}_i^d)$,

$$\mathcal{R}_i^n = \bigoplus_{m=0}^{\infty} H^{n,i}_{(2m,0,\ldots,0)} \qquad \text{and} \qquad \mathcal{T}_i^n = \bigoplus_{\substack{(2m_1,\ldots,2m_i,0,\ldots,0) \\ \lambda^{n,i}_{(2m_1,\ldots,2m_i,0,\ldots,0)} \neq 0}} H^{n,i}_{(2m_1,\ldots,2m_i,0,\ldots,0)}.$$

Our first objective is to establish the inclusion $\mathcal{R}_i^n \subset \mathcal{T}_i^n$. An important tool in our proof of this inclusion (and in later results) is the branching theorem (see Boerner [1963], Theorem 2.1). The branching theorem explains how to express the restriction of an irreducible representation of $SO(n)$ to $SO(n-1)$ as a sum of irreducible representations of $SO(n-1)$. In terms of the expansion in (2.9), it can be stated as follows

$$H^{2p,i}_{(m_1,\dots,m_p)}|_{\mathcal{L}_i^{2p-1}} \simeq \bigoplus_{m_1\geq m_1'\geq\cdots\geq m_{p-1}\geq m_{p-1}'\geq|m_p|} H^{2p-1,i}_{(m_1',\dots,m_{p-1}')}$$

and

$$H^{2p+1,i}_{(m_1,\dots,m_p)}|_{\mathcal{L}_i^{2p}} \simeq \bigoplus_{m_1\geq m_1'\geq\cdots\geq m_{p-1}\geq m_{p-1}'\geq m_p\geq|m_p'|} H^{2p,i}_{(m_1',\dots,m_p')}.$$

LEMMA 2.1. *The inclusion* $\mathcal{R}_i^n \subset \mathcal{T}_i^n$ *holds for all* $1 \leq i \leq n-1$ *and is strict precisely when* $1 < i < n-1$.

Proof. The first step in our proof is to show that, for $1 \leq i \neq j \leq n-1$, we have $\nu^{n,i,j}_{(2m,0,\dots,0)} \neq 0$ for all $m = 0, 1, 2, \dots$. In the case $i < j$, we choose a nontrivial function $g \in H^{n,1}_{(2m,0,\dots,0)}$ and put $f = R^n_{1,i}g \in H^{n,i}_{(2m,0,\dots,0)}$. This gives

$$\nu^{n,i,j}_{(2m,0,\dots,0)}f = R^n_{i,j}f = R^n_{1,j}g$$

which is nontrivial, since $R^n_{1,j}$ is injective. In particular, $\nu^{n,i,j}_{(2m,0,\dots,0)} \neq 0$, as required. The proof for $i > j$ is analogous.

It follows from (2.7) that, for $m = 0, 1, 2, \dots,$

$$\nu^{n,i,1}_{(2m,0,\dots,0)}\lambda^{n,i}_{(2m,0,\dots,0)} = \alpha^{i,n-i}_{1,n-1}\lambda^{n,1}_{(2m,0,\dots,0)}\nu^{n,i,1}_{(2m,0,\dots,0)}$$

and so

$$\lambda^{n,i}_{(2m,0,\dots,0)} = \alpha^{i,n-i}_{1,n-1}\lambda^{n,1}_{(2m,0,\dots,0)} \neq 0.$$

Consequently $\mathcal{R}_i^n \subset \mathcal{T}_i^n$ for each $i = 1, \dots, n-1$. In the cases $i = 1,\ n-1$, (2.10) shows that $\mathcal{R}_i^n = \mathcal{T}_i^n = L^2(\mathcal{L}_1^n)$.

Our next objective is to prove that if $\lambda^{n,i}_{(2m_1,\dots,2m_i,0,\dots,0)} \neq 0$ for some highest weight $(2m_1, \dots, 2m_i, 0, \dots, 0)$ then $\lambda^{n+1,i}_{(2n_1,\dots,2n_i,0,\dots,0)} \neq 0$ for some highest weight

$$(2n_1, \dots, 2n_i, 0, \dots, 0) \succ (2m_1, \dots, 2m_i, 0, \dots, 0),$$

where $\succ$ refers to the usual lexicographical ordering. To this end, we choose a nontrivial function $g^n \in H^{n,i}_{(2m_1,\dots,2m_i,0,\dots,0)}$ and let μ^n be the measure on $\mathcal{L}_i^{n+1}$

defined by

$$\int_{\mathcal{L}_i^{n+1}} f(X)\,\mu^n(dX) = \int_{\mathcal{L}_i^n} f|_{\mathcal{L}_i^n}(Y) g^n(Y)\,\nu_i^n(dY) \qquad \text{for } f \in C(\mathcal{L}_i^{n+1}).$$

Here we are thinking of $\mathcal{L}_i^n$ as a subset of $\mathcal{L}_i^{n+1}$ in the usual way and we note that μ^n is a measure on $\mathcal{L}_i^{n+1}$ which is supported by $\mathcal{L}_i^n$. Now we define $h = T_i^{n+1}\mu^n \in C(\mathcal{L}_i^{n+1}) \subset L^2(\mathcal{L}_i^{n+1})$. We note that

$$h|_{\mathcal{L}_i^n} = T_i^n g^n = \lambda^{n,i}_{(2m_1,\ldots,2m_i,0,\ldots,0)} g^n \in H^{n,i}_{(2m_1,\ldots,2m_i,0,\ldots,0)}.$$

It follows from the branching theorem that h has a nontrivial Hilbert space projection into some $H^{n+1,i}_{(2n_1,\ldots,2n_i,0,\ldots,0)}$ with

$$(2n_1,\ldots,2n_i,0,\ldots,0) \succ (2m_1,\ldots,2m_i,0,\ldots,0).$$

So there is a function $f^{n+1} \in H^{n+1,i}_{(2n_1,\ldots,2n_i,0,\ldots,0)}$ such that

$$\int_{\mathcal{L}_i^{n+1}} f^{n+1}(X) h(X)\,\nu_i^{n+1}(dX) \neq 0,$$

which means

$$\int_{\mathcal{L}_i^{n+1}} f^{n+1}(X) \int_{\mathcal{L}_i^{n+1}} |\langle X, Y\rangle|\, \mu^n(dY)\nu_i^{n+1}(dX) \neq 0.$$

This shows, in particular, that $T_i^{n+1} f^{n+1} \neq 0$ and therefore $\lambda^{n+1,i}_{(2n_1,\ldots,2n_i,0,\ldots,0)} \neq 0$, as required.

It is proved in Goodey, Howard and Reeder [1996] that

$$\mathfrak{I}_2^4 = \bigoplus_{m_1=0}^{\infty} \bigoplus_{m_2=0}^{1} H^{4,2}_{(2m_1,2m_2)}.$$

So $\lambda^{4,2}_{(2m,2)} \neq 0$ for all $m \geq 1$. Consequently, for each dimension $n \geq 4$, there are infinitely many $m_1 \geq m_2 \geq 1$ for which $\lambda^{n,2}_{(2m_1,2m_2,0,\ldots,0)} \neq 0$. It follows that $\mathfrak{I}_2^n \neq \mathfrak{R}_2^n$ for all $n \geq 4$. Next, we note that $\mathfrak{I}_3^5 = \mathfrak{I}_2^5$ and so $\lambda^{5,3}_{(2m_1,2m_2)} \neq 0$ for infinitely many $m_1 \geq m_2 \geq 1$. Again, we can deduce that, for each dimension $n \geq 5$, there are infinitely many $m_1 \geq m_2 \geq 1$ for which $\lambda^{n,3}_{(2m_1,2m_2,0,\ldots,0)} \neq 0$. Continuing in this fashion, we deduce that $\mathfrak{I}_i^n \neq \mathfrak{R}_i^n$ for each $i = 2,\ldots,n-2$, as required. □

LEMMA 2.2. *For $1 \le i \ne j \le n-1$, the cosine and Radon transforms*

$$T_i^n\colon \mathfrak{T}_i^n \to \mathfrak{T}_i^n \qquad \text{and} \qquad R_{i,j}^n\colon \mathfrak{R}_i^n \to \mathfrak{R}_j^n$$

are injective.

Proof. For the cosine transform, this is an immediate consequence of the definition of $\mathfrak{T}_i^n$. For the Radon transform, the result follows from the fact that $\nu^{n,i,j}_{(2m,0,\dots,0)} \ne 0$ for all $m = 0, 1, 2, \dots$, which was established in the proof of the previous lemma. □

We note that it also follows from Lemma 2.1 that $T_i^n\colon \mathfrak{R}_i^n \to \mathfrak{R}_i^n$ is injective. We will now establish a corresponding surjectivity result for T_i^n.

It is shown in Strichartz [1981] that

$$\nu^{n,1,j}_{(2m,0,\dots,0)} = (-1)^m \frac{\Gamma(m+1/2)\Gamma(j/2)}{\sqrt{\pi}\,\Gamma(m+j/2)} = (-1)^m \frac{\kappa_{2m+j-2}}{\pi \kappa_{2m-1}\kappa_{j-2}}.$$

For $1 < i < j \le n-1$, we use (2.2) to deduce that

$$\nu^{n,1,j}_{(2m,0,\dots,0)} = \nu^{n,i,j}_{(2m,0,\dots,0)} \nu^{n,1,i}_{(2m,0,\dots,0)}$$

and so

$$\nu^{n,i,j}_{(2m,0,\dots,0)} = \frac{\kappa_{i-2}\kappa_{2m+j-2}}{\kappa_{j-2}\kappa_{2m+i-2}}.$$

It follows immediately that, for $1 \le j < i < n-1$, we have

$$\nu^{n,i,j}_{(2m,0,\dots,0)} = \frac{\kappa_{n-i-2}\kappa_{2m+n-j-2}}{\kappa_{n-j-2}\kappa_{2m+n-i-2}}.$$

Schneider [1967] showed that

$$\begin{aligned}\lambda^{n,1}_{(2m,0,\dots,0)} &= (-1)^{m-1} \frac{\Gamma(2m-1)\Gamma(n/2)}{2^{2m-1}\sqrt{\pi}\,\Gamma(m)\Gamma(m+(n+1)/2)} \\ &= (-1)^{m-1} \frac{\kappa_{2m-2}\kappa_{2m+n-1}}{\pi^2 2^{2m-1}\kappa_{4m-4}\kappa_{n-2}}.\end{aligned}$$

Moreover, he gave the estimate

$$|\lambda^{n,1}_{(2m,0,\dots,0)}|^{-1} = O\big(m^{(n+2)/2}\big) \qquad \text{as } m \to \infty,$$

so we can immediately deduce that

$$|\lambda^{n,i}_{(2m,0,\dots,0)}|^{-1} = O\big(m^{(n+2)/2}\big) \qquad \text{as } m \to \infty. \tag{2.11}$$

The required surjectivity result will be simply a matter of transferring Schneider's techniques to the current setting.

LEMMA 2.3. *For* $1 \le i \le n-1$, *the cosine transform*

$$T_i^n \colon \mathcal{R}_i^n \cap C^\infty(\mathcal{L}_i^n) \to \mathcal{R}_i^n \cap C^\infty(\mathcal{L}_i^n)$$

is bijective.

Proof. We remark that this result will be sufficient for the sequel but that we could use the methods of Strichartz [1981] to obtain stronger results in terms of Sobolev spaces and that similar results could be obtained for the Radon transforms.

As mentioned above, the injectivity follows from Lemma 2.1 and so we will only prove the surjectivity. We denote by Δ the Laplace Beltrami operator on $\mathcal{L}_i^n$ and assume $f \in \mathcal{R}_i^n \cap C^\infty(\mathcal{L}_i^n)$. If f has harmonic expansion $f = \sum_{m=0}^\infty f_m$ where $f_m \in H^{n,i}_{(2m,0,\dots,0)}$ then Δf has expansion $\Delta f = -\sum_{m=0}^\infty (2m)(2m+n-2) f_m$, see, for example, Strichartz [1981]. We deduce that, if $f \in \mathcal{R}_i^n \cap C^\infty(\mathcal{L}_i^n)$ then $\sum_{m=0}^\infty m^p \|f_m\|^2$ converges for all $p = 1, 2, \dots$; here $\|f\|$ denotes the L^2-norm of f. It follows from (2.11) that the function g defined by $g = \sum_{m=0}^\infty (\lambda^{n,i}_{(2m,0,\dots,0)})^{-1} f_m$ is in $\mathcal{R}_i^n \cap C^\infty(\mathcal{L}_i^n)$. Clearly $f = T_i^n g$ and so the proof is complete. □

Our applications will concern convex bodies of revolution, so we will be interested in the action of the cosine and Radon transforms on functions with certain symmetry properties. We imagine $\mathbb{E}^{n-1}$ as being embedded in $\mathbb{E}^n$ and then think of $SO(n-1)$ as a subgroup of $SO(n)$. If $f \in H^{n,i}_{(m_1,\dots,m_p)}$ is nontrivial and rotationally symmetric, in the sense that $\rho f = f$ for all $\rho \in SO(n-1)$, then the restriction of f to $\mathcal{L}_i^{n-1}$ is a nonzero constant and therefore a member of $H^{n-1,i}_{(0,\dots,0)}$. The branching theorem, therefore, implies that $m_1 \ge 0 \ge m_2 \ge \cdots \ge 0$ and so $m_2 = \cdots = m_p = 0$. It follows that any rotationally symmetric function $f \in L^2(\mathcal{L}_i^n)$ is a member of $\mathcal{R}_i^n$; of course, this is true no matter what the axis of rotational symmetry of f might be.

3. Shephard's problem and bodies of revolution. As we remarked in the Introduction, the case $i = n-1$ of the generalized Shephard problem was solved by Petty [1967] and Schneider [1967]. The case $i = 1$ is trivial, since the hypothesis then implies that K contains some translate of L and so clearly $V_n(K) \ge V_n(L)$. Consequently, we will focus our attention on the cases $2 \le i \le n-2$.

It is natural to investigate the classes $\mathcal{K}(j)$, $(j = 1, \dots, n-1)$ of centrally symmetric convex bodies (see Schneider and Weil [1983] or Goodey and Weil [1993]). For $1 \le j \le n-1$, these are the centrally symmetric bodies K for which there is a positive measure $\rho_j(K,\cdot)$ on $\mathcal{L}_j^n$ such that

$$V_j(K|E) = \int_{\mathcal{L}_j^n} |\langle E,F\rangle|\,\rho_j(K,dF) \qquad \text{for each } E \in \mathcal{L}_j^n.$$

We note that $\mathcal{K}(1)$ is the class of zonoids and that $\mathcal{K}(n-1)$ comprises all centrally symmetric convex bodies. In fact, $\mathcal{K}(j)$ is the class of centrally symmetric bodies K such that the jth projection function $V_j(K|\cdot)$ is the cosine transform T_j^n of a positive measure on $\mathcal{L}_j^n$. Weil [1982] showed that $\mathcal{K}(1) \subset \mathcal{K}(j) \subset \mathcal{K}(n-1)$ for all $j = 1,\dots,n-1$. So the classes $\mathcal{K}(j)$ form a natural hierarchy ranging from the zonoids to all centrally symmetric convex bodies.

Our first objective is to prove that the generalized Shephard problem has a positive answer for $K \in \mathcal{K}(n-i)$. The proof will involve mixed volumes $V(L[i],K[n-i])$ of convex bodies K and L and, in particular, the Minkowski Inequality

$$V(L[i],K[n-i]) \geq V_n(L)^{i/n} V_n(K)^{(n-i)/n}. \tag{3.1}$$

We refer the reader to Schneider [1993] for information on mixed volumes and other fundamental notions in convexity.

THEOREM 3.1. *Let K, L be centrally symmetric convex bodies in $\mathbb{E}^n$ and assume $1 \leq i \leq n-1$. If $K \in \mathcal{K}(n-i)$ and*

$$V_i(K \mid E) \geq V_i(L \mid E) \qquad \textit{for all } E \in \mathcal{L}_i^n,$$

then $V_n(K) \geq V_n(L)$.

Proof. If μ is a measure on $\mathcal{L}_j^n$, we denote by $\mu^\perp$ the measure on $\mathcal{L}_{n-j}^n$ defined by

$$\int_{\mathcal{L}_{n-j}^n} f(E)\,\mu^\perp(dE) = \int_{\mathcal{L}_j^n} f^\perp(F)\,\mu(dF) \qquad \text{for } f \in C(\mathcal{L}_{n-j}^n).$$

It is explained in Schneider and Weil [1983] that, if $K \in \mathcal{K}(n-i)$ and if L is centrally symmetric, then

$$V(L[i],K[n-i]) = \binom{n}{i}^{-1} \int_{\mathcal{L}_i^n} V_i(L \mid E)\,\rho_{n-i}^\perp(K,dE).$$

Consequently, the positivity of the measure $\rho_{n-i}(K,\cdot)$ gives

$$V_n(K) - V(L[i],K[n-i]) = \binom{n}{i}^{-1} \int_{\mathcal{L}_i^n} \big(V_i(K \mid E) - V_i(L \mid E)\big)\,\rho_{n-i}^\perp(K,dE) \geq 0.$$

Combining this with Minkowski's Inequality (3.1) gives $V_n(K) \geq V_n(L)$, as required. □

We note that Theorem 3.1 is true for arbitrary bodies L and not just those that are centrally symmetric. It is shown in Schneider [1997] that, if $K \in \mathcal{K}(n-i)$, then

$$V(L[i], K[n-i]) = \binom{n}{i}^{-1} \int_{\mathcal{L}_i^n} V_i(L \mid E)\, \rho_{n-i}^{\perp}(K, dE)$$

for all convex bodies L. This result, which allows us to suppress the central symmetry condition on L, is based on a result of Klain [1997+]. We are grateful to Rolf Schneider for pointing out this strengthening of Theorem 3.1 and for agreeing to its inclusion here. We note also that a related result appears in Chakerian and Lutwak [1996].

THEOREM 3.2. *Let K be a centrally symmetric and rotationally symmetric convex body whose boundary is C^∞ and which has strictly positive curvature at all points. If $2 \leq i \leq n-1$ and $K \notin \mathcal{K}(n-i)$ then there is a centrally symmetric and rotationally symmetric convex body L such that*

$$V_i(K \mid E) < V_i(L \mid E) \quad \textit{for all } E \in \mathcal{L}_i^n \qquad \textit{but} \qquad V_n(K) > V_n(L).$$

Proof. We let $\mathcal{M}_1^+$ denote the positive measures on $\mathcal{L}_{n-i}^n$ which are rotationally symmetric and have the same axis of symmetry as K; we choose coordinates so that this axis is orthogonal to $\mathbb{E}^{n-1} \subset \mathbb{E}^n$. If $Q = \{f \in C(\mathcal{L}_{n-i}^n)\colon f = T_{n-i}^n \mu \text{ for some } \mu \in \mathcal{M}_1^+\}$ then Q is a convex cone in $C(\mathcal{L}_{n-i}^n)$. First, we will use the techniques of Weil [1976] to prove that Q is closed in $C(\mathcal{L}_{n-i}^n)$.

To this end, we assume that $f_m \to f \in C(\mathcal{L}_{n-i}^n)$ uniformly as $m \to \infty$ and that, for each $m = 1, 2 \ldots$, there is a $\mu_m \in \mathcal{M}_1^+$ with $f_m = T_{n-i}^n \mu_m$. Now let $g \in C^\infty(\mathcal{L}_{n-i}^n)$ and denote by $\tilde{g}$ the function obtained by rotating g about the axis of K. By this, we mean that, for $E \in \mathcal{L}_{n-i}^n$, $\tilde{g}(E)$ is the average of $g(\rho E)$ for all $\rho \in SO(n-1)$. Similarly, we let $\tilde{\mu}$ denote the measure obtained by rotating μ about this axis. This is equivalent to defining $\tilde{\mu}$ by

$$\int_{\mathcal{L}_{n-i}^n} g(E)\, \tilde{\mu}(dE) = \int_{\mathcal{L}_{n-i}^n} \tilde{g}(E)\, \mu(dE) \qquad \text{for each } g \in C^\infty(\mathcal{L}_{n-i}^n).$$

We note that, $\tilde{\mu} = \mu$ for each $\mu \in \mathcal{M}_1^+$. Now $\tilde{g} \in \mathcal{R}_{n-i}^n \cap C^\infty(\mathcal{L}_{n-i}^n)$ and so we use

Lemma 2.3 to choose $h \in \mathcal{R}^n_{n-i} \cap C^\infty(\mathcal{L}^n_{n-i})$ with $\tilde{g} = T^n_{n-i}h$. Then

$$\begin{aligned}\int_{\mathcal{L}^n_{n-i}} g(X)\,\mu_m(dX) &= \int_{\mathcal{L}^n_{n-i}} g(X)\,\tilde{\mu}_m(dX) = \int_{\mathcal{L}^n_{n-i}} \tilde{g}(X)\,\tilde{\mu}_m(dX)\\ &= \int_{\mathcal{L}^n_{n-i}} (T^n_{n-i}h)(X)\,\mu_m(dX) = \int_{\mathcal{L}^n_{n-i}} h(X)\,(T^n_{n-i}\mu_m)(X)\,\nu^n_{n-i}(dX)\\ &\to \int_{\mathcal{L}^n_{n-i}} h(X)f(X)\,\nu^n_{n-i}(dX) \qquad \text{as } m \to \infty.\end{aligned}$$

Consequently, the mapping

$$g \mapsto \int_{\mathcal{L}^n_{n-i}} h(X)f(X)\,\nu^n_{n-i}(dX)$$

is a distribution (see Schwartz [1955]). The positivity of the measures μ_m shows that it is a positive distribution and therefore a positive measure μ say, with $\tilde{\mu} = \mu$. So we have, for any $h \in \mathcal{R}^n_{n-i} \cap C^\infty(\mathcal{L}^n_{n-i})$,

$$\begin{aligned}\int_{\mathcal{L}^n_{n-i}} h(X)\,(T^n_{n-i}\mu)(X)\,\nu^n_{n-i}(dX) &= \int_{\mathcal{L}^n_{n-i}} (T^n_{n-i}h)(X)\,\mu(dX)\\ &= \int_{\mathcal{L}^n_{n-i}} h(X)f(X)\,\nu^n_{n-i}(dX).\end{aligned}$$

It follows that $f = T^n_{n-i}\mu$ with $\mu \in \mathcal{M}^+_1$, and so Q is closed, as required.

The hypothesis of the theorem implies that $V_{n-i}(K|\cdot) \notin Q$. So the Hahn-Banach theorem implies that there is a measure σ on $\mathcal{L}^n_{n-i}$ such that

$$\int_{\mathcal{L}^n_{n-i}} f(E)\,\sigma(dE) \geq 0 \quad \text{for all } f \in Q \qquad \text{but} \qquad \int_{\mathcal{L}^n_{n-i}} V_{n-i}(K|E)\,\sigma(dE) < 0.$$

Clearly, we can assume that the measure σ has the same axis of rotational symmetry as K. Consequently, we have a measure σ on $\mathcal{L}^n_{n-i}$ such that

$$T^n_{n-i}\sigma \geq 0 \qquad \text{but} \qquad \int_{\mathcal{L}^n_{n-i}} V_{n-i}(K \mid E)\,\sigma(dE) < 0.$$

The measure σ can be weakly approximated by functions in $C^\infty(\mathcal{L}^n_{n-i})$ having the same rotational axis of symmetry as K. So we can choose such a function $g \in C^\infty(\mathcal{L}^n_{n-i})$ with

$$\text{(3.2)} \qquad T^n_{n-i}g > 0 \qquad \text{but} \qquad \int_{\mathcal{L}^n_{n-i}} V_{n-i}(K \mid E)g(E)\,\nu^n_{n-i}(dE) < 0.$$

We denote by $f_i(K,\cdot)$ the ith curvature function of K. This is the density of the ith surface area measure, $S_i(K,\cdot)$ with respect to spherical Lebesgue measure λ_{n-1} on S^{n-1} (see, for example, Schneider [1993]). Zhang [1994a] shows that if $\epsilon > 0$ is sufficiently small, then $f_i(K,\cdot) + \epsilon R^n_{n-i,1}g$ is the ith curvature function of a convex body K_ϵ. Clearly K_ϵ has the same rotational axis of symmetry as K and is also centrally symmetric. So

$$S_i(K_\epsilon,\cdot) - S_i(K,\cdot) = \epsilon R^n_{n-i,1}g. \tag{3.3}$$

The smoothness of K and K_ϵ guarantees that they are generalized zonoids (see Schneider [1967]). In fact, their generating measures are C^∞ functions on $\mathcal{L}^n_1$. It follows that there are functions $\rho_i(K,\cdot)$, $\rho_i(K_\epsilon,\cdot) \in \mathcal{R}^n_i \cap C^\infty(\mathcal{L}^n_i)$ such that

$$V_i(K \mid \cdot) = T^n_i \rho_i(K,\cdot) \qquad \text{and} \qquad V_i(K_\epsilon \mid \cdot) = T^n_i \rho_i(K_\epsilon,\cdot),$$

see Weil [1982]. It is shown in Goodey and Weil [1991] that we have

$$S_i(K,\cdot) = n\kappa_{n-i}\binom{n}{i}^{-1} R^n_{n-i,1}\rho_i^\perp(K,\cdot)$$

and

$$S_i(K_\epsilon,\cdot) = n\kappa_{n-i}\binom{n}{i}^{-1} R^n_{n-i,1}\rho_i^\perp(K_\epsilon,\cdot).$$

Combining this with (3.3) and the injectivity of $R^n_{n-i,1}$ gives

$$\rho_i^\perp(K_\epsilon,\cdot) - \rho_i^\perp(K,\cdot) = \frac{\epsilon}{n\kappa_{n-i}}\binom{n}{i} g$$

which is the same as

$$\rho_i(K_\epsilon,\cdot) - \rho_i(K,\cdot) = \frac{\epsilon}{n\kappa_{n-i}}\binom{n}{i} g^\perp. \tag{3.4}$$

Applying the cosine transform T^n_i to both sides of this equation gives

$$T^n_i\rho_i(K_\epsilon,\cdot) - T^n_i\rho_i(K,\cdot) = \frac{\epsilon}{n\kappa_{n-i}}\binom{n}{i} T^n_i g^\perp = \frac{\epsilon}{n\kappa_{n-i}}\binom{n}{i} (T^n_{n-i}g)^\perp > 0,$$

which gives

$$V_i(K_\epsilon \mid E) > V_i(K \mid E) \qquad \text{for all } E \in \mathcal{L}^n_i.$$

Combining (3.2) and (3.4) gives

$$\begin{aligned} V(K_\epsilon[i], K[n-i]) - V_n(K) &= \binom{n}{i}^{-1} \int_{\mathcal{L}^n_{n-i}} V_{n-i}(K \mid E)\big(\rho_i^\perp(K_\epsilon, dE) - \rho_i^\perp(K, dE)\big) \\ &= \frac{\epsilon}{n\kappa_{n-i}} \int_{\mathcal{L}^n_{n-i}} V_{n-i}(K \mid E) g(E)\, \nu_{n-i}(dE) < 0. \end{aligned}$$

Minkowski's inequality now shows that $V_n(K_\epsilon) < V_n(K)$ and so the proof is completed by putting $L = K_\epsilon$. □

COROLLARY 3.3. *If there is a centrally symmetric body, with a rotational axis of symmetry which is not in* $\mathcal{K}(n-i)$, *then the generalized Shephard problem has a negative answer.*

Proof. First, we note that the classes $\mathcal{K}(j)$, for $j = 1, \ldots, n-1$, are closed. For example, this is a consequence of Theorem 5.1 of Goodey and Weil [1991] or see Zhang [1995] for a more general result. If there were a body K satisfying the hypotheses of the Corollary, it could be approximated, in the Hausdorff metric, by centrally symmetric bodies all with the same axis of rotational symmetry as K and whose boundaries are C^∞ with strictly positive curvature at all points. It follows that at least one of these bodies is not in $\mathcal{K}(n-i)$ and so Theorem 3.2 gives the required result. □

Our next objective is to show that the double cone D defined by

$$(x_1^2 + x_2^2 + \cdots + x_{n-1}^2)^{\frac{1}{2}} + |x_n| \le 1$$

is not in any of the classes $\mathcal{K}(j)$ for $1 \le j < n-1$. In order to find the ith surface area measure of D, we compute the $(n-1)$-st surface area measure of D_ϵ, the outer parallel body of D at distance $\epsilon > 0$. To this end, we consider the following partition of the unit sphere

$$S^{n-1} = S_1 \cup S_2 \cup S_3,$$

where

$$\begin{aligned} S_1 &= \left\{ u \in S^{n-1} \colon\ |\langle u, e_n \rangle| > \frac{1}{\sqrt{2}} \right\}, \\ S_2 &= \left\{ u \in S^{n-1} \colon\ |\langle u, e_n \rangle| = \frac{1}{\sqrt{2}} \right\}, \\ S_3 &= \left\{ u \in S^{n-1} \colon\ |\langle u, e_n \rangle| < \frac{1}{\sqrt{2}} \right\}. \end{aligned}$$

The surface area measure of D_ϵ is a sum $\mu_1 + \mu_2 + \mu_3$, where μ_i is supported by S_i.

For $u \in S^{n-1}$, let $u = (u_1 \cos\theta, \sin\theta)$, $u_1 \in S^{n-2}$. If λ_{n-1} denotes the surface measure of S^{n-1}, then

$$\mu_1 = \epsilon^{n-1}\lambda_{n-1}|_{S_1}. \tag{3.5}$$

If $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$, the principle curvature of radii of D_ϵ are

$$r_1(\theta) = \epsilon,\ r_2(\theta) = r_3(\theta) = \cdots = r_{n-1}(\theta) = \frac{1 + \epsilon\cos\theta}{\cos\theta}.$$

Therefore, the curvature function of D_ϵ is

$$r_1(\theta)r_2(\theta)\cdots r_{n-1}(\theta) = \epsilon\left(\frac{1}{\cos\theta} + \epsilon\right)^{n-2}.$$

It follows that, if $f \in C(S^{n-1})$, we have

$$\begin{aligned} \int_{S^{n-1}} f(u)\,\mu_3(du) &= \epsilon\int_{S^{n-2}}\int_{-\pi/4}^{\pi/4} f((u_1\cos\theta, \sin\theta))\left(\frac{1}{\cos\theta} + \epsilon\right)^{n-2} \\ &\qquad \cos^{n-2}\theta\, d\theta\lambda_{n-2}(du_1) \\ &= \epsilon\int_{S^{n-2}}\int_{-\pi/4}^{\pi/4} f((u_1\cos\theta, \sin\theta)) \\ &\qquad (1 + \epsilon\cos\theta)^{n-2}\, d\theta\lambda_{n-2}(du_1). \end{aligned} \tag{3.6}$$

To find μ_2, we will calculate the partial surface area $A(\epsilon)$ of D_ϵ corresponding to S_2, that is, $A(\epsilon) = S_{n-1}(D_\epsilon, S_2)$. Consideration of the line

$$x_1 + x_n = 1 + \epsilon\sqrt{2}$$

helps us see that

$$\begin{aligned} A(\epsilon) &= 2\sqrt{2}(n-1)\kappa_{n-1}(1 + \epsilon\sqrt{2})^{n-2}\int_{\frac{\epsilon}{\sqrt{2}}}^{1+\frac{\epsilon}{\sqrt{2}}}\left(1 - \frac{x}{1 + \epsilon\sqrt{2}}\right)^{n-2} dx \\ &= 2\kappa_{n-1}\sqrt{2}\left[\left(1 + \frac{\epsilon}{\sqrt{2}}\right)^{n-1} - \left(\frac{\epsilon}{\sqrt{2}}\right)^{n-1}\right]. \end{aligned}$$

Therefore, for an $SO(n-1)$ invariant continuous function g on S^{n-1}, we have

$$\int_{S^{n-1}} g(u)\,\mu_2(du) = A(\epsilon)g\left(\frac{\pi}{4}\right). \tag{3.7}$$

If $K_1, \ldots, K_{n-1}$ are convex bodies in $\mathbb{E}^n$ and $f \in C(S^{n-1})$ we will write

$$V(K_1, \ldots, K_{n-1}, f) = \frac{1}{n} \int_{S^{n-1}} f(u)\, S(K_1, \ldots, K_{n-1}, du).$$

If $V_f(\epsilon) = V(D_\epsilon[n-1], f)$ then

$$\begin{aligned} V_f(\epsilon) &= V(D[n-1], f) + (n-1)V(D[n-2], B_n, f)\epsilon + \cdots \\ &+ \binom{n-1}{i-1} V(D[n-i], B_n[i-1], f)\epsilon^{i-1} + \cdots + V(B_n[n-1], g). \end{aligned}$$

So the $(i-1)$-st derivative of $V_f(\epsilon)$ at $\epsilon = 0$ is given by

$$\begin{aligned} V_f^{(i-1)}(0) &= \frac{(n-1)!}{(n-i)!} V(D[n-i], B_n[i-1], f) \\ &= \frac{(n-1)!}{n(n-i)!} \int_{S^{n-1}} f(u)\, S_{n-i}(D, du). \end{aligned} \tag{3.8}$$

We can now use (3.5), (3.6) and (3.7) to show that, if g is an even function in $C(S^{n-1})$ which has the same axis of rotational symmetry as D, then

$$\begin{aligned} V_g(\epsilon) &= \frac{1}{n} \int_{S^{n-1}} g(u)(\mu_1 + \mu_2 + \mu_3)(du) \\ &= \frac{\epsilon^{n-1}}{n} \int_{S_1} g(u)\lambda_{n-1}(du) + \frac{1}{n} A(\epsilon) g\left(\frac{\pi}{4}\right) \\ &+ \frac{2(n-1)\kappa_{n-1}}{n} \int_0^{\frac{\pi}{4}} g(\theta)\epsilon(1 + \epsilon\cos\theta)^{n-2}\, d\theta. \end{aligned}$$

Consequently we have, for $1 < i < n$,

$$\begin{aligned} V_g^{(i-1)}(0) &= \frac{2\kappa_{n-1}}{2^{(i-2)/2} n} \frac{(n-1)!}{(n-i)!} g\left(\frac{\pi}{4}\right) \\ &+ \frac{2\kappa_{n-1}}{n} \frac{(n-1)!}{(n-i)!} (i-1) \int_0^{\frac{\pi}{4}} g(\theta) \cos^{i-2}\theta\, d\theta \\ &= \frac{2\kappa_{n-1}}{n} \frac{(n-1)!}{(n-i)!} \left[\frac{1}{(\sqrt{2})^{i-2}} g\left(\frac{\pi}{4}\right) + (i-1) \int_0^{\frac{\pi}{4}} g(\theta) \cos^{i-2}\theta\, d\theta \right]. \end{aligned} \tag{3.9}$$

Comparing (3.8) and (3.9), we obtain for $1 < i < n$,

$$\begin{aligned} &\int_{S^{n-1}} g(u)\, S_{n-i}(D, du) \\ &= 2\kappa_{n-1} \left[\frac{1}{(\sqrt{2})^{i-2}} g\left(\frac{\pi}{4}\right) + (i-1) \int_0^{\frac{\pi}{4}} g(\theta) \cos^{i-2}\theta\, d\theta \right]. \end{aligned} \tag{3.10}$$

LEMMA 3.4. *Let K be a convex body in $\mathbb{E}^n$ which is centrally symmetric. If $1 \le i \le n-1$, $K \in \mathcal{K}(n-i)$ with $\dim K \ge n-i+1$ and $g \in C(\mathcal{L}_1^n)$, then*

$$R_{1,i}^n g \ge 0 \Longrightarrow \int_{S^{n-1}} g(u)\, S_{n-i}(K, du) \ge 0. \tag{3.11}$$

Moreover, in the case $i = n-1$, (3.11) implies that K is a zonoid, that is, $K \in \mathcal{K}(1)$.

Proof. It is shown in Goodey and Weil [1991] that $S_{n-i}(K, \cdot)$ is the Radon transform $R_{i,1}^n$ of a positive measure μ on $\mathcal{L}_i^n$. Consequently

$$\int_{S^{d-1}} g(u)\, S_{n-i}(K, du) = \int_{\mathcal{L}_i^n} (R_{1,i}^n g)(X)\, \mu(dX),$$

which gives (3.11).

For the case $i = n-1$, we recall from Goodey and Weil [1992] that, if K is centrally symmetric with generating distribution T_K (see Weil [1976]) then

$$S_1(K, \cdot) = 2\kappa_{n-1} R_{n-1,1}^n T_K^{\perp}.$$

Now let $f \in C^\infty(\mathcal{L}_1^n)$ with $f \ge 0$. Then we can choose $g \in C^\infty(\mathcal{L}_1^n)$ with $f = R_{n-1,1}^n g^\perp$. It follows from (3.11) that

$$\begin{aligned} T_K(f) &= T_K(R_{n-1,1}^n g^\perp) = (R_{1,n-1}^n T_K)(g^\perp) = (R_{n-1,1}^n T_K^\perp)(g) \\ &= \frac{1}{2\kappa_{n-1}} \int_{S^{n-1}} g(u)\, S_1(K, du) \ge 0. \end{aligned}$$

Consequently the distribution T_K is a positive measure and therefore K is a zonoid, as required. □

THEOREM 3.5. *The double cone D is not contained in any of the classes $\mathcal{K}(j)$ for $1 \le j < n-1$.*

Proof. We choose $g \in C(\mathcal{L}_1^n)$ to have the same axis of rotational symmetry as D. It is shown in Zhang [1994b and 1996] that the inequality $R_{1,i}^n g \ge 0$ is equivalent to

$$\int_0^\phi g(\theta) \left(1 - \frac{\sin^2\theta}{\sin^2\phi}\right)^{\frac{i-3}{2}} \cos\theta\, d\theta \ge 0, \quad 0 \le \phi \le \frac{\pi}{2}. \tag{3.12}$$

Using (3.10), we see that the inequality

$$\int_{S^{n-1}} g(u)\, S_{n-i}(D, du) \ge 0$$

is equivalent to

$$\frac{1}{(\sqrt{2})^{i-2}} g\left(\frac{\pi}{4}\right) + (i-1)\int_0^{\frac{\pi}{4}} g(\theta)\cos^{i-2}\theta\, d\theta \geq 0. \tag{3.13}$$

We note that (3.12) and (3.13) are both dependent only on i and not on n. If one chooses $n = i+1$, then $\mathcal{K}(n-i)$ is the class of zonoids. But the double cone is not a zonoid (see Petty [1967] or Gardner [1995]). So it follows from Lemma 3.4 that there exists $g \in C(\mathcal{L}_1^n)$ so that (3.12) holds but (3.13) is reversed. Therefore, the double cone D is not contained in any class $\mathcal{K}(j)$, $1 < j < n$. □

THEOREM 3.6. *The generalized Shephard problem has a negative answer.*

4. Other results. If K is a convex body in $\mathbb{E}^n$ and $1 \leq i \leq n-1$, the function $V_i(K \mid E)$, defined for all $E \in \mathcal{L}_i^n$ is called the ith projection function of K and the function $V_i(K \mid F)$ defined for all $F \in \mathcal{L}_{n-1}^n$ is called the ith brightness function of K. It follows immediately from the Cauchy-Kubota formulas (2.3) that, if the ith projection function of K is constant, then so is its ith brightness function. Firey [1970] investigated whether the reverse implication is true and gave an affirmative answer, in the case of convex bodies K having an axis of rotational symmetry. We can now use our earlier observations to provide an extension of Firey's result.

THEOREM 4.1. *Let K and L be convex bodies in $\mathbb{E}^n$ whose ith projection functions are in $\mathcal{R}_i^n$ where $1 < i < n-1$. If $V_i(K \mid u^\perp) = V_i(L \mid u^\perp)$ for all $u \in S^{n-1}$, then $V_i(K \mid E) = V_i(L \mid E)$ for all $E \in \mathcal{L}_i^n$.*

Proof. We note that the result is an immediate consequence of the Cauchy-Kubota formulas and Lemma 2.2. □

Of course, the ith projection function of a rotationally symmetric convex body is itself rotationally symmetric and therefore a member of the space $\mathcal{R}_i^n$. Consequently Firey's result follows from Theorem 4.1 if we let L be an appropriate multiple of B_n.

For our next application, we will use ideas of Weil [1976] to obtain a distributional representation of the projection functions of a class of bodies that includes the bodies of revolution. We note, in particular, that this class includes bodies which are not necessarily centrally symmetric.

THEOREM 4.2. *Let K be a convex body in $\mathbb{E}^n$ whose ith projection function is in $\mathcal{R}_i^n$ where $1 < i < n-1$. Then, there is a distribution $T_{K,i}$ on $\mathcal{L}_i^n$ such that $V_i(K \mid \cdot) = T_i^n T_{K,i}$.*

Proof. If $f \in L^2(\mathcal{L}_i^n)$ we denote by f_1 the Hilbert space projection of f onto $\mathcal{R}_i^n$. We note that if $f \in C^\infty(\mathcal{L}_i^n)$ then $f_1 \in \mathcal{R}_i^n \cap C^\infty(\mathcal{L}_i^n)$ and so we may use

Lemma 2.3 to see that there is a unique $g \in \mathcal{R}_i^n \cap C^\infty(\mathcal{L}_i^n)$ with $f_1 = T_i^n g$. We will define the distribution $T_{K,i}$ by

$$T_{K,i}f = \int_{\mathcal{L}_i^n} V_i(K \mid E) g(E) \nu_i^n(dE) \qquad \text{for } f \in C^\infty(\mathcal{L}_i^n).$$

The harmonic expansion of f corresponding to (2.9) will be denoted by

$$\sum_{(2m_1,\ldots,2m_i,0,\ldots,0)} f_{(2m_1,\ldots,2m_i,0,\ldots,0)}.$$

We note that g has harmonic expansion

$$\sum_{m=0}^{\infty} \left(\lambda_{(2m,0\ldots,0)}^{n,i}\right)^{-1} f_{(2m,0\ldots,0)}.$$

It follows from (2.11) that, if p is an integer greater than $(n+2)/4$, there are constants c_1 and c_2 such that

$$\|g\|^2 \leq c_1 \sum_{m=0}^{\infty} m^{n+2} \|f_{(2m,0,\ldots,0)}\|^2 \leq c_2 \|\Delta^p f_1\|^2 \leq c_2 \|\Delta^p f\|^2.$$

Consequently

$$|T_{K,i}f| \leq c \|V_i(K \mid \cdot)\| \|\Delta^p f\|,$$

for some constant c, and so $T_{K,i}$ is, indeed, a distribution.

To see the connection between this distribution and the projection function, we choose $f \in C^\infty(\mathcal{L}_i^n)$. Then, since $(T_i^n f)_1 = T_i^n f_1$, we have

$$(T_i^n T_{K,i})(f) = T_{K,i}(T_i^n f) = \int_{\mathcal{L}_i^n} V_i(K \mid E) f_1(E)\, \nu_i^n(dE) = \int_{\mathcal{L}_i^n} V_i(K \mid E) f(E)\, \nu_i^n(dE),$$

as required. □

If K is a convex body in $\mathbb{E}^n$, we denote by K^* its reflection in the origin. Then, for $i = 1, \ldots, n-1$, we can think of $S_i(K, \cdot) + S_i(K^*, \cdot)$ as a measure on $\mathcal{L}_{n-1}^n$ (or $\mathcal{L}_1^n$) with total measure $\binom{n}{i}^{-1} \kappa_{n-i} V_i(K)$. With this normalization, (2.3) can be reformulated to give

$$R_{i,n-1}^n V_i(K \mid \cdot) = \frac{n\kappa_i}{2\kappa_{n-1}} T_{n-1}^n \left(S_i(K, \cdot) + S_i(K^*, \cdot)\right).$$

If the ith projection function of K is in $\mathcal{R}_i^n$, we can combine this with (2.7) and Theorem 4.2 to get

$$\begin{aligned} T_{n-1}^n\big(S_i(K,\cdot)+S_i(K^*,\cdot)\big) &= \frac{2\kappa_{n-1}}{n\kappa_i} R_{i,n-1}^n T_i^n T_{K,i} \\ &= \binom{n}{i}^{-1} \kappa_{n-i} T_{n-1}^n R_{i,n-1}^n T_{K,i}. \end{aligned}$$

The injectivity of T_{n-1}^n gives

$$S_i(K,\cdot)+S_i(K^*,\cdot) = \binom{n}{i}^{-1} \kappa_{n-i} R_{i,n-1}^n T_{K,i}.$$

In particular, if K is also centrally symmetric, then

$$S_i(K,\cdot) = \frac{1}{2}\binom{n}{i}^{-1} \kappa_{n-i} R_{i,n-1}^n T_{K,i}.$$

It follows that, if K is centrally symmetric and rotationally symmetric and if $S_i(K,\cdot)$ is the Radon transform $R_{n-i,1}^n$ of a measure μ on $\mathcal{L}_{n-i}^n$, then

$$\mu = \frac{1}{2}\binom{n}{i}^{-1} \kappa_{n-i} T_{K,i}^{\perp}.$$

This can be used to deduce, for example, that all the classes $\mathcal{P}_S(j,k)$, for $k = 1,\dots,n-j$, discussed in Goodey and Weil [1991] contain the same rotationally symmetric members and that these are just the rotationally symmetric members of $\mathcal{K}(j)$ of dimension at least $j+1$.

THEOREM 4.3. *Let f be a rotationally symmetric member of $C^\infty(\mathcal{L}_i^n)$ for some $1 \le i \le n-1$. Then there are rotationally symmetric and centrally symmetric convex bodies K and L with $f = V_i(K \mid \cdot) - V_i(L \mid \cdot)$.*

Proof. We note that $f \in \mathcal{R}_i^n \cap C^\infty(\mathcal{L}_i^n)$ and so, by Lemma 2.3, there is a $g \in \mathcal{R}_i^n \cap C^\infty(\mathcal{L}_i^n)$ with $f = T_i^n g$. Now $R_{i,n-1}^n g \in C^\infty(\mathcal{L}_{n-1}^n)$. It follows from Corollary 6.9 of Zhang [1994a] that there are rotationally symmetric and centrally symmetric bodies K, L such that

$$R_{i,n-1}^n g = S_i(K,\cdot) - S_i(L,\cdot) = \frac{1}{2}\binom{n}{i}^{-1} \kappa_{n-i}(R_{i,n-1}^n T_{K,i} - R_{i,n-1}^n T_{L,i}).$$

The injectivity of $R^n_{i,n-1}$ on $\mathcal{R}^n_i$ shows that

$$f = T^n_i g = \frac{1}{2}\binom{n}{i}^{-1} \kappa_{n-i} T^n_i (T_{K,i} - T_{L,i}) = \frac{1}{2}\binom{n}{i}^{-1} \kappa_{n-i}(V_i(K \mid \cdot) - V_i(L \mid \cdot)),$$

which gives the required result. □

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